2. *Besov Spaces and Bessel Potential Spaces on Certain Groups*

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§0. **Introduction.** The theory of Besov spaces ($\cong$ generalized Lipschitz spaces) and Sobolev spaces ($\cong$ Bessel potential spaces) on $\boldsymbol{R}^n$ is well developed (but not complete) [2] [4] [8] [10] [11] [14] [15], etc. The idea to generalize the theory to the local field setting was suggested by M. H. Taibleson, see [11, p. 259], also [12, p. 180, Theorem (2.2)]. The Lipschitz spaces on (compact) Vilenkin groups, which are more general than the additive groups of local fields, have been studied for some time. But those earlier works seem to be limited in scope and not oriented towards a systematic treatment of the subject like the euclidean case. Here, the author would like to report his development of the theory of Besov spaces on (non-compact) Vilenkin groups. Most of these results are in [6] and the details together with further results will be published elsewhere. The recent paper [7] also deals with the same subject in the same spirit but covers the topics different from the ones in the present notes.

In what follows, let G be a locally compact, non-compact abelian group and assume that there exists a sequence $(G_n)_{n=-\infty}^{+\infty}$ of compact open subgroups of G such that $G_{n+1}\subsetneq G_n$, $\bigcap_{n=-\infty}^{+\infty} G_n=\{0\}$, $\bigcup_{n=-\infty}^{+\infty} G_n=G$, and $\sup_n [G_n : G_{n+1}]<+\infty$. Let μ be the Haar measure on G such that $\mu(G_0)=1$. Then let $m_n=\mu(G_n)^{-1}$. Also let $S(G)$ and $S^*(G)$ be the spaces of testing functions and distributions, respectively, on G. Finally, let $\Delta_n(x)=m_n$ $(x\in G_n)$, $\Delta_n(x)=0$ $(x\notin G_n)$.

§1. **Besov spaces and Bessel potential spaces.** Let α, p, and s be such that $-\infty<\alpha<+\infty$, $0<p\leq+\infty$, and $0<s\leq+\infty$, respectively. Then the (inhomogeneous) Besov spaces $B(\alpha, p, s)$ are the (quasi-) Banach spaces of all distributions $f\in S^*(G)$ such that

$$\|f\|_{B(\alpha,p,s)}:=\left\{\|f*\Delta_0\|_p^s+\sum_{j=1}^{+\infty}(m_j^\alpha\|f*(\Delta_j-\Delta_{j-1})\|_p)^s\right\}^{1/s}<+\infty,$$

with the usual modification when $s=+\infty$.

Similarly, for α and p with $-\infty<\alpha<+\infty$ and $0<p<+\infty$, respectively, the Bessel potential spaces $h_\alpha^p(G)$ are the (quasi-) Banach spaces of all distributions $f\in S^*(G)$ such that

$$\|f\|_{h_\alpha^p}:=\left\|\left\{|f*\Delta_0|^2+\sum_{j=1}^{+\infty}(m_j^\alpha|f*(\Delta_j-\Delta_{j-1})|)^2\right\}^{1/2}\right\|_p<+\infty.$$

Notice that $h_0^p(G)=h^p(G)$ is the analogue of Goldberg's local Hardy spaces $h^p(\boldsymbol{R}^n)$ ([3]) and that if $1<p<+\infty$, $h_\alpha^p(G)=L_\alpha^p(G)$ with equivalent norms, where $L^p_\alpha(G)$ are the Bessel potential spaces introduced by M. H. Taibleson [12, p. 136]. Then the basic properties are summarized as:

Proposition 1.1. (1) *Let* $-\infty<\alpha<+\infty$. *Then the space* $S(G)$ *is dense in* $B(\alpha, p, s)$ $(0<p, s<+\infty)$, *and in* $h^p_\alpha(G)$ $(0<p<+\infty)$.

(2) *Let* $-\infty<\alpha, \beta<+\infty$. *Then the space* $B(\alpha+\beta, p, s)$ *is isometric with the space* $B(\alpha, p, s)$ *for* $0<p, s\leq+\infty$. *The same is true between* $h^p_{\alpha+\beta}(G)$ *and* $h^p_\alpha(G)$ $(0<p<+\infty)$.

Remark 1.2. Let $-\infty<\alpha<+\infty$ and $0<p, s\leq+\infty$. Then if $p=+\infty$ or $s=+\infty$, $S(G)$ is not dense in $B(\alpha, p, s)$.

§2. Duality and interpolation. In order to state the theorem on duality, for each positive number t, define t^* by $t^*=t/(t-1)$ $(1<t<+\infty)$, $t^*=+\infty$ $(0<t\leq 1)$. Then we have:

Theorem 2.1. (1) *If* $-\infty<\alpha<+\infty$ *and* $0<s<+\infty$, *then*

$$[B(\alpha, p, s)]^*\cong\begin{cases}B(-\alpha, p^*, s^*) & (1<p<+\infty),\\ B(\alpha-1+1/p, \infty, s^*) & (0<p\leq 1).\end{cases}$$

(2) *If* $-\infty<\alpha<+\infty$, *then*

$$[h^p_\alpha(G)]^*\cong\begin{cases}[L^p_\alpha(G)]^*\cong L^{p^*}_{-\alpha}(G)\cong h^{p^*}_{-\alpha}(G) & (1<p<+\infty),\\ B(-\alpha-1+1/p, \infty, \infty) & (0<p\leq 1).\end{cases}$$

J. Peetre's real [resp. A. P. Calderón's complex] interpolation space is denoted by $(X_0, X_1)_{\theta,q}$ $(0<\theta<1, 0<q<+\infty)$ [resp. $[X_0, X_1]_\theta$ $(0<\theta<1)$] for the given interpolation couple (X_0, X_1) of quasi-Banach [resp. Banach] spaces X_0 and X_1. Then our results are as follows:

Theorem 2.2. *If* $-\infty<\alpha_0, \alpha_1<+\infty$, $0<p_0, p_1, s_0, s_1\leq+\infty$, *and* $0<\theta<1$, *define* α', p', s' *by*

$$\alpha'=(1-\theta)\alpha_0+\theta\alpha_1, \quad \frac{1}{p'}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1}, \quad \frac{1}{s'}=\frac{1-\theta}{s_0}+\frac{\theta}{s_1}.$$

(1) *If* $-\infty<\alpha_0, \alpha_1<+\infty, \alpha_0\neq\alpha_1, 0<p, s_0, s_1, s\leq+\infty$, *and* $0<\theta<1$, *then*

$$(B(\alpha_0, p, s_0), B(\alpha_1, p, s_1))_{\theta,s}=B(\alpha', p, s).$$

(2) *If* $-\infty<\alpha<+\infty$, $0<p<+\infty$, $0<s_0, s_1\leq+\infty$, $s_0\neq s_1$, *and* $0<\theta<1$, *then*

$$(B(\alpha, p, s_0), B(\alpha, p, s_1))_{\theta',s'}=B(\alpha, p, s').$$

(3) *If* $-\infty<\alpha_0, \alpha_1<+\infty$, $0<p_0, p_1, s_0, s_1<+\infty$, $p_0\neq p_1$, $0<\theta<1$, *and* $p'=s'$, *then*

$$(B(\alpha_0, p_0, s_0), B(\alpha_1, p_1, s_1))_{\theta,p'}=B(\alpha', p', s').$$

(4) *If* $-\infty<\alpha_0, \alpha_1<+\infty$, $1<p_0, p_1<+\infty$, $1\leq s_0, s_1<+\infty$, *and* $0<\theta<1$, *then*

$$[B(\alpha_0, p_0, s_0), B(\alpha_1, p_1, s_1)]_\theta=B(\alpha', p', s').$$

(5) *If* $-\infty<\alpha_0, \alpha_1<+\infty$, $\alpha_0\neq\alpha_1$, $0<p<+\infty$, $0<s\leq+\infty$, *and* $0<\theta<1$, *then*

$$(h^p_{\alpha_0}(G), h^p_{\alpha_1}(G))_{\theta,s}=B(\alpha, p, s).$$

§3. Embedding theorems. One way to know the relationships among the spaces defined in §1 is through embedding theorems. We have the following results.

Theorem 3.1. (1) *If* $-\infty<\alpha<\beta<+\infty$, $0<p\leq+\infty$, *and* $0<s\leq t\leq+\infty$, *then*

$$S(G)\hookrightarrow B(\beta, p, \infty)\hookrightarrow B(\alpha, p, s)\hookrightarrow B(\alpha, p, t)\hookrightarrow S^*(G).$$

(2) *If* $-\infty<\alpha<+\infty$, $0<p<q\leq+\infty$, *and* $0<s\leq t\leq+\infty$, *then*

$$B(\alpha, p, s)\hookrightarrow B(\alpha-1/p+1/q, q, t).$$

Remark 3.2. The second and third inclusions of Theorem 3.1 (1) are the best possible in the following sense:

(1) *If* $-\infty<\alpha<\beta<+\infty$ and $0<p, s, t\leq+\infty$, then $B(\alpha, p, s)\not\hookrightarrow B(\beta, p, t)$.

(2) *If* $-\infty<\alpha<+\infty$, $0<p\leq+\infty$, and $0<t<s<+\infty$, then $B(\alpha, p, s)\not\hookrightarrow B(\alpha, p, t)$.

Theorem 3.3. (1) *If* $1\leq p\leq q\leq+\infty$ *and* $\alpha-1/p>\beta-1/q$, *then* $L^p_\alpha(G)\hookrightarrow L^q_\beta(G)$.

(2) *If* $1<p<q<+\infty$ *and* $\alpha-1/p=\beta-1/q$, *then* $L^p_\alpha(G)\hookrightarrow L^q_\beta(G)$.

(3) *If* $0<p<q<+\infty$ *and* $\alpha-1/p=\beta-1/q$, *then* $h^p_\alpha(G)\hookrightarrow h^q_\beta(G)$.

Remark 3.4. (1) *If* $p=1$ or $q=+\infty$ in Theorem 3.3 (2), then the embedding fails.

(2) *If* $p=1$ and $\beta=0$ in Theorem 3.3 (2), a weaker result holds: $L^1_\alpha(G)\hookrightarrow$ weak $L^q(G)$.

Theorem 3.5. (1) *If* $1\leq p\leq+\infty$ *and* $-\infty<\alpha<+\infty$, *then*

$$B(\alpha, p, 1)\hookrightarrow L^p_\alpha(G)\hookrightarrow B(\alpha, p, \infty).$$

(2) *If* $1<p<+\infty$ *and* $-\infty<\alpha<+\infty$, *then*

$$B(\alpha, p, 2\wedge p)\hookrightarrow L^p_\alpha(G)\hookrightarrow B(\alpha, p, 2\vee p).$$

(3) *If* $0<p<+\infty$ *and* $-\infty<\alpha<+\infty$, *then*

$$B(\alpha, p, 2\wedge p)\hookrightarrow h^p_\alpha(G)\hookrightarrow B(\alpha, p, 2\vee p).$$

(4) *If* $1<p<q\leq+\infty$, *then*

$$L^p(G)\hookrightarrow B(1/q-1/p, q, p).$$

(5) *If* $1\leq p<q<+\infty$, *then*

$$B(1/p-1/q, p, q)\hookrightarrow L^q(G).$$

Remark 3.6. In view of the third inclusion of Theorem 3.1 (1), Theorem 3.5 (2) is sharp in the following sense.

(1) If $1<p<+\infty$, $s>p\wedge 2$, and $-\infty<\alpha<+\infty$, then $B(\alpha, p, s)\not\hookrightarrow L^p_\alpha(G)$.

(2) If $1<p<+\infty$, $s<p\vee 2$, and $-\infty<\alpha<+\infty$, then $L^p_\alpha(G)\not\hookrightarrow B(\alpha, p, s)$.

§4. Translation-invariant operators. For the quasi-Banach subspaces X and Y of $S^*(G)$ such that $S(G)$ is dense in each of them and such that they are closed under translations, let $Cv(X, Y)$ be the quasi-Banach space of all $k\in S^*(G)$ such that $\|k*\varphi\|_Y\leq C\|\varphi\|_X$ for all $\varphi\in S(G)$. [Below if $S(G)$ is not dense in the given space X or Y, we interpret X or Y as the closure of $S(G)$ with respect to the quasi-norm of X or Y.] We are interested in characterizing $Cv(X, Y)$ for various choices of the pairs (X, Y). We have the following results.

Theorem 4.1. (1) *If* $-\infty<\alpha<\beta<+\infty$, $1\leq q\leq+\infty$, *and* $1\leq s\leq t\leq+\infty$, *then*

$$Cv(B(\alpha,1,s),\ B(\beta,q,t))=B(\beta-\alpha,q,\infty).$$

(2) *If* $-\infty<\alpha$, $\beta<+\infty$, $1\leq p$, $s<+\infty$, $1<t\leq+\infty$, *and* $s\leq t$, *then*

$$Cv(B(\alpha,p,s),\ B(\beta,\infty,t))=B(\beta-\alpha,p^*,\infty).$$

(3) *If* $-\infty<\beta<+\infty$, $1\leq q\leq+\infty$, *and* $0<p<1<s<+\infty$ *or* $p=1, 2\leq s\leq+\infty$, *then*

$$Cv(h^p(G),\ B(\beta,q,s))=B(\beta-1+1/p,q,\infty).$$

(4) *If* $2\leq p<+\infty$, *then*

$$Cv(h^1(G),\ L^p(G))=B(0,p,\infty).$$

(5) *If* $0<p<1\leq q<+\infty$ *or* $p=1<2\leq q<+\infty$, *then*

$$Cv(h^p(G),L^q(G))=B(-1+1/p,q,\infty).$$

We remark that we also have several partial results as in [5] [9]. When $X=Y=B(\alpha,p,s)$, we also have the following:

Theorem 4.2. (1) *For each* p *with* $0<p\leq+\infty$, $Cv(B(\alpha,p,s),\ B(\alpha,p,s))$ *is independent of* α *or* $s(-\infty<\alpha<+\infty,\ 0<s\leq+\infty)$. *Thus, we write* $Cv_p=Cv(B(\alpha,p,s),\ B(\alpha,p,s))$.

(2) $Cv_p=B(-1+1/p,p,\infty)$ $\qquad (0<p\leq1)$,
$Cv_1=Cv_\infty=B(0,1,\infty)$,
$Cv_2=Cv(L^2(G),L^2(G))$.

(3) $Cv_p=Cv_{p^*}$ $\qquad (1\leq p\leq+\infty)$.

(4) $Cv(L^p(G),L^p(G))\subsetneqq Cv_p$, $\qquad (1<p<+\infty,\ p\neq2)$.

(5) $Cv(L^p(G),L^p(G))$
$=\{k\in Cv_p$: *supp* $\hat{k}$ *is compact*$\}$ $\qquad (1\leq p\leq+\infty)$.

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