Besov spaces in theory of approximation.

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Summary. - *In this paper we apply the theory of interpolation spaces to different parts of Approximation theory. We study the rate of convergence of summation processes of Fourier series and Fourier integrals. The main body of the paper is devoted to a study of the rate of convergence of solutions of difference schemes for parabolic initialvalue problems with constant coefficients and to related problems.*

0. - Introduction.

The theory of interpolation spaces has applications to many branches of Analysis, in particular to Approximation Theory; see in particular LÖFSTRÖM [16], of which paper the present one is to some extent a sequel. Our main intention is to apply the techniques of interpolation spaces (actually disguised as BESOV spaces), to some problems related to finite difference approximations for partial differential equations. In doing so we extend and complement previous work by PEETRE-THOMÉE [24], HEDSTRÖM [9], WIDLUND [35].

We shall work within a rather general framework, which we shall now explain. We shall consider two families $E_h(t)$ and $E(t)$ ($0 < h < 1$, t in a given set I_h, depending on h) of translation invariant, bounded linear operators on $L_p = L_p[R^d]$. We consider the generalized LIPSCHITZ space Λ_σ of all functions $f \in L_p$, such that

$$\sup_{0<h<1} \sup_{t \in I_h} h^{-\sigma} \| E_h(t)f - E(t)f \|_{L_p}$$

is finite. Of particular interest to us is the study of Λ_s, when s is the least upper bound for set of numbers σ with $\Lambda_\sigma \neq 0$.

We say that $E_h(t)$ is a saturated approximation of $E(t)$ with order s if

$$\Lambda_s \neq 0, \tag{0.1}$$

$$\lim_{h \to 0} \sup_{t \in I_h} h^{-s} \| E_h(t)f - E(t)f \|_{L_p} = 0 \quad \text{implies} \quad f = 0, \tag{0.2}$$

In particular $\Lambda_{s+\varepsilon} = 0$, $\varepsilon > 0$ (c.f. FAVARD [8], BUTZER [5], BUTZER-BERENS [7]). The approximation is said to be non-saturated if either (0.1) or (0.2) is violated. We give simple criterions for saturation, as well as for non-saturation.

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Our main object is however to compare the LIPSCHITZ spaces Λ_s with the BESOV spaces introduced by BESOV [4] and PEETRE [19], [22], [23]. These spaces are usually defined by means of certain moduli of continuity, but here we shall use the alternative definition of PEETRE [19], [23]. We shall develop criterions for continuous inclusions of the types $B \subseteq \Lambda_s$ and $\Lambda_s \subseteq B$, where B stands for suitable BESOV spaces.

The general results described above will be applied to two special cases. The first one is characterized by the facts, that I_h consists of one single point t_0, that $E(t_0)$ is the identity operator and that $E_h = E_h(t_0)$ is given by the « singular integral »

$$E_h f(x) = h^{-d} \int_{R^d} k(h^{-1}y) f(x - y) dy.$$

Here we get generalizations of precious works by BUTZER [6], LÖFSTRÖM [16], NESSEL [18], SHAPIRO [26]. Our results are particularly useful in the non saturated cases.

We shall also discuss similar problems on the d-dimensional torus T^d. We consider the RIESZ means operator

$$E_h f(x) = \sum_{\xi \in Z^d} (1 - hP(\xi))_+^\alpha f^\wedge(\xi) \exp(i < x, \xi >),$$

where Z is the set of integers, $f^\wedge(\xi)$ the FOURIER coefficients of f and $P(\xi)$ any homogenous function of positive order, which is positive and infinitely differentiable outside $\xi = 0$. Here

$$(1 - u)_+ = \begin{cases} 1 - u, & 0 < u < 1, \\ 0, & u > 1. \end{cases}$$

We shall prove that

$$\| E_h f(x) \|_{L_p[T^d]} \leq C \| f \|_{L_p[T^d]} \tag{0 3}$$

if $\alpha > (d - 1) \left| \frac{1}{p} - \frac{1}{2} \right|$ and discuss the rate of convergence of E_h to the identity operator. In doing so we generalize the work of STEIN [29] who proved (0.3) for $P(\xi) = |\xi|^2$, and LÖFSTRÖM [16], where the corresponding problems on R^d were considered.

We shall also discuss similar questions for operators E_h defined on $L_p[T^d]$ by

$$E_h f(x) = (2\pi)^{-d} \sum_{\xi \in Z^d} \varphi(hP(\xi)) f^\wedge(\xi) \exp(i < x, \xi >),$$

where φ is a given function, satisfying certain regularity conditions. Here we get generalizations of some of te results of WAINGER [33].

In our second application, $E(t)f$ is the solution of the initial value problem

$$\begin{cases} \frac{\partial u}{\partial t} + P(D)u = 0, & x \in R^d, \quad 0 < t < \infty, \\ u = f, & x \in R^d, \quad t = 0. \end{cases} \tag{0.4}$$

Here $P(D)$ is an elliptic differential operator with constant coefficients. We assume that (0.4) is correctly posed in L_p, so that the operators $E(t)$, $0 < t < \infty$ form a strongly continuous semi-group of operators on $L_p[R^d]$. For simplicity we also suppose that the polynomial $P(\xi)$ is homogenous of order m and positive for $\xi \neq 0$. The operator $E(t)$ is formally defined by

$$E(t) = \exp(-tP(D)).$$

Now consider the discrete initial value problem

$$\begin{cases} u_h(x,\ t+k) = \sum_\alpha e_\alpha u(x+\alpha h,\ t), & k = \lambda h^m, \quad T = Nk, \\ u_h(x,\ 0) = f(x). \end{cases} \tag{0.5}$$

Here $x \in h \cdot Z^d$, Z being the set of integers, and $0 < t < \infty$; λ is a constant. The solution at time k can be written formally

$$E_h(k) = \exp(-kP_h(D)), \tag{0.6}$$

where $P_h(\xi)$ is a suitable function. We assume that the difference operator (0.6) is stable (see RICHTMYER–MORTON [25]). We can then define $E_h(t)$, $0 < t < \infty$ by

$$E_h(t) = \exp(-tP_h(D)), \qquad t \in I_h,$$

where $I_h = \{t;\ t = Nk,\ N = 0,\ 1,\ 2,\ \dots\}$. Now our general theory gives rather exact information about the rate of convergence of the solution of the discrete initial value problem (0.5) to the solution of (0.4). In fact we give necessary and sufficient conditions for

$$\sup_{0<t=Nk<\infty} t^{-\theta} \| E_h(t)f - E(t)f \|_{L_p} = 0(h^s), \qquad h \to 0, \qquad 0 \leq \theta,$$

thus generalizing results by HEDSTROM [9], PEETRE–THOMÉE [24], WIDLUND [35] and others. In the case $p = 2$ we also discuss briefly the situation when $\lambda = kh^{-m}$ is a non-konstant function of h.

We shall also treat the case when $P(\xi)$ is any function, infinitely differentiable and positive for $\xi \neq 0$ and homogeneous of order $m > 0$, and $E(t)$ is defined by

$$E(t) = \varphi(tP(D)), \tag{0.6}$$

with a function φ satisfying certain regularity assumptions. We suppose that

$$E_h(t) = \varphi(tP_h(D)). \tag{0.7}$$

where $P_h(D)$ is an operator of a special type which approximates $P(D)$. The stability of this operator, i.e. the inequality

$$\| E_h(t)f \|_{L_p} \leq C \| f \|_{L_p}, \tag{0.8}$$

now represents a new difficulty. On the basis of the theory of interpolation spaces we shall gfve a simple, but rather restrictive condition for (0.8). By the same technique we can prove a stability theorem for difference operators, which is a d-dimensional analogue of a stability thoorem of STRANG [30].

With the aid of the stability theorem for the operator (0,7) we can study the rate of convergence of $\varphi(tP_h(D))$ to $\varphi(tP(D))$. Our results are analogous to those mentioned above. In particular we get a result for the rate of convergence of the resolvent of $P_h(D)$.

The paper consists of three parts. The first one consists of two introductory sections. In section 1 we list some basic facts about FOURIER multipliers. Our main source here is HÖRMANDER [12]. Following PEETRE [19], [22], [23] we give in section 2 the necessary preliminaries on BESOV spaces, and some auxiliary spaces.

The second part of the paper (sections 3 — 5) contains the general theory. After having introduced some notations and definitions in section 3, we give in section 4 criterions for saturation and non-saturation. In section 5 we give three simple theorems concerning the comparision of the LIPSCHITZ spaces and the BESOV spaces and certain related spaces.

Our paper concludes with five sections, containing the applications described above. In section 6 we consider singular integrals, while section 7 contains the results on the rate of convergence of the difference scheme (0.5). Finally, in section 8 we study the stability of the operator $\varphi(tP_h(D))$ and in section 9 we study its rate of convergence. In section 10 we consider the RIESZ mean operator and other summation methods for FOURIER series.

1. - Preliminaries on Fourier multipliers.

Let $L_p = L_p(R^d)$, $1 \leq p < \infty$ denote the BANACH space of all complex-valued locally integrable functions f on d-dimensional Euclidian space R^d for which the norm

$$\| f \|_{L_p} = \left(\int_{R^d} | f(x) |^p dx \right)^{1/p} < \infty.$$

The space of all continuous functions f on R^d such that

$$f(x) \to 0, \qquad | x | \to \infty,$$

is denoted by L_∞. It is a BANACH space with norm

$$\| f \|_{L_\infty} = \sup_{x \in R^d} | f(x) | .$$

A function $f \in L_p$ can be considered as a tempered distribution, and we can therefore speak about its FOURIER transform $\mathfrak{F} f$, which is a tempered distribution. Formally we can define $\mathfrak{F} f$ by

$$(\mathfrak{F} f)(\xi) = f^\wedge(\xi) = \int_{R^d} \exp(-i < x, \xi >) f(x) dx.$$

Here $< x, \xi > = x_1\xi_1 + ... + x_d\xi_d$ if $x = (x_0, ..., x_d)$ and $\xi = (\xi_1, ..., \xi_d)$. If g is any tempered distribution we define

$$g^\vee(x) = (2\pi)^{-d} \int_{R^d} \exp(i < x, \xi >) g(\xi) d\xi.$$

Then holds the inversion formula

$$f(x) = (\mathfrak{F}^{-1} g)(x) = g^\vee(x), \qquad g = f^\wedge.$$

In the sequel we shall let x, y, ... denote the variables of the function, while ξ, η, ... will denote the variables of the FOURIER transform.

If f is any tempered distribution we let supp f be the support of f. The support of the FOURIER transform $f^\wedge$ will be called the spectrum of f and is denoted spec f, i.e.

$$\operatorname{spec} f = \operatorname{supp} f^\wedge.$$

The space of all infinitely differentiable functions with compact supports is denoted by $\mathfrak{D}$.

We are now ready for the definition of the concept of FOURIER multipliers.

DEFINITION 1.1. - A tempered distribution $\varphi(\xi)$ on R^d is called a FOURIER multiplier on L_p if

$$\| \varphi^v * g \|_{L_p} \leq C \| g \|_{L_p}, \tag{1.1}$$

for all functions $g \in \mathfrak{D}$. The infinum of the constant C in (1.1) is deroted

$$\| \varphi(\xi) \|_{M_p}, \tag{1.2}$$

and the set of all such distributions φ is denoted M_p.

Suppose $\varphi \in M_p$. Then the linear operator

$$\mathfrak{D} \ni g \rightarrow \varphi^v * g \in L_p$$

in bounded with norm $\| \varphi(\xi) \|_{M_p}$. By closing this operator in L_p we get a new linear operator T with the same norm. We shall write

$$Tg = \varphi^v * g,$$

for g in L_p.

We now collect a few basic facts about the spaces M_p. For the details in the proof we refer the reader to HÖRMANDER [12].

By means of PARSEVALS relation it follows that

$$M_2 = L_\infty \tag{1.3}$$

It is also easy to show that

$$M_p = M_{p'}, \qquad (p')^{-1} + p^{-1} = 1. \tag{1.4}$$

The relations (1.3) and (1.4) hold with equal norms. From RIESZ-THORIN'S convexity theorem it follows that

$$M_p \subseteq M_q, \qquad 1 \leq p \leq q \leq 2, \tag{1.5}$$

in the sense that

$$\| \varphi \|_{M_q} \leq \| \varphi \|_{M_p}, \qquad 1 \leq p \leq q \leq 2. \tag{1.6}$$

In particular

(1.7) $$M_p \subseteq L_\infty, \qquad 1 \leq p \leq \infty.$$

In view of (1.7) we can form the product of two FOURIER multipliers φ and ψ on L_p. This product belongs to M_p and

(1.8) $$\| \varphi(\xi)\psi(\xi) \|_{M_p} \leq \| \varphi(\xi) \|_{M_p} \| \psi(\xi) \|_{M_p},$$

and since M_p is a BANACH space we see that M_p is a commutative BANACH algebra under pointwise multiplication. It is clear that M_p has a unit element (the constant function 1).

A very important fact about the M-spaces is that they are invariant for homotheties. This means that if $\varphi(\xi) \in M_p$, then $\varphi_t(\xi) = \varphi(t\xi) \in M_p$ and

(1.9) $$\| \varphi(t\xi) \|_{M_p} = \| \varphi(\xi) \|_{M_p}.$$

This follows easily from

(1.10) $$\varphi_t^v(x) = t^{-d}\varphi^v(t^{-1}x).$$

Le $\mathfrak{M}$ denote the space of bounded measure and $\mathcal{F}\mathfrak{M}$ the space of their FOURIER transforms. It follows directly from the definition that $M_\infty = \mathcal{F}\mathfrak{M}$. Thus, in view of (1.4) and (1.7)

(1.11) $$\mathcal{F}\mathfrak{M} = M_1 = M_\infty;$$

(1.12) $$\mathcal{F}\mathfrak{M} \subseteq M_p, \qquad 1 \leq p \leq \infty.$$

Since $L_1 \subseteq \mathfrak{M}$ we conclude

(1.13) $$\mathcal{F}L_1 \subseteq M_p, \qquad 1 \leq p \leq \infty.$$

This holds in the metrical sense, i.e.

(1.14) $$\| \varphi(\xi) \|_{M_p} \leq \| \varphi^v \|_{L_1} = \| \varphi(\xi) \|_{\mathcal{F}L_1}, \qquad 1 \leq p \leq \infty.$$

In order to show that a given function φ belongs to the space $\mathcal{F}L_1$ we shall sometimes use the following simple lemma.

LEMMA 1.1. - Suppose that

(1.15) $$\int_{R^d} | \varphi(\xi) |^2 d\xi + \int_{R^d} | D^L\varphi(\xi) |^2 d\xi \leq A^2,$$

for some $L > d/2$. Then $\varphi \in \mathcal{F}L_1$ and

$$\|\varphi\|_{\mathcal{F}L_1} = \|\varphi^v\|_{L_1} \leq CA,$$

where C is a constant depending only on the dimension d.
(Here and in the sequel D^L denotes any generalized derivative of order L).

PROOF. - Suppose $\varphi \in \mathfrak{D}$. By the CAUCHY-SCHWARTZ inequality and PARSEVAL'S formula

$$\int_{R^d} |\varphi^v(x)|\, dx \leq \Big(\int_{R^d} (1 + |x|^{2L})^{-1} dx\Big)^{1/2} \Big(\int_{R^d} (1 + |x|^{2L}) |\varphi^v(x)|^2 dx\Big)^{1/2} \leq$$

$$\leq A\Big(\int_{R^d} (1 + |x|^{2L})^{-1} dx\Big)^{1/2}.$$

Since the integral on the right hand side converges if $2L > d$, the conclusion follows.

We shall often work with local FOURIER multipliers.

DEFINITION 1.2. - Let V be any (open) subset of R^d. Let two tempered distributions belong to the same equivalence class if they are equal on V. Then the space $M_p(V)$ of (local) FOURIER multipliers on V is the space of all equivalence classes of tempered distributions, which agree on V with a multiplier on L_p. For convenience we shall not distinguish between the distribution φ and the equivalence class to which it belongs. The norm on $M_p(V)$ is

$$\|\varphi(\xi)\|_{M_p(V)} = \inf_{\chi} \|\chi(\xi)\|_{M_p}, \tag{1.16}$$

where $\chi \in M_p$ and $\chi = \varphi$ on V. We denote by $\mathcal{F}L_1(V)$ the subalgebra of $M_1(V)$ consisting of all (equivalence classes of) tempered distributions, which agree on V with a function $\chi \in \mathcal{F}L_1$.

It is clear that $M_p(V)$ and $\mathcal{F}L_1(V)$ are BANACH algebras (with unit element) under pointwise multiplication. It is also quite clear how the relations (1.3)-(1.9) are inherited to the spaces $M_p(V)$. In particular, (1.9) corresponds to

$$\|\varphi(t\xi)\|_{M_p(V)} = \|\varphi(\xi)\|_{M_p(tV)}, \tag{1.17}$$

where

$$tV = \{t\xi \mid \xi \in V\}. \tag{1.18}$$

If specf. $\subseteq V$, then clearly

$$\|\varphi^v * f\|_{L_p} \leq \|\varphi\|_{M_p(V)} \|f\|_{L_p}. \tag{1.19}$$

It is also obvious that if $\operatorname{supp}\chi \subseteq V$ and $\chi \in M_p$, then $\chi(\xi)\varphi(\xi) \in M_p$ and

$$\|\chi(\xi)\varphi(\xi)\|_{M_p} \leq \|\chi(\xi)\|_{M_p} \|\varphi(\xi)\|_{M_p(V)}. \tag{1.29}$$

If W ir an open subset of V and $\varphi \in M_p(V)$, then $\varphi \in M_p(W)$ and

$$\|\varphi(\xi)\|_{M_p(W)} \leq \|\varphi(\xi)\|_{M_p(V)}. \tag{1.21}$$

The following lemma will be very useful to us. (C.f. MICHLIN'S multiplier theorem, see HÖRMANDER [12]).

LEMMA 1.2. - Let U_r denote the annulus $2^{-1}r < |\xi| < 2r$. Suppose that

$$|\xi|^J |D^J\varphi(\xi)| \leq A, \qquad 4^{-1}r < |\xi| < 4r, \qquad 0 \leq J \leq L, \tag{1.22}$$

for some $L > d/2$. Then $\varphi \in \mathcal{F}L_1(U_r)$ and consequently $\varphi \in M_p(U_r)$, $1 \leq p \leq \infty$. Moreover

$$\|\varphi(\xi)\|_{\mathcal{F}L_1(U_r)} \leq CA,$$

where C depends on the dimension d only.

PROOF. - Choose $\psi \in \mathfrak{D}$ so that $\psi(\xi) = 1$ on $2^{-1} \leq |\xi| \leq 2$. and $\psi(\xi) = 0$ outside $4^{-1} < |\xi| < 4$. Write

$$\chi(\xi) = \psi(\xi/r)\varphi(\xi).$$

Then $\chi(\xi) = \varphi(\xi)$ for $\xi \in U_r$, so we have only to prove $\chi \in \mathcal{F}L_1$ and $\|\chi^v\|_{L_1} \leq CA$.
As in the proof of lemma 1.1 we get

$$\int\limits_{|x| \geq r^{-1}} |\chi^v(x)|\, dx \leq \Big(\int\limits_{|x| \geq r^{-1}} |x|^{-2L} dx\Big)^{1/2} \Big(\int\limits_{R^d} |x|^{2L} |\chi^v(x)|^2 dx\Big)^{1/2} \leq$$

$$\leq Cr^{L-d/2} \max\Big(\int\limits_{R^d} |D^L\chi(\xi)|^2 d\xi\Big)^{1/2}$$

where the maximum is taken over all derivatives of order L. But $D^L\chi(\xi)$ is a finite sum of terms of the type $r^{-J}(D^J\psi)(\xi/r)D^{L-J}\varphi(\xi)$, and since $|D^{L-J}\varphi(\xi)| \leq CAr^{J-L}$, $4^{-1}r < |\xi| < 4r$ we conclude

$$|D^L\chi(\xi)| \leq CAr^{-L},$$

Thus

$$\int_{|x|\geq r^{-1}} |\chi^v(x)|\, dx \leq CAr^{-d/2}\Big(\int_{\operatorname{supp}\chi} d\xi\Big)^{1/2} \leq C'A.$$

We also have

$$\int_{|x|\leq r^{-1}} |\chi^v(x)|\, dx \leq Cr^{-d/2}\Big(\int_{R^d} |\chi(\xi)|^2 d\xi\Big)^{1/2} \leq CA.$$

It follows that

$$\int_{R^d} |\chi^v(x)|\, dx \leq CA,$$

and the lemma is proved.

We shall also need the following local version of a well-known theorem by WIENER.

LEMMA 1.3. - Suppose that $\varphi \in \mathfrak{F}L_1(V)$ where V is an open, bounded set and $\varphi(\xi) \neq 0$ on the closure of V. Then $\varphi(\xi)^{-1} \in \mathfrak{F}L_1(V)$, and thus $\varphi(\xi)^{-1} \in M_p(V)$, $1 \leq p \leq \infty$.

PROOF. - Let F be any continuous character on $\mathfrak{F}L_1(V)$. We shall prove $F(\varphi) \neq 0$, since then follows that φ has an inverse in $L_1(V)$. Let $f \in \mathfrak{F}L_1$ and $\bar{f}$ be the equivalence class of all functions, which agrees on V with f. Write $G(f) = F(\bar{f})$. Then G is a continuous character on $\mathfrak{F}L_1$, thus of the form $f \to f(\xi)$. Let $\chi \in \mathfrak{D}$ be identically 1 on V. Then

$$\chi(\xi)f(\xi) = G(\chi f) = F(\overline{\chi f}) = F(\bar{f}) = G(f) = f(\xi),$$

and we conclude that ξ belongs to the closure of V. Thus $F(\varphi) = \varphi(\xi) \neq 0$.

We conclude this section with

LEMMA 1.4. - Suppose that $P(\xi)$ is a homogeneous function of order $m > 0$, i.e. $P(t\xi) = t^m P(\xi)$, $0 < t < \infty$ and that $P(\xi)$ is positive and infinitely differentiable for $\xi \neq 0$. Assume that $\varphi(u)$ is an infinitely differentiable function on $0 < u < \infty$ and that

$$|\varphi(u) - \varphi(0)| \leq C_0 u^{\alpha}, \qquad 0 < u < 1, \tag{1.23}$$

$$|\varphi(u)| \leq C_0 u^{-\beta}, \qquad 1 < u < \infty, \tag{1.24}$$

$$|D^j\varphi(u)| \leq C_j \min(u^{\alpha-j}, \quad u^{-\beta-j}), \qquad 0 < u < \infty \tag{1.25}$$

where $J = 1, 2, \dots, N$, $N > \frac{d}{2}$ and $\alpha, \beta > 0$. Then

$$\varphi(P(\xi)) \in \mathcal{F}L_1 .$$

This lemma is proved in LÖFSTRÖM [15]. We give an alternative proof, based on lemma 1.2. We need here and on several occasions in section 8, the following well-known formula for differentiation of composite functions

LEMMA 1.5. - Let f be a real valued function on a domain $\Omega \subset R^d$, let g be defined on the range of f and suppose f and g are sufficiently differentiable. Then

$$D^L g(f(\xi)) = \Sigma c_{L,\alpha} (Df(\xi))^{\alpha_1} \dots (D^{\alpha_L} f(\xi))^{\alpha_L} g^{(K)}(f(\xi)).$$

Here $g^{(K)} = D^K g$, $\alpha = (\alpha_1, \dots, \alpha_L)$, $1 \leq K \leq L$ and

$$\alpha_1 + \alpha_2 + \dots + \alpha_L = K,$$

$$\alpha_1 + 2\alpha_2 + \dots + L\alpha_L = L.$$

Lemma 1.5 is easily proved by induction. We leave the details to the reader.

PROOF OF LEMMA 1.4. - We may assume without loss of generality that $\varphi(0) = 0$. Let $\Phi \in \mathfrak{D}$ satisfy

$$\operatorname{supp} \Phi = \{ \xi \mid 2^{-1} \leq |\xi| \leq 2 \}, \tag{1.26}$$

$$\sum_{-\infty}^{+\infty} \Phi(2^{-k}\xi) = \begin{cases} 1, & \xi \neq 0 \\ 0, & \xi = 0. \end{cases} \tag{1.27}$$

(For the existence of such a function see HÖRMANDER [12], c.f. lemma 2.3 below). We shall prove that

$$\sum_{-\infty}^{+\infty} \| \Phi(2^{-k}\xi)\varphi(P(\xi)) \|_{\mathcal{F}L_1} < \infty .$$

Since $\mathcal{F}L_1$ is a BANACH space this implies that

$$\sum_{-\infty}^{+\infty} \Phi(2^{-k}\xi)\varphi(P(\xi)) \in \mathcal{F}L_1 ,$$

which give the conclusion, in view of (1.27).

From the invariance under homotheties (1.9) and (1.20) we get

$$\| \Phi(2^{-k}\xi)\varphi(P(\xi)) \|_{\mathfrak{F}L_1} \leq \| \Phi \|_{\mathfrak{F}L_1} \| \varphi(P(\xi)) \|_{M_1(U_{2^k})} .$$

Since

$$| D^J P(\xi) | \leq C_J P(\xi)^{1-J/m}$$

lemma 1.5 gives

$$| D^L \varphi(P(\xi)) | \leq C \sum_{K=1}^{L} P(\xi)^{K-L/m} | \varphi^{(K)}(P(\xi)) | .$$

Thus the assumptions and the fact that

$$| \xi |^L \leq C P(\xi)^{L/m}$$

gives for $L = 0, 1, 2, \dots$

$$| \xi |^L | D^L \varphi(P(\xi)) | \leq C \min (P(\xi)^\alpha ; \; P(\xi)^{-\beta})$$

We conclude

$$\| \varphi(P(\xi)) \|_{\mathfrak{F}L_1(U_{2^k})} \leq C \min (2^{\alpha k m}, \; 2^{-\beta k m}).$$

Thus

$$\sum_{-\infty}^{+\infty} \| \Phi(2^{-k}\xi)\varphi(P(\xi)) \|_{\mathfrak{F}L_1} \leq C \sum_{-\infty}^{+\infty} \min (2^{\alpha k m} ; \; 2^{-\beta k m})$$

and since the series on the right hand side converges for α, $\beta > 0$, we get the conclusion.

REMARK 1.1. - The concept of FOURIER multipliers can be generalized to other situations than the one described above. We can for instance replace R^d by the d-dimensional torus T^d. The FOURIER transform should then be replaced by the operator

$$f \to f^\wedge(\xi) = \int_{T^d} f(x) \exp (- i < \xi, \; x >) dx, \tag{1.28}$$

where $\xi \in Z^d$, Z being the space of integers. Then the inversion formula takes the form

$$f(x) = g^\vee(x) = (2\pi)^{-d} \sum_{\xi \in Z^d} g(\xi) \exp \; i < \xi, \; x >, \tag{1.29}$$

where $g(\xi) = f^\wedge(\xi)$.

The space m_p of FOURIER multipliers on $L_p(T^d)$ is defined by the inequality

$$\| \varphi^v * g \|_{L_p(T^d)} \leq \| \varphi(\xi) \|_{m_p} \| g \|_{L_p(T^d)},$$

where

$$(\varphi^v * g)\hat{}(\xi) = \varphi(\xi) g^\wedge(\xi), \qquad \xi \in Z^d,$$

i.e.

$$\varphi^v * g(x) = \int_{T^d} \varphi^v(x - y) g(y) dy.$$

A general approach is possible by means of the concept of direct integrals of HILBERT spaces and the spectral theorem for commating self-adjoint operators $A_1, \ldots, A_d$ on a HILBERT space H. (See GÅRDING'S article in BERS-JOHN-SCHECHTER [3]). In fact we can find a direct integral $L_2(R^d, \sigma, \beta)$, (σ being a positive measure and β a dimension function), and a unitary operator

$$\mathcal{F} : H \to L_2(R^d, \sigma, \beta),$$

such that

$$(\mathcal{F} A_k f)(\xi) = \xi_k (\mathcal{F} f)(\xi).$$

If H itself is a direct integral; $H = L_2(\Omega, \mu, \alpha)$ (Ω is a manifold with density μ and α a dimension), we write

$$L_p = L_p(\Omega, \mu, \alpha)$$

and define the space of FOURIER multipliers on L_p by means of the inequality

$$\| \varphi(A) f \|_{L_p} \leq \| \varphi(\xi) \|_{M_p} \| f \|_{L_p},$$

where

$$(\mathcal{F} \varphi(A) f)(\xi) = \varphi(\xi) (\mathcal{F} f)(\xi).$$

Here $\varphi(\xi)$ is a square matrix with $\beta(\xi)$ rows and columns.

For

$$A_k = i^{-1} \frac{\partial}{\partial x_k}, \qquad \mu = dx, \qquad \alpha(x) = 1, \qquad \Omega = R^d$$

we get back the situation described in this section. For $\Omega = T^d$ the diagonalizing operator is given by (1.28) and (1.29). As a final example take $d = 1$ and let A be a self-adjoint elliptic differential operator on $L_2(\Omega, \mu, \alpha)$, $(\alpha = 1)$. Assume for simplicity that A has pure point spectrum consisting of points denoted ξ. Let the multiplicity of ξ be $\beta(\xi) = 1$. The corresponding direct integral is $L_2(R, \sigma, 1)$ where

$$\sigma = \sum_{\xi} \delta_\xi,$$

with $\delta_\xi(f) = f(\xi)$.

The FOURIER transform is

$$(\mathcal{F}f)(\xi) = f^\wedge(\xi) = (f, \Phi(\xi)),$$

where (,) is the inner product on $L_2(\Omega, \mu, 1)$, and $\Phi(\xi)$ is the eigenfunction corresponding to the eigenvalue ξ;

$$A\Phi(\xi) = \xi\Phi(\xi).$$

The inversion formula takes the form

$$f = \Sigma f^\wedge(\xi)\Phi(\xi),$$

whice is the eigenfunction expansions of f.

In all these cases we have also analogues of lemmata 1.1., 1.2. and 1.4. (See SPANNE [26], c.f. also section 10 below).

2. - Preliminaries on Besov spaces.

Throughout the rest of the paper Φ shall denote an infinitely differentiable function with the following properties; Φ is positive on the annulus $2^{-1} < |\xi| < 2$ and vanishes outside. Moreover

$$\sum_{-\infty}^{\infty} \Phi(2^{-k}\xi) = 1, \qquad \xi \neq 0.$$

(cf. (1.27). We shall write

$$\Phi_k(\xi) = \Phi(2^{-k}\xi), \qquad k = 0, \pm 1, \pm 2, \ldots,$$

$$\Psi(\xi) = 1 - \sum_{k=1}^{\infty} \Phi_k(\xi).$$

By means of the functions Ψ, Φ_1, Φ_2, ..., we define the BESOV space $B_p^{m,q}$ $(-\infty < m < \infty,\ 1 \leq q \leq \infty,\ 1 \leq p \leq \infty)$ as the BANACH space corresponding to the norm

$$\|f\|_{B_p^{m,q}} = \left(\sum_{k=0}^{\infty} (2^{mk}\|f_k\|_{L_p})^q\right)^{1/q}, \tag{2.1}$$

with

$$f_k = \Phi_k^v * f, \qquad k = 1, 2, \ldots, \qquad f_0 = \Psi^v * f. \tag{2.2}$$

(See PEETRE [22], [23]). This definition is rather implicit. However if $m > 0$ it is possible to give a more explicit alternative definition in terms of the modulus of continuity

$$\omega_p(t, f) = \omega_p^1(t, f) = \sup_{|h|<t} \| T_h f - f \|_{L_p},$$

where

$$T_h f(x) = f(x + h).$$

In fact, for $m = J + \alpha$, $0 < \alpha < 1$, (J integer ≥ 0), the BESOV space $B_p^{m, q}$ can be defined by the (equivalent) norm

$$\sum_{K=0}^{J} \| D^K f \|_{L_p} + \left(\int_0^\infty \left(\frac{\omega_p(t, D^J f)}{t^\alpha} \right)^q \frac{dt}{t} \right)^{1/q}. \tag{2.3}$$

For $\alpha = 1$ one has to modify (2.3). Let us write

$$\omega_p^2(t, f) = \sup_{0<|h|<t} \| T_h f - 2f + T_{-h} f \|_{L_p}.$$

Then $B_p^{J+1, q}$ is defined by the norm

$$\sum_{K=0}^{J} \| D^K f \|_{L_p} + \left(\int_0^\infty \left(\frac{\omega_p^2(t, D^J f)}{t} \right) q \frac{dt}{t} \right)^{1/q}. \tag{2.4}$$

Thus $B_p^{J+1, q}$ is defined by means of a ZYGMUND condition, while $B_p^{J+\alpha, q}$, $0 < \alpha < 1$ is defined by a LIPSCHITZ condition.

We shall denote by H_p^{J+1} the LIPSCHITZ space, which corresponds to $B_p^{J+1, \infty}$, This means that H_p^L is the BANACH space defined by the norm

$$\sum_{K=0}^{L-1} \| D^K f \|_{L_p} + \sup_{0<t<\infty} \frac{\omega_p(t, D^{L-1} f)}{t}. \tag{2.5}$$

For $p \neq 2$, H_p^L is not a BESOV space, but one can show

$$B_2^{L, 2} = H_2^L.$$

However, for $1 \leq p \leq \infty$

$$B_p^{L, 1} \subset H_p^L \subset B_p^{L, \infty}, \tag{2.6}$$

with strict inclusions.

Occasionally we shall also work with a space H_p^{*m} closely connected with H_p^L. This space is defined in the following way.

Let $G_m(t)$ be the operator on L_p defined by

$$(\mathcal{F} G_m(t) f)(\xi) = \exp(-t |\xi|^m) f^\wedge(\xi), \qquad t > 0. \tag{2.7}$$

Clearly $G_m(t)$ is a semi-gruop of operators. By the invariance for homotheties (1.9)

$$\| G_m(t)f \|_{L_p} \leq \| \exp(-t |\xi|^m) \|_{M_p} \| f \|_{L_p} \leq \| \exp(-|\xi|^m) \|_{\mathfrak{F}L_1} \| f \|_{L_p}.$$

Clearly $\varphi(u) = \exp(-u)$ satisfies the assumptions of lemma 1.4. Thus $G_m(t)$ is a strongly continuous semi-group of operators on L_p. The infinitesimal generator of $G_m(t)$ is the operator $-|D|^m$, defined by

$$(\mathfrak{F} |D|^m f)(\xi) = |\xi|^m f^\wedge(\xi). \tag{2.8}$$

(C.f. BUTZER-BERENS [7]).

The space H_p^{*m}, $(m > 0)$ is now defined as the BANACH space corresponding to the norm

$$\| f \|_{H_p^{*m}} = \| f \|_{L_p} + \sup_{0<t<1} t^{-1} \| G_m(t)f - f \|_{L_p}. \tag{2.9}$$

For $1 < p < \infty$, H_p^{*m} is the domain of the operator $|D|^m$ in L_p. For $p = 1$, $p = \infty$ one can characterize H_p^{*m} as an interpolation space between, the domain of $|D|^m$ and L_p, (see BUTZER-BERENS [7], LÖFSTRÖM [14], [15] and PEETRE [17]). We have

$$H_p^{*L} = H_p^L, \qquad L \text{ integer}, \qquad 1 < p < \infty.$$

Moreover

$$B_p^{m,1} \subset H_p^{*m} \subset B_p^{m,\infty}. \tag{2.10}$$

It can also be proved that

$$\| f \|_{L_p} + \sup_{t>0} \| |D|^m G_m(t)f \|_{L_p} \tag{2.11}$$

is an equivalent norm on H_p^{*m}, (see BUTZER-BERENS [7], LÖFSTRÖM [16] and PEETRE [19]).

For the proof of the alternative definitions (2.3) and (2.4) of the BESOV spaces and of the inequality (2.6) we refer the reader to PEETRE [19], [22] and [23], where he also can find a more detailed study of the BESOV spaces. See also LIONS-LIZORKIN-NIKOLSKIJ [14] and BESOV [4].

Throughout the paper we shall work mostly with definition (2.1). From this definition it follows immediately that

$$B_p^{m'',q} \subseteq B_p^{m',q}, \qquad m' \leq m'', \tag{2.12}$$

$$B_p^{m,q'} \subseteq B_p^{m,q''}, \qquad q' \leq q'', \tag{2.13}$$

with continuous injections. We also have the following simple interpolation theorem.

LEMMA 2.1. - Suppose that T is a continuous linear operator from L_p into L_p and from B_p^{m,q_0} into L_p, wilh norms M_0 and M_1, respectively. Then T maps $B_p^{\theta m,q}$, $(1 \leq q \leq \infty)$ continuously into L_p, for $0 < \theta < 1$, with norm

$$M_\theta \leq C_m(\theta(1-\theta))^{-1+1/q} M_0^{1-\theta} M_1^\theta, \qquad 0<\theta<1. \tag{2.14}$$

REMARK 2.1. - By means of the results of HOLMSTEDT [11] one can prove

$$M_\theta \leq C_m \theta^{-1+1/q}(1-\theta)^{\min(1/q-1/q_0;\,0)} M_0^{1-\theta} M_1^\theta. \tag{2.15}$$

REMARK 2.2. - The conclusion (2.14) also holds if T maps H_p^{*m} into L_p with norm M_1. This follows from (2.10). It is also easy to prove that if T maps $B_p^{m_j,q_j}$ into L_p with norm M_j $(j=0,\ 1)$ then T maps $B_p^{m,q}$ into L_p with norm less than $C^{m_0,m_1}(\theta(1-\theta))^{-1+1/q} M_0^{1-\theta} M_1^\theta$, if $m = (1-\theta)m_0 + \theta m_1$, $0<\theta<1$.

Lemma 2.1. is a consequence of a general interpolation theorem (see PEETRE [19], [20], [22]), but for the convenience of the reader we give a direct proof (c.f. PEETRE-THOMÉE [24]).

PROOF. - Write $f_k = \Phi_k^v * f$, $f_0 = \Psi^v * f$. Then the assumptions give

$$\| Tf_k \|_{L_p} \leq \min(M_0 \| f_k \|_{L_p}, \qquad M_1 \| f_k \|_{B_p^{s,q_0}}).$$

Since however f_k has its spectrum in the annulus $2^{k-1} \leq |\xi| \leq 2^{k+1}$, $(k \geq 1)$ we have

$$f_{kj} = \Phi_j^v * f_k = 0$$

except for $j = k-1,\ k,\ k+1$. In the same way

$$f_{0,j} = \Phi_j^v * f_0 = 0, \qquad j \neq 0, \qquad j \neq 1,$$

$$f_{k,0} = \Psi^v * f_k = 0, \qquad k \neq 0, \qquad k \neq 1.$$

Thus easily

$$\| f_k \|_{B_p^{s,q_0}} = (\sum_{j=0}^{\infty} (2^{jm} \| f_{kj} \|_{L_p})^{q_0})^{1/q_0} \leq C_m 2^{km} \| f_k \|_{L_p},$$

since $\| \Phi_k(\xi) \|_{M_p} = \| \Phi(\xi) \|_{M_p}$, by (1.9) and $\Psi \in M_p$. Consequentiy

$$\| Tf_k \|_{L_p} \leq C_m \min(M_0,\ 2^{km} M_1) \| f_k \|_{L_p}.$$

Since we can suppose that $f \in \mathfrak{D}$, we have

$$f = \sum_{k=0}^{\infty} f_k$$

(with convergence in L_p). Thus

$$\| Tf \|_{L_p} \leq \sum_{k=0}^{\infty} \| Tf_k \|_{L_p} \leq C_m \sum_{k=0}^{\infty} \min(M_0, 2^{km} M_1) \| f_k \|_{L_p}.$$

From HÖLDER's inequality it follows

$$\| Tf \|_{L_p} \leq C_m \Big(\sum_{k=0}^{\infty} (2^{-k\theta m} \min(M_0, 2^{km} M_1))^{q'} \Big)^{1/q'} \| f \|_{B_p^{\theta m, q}} \leq$$

$$\leq C \Big(\int_0^{\infty} (x^{-\theta m} \min(M_0, x^m M_1))^{q'} \frac{dx}{x} \Big)^{1/q'} \| f \|_{B_p^{\theta m, q}}.$$

Here $(q')^{-1} + q^{-1} = 1$. Evaluating the integral we get

$$\| Tf \|_L \leq C'_m (\theta(1-\theta))^{-1+1/q} M_0^{1-\theta} M_1^{\theta} \| f \|_{B_p^{\theta m, q}},$$

which is the desired inequality.
For technical reasons we shall work certain modified BESOV spaces $\dot{B}_p^{m,q}$, $(-\infty < m < \infty,\ 1 \leq p \leq \infty)$, which are defined by the semi-norm

$$\| f \|_{\dot{B}_p^{m,q}} = \Big(\sum_{-\infty}^{+\infty} (2^{mk} \| \Phi_k^{\nu} * f \|_{L_p})^q \Big)^{1/q}, \tag{2.16}$$

(See PEETRE [22], [23]). It is easy to see that all polynomials belong to $\dot{B}_p^{m,q}$ and have semi-norm 0. From the proof above it is clear that lemma 2.1 remains true if we replace all BESOV spaces by modified BESOV spaces. Note however that the inequalities (2.12) and (2.13) do not hold for the modified spaces.

The connection between the modified BESOV spaces $\dot{B}_p^{m,q}$ and the BESOV spaces $B_p^{m,q}$ is given by

LEMMA 2.2. - For $0 < m < \infty$ we have $L_p \cap \dot{B}_p^{m,q} = B_p^{m,q}$ in the sense that

$$f \to \| f \|_{L_p} + \| f \|_{\dot{B}_p^{m,q}},$$

is an equivalent norm on $B_p^{m,q}$.

The proof is trivial and will be omitted.

We shall conclude this section with a brief discussion of definition (2.1). It is natural to ask what special role is played by the number 2, which appears in the conditions on Φ. Could it be replaced by any number $\gamma > 1$? MORE generally, is it possible to replace the annulus $2^{-1} < |\xi| < 2$, where $\Phi(\xi) > 0$, with a more general type of annulus?

Let $\Phi^* \in \mathfrak{D}$ have the following properties:

$$\text{i)} \quad 0 \leq \Phi^*(\xi), \qquad \xi \in R^d,$$

$$\text{(2.17)} \qquad \text{ii)} \quad 0 \notin \operatorname{supp} \Phi^*,$$

iii) there exists a number $\gamma > 1$, such that for every $\xi \in R^d$, $\xi \neq 0$, we can find an integer k, such that $\Phi^*(\gamma^{-k}\xi) > 0$.

From ii) we easily get that the series

$$\sum_{j=-\infty}^{+\infty} \Phi^*(\gamma^{-j}\xi),$$

is finite for every $\xi \neq 0$. If $\xi \neq 0$ at least one term in the series is positive, so we have

$$\sum_{j=-\infty}^{+\infty} \Phi^*(\gamma^{-j}\xi) > 0, \qquad \xi \neq 0.$$

Write

$$\Phi(\xi) = \Phi^*(\xi)\Big(\sum_{j=-\infty}^{+\infty} \Phi^*(\gamma^{-j}\xi)\Big)^{-1}.$$

Then clearly $\Phi \in \mathfrak{D}$, $\operatorname{supp} \Phi = \operatorname{supp} \Phi^*$ and

$$\sum_{k=-\infty}^{+\infty} \Phi(\gamma^{-k}\xi) = 1, \qquad \xi \neq 0.$$

If we write

$$\Phi_k(\xi) = \Phi(\gamma^{-k}\xi), \qquad k = 0, \pm 1, \dots,$$

$$\Psi(\xi) = 1 - \sum_{k=1}^{\infty} \Phi_k(\xi),$$

we can define the Bosov spaces $B_p^{m,q}(\gamma, \Phi^*)$ and $\dot{B}_p^{m,q}(\gamma, \Phi^*)$ by means of (2.1) and (2.16) with 2^{mk} replaced by γ^{mk}. The annulus $2^{k-1} < |\xi| < 2^{k+1}$ is replaced replaced by $\gamma^k \cdot \operatorname{supp} \Phi^* = \{\gamma^k\xi \mid \xi \in \operatorname{supp} \Phi^*\}$.

We shall prove

$$\text{(2.18)} \qquad B_p^{m,q}(\gamma, \Phi^*) = B_p^{m,q},$$

$$\text{(2.19)} \qquad \dot{B}_p^{m,q}(\gamma, \Phi^*) = B_p^{m,q},$$

with eqivalent norms. In view of lemma 2.2 (which also holds for the modified spaces $B_p^{m,q}(\gamma, \Phi^*)$ and $\dot{B}_p^{m,q}(\gamma, \Phi^*)$), it is enough to prove (2.19), which follows from

LEMMA 2.3 - Let Φ^* and Ψ^* be two functions, satisfying (2 17) with number γ and δ. Then

$$\|f\|_{\dot{B}_p^{m,q}(\gamma, \Phi^*)} \leq C \|f\|_{\dot{B}_p^{m,q}(\delta, \Psi^*)}, \tag{2.20}$$

where C is independent p and q, and depends continuously on m.

PROOF. - Let Φ_k and Ψ_k, $k = 0, \pm 1, \ldots$, be the functions that appear in the definitions of $\dot{B}_p^{m,q}(\gamma, \Phi^*)$ and $\dot{B}_p^{m,q}(\delta, \Psi^*)$, respectively. Then

$$\Phi_k^v * f = \sum_j \Phi_k^v * \Psi_j^v * f, \qquad f \in \mathfrak{D}.$$

In the sum only a finite number of terms can be non-vanishing, namely the terms that correspond to such indices j, for which

$$\gamma^k \cdot \operatorname{supp} \Phi^* \cap \delta^j \cdot \operatorname{supp} \Psi^* \neq 0. \tag{2.21}$$

Suppose

$$\operatorname{supp} \Phi^* \cup \operatorname{supp} \Psi^* \subset \{ \xi \mid R^{-1} < |\xi| < R \}.$$

Then it is easy to see that (2.21) implies

$$R^{-2}\gamma^k < \delta^j < R^2\gamma^k. \tag{2.22}$$

Consequently

$$\gamma^{km} \| \Phi_k^v * f \|_{L_p} \leq R^{2m} \Sigma \delta^{jm} \| \Phi_k^v * \Psi_j^v * f \|_{L_p} \leq$$
$$\leq C R^{2m} \Sigma \delta^{jm} \| \Psi_j^v * f \|_{L_p} \leq C R^{2m} (\Sigma 1)^{1/q'} (\Sigma (\delta^{jm} \| \Psi_j^v * f \|_{L_p})^q)^{1/q}$$

(summation of j such that (2.22) holds). Since the first sum in the last expression is bounded by a number N we get

$$\|f\|_{\dot{B}_p^{m,q}(\gamma, \Phi^*)} \leq C N^{1/q'} R^{2m} \Big(\sum_{-\infty}^{\infty} \sum_{R^{-2}\gamma^k < \delta^j < R^2\gamma^k} (\delta^{jm} \| \Psi_j^v * f \|_{L_p})^q \Big)^{1/q} \leq$$
$$\leq C N R^{2m} \|f\|_{\dot{B}_p^{m,q}(\delta, \Psi^*)}.$$

REMARK 2.3. - The definition of BESOV spaces can be generalized to the more general situation described is remark 1.1. In fact, suppose

$$\sum_{-\infty}^{+\infty} \Phi_k(\xi) = 1, \qquad \xi \neq 0,$$

and

(2.23) $$\|\Phi_k(\xi)\|_{M_p} \leq C, \qquad -\infty < k < \infty.$$

Then we define $\dot{B}_p^{s,q}[\Omega, A]$ by means of the norm

$$\|f\|_{\dot{B}_p^{s,q}[\Omega, A]} = \left(\sum_{-\infty}^{+\infty} (2^{-ks} \|\Phi_k(A)f\|_{L_p})^q\right)^{1/q}.$$

(Here $A = (A_1, \ldots, A_d)$).

For instance we can take $\Omega = T^d$ and $A_k = i^{-1}\dfrac{\partial}{\partial x_k}$. Then one can prove (2.23) and thus we can form the space $\dot{B}_p^{s,q}[T^d] = \dot{B}_p^{s,q}[T^d, A]$. (See SPANNE [28] and section 8 and 10 below).

Most of the general theory developed in section 3–5 carries over to the general situation indicated here.

3. – The Lipschitz' spaces.

The object of the rest of the paper is to study the following situation. Let $E_h(t)$ and $E(t)$ be two families of bounded operators on the BANACH space $L = L_p$, given by

(3.1) $$(\mathcal{F}E_h(t)f)(\xi) = e_h(t, \xi)f^\wedge(\xi), \qquad e_h(t, \xi) \in M_p,$$

(3.2) $$(\mathcal{F}E(t)f)(\xi) = e(t, \xi)f^\wedge(\xi), \qquad e(t, \xi) \in M_p.$$

Here $0 < h < 1$ and t belongs to a given set I_h, which depends on h. We shall assume that $E_h(t)$ and $E(t)$ are uniformly bounded:

(3.3) $$\|E_h(t)f\|_L \leq C\|f\|_L, \qquad 0 < h < 1, \qquad t \in I_h,$$

(3.4) $$\|E(t)f\|_L \leq C\|f\|_L, \qquad 0 < h < 1, \qquad t \in I_h,$$

This is equivalent to

(3.5) $$\|e_h(t, \xi)\|_{M_p} \leq C, \qquad 0 < h < 1, \qquad t \in I_h,$$

(3.6) $$\|e(t, \xi)\|_{M_p} \leq C, \qquad 0 < h < 1, \qquad t \in I_h.$$

However the preliminary discussions in this section carries over to the more general case, when L is any BANACH space and $E_h(t)$ and $E(t)$ satisfies (3.3) and (3.4) ($\| \ \|_L$ being the norm on L).

The (generalized) LIPSCHITZ' space Λ_σ, $(0 \leq \sigma < \infty)$ is defined as follows. It is the space of all $f \in L$, such that

$$\sup_{0<h<1} \sup_{t \in I_h} h^{-\sigma} \| (E_h(t) - E(t))f \|_L < \infty .$$

It is BANACH space with norm

$$\| f \|_{\Lambda_\sigma} = \| f \| + \sup_{0<h<1} \sup_{t \in I_h} h^{-\sigma} \| (E_h(t) - E(t))f \|_L .$$

EXAMPLE 3.1. - Consider $L = L_p$ and let I_h consist of one single point t_0 and assume

$$E(t_0)f = f.$$

Suppose that the dimension $d = 1$ and

$$E_h(t_0)f(x) = T_h f(x) = f(x + h).$$

Then

$$\| f \|_{\Lambda_\sigma} = \| f \|_{L_p} + \sup_{0<h<1} h^{-\sigma} \| T_h f - f \|_{L_p},$$

so that (in the notations of section 2)

$$\Lambda_\sigma = B_p^{\sigma, \infty}, \qquad 0 < \sigma < 1,$$

$$\Lambda_1 = H_p^1 .$$

Thus Λ_σ is a LIPSCHITZ' space in the classical sense.

We now return to the general case. It is clear that

$$\Lambda_0 = L,$$

(with equivalent norms). Obviously Λ_σ is continuously embedded in L and more generally we have

$$\Lambda_{\sigma''} \subseteq \Lambda_{\sigma'}, \qquad 0 \leq \sigma' \leq \sigma'', \tag{3.7}$$

with continuous injections.

We also introduce the space N_σ of all elements $f \in L$ for which

$$\lim_{h \to 0} \sup_{t \in Ih} h^{-\sigma} \| (E_h(t) - E(t))f \|_L = 0.$$

Then we have the following inclusions

$$N_\sigma \subseteq \Lambda_\sigma, \qquad 0 \leq \sigma < \infty, \tag{3.8}$$

$$\Lambda_{\sigma''} \subseteq N_{\sigma'}, \qquad 0 \leq \sigma' < \sigma'' < \infty, \tag{3.9}$$

$$N_{\sigma''} \subseteq N_{\sigma'}, \qquad 0 \leq \sigma' \leq \sigma'' < \infty. \tag{3.10}$$

From (3.7) we see that the set $\{\sigma;\ \Lambda_\sigma \neq 0\}$ must be an interval of the form $0 \leq \sigma < \sigma_0$ or $0 \leq \sigma \leq \sigma_0$, where

$$\sigma_0 = \sup\{\sigma;\ \Lambda_\sigma \neq 0\}.$$

In the same way (3.10) shows that $\{\sigma;\ N_\sigma = 0\}$, is either empty for all values of σ or of the form $\sigma_1 < \sigma < \infty$ or $\sigma_1 \leq \sigma < \infty$, where

$$\sigma_1 = \inf\{\sigma;\ N_\sigma = 0\}.$$

We put $\sigma_1 = \infty$ if $N_\sigma \neq 0$ for all σ. Then easily

$$\sigma_0 = \sigma_1. \tag{3.11}$$

In fact, suppose $\sigma_1 < \sigma_0$. Then we can find σ' and σ'', such that $\sigma_1 < \sigma' < \sigma'' < \sigma_0$. Therefore $N_{\sigma'} = 0$, $\Lambda_{\sigma''} \neq 0$, which contradicts (3.9). Consequently we must have $\sigma_0 \leq \sigma_1$. But if $\sigma_0 < \sigma_1$, we take σ such that $\sigma_0 < \sigma < \sigma_1$. Then $\Lambda_\sigma = 0$, so that by (3.8), $N_\sigma = 0$. This contradicts $\sigma < \sigma_1$.

We now reformulate (3.11) in the following theorem.

THEOREM 3.1. - Let $s = \sigma_0$ be finite. Then there are three possibilities:

$$\Lambda_s = 0, \qquad N_s = 0, \tag{3.12}$$

$$\Lambda_s \neq 0, \qquad N_s = 0, \tag{3.13}$$

$$\Lambda_s \neq 0, \qquad N_s \neq 0. \tag{3.14}$$

If (3.13) holds for some s, then s is the largest number for which $\Lambda_s \neq 0$.

In case (3.13) we shall say that $E_h(t)$ is a saturated approximation of $E(t)$ and s is called the order of the saturation. (See FAVARD [8], BUTZER [5], LÖFSTRÖM [16]). We shall refer to the cases (3.12) and (3.14) as the non-saturated cases. Note that the order of saturation s is to a certain extent arbitrary, since if we replace $E_h(t)$ by $E_{h^\gamma}(t)$ then s is replaced by $s \cdot \gamma$.

4. - A criterion for saturation.

In this section we shall consider $L = L_p$ and two families $E_h(t)$, $0 < h < 1$, $t \in I_h$ and $E(t)$, $t \in I_h$ of operators defined by (3.1) and (3.2), where the functions $e_h(t, \xi)$ and $e(t, \xi)$ satisfies (3.5) and (3.6) respectively. We shall give a simple criterion for saturation, i.e. for (3.13) in terms of $e_h(t, \xi)$ and $e(t, \xi)$.

We shall let U_r denote the annulus

$$U_r = \{\xi;\ 2^{-1}r < |\xi| < 2r\}. \tag{4.1}$$

and write

$$\|g(\xi)\|_n = \|g(\xi)\|_{M_p(U_{2^n})}. \tag{4.2}$$

THEOREM 4.1. - Suppose

$$\overline{\lim_{h\to 0}}\ \sup_{t\in I_h} h^{-s}\|e_h(t, \xi) - e(t, \xi)\|_n < \infty \tag{4.3}$$

for *some* integer n. Then $\Lambda_s \neq 0$. If the limit (4.3) is zero it also follows that $N_s \neq 0$.

PROOF. - Let $\chi \in \mathfrak{D}$ have its support in the annulus U_{2^n}. Then $\chi \in M_p$. Let $g \in L_p$ and put $f = \chi^v * g$. Then spec $f \subset U_{2^n}$ and by (1.20),

$$\|(E_h(t) - E(t))f\|_{L_p} \leq \|(e_h(t, \xi) - e(t, \xi))\chi(\xi)\|_{M_p}\|g\|_{L_p} \leq$$

$$\leq \|\chi\|_{M_p}\|e_h(t, \xi) - e(t, \xi)\|_n\|g\|_{L_p}$$

Multiplying by h^{-s} we conclude

$$\overline{\lim_{h\to 0}}\ \sup_{t\in I_h} h^{-s}\|(E_h(t) - E(t))f\|_{L_p} < \infty.$$

It follows that $f \in \Lambda_s$: If the limit (4.3) is zero we see that $f \in N_s$. Since we can assume $f \neq 0$, the conclusion follows.

REMARK 4.1. - Clearly the conclusions of theorem 4.1 hold if we replace the annulus U_{2^n} with any open set V.

THEOREM 4.2. - Suppose that for every h and every integer n there exists a number $t \in I_h$ such that

$$[e_h(t;\ \xi) - e(t;\ \xi)]^{-1} \in M_p(U_{2^n}).$$

Assume

$$\varliminf_{h\to 0}\ \inf_{t\in I_h} h^s \| [e_h(t;\ \xi) - e(t;\ \xi)]^{-1} \|_n < \infty \tag{4.4}$$

for *all* integers n. Then $N_s = 0$. If the limit (4.4) is zero (for every n), then $\Lambda_s = 0$.

PROOF. - We have

$$\| \Phi_n^v * f \|_{L_p} \leq C h^s \| [e_h(t;\ \xi) - e(t;\ \xi)]^{-1} \|_n h^{-s} \| E_h(t)f - E(t)f \|_{L_p}.$$

Thus if (4.4) is finite the right hand side tends to zero for all $f \in N_s$. Thus $\Phi_n^v * f = 0$ for all integers n, which gives $f = 0$. If the limit (4.4) is zero the same conclusion holds for all $f \in \Lambda_s$.

COROLLARY 4.1. - Suppose that the limit (4.3) is finite for some integer n and that (4.4) is finite for all integers n. Then $E_h(t)$ is a saturated approximation of $E(t)$ and s is the order of the saturation.

5. - Comparision of Lipschitz spaces and Besov spaces.

In this section we shall consider the same general situation as in section 4. Our object is now to compare the LIPSCHITZ spaces Λ_σ with the BESOV spaces $B_p^{m,\,q}$ and the spaces H_p^{*m}.

With the notations of section 4 we have

THEOREM 5.1. - Suppose that (3.5) and (3.6) hold. Then a sufficient for

$$\| f \|_{\Lambda_s} \leq C \| f \|_{B_p^{m,\,q_0}}, \tag{5.1}$$

is that for some $\varepsilon > 0$

$$\{ \sum_{h^s 2^{nm} \leq \varepsilon} \{ h^{-s} 2^{-nm} \| e_h(t;\ \xi) - e(t;\ \xi) \|_n \}^{q_0'} \}^{1/q_0'} \leq C, \tag{5.2}$$

for $0 < h < 1$, $t \in I_h$, and $(q_0')^{-1} + q_0^{-1} = 1$. For $q_0 = 1$ $(q_0' = \infty)$, condition (5.2) is necessary for (5.1).

PROOF. - From (3.5) and (3.6) we see that

$$\{ \sum_{h^s 2^{nm} > \varepsilon} \{ h^{-s} 2^{-nm} \| e_h(t;\ \xi) - e(t;\ \xi) \|_n \}^{q_0'} \}^{1/q_0'} \leq$$

$$\leq C \{ \sum_{h^s 2^{nm} > \varepsilon} \{ h^{-s} 2^{-nm} \}^{q_0'} \}^{1/q_0'} \leq C' \varepsilon^{-1}.$$

Thus

$$\{\sum_{-\infty}^{+\infty}\{h^{-s}2^{-nm}\|e_h(t;\ \xi)-e(t;\ \xi)\|_n\}^{q_0'}\}^{1/q_0'}\leq C.$$

It follows that

$$h^{-s}\|(E_h(t)-E(t))f\|_{L_p}\leq\sum_{-\infty}^{+\infty}h^{-s}\|(E_h(t)-E(t))\Phi_n^v*f\|_{L_p}\leq$$

$$\leq\sum_{-\infty}^{+\infty}h^{-s}2^{-nm}\|e_h(t;\ \xi)-e(t;\ \xi)\|_n2^{nm}\|\Phi_n^v*f\|_{L_p},$$

since Φ_n vanishes outside annulus U_{2^n}. In view of the definition of $\dot{B}_p^{m,q_0}$ we get from HÖLDER'S inequality

$$h^{-s}\|E_h(t)f-E(t)f\|\leq C\|f\|_{\dot{B}_p^{m,q_0}}.$$

Thus we have

$$\|f\|_{\Lambda_s}\leq C'(\|f\|_{L_p}+\|f\|_{\dot{B}_p^{m,q_0}}),$$

and the conclusion follows from lemma 2.2.

To prove the necessity, assume that (5.1) holds, let $\chi\in\mathfrak{D}$ be identically 1 on $2^{-1}\leq|\xi|\leq 2$, and suppose that χ vanishes outside $4^{-1}\leq|\xi|\leq 4$. Write $\chi_n(\xi)=\chi(2^{-n}\xi)$. Since the operator on L_p

$$f\to h^{-s}(E_h(t)-E(t))\chi_n^v*f$$

has norm

$$h^{-s}\|(e_h(t;\ \xi)-e(t;\ \xi))\chi_n(\xi)\|_{M_p},$$

we can find a function $f\in L_p$, $\|f\|_{L_p}=1$, so that

$$h^{-s}\|(e_h(t;\ \xi)-e(t;\ \xi))\chi_n(\xi)\|_{M_p}\leq 2h^{-s}\|(E_h(t)-E(t))\chi_n^v*f\|_{L_p}$$

However $\chi_n^v*f\in B_p^{m,1}$ and

$$\|\chi_n^v*f\|_{B_p^{m,1}}\leq C2^{nm}.$$

Thus by the definition of local FOURIER multipliers we get from (5.1)

$$h^{-s}\|e_n(t;\ \xi)-e(t;\ \xi)\|_n\leq C2^{nm}$$

which gives (5.2) for $q_0=1$.

We shall also need

THEOREM 5.2. - Suppose (3.5) and (3.6) hold. Then a sufficient condition for

$$\| f \|_{\Lambda_s} \leq C \| f \|_{H_p^{*m}} \tag{5.3}$$

is that for some $\varepsilon > 0$

$$\| h^{-s} | \xi |^{-m} (e_h(t; \xi) - e(t; \xi)) \|_{M_p(B_{h,\varepsilon})} \leq C, \tag{5.4}$$

where $0 < h < 1$, $t \in I_h$ and

$$B_{h,\varepsilon} = \{ h^s | \xi |^m \leq \varepsilon \}.$$

(Note that (5.2) $q_0' = 1$ implies (5.4)).

PROOF. - We shall prove that

$$\| (e_h(t; \xi) - e(t; \xi))[\exp(-h^s | \xi |^m) - 1]^{-1} \|_{M_p} \leq C \tag{5.5}$$

for this implies (in the notation of section 2)

$$\| E_h(t) f - E(t) f \|_{L_p} \leq C \| G_m(h^s) f - f \|_{L_p}.$$

To prove (5.5) we note that (5.4) implies

$$\| (e_h(t; \xi) - e(t; \xi))[\exp(-h^s | \xi |^m) - 1]^{-1} \|_{M_p(B_{h,\varepsilon})} \leq$$
$$\leq C \| h^s | \xi |^m [\exp(-h^s | \xi |^m) - 1]^{-1} \|_{M_p(B_{h,\varepsilon})}.$$

Using the invariance under homotheties, we see that it suffices to show

$$| \xi |^m [\exp(-| \xi |^m) - 1]^{-1} \in M_p(B_{1,\varepsilon}). \tag{5.6}$$

But this follows from lemma 1.3. In fact, we have (by lemma 1.4)

$$| \xi |^{-m} [\exp(-| \xi |^m) - 1] = \int_0^1 \exp(-r | \xi |^m) dr \in \mathcal{F} L_1(B_{1,\varepsilon})$$

and

$$| \xi |^{-m} [\exp(-| \xi |^m) - 1] \neq 0, \qquad \xi \in B_{1,\varepsilon},$$

which gives (5.6).

Write

$$B_{h,\varepsilon}^* = \{ \xi \mid h^s | \xi |^m \geq \varepsilon/2 \}.$$

Then it remains to show

$$\| (e_h(t;\ \xi) - e(t;\ \xi))[\exp(-h^s |\xi|^m) - 1]^{-1} \|_{M_p(B^*_{h,\varepsilon})} \leq C.$$

Using (3.5) and (3.6) we see however that

$$\| (e_h(t;\ \xi) - e(t;\ \xi))[\exp(-h^s |\xi|^m) - 1]^{-1} \|_{M_p(B^*_{h,\varepsilon})} \leq$$
$$\leq C \| [\exp(-h^s |\xi|^m) - 1]^{-1} \|_{M_p(B^*_{h,\varepsilon})}.$$

By the invariance under homotheties it suffices to show

$$[\exp(-|\xi|^m) - 1]^{-1} \in M_p(B^*_{1,\varepsilon}) \tag{5.7}$$

But

$$[\exp(-|\xi|^m) - 1]^{-1} = -1 + [1 - \exp|\xi|^m]^{-1}$$

and from lemma 1.1 (or 1.4) it is easy to see that

$$[1 - \exp|\xi|^m]^{-1} \in M_p(B^*_{1,\varepsilon}).$$

Now (5.7) follows and theorem 5.2 proved

COROLLARY 5.1. - Suppose that either (5.2) or (5.4) holds. Then follows

$$\| f \|_{\Lambda_{\theta s}} \leq C_{\theta, q} \| f \|_{B_p^{\theta m, q}}, \qquad 0 < \theta < 1 \tag{5.8}$$

Here $C_{\theta, q} \leq C(\theta(1-\theta))^{-1/q'}$.
(The estimate of the constant $C_{\theta, q}$ can be improved for certain values of q_0 and q, if (5.2) holds, by means of the results of HOLMSTEDT [11], c.f. remark 2.1.).

PROOF. - Put $T = E_h(t) - E(t)$. Then (3.3) and (3.4) gives

$$\| Tf \|_{L_p} \leq C \| f \|_{L_p}.$$

Theorem 5.1 and (2.13) (or theorem 5.2 and (2.10)) gives

$$\| Tf \|_{L_p} \leq Ch^s \| f \|_{B_p^{m, 1}}$$

Using the interpolation lemma (lemma 2.1) we get

$$\| Tf \|_{L_p} \leq C_{\theta, q} h^{\theta s} \| f \|_{B_p^{\theta m, q}},$$

which is the desired inequality.

We now proceed to the converse of (5.1).

THEOREM 5.3. - A sufficient condition for

$$\| f \|_{B_p^{m,\infty}} \leq C \| f \|_{\Lambda_s}, \qquad m = s \cdot \nu \tag{5.9}$$

is that there exists a number $l > 0$ and a sequence $t_n \in I_h$, with $h = l2^{-n\nu}$, such that

$$\| (e_h(t, \xi) - e(t, \xi))^{-1} \|_n \leq D_l, \qquad n = 1, 2, \ldots, \tag{5.10}$$

for $t = t_n$, $h = l \cdot 2^{-n\nu}$.

PROOF. - It is clear that

$$\| \Phi_n^v * f \|_{L_p} \leq Ch^s \| (e_h(t, \xi) - e(t, \xi))^{-1} \|_n \| f \|_{\Lambda_s}$$

for all $t \in I_h$ and $0 < h < 1$. Thus if we take $t = t_n$, $h = l \cdot 2^{-n\nu}$, (i.e. $h^s = l^s 2^{-nm}$), we get

$$\| \Phi_n^v * f \|_{L_p} \leq D_l l^s 2^{-nm} \| f \|_{\Lambda_s}, \qquad n = 1, 2, \ldots,$$

which gives the conclusion, since $\| \Psi^v * f \|_{L_p} \leq C \| f \|_{L_p}$.

REMARK. 5.2. - It is clear that we can replace the annulus $U_{2^k} = 2^k \cdot U_1$, $U_1 = \{ \xi \mid 2^{-1} < |\xi| < 2^1 \}$, with the generalized annuli $\gamma^k \operatorname{supp} \Phi^*$, Φ^* satisfying (2.17). In view of the equivalence of $B_p^{s,q}$ and $B_p^{s,q}(\gamma, \Phi^*)$ we get the same conclusions in theorems 5.1 and 5.3. In theorem 5.3 we shall choose $h = l\gamma^{-n\nu}$.

REMARK 5.4. - A sufficient condition for the converse of (5.3), i.e. for

$$\| f \|_{H_p^{*m}} \leq C \| f \|_{\Lambda_s} \tag{5.11}$$

is that for every h there exists $t_h \in I_h$ such that

$$\| h^s |\xi|^m \exp(-h^s |\xi|^m)[e_h(t_h; \xi) - e(t_h; \xi)]^{-1} \|_{M_p} \leq C.$$

This follows easily from (2.11), but we leave the details to the reader.

6. - Convergence of singular integrals.

Let I_h consist of one single point t_0 and suppose that $E(t_0)$ is the identity operator; $E(t_0)f = f$. We shall suppose that $E_h = E_h(t_0)$ is given by the function

$$e_h(t_0, \xi) = K(h\xi).$$

If K is the FOURIER transform of k, then $K(h\xi)$ is, by (1.10) the FOURIER transform of $h^{-d}k(h^{-1}x)$. Thus E_h is given by the singular integral

$$E_h f(x) = h^{-d}\int_{R^d} k(h^{-1}y)f(x-y)dy = \int_{R^d} k(y)f(x-hy)dy.$$

(c.f. BUTZER [6]).

Since M_p is invariant under homotheties the condition (3.5) reduces to

$$K \in M_p .$$

Occasionally we shall assume that

$$K(\xi) \in \mathcal{F}L_1(U_r), \tag{6.1}$$

for every annulus $U_r = \{\, \xi \mid 2^{-1}r < |\,\xi\,| < 2r \,\}$, $0 < r < \infty$.

First we shall apply theorems 4.1 and 4.2

THEOREM 6.1. - i) Suppose that $K \in M_p$ is continuous for $\xi \neq 0$ and that there is a function $L(\xi)$ such that for $J = 0, 1, 2, \ldots N$, $N > \frac{d}{2}$,

$$D^J(h^{-\sigma}(K(h\xi) - 1) - L(\xi)) \rightarrow 0, \qquad h \rightarrow 0 \tag{6.2}$$

uniformly on some annulus U_{2^n}. Then $\Lambda_\sigma \neq 0$. If $L(\xi) = 0$ on some open set $V \subset U_{2^n}$ we also have $N_\sigma \neq 0$.

ii) Suppose that $K(\xi) \neq 1$, $0 < |\,\xi\,| < \varepsilon$ for some $\varepsilon > 0$, that (6.1) holds and that for $J = 0, 1, \ldots, N$,

$$D^J(h^{\sigma}(K(h\xi) - 1)^{-1} - L(\xi)^{-1}) \rightarrow 0, \qquad h \rightarrow 0 \tag{6.3}$$

uniformly on every annulus U_{2^n}. Then $N_\sigma = 0$. If $L(\xi) = \infty$ for all $\xi \neq 0$, then $\Lambda_\sigma = 0$.

COROLLARY 6.1. - Suppose (6.1) and that either (6.2) or (6.3) holds for every annulus U_{2^n} and that

$$0 < |\,L(\xi)\,| < \infty, \qquad \xi \neq 0, \tag{6.4}$$

Then E_h is a saturated approximation of the identity operator and the order of the saturation is σ.

PROOF. - From (6.2) and lemma 1.2 we get $L(\xi) \in M_p(U_{2^n})$ and

$$h^{-\sigma}\,\|\,K(h\xi) - 1\,\|_n \rightarrow \|\,L(\xi)\,\|_n .$$

Thus, by theorem 4.1, $\Lambda_\sigma \neq 0$. If $L(\xi) = 0$ on some open set V, we conclude

$$\lim_{h \to 0} h^{-\sigma} \| K(h\xi) - 1 \|_{M_p(V)} = 0,$$

and thus, by remark 4.1, $N_\sigma \neq 0$.

If (6.3) holds and $K(\xi) \neq 1$, $0 < |\xi| < \varepsilon$ we get in the same way

$$h^\sigma \| (K(h\xi) - 1)^{-1} \|_n \to \| L(\xi)^{-1} \|_n .$$

Here we have used that $(K(h\xi) - 1)^{-1} \in M_p(U_{2^n})$ if h is sufficiently small. This follows from lemma 1.3. In fact, since $K(\xi) - 1 \in \mathcal{F}L_1(U_r)$ $r = h2^n < \varepsilon$ and $K(\xi) - 1 \neq 0$, for ξ in the closure of U_r we get $(K(\xi) - 1)^{-1} \in \mathcal{F}L_1(U_r)$ and thus, by (1.17), $(K(h\xi) - 1)^{-1} \in M_p(U_{2^n})$.

By theorem 4.2 we now conclude $N_\sigma = 0$. If $L(\xi)^{-1} = 0$ on all annuli U_{2^n}, we get by the same theorem, $\Lambda_\sigma = 0$.

For the proof of corollary 6.1 we have only to note that if (6.4) holds, then (6.2) and (6.3) are equivalent.

As an illustration we consider

EXAMPLE 6.1. - Suppose that $\chi_0(u)$ is infinitely differentiable on $0<u<+\infty$ and satisfies

$$| \chi_0(u) - 1 | \leq C_0 u^{\alpha-1}, \qquad 0 < u < 1,$$

$$| D^J \chi_0(u) | \leq C_J u^{\alpha-1-J}, \qquad J = 1, 2, \ldots, \qquad 0 < u < 1,$$

$$| D^J \chi_0(u) | \leq C_J u^{-\beta-1-J}, \qquad 1 < u < \infty, \qquad J = 0, 1, 2, \ldots .$$

Suppose that $\omega_\rho(u)$ is infinitely differentiable on $0 < u < \infty$ and

$$\omega_\rho(u) = (\log 1/u)^\rho, \qquad 0 < u < 1/4,$$

$$\omega_\rho(u) = (\log 1/2)^\rho, \qquad 1/2 < u < \infty.$$

Put

$$K(\xi) = 1 + | \xi | \, \omega_\rho(| \xi |) \chi_0(| \xi |) \tag{6.5}$$

Since $u\omega_\rho(u)\chi_0(u)$ satisfies the assumptions of lemma 1.4 if α, $\beta > 0$, we have

$$K \in \mathcal{F}L_1 .$$

Now

$$h^{-\sigma}(K(h\xi) - 1) = h^{1-\sigma} | \xi | (\log 1/h | \xi |)^\rho \chi_0(h | \xi |), \qquad 0 < h | \xi | < 1/4,$$

so that (6.2) holds for $\rho \leq 0$, and (6.3) is satisfied for $\rho \geq 0$, with the function $L(\xi)$ given by

$$L(\xi) = \begin{cases} \infty & \text{if } \sigma > 1 \\ \infty & \text{if } \rho > 0, \quad \sigma = 1, \\ |\xi| & \text{if } \rho = 0, \quad \sigma = 1, \\ 0 & \text{if } \rho < 0, \quad \sigma = 1, \\ 0 & \text{if } \sigma < 1. \end{cases}$$

Consequently $s = \sup\{\sigma;\ \Lambda_\sigma \neq 0\} = 1$ and

$$\Lambda_1 \neq 0, \qquad N_1 \neq 0 \quad \text{if and only if} \quad \rho < 0,$$

$$\Lambda_1 \neq 0, \qquad N_1 = 0 \quad \text{if and only if} \quad \rho = 0,$$

$$\Lambda_1 = 0, \qquad N_1 = 0 \quad \text{if and only if} \quad \rho > 0.$$

In particular the approximation is saturated if and only if $\rho = 0$.

It should be noted, however, that if Λ_1^ρ and N_1^ρ are defined by the conditions

$$\sup_{0<h<1} h^{-1}(\log 1/h)^{-\rho} \| E_h f - f \|_{L_p} < \infty, \qquad f \in L_p,$$

$$\lim_{h \to +0} h^{-1}(\log 1/h)^{-\rho} \| E_h f - f \|_{L_p} = 0, \qquad f \in L_p,$$

then it follows from the proofs of theorems of theorems 4.1 and 4.2 that

$$\Lambda_1^\rho \neq 0, \qquad N_1^\rho = 0.$$

We now turn to the application of theorems 5.1 and 5.2 to the general situation of this section.

THEOREM 6.2. - Suppose $K \in M_p$. Then

$$B_p^{s,1} \subseteq \Lambda_s, \tag{6.6}$$

if and only if for some $\varepsilon > 0$

$$\| |\xi|^{-s}(K(\xi) - 1) \|_{M_p(U_r)} \leq C, \qquad 0 < r \leq \varepsilon. \tag{6.7}$$

If moreover for some $\varepsilon > 0$

(6.8) $$\| \, |\xi|^{-s}(K(\xi) - 1) \|_{M_p(B_\varepsilon)} < \infty$$

$$B_\varepsilon = \{ \xi \, | \, |\xi| \leq \varepsilon \}$$

then

(6.9) $$H_p^{*s} \subseteq \Lambda_s .$$

If (6.7) or (6.8) holds we have

(6.10) $$B_p^{\sigma,\infty} \subseteq \Lambda_\sigma , \qquad 0 < \sigma < s,$$

(Continuous inclusions).

PROOF. - For $0 < r < \infty$ and $-\infty < a < +\infty$ we have by (1.17) and lemma 1.2.

(6.11) $$\| \, |\xi|^a \|_{M_p(U_r)} = Cr^a$$

where the constant C depends on a and the dimension. Thus, using (1.17) we get from (6.7)

$$(h2^n)^{-s} \| K(h\xi) - 1 \|_n \leq C \| (h \, |\xi|)^{-s}(K(h\xi) - 1) \|_n =$$
$$= C \| \, |\xi|^{-s}(K(\xi) - 1) \|_{M_p(U_r)} \leq C'$$

if $r = h2^n \leq \varepsilon$. Thus theorem 5.1 gives (6.6). If conversely (6.6) holds we must have

$$(h2^n)^{-s} \| K(h\xi) - 1 \|_n \leq C$$

and thus, by (6.11),

$$\| (h \, |\xi|)^{-s}(K(h\xi) - 1) \|_n \leq C'.$$

Using again the invariance under homotheties we get (6.7).

If (6.8) holds we can use theorem 5 2, from which (6.9) follows immediately. The inclusion (6.10) follows from corollary 5.1.

REMARK 6.1. - Note that (6.10) holds only in the sense

$$\| f \|_{\Lambda_\sigma} \leq C_\sigma \| f \|_{B_p^{\sigma,\infty}}, \qquad 0 < \sigma < s.$$

Here $C_\sigma = 0(\sigma^{-1}(s - \sigma)^{-1})$, $\sigma \to s$ according to corollary 5.1 (c.f however remark 2.1). We also note that if (6.7) holds

$$\| f \|_{\Lambda_\sigma} \leq C \| f \|_{B_p^{\sigma,1}}. \qquad 0 < \sigma \leq s,$$

where C is independent of σ.

The following theorem is a slight generalization of one of the results in SHAPIRO [24].

THEOREM 6.3. – Suppose that there is a compact set Γ not containing the origin, such that for all $\xi \in R^d$, $\xi \neq 0$ there exists a positive number C such that $C\xi \in \Gamma$. Assume that there is a neighbourhood 0 of Γ such that $K(\xi) \in \mathfrak{F}L_1(0)$ suppose

$$K(\xi) \neq 1, \qquad \xi \in \Gamma, \tag{6.12}$$

Then

$$\| f \|_{B_p^{\sigma,\infty}} \leq C \| f \|_{\Lambda_\sigma}, \qquad 0 < \sigma$$

where C is independent of σ.

The set Γ can, for example, be a surface which is homeomorphic to the unit sphere $|\xi| = 1$, provided that the interior of Γ (which exists according to JORDAN-BROWER'S separation theorem) does contain the origin. Note also that the assumption on K can be relaxed. It suffices to assume that K is on 0 the limit in M_p of functions in SCHWARTZ class $\mathcal{S}$.

PROOF. – Since $K(\xi) \in \mathfrak{F}L_1(0)$, $K(\xi)$ is continuous on 0, because the FOURIER transform of an integrable function is continuous. Thus $K(\xi) \neq 1$ on an open set Ω containing Γ. Now let $\Phi^* \in \mathfrak{D}$ have support in Ω and suppose $\Phi^*(\xi) = 1$, $\xi \in \Gamma$. We can also assume $0 \leq \Phi^*(\xi)$, and that $0 \notin \operatorname{supp} \Phi^*$. It is easy to see that there exists a number $\gamma > 1$, such that if $\xi \in \Gamma$, $\eta = \rho\xi$, $\gamma^{-1} \leq \rho \leq \gamma$, then $\Phi^*(\eta) > 0$. In fact, if this is not the case we can find a sequence $\xi_n \in \Gamma$, $\eta_n = \rho_n \xi_n$, $\gamma_n^{-1} \leq \rho_n \leq \gamma_n$, where $\gamma_n \to 1$, $\Phi^*(\eta_n) = 0$. By compactness, we can pick out a subsequence n′, such that $\xi_{n'} \to \xi$. Thus $\eta_{n'} = \rho_{n'}\xi_{n'} \to \xi$. But $\xi \in \Gamma$ implies $\Phi^*(\xi) = 1$ and thus $\Phi^*(\eta_{n'}) = 0$ for all n' leads to a contradiction.

Now Φ^* satiesfies the condition (2.17). In fact, take $\xi \in R^d$, $\xi \neq 0$. Then $\xi = \lambda\xi_0$, for some $\xi_0 \in \Gamma$. Choose k so that $\gamma^{-1} \leq \gamma^{-k}\lambda \leq \gamma$. Then, according to what we have just proved, $\gamma^{-k}\xi = \gamma^{-k}\lambda \cdot \xi_0$ satisfies $\Phi^*(\gamma^{-k}\xi) > 0$.

By theorem 5.3 and remark 5.3 we see that it suffices to show

$$\| (K(h\xi) - 1)^{-1} \|_{M_p(\gamma^n \operatorname{supp} \Phi^*)} \leq D \tag{6.13}$$

for $n = 1, 2, \ldots$ and $h = \gamma^{-n}$. But this is equivalent to

$$(K(\xi) - 1)^{-1} \in M_p(\operatorname{supp} \Phi^*). \tag{6.14}$$

However $K(\xi) \neq 1$ on $\operatorname{supp} \Phi^*$. Moreover $K(\xi) - 1 \in \mathfrak{F}L_1(\operatorname{supp} \Phi^*)$, because $\operatorname{supp} \Phi^*$ is contained in 0 and the constant functions are in $\mathfrak{F}L_1 0)$. We conclude from lemma 1.3 that (6.14) must hold, and the theorem is proved.

As an illustration we shall apply theorems 6.2 and 6.3 to example 6.1.

EXAMPLE 6.2. - Define K by (6.5). Then

$$|\xi|^{-1}(K(\xi)-1)=(\log 1/|\xi|)^{\rho}\chi_0(|\xi|),\qquad 0<|\xi|<1/4.$$

Supposing $\alpha\geq 1$ we get from lemma 1.5

$$|\xi|^{J}|D^{J}|\xi|^{-1}(K(\xi)-1)|\leq C_J(\log 1/|\xi|)^{\rho},\qquad 0<|\xi|<1/4,$$

so that for $0<r<1/8$,

$$\||\xi|^{-1}(K(\xi)-1)\|_{M_p(U_r)}\leq C(\log 1/r)^{\rho}. \tag{6.15}$$

Therefore

$$\||\xi|^{-1}(K(\xi)-1)\|_{M_p(U_r)}\leq C,\qquad 0<r<1/8$$

for $\rho\leq 0$. It follows that

$$B_p^{1,1}\subseteq\Lambda_1,\qquad \rho\leq 0.$$

Using theorem 5.1 this inclusion can be improved, for $\rho<0$. We have

$$B_p^{1,q_0}\subseteq\Lambda_1\quad\text{if}\quad \rho<-1/q_0',\qquad q_0<\infty.$$

In fact, this follows from theorem 5.1 if we can prove

$$\sum_{h2^n<1/2}((h2^n)^{-1}\|K(h\xi)-1\|_n)^{q_0'}\leq C. \tag{6.16}$$

But according to (6.11)

$$(h2^n)^{-1}\|K(h\xi)-1\|_n\leq C\||\xi|^{-1}(K(\xi)-1)\|_{M_p(U_{h2^n})}\leq C(\log 1/h2^n)^{\rho}.$$

Since

$$\sum_{h2^n<1/4}(\log 1/h2^n)^{\rho q_0'}\leq C\int_0^{1/4}(\log 1/x)^{\rho q_0'}\frac{dx}{x}<\infty$$

if $\rho q_0'<-1$, we get (6.16).

For $\rho=0$ we have

$$|\xi|^{-1}(K(\xi)-1)=\chi_0(|\xi|),\qquad |\xi|<1/4.$$

If we suppose that $\alpha>1$, we have $\chi_0(|\xi|)\in\mathfrak{F}_{L_1}$ by lemma 1.4. Thus

$$|\xi|^{-1}(K(\xi)-1)\in M_p\,B_{1/4}).$$

Thus for $\rho = 0$

$$H_p^{*1} \subseteq \Lambda_1 .$$

For $\rho > 0$ inclusion of this type are impossible, since we know from example 6.1, that $\Lambda_1 = 0$. However, by (6.15) we have for $0 < s < 1$

$$r^{1-s} \| |\xi|^{-1}(K(\xi) - 1)\|_{M_p(U_r)} \leq C, \ 0 < r < 1/4$$

and thus, by (6.11)

$$\| |\xi|^{-s}(K(\xi) - 1)\|_{M_p(U_r)} \leq C.$$

Consequently we get from (6.10)

$$B_p^{\sigma,\infty} \subseteq \Lambda_\sigma, \ 0 < \sigma < 1,$$

for all values of ρ, $(\alpha > 0)$.

Finally we see that the condition of theorem 6.3 is satisfied for any value of ρ. Therefore

$$\Lambda_\sigma \subseteq B_p^{\sigma,\infty}, \ 0 < \sigma.$$

For the set Γ in theorem 6.3 we can take $|\xi| = r$ for any r, $0 < r < 1/4$ if $\chi_0(r) \neq 0$.

We have for $\rho = 0$

$$h|\xi| \exp(-h|\xi|)[K(h\xi) - 1]^{-1} = \exp(-h|\xi|)\chi_0(h|\xi|)^{-1}.$$

Assuming for instance that $\chi_0(u)$ satisfies

$$\chi_0(u) \geq \delta \exp(-\theta u),$$

where $0 < \theta < 1$, we see that

$$\exp(-u)(\chi_0(u))^{-1}$$

satisfies the assumptions of lemma 1.4. Thus

$$\| |h| \exp(-h|\xi|)[K(h\xi) - 1]^{-1} \|_{M_p} \leq C.$$

For remark 5.4 we therefore get

$$\Lambda_1 \subseteq H_p^{*1}, \ (\rho = 0).$$

We have proved that, under the assumptions above

$$\Lambda_\sigma = B_p^{\sigma,\infty}, \ 0 < \sigma < 1, \ -\infty < \rho < +\infty,$$

$$\Lambda_1 = B_p^{1,\infty}, \ \rho < -1$$

$$B_p^{1,q} \subseteq \Lambda_1 \subseteq B_p^{1,\infty}, \ -1 \leq \rho < -1/q',$$

$$\Lambda_1 = H_p^{*1}, \ \rho = 0,$$

$$\Lambda_1 = 0, \qquad \rho > 0.$$

$$\Lambda_\sigma = 0, \qquad \sigma > 1, \ -\infty < \rho < +\infty,$$

For other examples, see BUTZER [6], BUTZER-BERHENS [7], LÖFSTRÖM [16] and NESSEL [18]. See also section 10 below, where similar problems for FOURIER series are discussed.

7. - The rate of convergence of difference operators.

In this section we shall let $E(t)$ be the solution operator for the initial value problem

$$\left\{\begin{aligned} &\frac{\partial u}{\partial t} + P(D)u = 0, \ x \in R^d, \ 0 < t < \infty, \\ &\qquad u = f, \ x \in R^d, \ t = 0. \end{aligned}\right. \tag{7.1}$$

We shall assume that the differential operator $P(D)$ has constant coefficients and that the polynomial $P(\xi)$ is homogenous of order m and positive for $\xi \neq 0$, i.e.

$$tP(\xi) = P(t^{1/m}\xi). \tag{7.2}$$

It is easy to see that (7.1) is correctly posed, i.e.

$$\| E(t)f \|_{L_p} \leq C \| f \|_{L_p}, \qquad 0 < t < \infty. \tag{7.3}$$

In fact, it is clear that $E(t)$ is given by (3.2) with

$$e(t, \ \xi) = \exp(-tP(\xi)) = \exp(-P(t^{1/m}\xi)). \tag{7.4}$$

By the invariance for homotheties (1.9) it suffices to prove

$$\exp(-P(\xi)) \in \mathfrak{F}_{L_1}. \tag{7.5}$$

This follows however from lemma 1.4, but it can also be proved by means of lemma 1.1. In fact, since $P(\xi)$ is homogenous of order m and positive on $\xi \neq 0$,

$$A\,|\,\xi\,|^m \leq P(\xi) \leq A^{-1}\,|\,\xi\,|^m. \tag{7.6}$$

It follows

$$|D^J \exp(-P(\xi))| \leq C_J \exp\left(-\frac{A}{2}|\xi|^m\right),$$

and thus by lemma 1.1, (7.5) follows.

The family $E(t)$, $0 < t < \infty$ form a strongly continuous semi-group of operators on L_p, i.e.

$$E(t+s)f = E(t)(E(s)f), \tag{7.7}$$

$$E(t)f \to f, \qquad t \to 0. \tag{7.8}$$

(See BUTZER-BERENS [7]). This is true for $1 \leq p < \infty$. For $p = \infty$ the same statement holds if we let L_∞ denote the space of continuous functions f for which $f(x) \to 0$, $|x| \to \infty$. The infinitesimal generator of the semi-group $E(t)$ is $-P(D)$.

We shall approximate the semi-group $E(Nk)$, $N = 0, 1, 2, \ldots$, $(k = \lambda h^m$, λ given $> 0)$ by means of a family of «discrete» semi-group $E_h(Nk)$, $N = 0, 1, 2, \ldots$, $0 < h < 1$, defined by

$$E_h(k)f(x) = \sum_\alpha e_\alpha(\lambda) f(x + \alpha h), \qquad k = \lambda h^m, \tag{7.9}$$

$$E_h(Nk) = E_h(k)^N. \tag{7.10}$$

Here $e_\alpha(\lambda)$ are given numbers, independent of h. The sum can be finite or infinite. In the first case $E_h(k)$ is called an explicit difference operator, while it is called implicit in the second case. The function

$$u_h(x, Nk) = E_h(k)^N f(x)$$

is the solution of the discrete initial-value problem

$$\begin{cases} u_h(x, Nk + k) = \sum_\alpha e_\alpha(\lambda) u_h(x, Nk), & x \in hZ^d, \\ u_h(x, 0) = f(x), & x \in h \cdot Z^d, \end{cases}$$

where Z denotes the set of integers.

From (7.9) we get that $E_h(Nk)f$ is given by (3.1) with

$$e_h(k;\ \xi) = \sum_\alpha e_\alpha(\lambda) \exp i < \alpha, h\xi >, \qquad k = \lambda h^m.$$

Put

$$P_h(\xi) = -k^{-1} \log e_h(k;\ \xi).$$

We assume that $P_h(\xi)$ is well defined if $h|\xi|$ is sufficiently small. Then we see from (7.10) that

$$e_h(t;\ \xi) = \exp(-tP_h(\xi)), \qquad h|\xi| \ \text{small}, \quad t \in I_h,$$

where

$$I_h = \{t \,|\, t = Nk, \qquad N = 0,\ 1,\ 2,\ \ldots\}. \tag{7.11}$$

Our condition (3.5) is equivalent to

$$\|E_h(k)^N f\|_{L_p} \leq C\|f\|_{L_p}, \qquad N = 1,\ 2,\ \ldots,\ 0 < h < 1. \tag{7.12}$$

This means that $E_h(k)$ is a stable difference operator on L_h, (see RICHTMYER–MORTON [24]). We shall not discuss this condition in detail, but refer the reader to STRANG [30]. THOMÉE [31], [32], WIDLUND [34]. See however theorem 8.1 below.

REMARK 7.1. – Most of the subsequent analysis carries over to the case when the coefficients $e_\alpha(\lambda)$ depend on h too. In this case the stability condition (7.12) should be replaced by

$$\|E_h(k)^N f\|_{L_p} \leq C_T\|f\|_{L_p}, \qquad 0 < Nk < T, \qquad 0 < h < 1 \tag{7.12'}$$

(C_T might tend to ∞ as $T \to \infty$). The set I_h should then be defined by

$$I_h = \{t;\ t = Nk,\ 0 < N < T/k\}. \tag{7.11'}$$

In our case, however the (7.12) and (7.12′) are equivalent. In fact, (7.12′) implies with $T = 1$

$$\|e_h(k;\ \xi)^N\|_{M_p} \leq C_1, \qquad 0 < N < 1/k, \qquad 0 < h < 1.$$

But now $e_h(k;\ \xi)$ is a function of $h \cdot \xi$ so by (1.9)

$$\|e_h(\lambda;\ \xi)^N)\|_{M_p} \leq C_1, \qquad 0 < N < 1/k, \qquad 0 < h < 1,$$

with $k = \lambda h^m$. Thus

$$\|e_1(\lambda;\ \xi)^N\|_{M_p} \leq C_1, \qquad 0 < N < \infty.$$

From the invariance under homotheties we again get

$$\|e_h(k;\ \xi)^N\|_{M_p} \leq C_1, \qquad 0 < N < \infty, \qquad 0 < h < 1$$

which implies (7.12).

DEFINITION 7.1. - We say that $P_h(\xi)$ approximates $P(\xi)$ with degree $s > 0$, if

$$P_h(\xi) - P(\xi) = h^s |\xi|^{m+s} Q(h\xi), \tag{7.13}$$

where Q is infinitely differentiable on $0 < |\xi| < \varepsilon_0$ and has bounded derivatives there. The approximation is of order exactly s, if, in addition

$$|Q(\xi)| \geq Q_0 > 0, \qquad 0 < |\xi| < \varepsilon_0. \tag{7.14}$$

REMARK 7.2. - The conditions of the definition can easily be formulated in terms of the functions $e_h(t;\ \xi)$ and $e(t;\ \xi)$. In fact, $P_h(\xi)$ approximates $P(\xi)$ with degree exactly s, if and only if

$$e_h(k;\ \xi) - e(k;\ \xi) = kh^s |\xi|^{m+s} R(h\xi),$$

where $k = \lambda h^m$ for some $\lambda > 0$, and R is infinitely differentiable and has bounded derivatives for $\xi \neq 0$ and, for some $\varepsilon_1 > 0$,

$$|R(\xi)| \geq R_0 > 0, \qquad 0 < |\xi| < \varepsilon_1.$$

Before we apply our general theorems to this situation we give an example.

EXAMPLE 7.1. - Consider the initial value problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} = 0, & x \in R^1, \quad 0 < t < \infty, \\[2ex] u = f, & x \in R^1, \quad t = 0. \end{cases} \tag{7.15}$$

The clearly $P(\xi) = |\xi|^2$. Let us approximate the equation with the difference equation

$$\frac{u_h(x;\ t+k) - u(x;\ t)}{k} = \frac{\theta\delta^2 u_h(x;\ t+k) + (1-\theta)\delta^2 u_h(x;\ t)}{h^2}, \tag{7.16}$$

where $t = Nk$, $k = \lambda h^2$, $0 \leq \theta \leq 1$ and

$$\delta^2 u_h(x;\ t) = u_h(x+h;\ t) - 2u_h(x;\ t) + u_h(x-h;\ t).$$

Thus

$$u_h(x;\ t+k) = E_h(k) u_h(x;\ t),$$

where

$$E_h(k) - 1 = \lambda(\theta E_h(k) + 1 - \theta)\delta^2.$$

Fourier transforming bot sides of (7.16) we get

$$e_h(k;\ \xi) = \frac{1 - 4\lambda(1-\theta)\sin^2 h\xi/2}{1 + 4\lambda\theta \sin^2 h\xi/2}, \qquad k = \lambda h^2.$$

For $0 < \theta \leq 1$, $E_h(k)$ is an implicit difference operator, but for $\theta = 0$ it is explicit;

$$E_h(k)f(x) = f(x) + \lambda\delta^2 f(x), \qquad (\theta = 0).$$

It is easy to see that

$$|e_h(k;\ \xi)| < 1, \qquad 0 < |h\xi| < 2\pi,$$

if and only if

$$2\lambda(1 - 2\theta) \leq 1. \tag{7.17}$$

Thus $E_h(k)$ is a stable difference operator on L_2 if (7.17) holds. It can be proved (for instance by means of the stability theorem by STRANG [30]), that $E_h(k)$ is stable on L_p $(1 \leq p \leq \infty)$ if (7.17) holds. See also theorem 8.1 below. This is easy to see directly for $\theta = 0$, because then $E_h(k)$ means convolution with a measure with total mass 1. By (1.11) therefore have

$$\|e_h(k;\ \xi)^N\|_{M_1} \leq 1, \qquad N = 0,\ 1,\ \ldots,\ 0 < h < 1.$$

An easy calculation will show that

$$P_h(\xi) - P(\xi) = -\frac{1}{k}\log e_h(k;\ \xi) - |\xi|^2 =$$

$$= C_\lambda h^2|\xi|^4\left(1 + 12\lambda\left(\theta - \frac{1}{2}\right)\right) + 0(h^4|\xi|^6), \qquad h \to 0.$$

Therefore $P_h(\xi)$ approximates $P(\xi)$ with order

$$s = 2 \quad \text{if} \quad 1 + 12\lambda\left(\theta - \frac{1}{2}\right) \neq 0,$$

$$s = 4 \quad \text{if} \quad 1 + 12\lambda\left(\theta - \frac{1}{2}\right) = 0.$$

(This example is borrowed from RICHTMYER-MORTON [25]).

We now return to the general case.

THEOREM 7.1. - Suppose that $P_h(\xi)$ approximates $P(\xi)$ with degree exactly s and suppose $E_h(k)$ is a stable on L_p. Define I_h by (7.11). Then $E_h(t)$ is a saturated approximation of $E(t)$ and s is the order of the saturation.

Moreover

$$\Lambda_\sigma = B_p^{\sigma,\infty}, \qquad 0 < \sigma \leq s,$$

with equivalent norms.

REMARK 7.3. - Our theorem is a refinement of a result (valid also for variable coefficients) by PEETRE-THOMÉE [24]. They proved the inclusion $B_p^{\sigma,1} \subseteq \Lambda_\sigma$. For the particular initial-value problem (7.15), HEDSTROM [9], proved the inclusion $H_p^\sigma \subseteq \Lambda_\sigma$, $0 < \sigma \leq s$ $(p = \infty)$. HEDSTROM also showed the existence of a function f_σ in H_p^σ, for which $f_\sigma \notin \Lambda_\tau$, $\tau > \sigma$. Recently HEDSTROM'S results were generalized by WIDLUND [35] to parabolic systems with variable coefficients. However WIDLUND'S result holds for intervals I_h-which are bounded away from the origin. In the special case considered here our result is therefore sharper, since the origin is a limit point of our interval I_h, as $h \to 0$. We also work with the space $B_p^{\sigma,\infty}$, which is larger than H_p^σ. We can also prove the converse inclusion $\Lambda_\sigma \subseteq B_p^{\sigma,\infty}$.

REMARK 7.4. - Our proof carries over to more general parabolic initial-value problems than (7.1) (non-homogenous P) and also to certain systems, but we must insist on constant coefficients. We can also treat the case when the coefficients $e_\chi(\lambda)$ in (7.9) depends on h. The unbounded interval $0 < t < \infty$ must then be replaced by an interval $0 < \tau < T$, (c.f. remark 7.1). In case of systems, $P_h(\xi)$ and $P(\xi)$ are matrices. The remark 7.2 and theorem 7.1 holds true under the extra assumption that $P_h(\xi)$ and $P(\xi)$ commute.

PROOF OF THEOREM 7.1. - We begin by proving

$$\sum_{h2^n \leq \varepsilon} (h2^n)^{-s} \| e_h(t;\ \xi) - e(t;\ \xi) \|_n \leq C, \tag{7.18}$$

for $t \in I_h$, $0 < h < 1$ and ε sufficiently small. By theorem 5.1 and corollary 5.1 this shows

$$\|f\|_{\Lambda_\sigma} \leq C_\sigma \|f\|_{B_p^{\sigma,\infty}}, \qquad 0 < \sigma \leq s.$$

(Here $C_\sigma \leq C\sigma^{-1}$, according to remark 5.1).
We need the following lemmata.

LEMMA 7.1. - There are constants B and B' such that

$$\| \exp(-tP(\xi)) \|_n \leq B' \exp(-Bt2^{nm}). \tag{7.19}$$

PROOF OF LEMMA 7.1. - By formula (1.17) we see that

$$\| \exp(-tP(\xi)) \|_{M_p(U_r)} = \| \exp(-P(\xi)) \|_{M_p(U_r)}, \ r = t^{1/m} 2^n$$

But since

$$|\xi|^J |D^J \exp(-P(\xi))| \leq C_J \exp(-Ar^m)$$

for $2^{-2}r \leq |\xi| \leq 2^2 r$ we get from lemma 1.2

$$\| \exp(-P(\xi)) \|_{M_p(U_r)} \leq C \exp(-Ar^m),$$

which proves (7.19).

LEMMA 7.2. - Suppose that $H(\xi)$ is a homogenons function of order m, which is infinitely differentiable and positive for $\xi \neq 0$. Then

$$\|H(\xi)\|_{M_p(U_r)} = Cr^m. \tag{7.20}$$

PROOF OF LEMMA 7.2. - By (1.17) we have

$$\|H(\xi)\|_{M_p(U_r)} = \|H(r\xi)\|_{M_p(U_1)} = r^m \|H(\xi)\|_{M_p(U_1)}$$

and since by lemma 1.2 $H(\xi) \in M_p(U_1)$ the conclusion follows.

LEMMA 7.3. - Suppose that $f(\xi) \in M_p(U_{2^n})$. Then

$$\exp f(\xi) \in M_p(U_{2^n})$$

and

$$\|\exp f(\xi)\|_n \leq \exp \|f(\xi)\|_n .$$

PROOF OF LEMMA 7.3. - The conclusion follows at once from the series expansion

$$\exp f(\xi) = \sum_{j=0}^{\infty} \frac{1}{j!} f(\xi)^j .$$

We now proceed with the proof of (7.18). Write

$$e_h(t;\ \xi) - e(t;\ \xi) = \exp(-tP(\xi))[\exp(th^s|\xi|^{m+s}Q(h\xi)) - 1], \tag{7.21}$$

and

$$\exp(th^s|\xi|^{m+s}Q(h\xi)) - 1 = t|\xi|^m(h|\xi|)^s Q(h\xi) \int_0^1 \exp r(t|\xi|^m(h|\xi|)^s Q(h\xi))dr \tag{7.22}$$

By lemma 7.2 (or (6.11)) we have

$$\|t|\xi|^m\|_n \leq Ct2^{nm}$$

and

$$\|(h|\xi|)^s\|_n \leq C(h2^n)^s .$$

By the assumptions on Q and lemma 1.2

$$\|Q(h\xi)\|_n = \|Q(\xi)\|_{M_p(U_{h2^n})} \leq C, \qquad h2^n \leq \varepsilon, \qquad 2\varepsilon \leq \varepsilon_0 . \tag{7.23}$$

Using lemma 7.3 we therefore get

$$\left\| \int_0^1 \exp (rt \,|\, \xi \,|^m (h \,|\, \xi \,|)^s Q(h\xi)) dr \right\|_n \leq$$

$$\leq \int_0^1 \exp (rAt2^{nm}(h2^n)^s) dr \leq \int_0^1 \exp (rA\varepsilon^s t2^{nm}) dr$$

and therefore

$$\| \exp (th^s \,|\, \xi \,|^{m+s} Q(h\xi)) - 1 \|_n \leq \tag{7.24}$$

$$\leq Ct2^{nm}(h2^n)^s \int_0^1 \exp (rA\varepsilon^s t2^{nm}) dr \leq$$

$$\leq C\varepsilon^{-s}(h2^n)^s (\exp (A\varepsilon^s t2^{nm}) - 1).$$

From (7.21) and lemma 7.1 we therefore get

$$(h2^n)^{-s} \| e_h(t;\ \xi) - e(t;\ \xi) \|_n \leq$$

$$\leq C\varepsilon^{-s} \exp (-Bt2^{nm})(\exp (A\varepsilon^s t2^{nm}) - 1).$$

Now we choose ε so small that

$$B - A\varepsilon^s = C > 0,$$

and then we get

$$(h2^n)^{-s} \| e_h(t;\ \xi) - e(t;\ \xi) \|_n \leq$$

$$\leq C\varepsilon^{-s} [\exp (-Ct2^{nm}) - \exp (-Bt2^{nm})].$$

Thus

$$\sum_{h2^n \leq \varepsilon} (h2^n)^{-s} \| e_h(t;\ \xi) - e(t;\ \xi) \|_n \leq$$

$$\leq C\varepsilon^{-s} \sum_{h2^n \leq \varepsilon} [\exp (-Ct^{nm}) - \exp (-Bt2^{nm})] \leq$$

$$\leq C\varepsilon^{-s} \int_0^\infty [\exp (-Ctx^m) - \exp (-Btx^m)] \frac{dx}{x} =$$

$$\leq C'\varepsilon^{-s} \int_0^\infty [\exp (-Cy) - \exp (-By)] \frac{dy}{y}.$$

Since the integral converges we get (7.18).

The next step in the proof is to show that for $N = \lambda^{-1}l^{-m}$ $h = l2^{-n}$, $t_n = Nk = 2^{-nm}$,

$$\|(e_h(t;\ \xi) - e(t;\ \xi))^{-1}\|_n \leq Cl^{-s} \tag{7.25}$$

for $n = 1, 2, \ldots$. From this inequality we get by theorem 5.3, that

$$\|f\|_{B_p^{\sigma,\infty}} \leq C\|f\|_{\Lambda_\sigma}, \qquad 0 < \sigma < \infty.$$

But if (7.25) holds for all possible $l > 0$, we also get

$$\varliminf_{l\to 0}\ \inf_{t\in I_h} h^s\|(e_h(t;\ \xi) - e(t;\ \xi))^{-1}\|_n =$$

$$= \varliminf_{\substack{l\to 0\\ t=Nk\\ k=l2^{-n}}} (l2^{-n})^s\|(e_h(t;\ \xi) - e(t;\ \xi))^{-1}\|_n \leq C2^{-ns}.$$

Thus theorem 4.2 shows that $N_s = 0$ and since we have already proved $\Lambda_s \neq 0$, we conclude that the approximation is saturated of order s.

It remains to prove (7.25). First we note that

$$\|(e_h(t;\ \xi) - e(t;\ \xi))^{-1}\|_n \leq$$

$$\leq \|\exp tP(\xi)\|_n \|(\exp (th^s|\xi|^{m+s}Q(h\xi) - 1)\|_n.$$

With $t = Nk = 2^{-nm}$ we get from lemma 7.

$$\|\exp tP(\xi)\|_n = \|\exp P(\xi)\|_1 = C.$$

By the invariance for homotheties (1.17)

$$\|(\exp (th^s|\xi|^{m+s}Q(h\xi)) - 1)^{-1}\|_n = \|\exp (l^s|\xi|^{m+s}Q(l\xi)) - 1)^{-1}\|_1.$$

Now the assumptions on Q shows that the function

$$(\exp (l^s|\xi|^{m+s}Q(l\xi)) - 1)^{-1}$$

is infinitely differentiable on $4^{-1} < |\xi| < 4$, if $4\cdot l < \varepsilon_0$, and

$$|D^J(\exp (l^s|\xi|^{m+s}Q(1\xi)) - 1)^{-1}| \leq C_J l^{-s}. \tag{7.26}$$

But now lemma 1.2 gives the desired inequality. This concludes the proof.

REMARK 7.5. - It is clear that the theorem holds also in the following slightly more general case; let $P(\xi)$ be any homogenuous function of order

$m > 0$, which is infinitely differentiable and positive for $\xi \neq 0$. Suppose that $P_h(D)$ is any operator approximating $P(D)$ with order exactly s and assume that the operator

$$E_h(Nk) = \exp(-NkP_h(D))$$

is stable, in the sense that

$$\|\exp(-NkP_h(\xi))\|_{M_p} \leq C, \qquad N = 0, 1, \dots, \qquad 0 < h < 1.$$

Untill now we have supposed that $P(\xi)$ is positive for $\xi \neq 0$ and homogenous of order $m > 0$. This fact led to the estimate

$$\|\exp(-tP(\xi))\|_n \leq \exp(-At2^{nm}),$$

which was very essential in the proof of theorem 7.1. If we assume only that (7.1) is correctly posed on L_p:

$$\|\exp(-tP(\xi))\|_{M_p} \leq C, \qquad 0 < t < \infty \tag{7.27}$$

(c.f. (7.3)), then the proof of theorem 7.1 does not work. In the rest of this section we shall suppose that $P(\xi)$ is homogenous of order m, but not necessarily positive outside the origin. We shall also suppose that (7.27) holds. We assume that $P_h(\xi)$ approximates $P(\xi)$ with order order exactly s in the sense of definition 7.1. If E_h is stable it is possible to prove (see theorem 4.2 in PEETRE-THOMÉE [24]),

$$\| E_h(t)f - E(t)f\|_{L_p} \leq Cth^s \sum_{|\alpha| \leq m+s} \|D^\alpha f\|_{L_p}. \tag{7.28}$$

In view of (7.28) it is natural to consider the space $\Lambda_{\sigma,\theta}$, cerresponding to the norm

$$\|f\|_{\Lambda_{\sigma,\theta}} = \|f\|_{L_p} + \sup_{\substack{0<h<1 \\ t\in I_h}} h^{-\sigma}t^{-\theta}\|E_h(t)f - E(t)f\|_{L_p} =$$

$$= \|f\|_{L_p} + \lambda^{-\theta} \sup_{\substack{0<h<1 \\ N=1,2,\dots}} N^{-\theta}h^{-(\sigma+\theta m)}\|E_h(k)^N f - E(k)^N f\|_{L_p},$$

where as usual $k = \lambda h^m$. Here $0 \leq \theta$, $0 \leq \sigma$. Note that $\Lambda_{\sigma,0} = \Lambda_\sigma$.

We have

$$\Lambda_{\sigma,\theta} = 0 \quad \text{if} \quad \sigma + \theta m > s + m. \tag{7.29}$$

For if $f \in \Lambda_{\sigma,\theta}$, $\sigma + \theta m > s + m$ we get with $t = k$

$$\|E_h(k)f - E(k)f\|_{L_p} = o(h^{s+m}).$$

Now the argument that led to (7.26) also shows that

$$\| \{ \exp(\lambda l^{m+s} |\xi|^{m+s} Q(l\xi) - 1 \}^{-1} \|_1 \leq C l^{-s-m}.$$

Therefore

$$h^{s+m} \| \{ e_h(k;\ \xi) - e(k;\ \xi) \}^{-1} \|_1 \leq$$

$$\leq 2^{n(s+m)} \| \exp \lambda l^m P(\xi) \|_1 \, l^{s+m} \| \{ \exp(\lambda l^{m+s} |\xi|^{m+s} Q(l\xi)) - 1 \}^{-1} \|_1 \leq C_n.$$

Now apply theorem 4.2 with $I_h = \{ t_0 \}$, $E_k(t_0) = E_h(k) - E(k)$, $E(t_0) = 0$. This gives $f = 0$.

In a similar way we get

$$\Lambda_{\sigma, \theta} = 0 \quad \text{if} \quad \sigma > s. \tag{7.30}$$

In fact, for a fixed t, say $t = 1$, we have

$$\| E_h(1) f - E(1) f \|_{L_p} = o(h^s).$$

But it is easy to see that for $h = l \cdot 2^{-n}$, l small

$$h^s \| \{ e_h(1;\ \xi) - e(1;\ \xi) \}^{-1} \|_n \leq C_n.$$

Thus, by theorem 4.2, $f = 0$.

THEOREM 7.2. - Suppose that $P(\xi)$ is homogenous of order m, and that (7.1) is correctly posed. Assume that $P_h(\xi)$ approximates $P(\xi)$ with order (exactly, s and that $E_h(k)$ is stable. Then for $0 \leq \sigma \leq s$, $\sigma + \theta m = s + m$

$$H_p^{*(s+m)} \subseteq \Lambda_{\sigma, \theta}. \tag{7.31}$$

PROOF. - We shall prove

$$\| t^{-\theta} h^{-\sigma} |\xi|^{-(m+s)} (e_h(t;\ \xi) - e(t;\ \xi)) \|_{M_p(B_{h,\varepsilon})} \leq C, \tag{7.32}$$

if $B_{h,\varepsilon} = \{ \xi \,|\, t^\theta h^\sigma |\xi|^{m+s} \leq \varepsilon \}$. By the proof of theorem 5.2 this gives

$$\| (e_h(t;\ \xi) - e(t;\ \xi)) [\exp(-t^\theta h^\sigma |\xi|^{m+s}) - 1]^{-1} \|_{M_p} \leq C,$$

which immediately leads to (7.31).

To prove (7.32) we note that (7.21), (7.22) and the correctness of the initial value problem gives

$$\| t^{-\theta} h^{-\sigma} |\xi|^{-(m+s)} (e_h(t;\ \xi) - e(t;\ \xi)) \|_{M_p(B_{h,\varepsilon})} \leq$$

$$\leq C t^{1-\theta} h^{s-\sigma} \| Q(h\xi) \|_{M_p(B_{h,\varepsilon})} \int_0^1 \exp r t h^s \| |\xi|^{m+s} Q(h\xi) \|_{M_p(B_{h,\varepsilon})} dr$$

But $t = \lambda N h^m$ so that $\xi \in B_{h,\varepsilon}$ implies $(h|\xi|)^{m+s} \leq \varepsilon \lambda^{-\theta} N^{-\theta} \leq \varepsilon \lambda^{-\theta}$. If ε is small enough we conclude

$$\| Q(h\xi) \|_{M_p(B_{h,\varepsilon})} \leq C.$$

If ε is sufficiently small

$$t^{\theta} h^{\sigma} \| |\xi|^{m+s} \|_{M_p(B_{h,\varepsilon})} \leq \| \chi(|\xi|) |\xi|^{m+s} \|_{M_p},$$

where $\chi(u)$ is infinitely differentiable on $-\infty < u < +\infty$, $\chi(u) = 1$ for $u < 1$ and $\chi(u) = 0$ for $u > 1$. Using lemma 1.4 on the function $\varphi(u) = u^{m+s} \chi(u)$ we conclude

$$\chi(|\xi|) |\xi|^{m+s} \in M_p.$$

Thus follows

$$\| t^{-\theta} h^{-\sigma} |\xi|^{m+s} (e_h(t;\ \xi) - e(t;\ \xi)) \|_{M_p(B_{h,\varepsilon})} \leq$$

$$\leq C t^{1-\theta} h^{s-\sigma} \int_0^1 \exp r A t^{1-\theta} h^{s-\sigma} dr.$$

However $t^{1-\theta} h^{s-\sigma} = \lambda^{1-\theta} N^{1-\theta}$ and $\theta = 1 + (s-\sigma)/m \geq 1$. We conclude thet (7.32) holds.

COROLLARY 7.2. - Let (7.1) be correctly posed and $P(\xi)$ be homogenous of order $m > 0$. Assume that $E_h(k)$ is stable and $P_h(\xi)$ approximates $P(\xi)$ with order exactly s. Then for $0 \leq \sigma \leq \theta s$, $0 < \sigma + \theta m < s + m$,

$$\Lambda_{\sigma,\theta} = B_p^{\sigma+\theta m, \infty}$$

PROOF. - Let T be operator $E_h(t) - E(t)$. Then we know from the stability that T maps L_p continuously into L_p with norm $M_0 \leq C$. Put

$$\nu = (\sigma + \theta m)/(s + m), \qquad \sigma_0 = \nu^{-1}\sigma, \qquad \theta_0 = \nu^{-1}\theta.$$

Then $0 \leq \sigma_0 \leq s$ since $(s+m)\sigma \leq s(\sigma + \theta m)$, and $\sigma_0 + \theta_0 m = s + m$. Thus we get from theorem 7.2 that T maps $H_p^{*(s+m)}$ into L_p with norm $M_1 \leq C t^{\theta_0} h^{\sigma_0}$.

From the interpolation lemma 2.1 follows that T maps $B_p^{\nu(s+m),\infty} = B_p^{\sigma+\theta m,\infty}$ into L_p with norm $M \leq Ct^{\nu\theta_0}h^{\nu\sigma_0} = Ct^{\theta}h^{\sigma}$. This proves

$$B_p^{\sigma+\theta m,\infty} \subseteq \Lambda_{\sigma,\theta}.$$

To prove the converse inolusion it suffices in view of theorem 5.3 and remark to show

$$\|t^{\theta}[e_h(t;\xi) - e(t;\xi)]^{-1}\|_n \leq Cl^{-s}$$

for $t = 2^{-nm}$, $h = l\cdot 2^{-n}$, $n = 1, 2, \ldots$. But since $t^{\theta} = 2^{-n\theta m} \leq 1$ this follows immediately from (7.25).

COROLLARY 7.3. - Suppose that $P(\xi)$ is homogenous of order m and *positive outside* $\xi = 0$. Let $E_h(k)$ be stable and assume that $P_h(\xi)$ approximates $P(\xi)$ with order exactly s. Then for $0 < \sigma \leq s$, $0 < \sigma + \theta m < s + m$

$$\Lambda_{\sigma,\theta} = B_p^{\sigma+\theta m,\infty}.$$

PROOF. - The operator $T = E_h(t) - E(t)$ maps $B_p^{\sigma,\infty}$ (or L_p if $\sigma = 0$) into L_p with norm $M_0 \leq Ch^{\sigma}$ $(0 \leq \sigma \leq s)$. This follows from theorem 7.1. By theorem 7.2, T maps $H_p^{*(s+m)}$ into L_p with norm $M_1 \leq Ct^{\theta}h^{\sigma}$, $m\vartheta = m + s - \sigma$.

Thus, by interpolation (remark 2.2), T maps $B_p^{\eta\sigma+(1-\eta)(s+m),\infty}$ into L_p with $M \leq Ch^{\sigma}t^{(1-\eta)\theta}$. Since $\sigma + (1-\eta)\theta m = \eta\sigma + (1-\eta)(s+m)$, we conclude

$$B_p^{\sigma+\theta m,\infty} \subseteq \Lambda_{\sigma,\theta},\ 0 \leq \sigma \leq s,\ 0 < \sigma + \theta m < s + m.$$

The converse inequality follows from the proof of corollary 7.2.

To illustrate the corollaries, consider a (σ, θ)-plane (fig. 1).

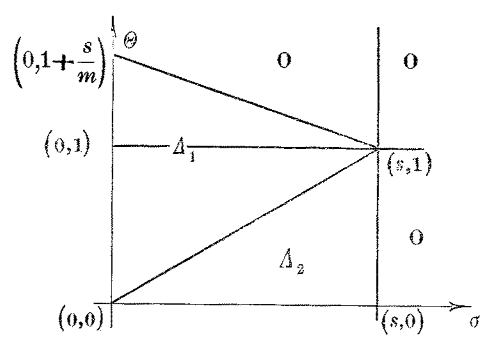

fig. 1

In the triangle Δ_1 we have complete infarmation on $\Lambda_{\sigma,\theta}$, if we know only that the initial value problem is correctly posed and $P(\xi)$ homogenous. On Δ_2 we know that $B_p^{\sigma+\theta m,\infty} \subseteq \Lambda_{\sigma,\theta}$. This follows from the proof of corollary 7.2. If in addition $P(\xi)$ is positiv for $\xi \neq 0$ then we have a complete description of $\Lambda_{\sigma,\theta}$ on Δ_2, too.

We shall conclude this section with a few remarks concerning the relation between the steplength k in the t-direction and the steplength h in the «space direction». Above we considered only the case $k = \lambda h^m$, where λ is a fixed constant. Now we shall let λ be a function of h. Since we do not want to discuss the question of stability in the general L_p case, we consider only L_2-norm. Let us first look at a simple example.

Consider again the initial value problem (7.14) and the difference schema (7.16), but now with $\lambda = \lambda h^{-2} = \lambda(h)$. Then

$$e_h(k;\,\xi) = \frac{1 - 4\lambda(1-\theta)\sin^2 h\xi/2}{1 + 4\lambda\theta \sin^2 h\xi/2}, \qquad 0 \leq \theta \leq 1,$$

and thus we have stability in L_2 if and only if $2\lambda(1-2\theta) \leq 1$. If we suppose that $\lambda = \lambda(h) \to +\infty$, $h \to 0$ we get $\theta \geq \frac{1}{2}$, which conversely garantees stability in L_2. Put

$$P_h^\lambda(\xi) = -k^{-1} \log e_h(k;\,\xi).$$

Expanding $P_h^\lambda(\xi)$ in TAYLOR series for α small, we get

$$P_h^\lambda(\xi) - \xi^2 = \lambda h^2 \xi^4 \left(\theta - \frac{1}{2} + A\lambda^{-1} + O(\lambda h^2 \xi^2)\right), \tag{7.33}$$

where $A \neq 0$.

Inspired by this example we consider now the initial value problem (7.1), with $P(\xi)$ homogenous of order m and positive for $\xi \neq 0$. Let $e_h(k;\,\xi)$ be the symbol for a stable difference schema, with $k = \lambda h^m$, $\lambda = \lambda(h) \to \infty$, $h \to 0$. Suppose that $P_h^\lambda = P_h$ satisfies

$$P_h^\lambda(\xi) - P(\xi) = \lambda(h\,|\xi|)^s |\xi|^m Q(h,\,\xi), \tag{7.35}$$

where

$$Q(h,\,\xi) = \lambda^{-q_0} Q_0(\lambda^{-1}) + O(\lambda(h|\xi|)^s). \tag{7.36}$$

Here Q_0 is a polynomial, $Q_0(0) \neq 0$. In our example we have $q_0 = 0$ if $\theta > 1/2$ and $q_0 = 1$ if $\theta = 1/2$.

Now it is clear that we can repeate the proof of theorem 7.1. For $t = Nk$ we have

$$\|\exp(-tP_h^\lambda(\xi)) - \exp(-tP(\xi))\|_n \leq$$
$$\leq C\varepsilon^{-1}\lambda h^s 2^{ns}(\exp(-tB2^{nm}) - \exp(-tA2^{nm})),$$

if $\lambda h^s 2^{ns} \leq \varepsilon$. Thus

$$\sum_{\lambda h^s 2^{ns} \leq \varepsilon} (\lambda h^s 2^{ns})^{-1} \|\exp(-tP_h^\lambda(\xi)) - \exp(-tP(\xi))\|_n \leq C\varepsilon^{-1}$$

and therefore theorem 5.1 gives

$$\|E_h(t)f - E(t)f\|_{L_2} \leq C\lambda h^s \|f\|_{B_2^{s,\infty}}.$$

and by interpolation

$$\|E_h(t)f - E(t)f\|_{L_2} \leq C\lambda^\eta h^{\eta s}\|f\|_{B_2^{\eta s,\infty}}, \tag{7.37}$$

for $0 < \eta \leq 1$.

For the converse of (7.37) we can use the same argument as in the proof of theorem 7.1. Take $t = 2^{-nm}$, $\lambda^{1-q_0}h^s = l^s 2^{-ns}$. Then for $2^{n-1} \leq |\xi| \leq 2^{n+1}$

$$|\exp(-t\lambda(h|\xi|)^s|\xi|^m Q(h, \xi)) - 1|^{-1} \leq$$
$$\leq C(t\lambda^{1-q_0}(h2^n)^s 2^{nm})^{-1} \leq Cl^{-s}.$$

if l is sufficiently small. Thus theorem 5.3 gives

$$\|f\|_{B_2^{\eta s,\infty}} \leq C \sup_{h,t} \lambda^{-\eta(1-q_0)}h^{-\eta s}\|E_h(t)f - E(t)f\|_{L_2}, \tag{7.38}$$

and we also see that

$$\lim_{h\to 0} \lambda^{-\eta(1-q_0)}h^{-\eta s}\|E(t)f - E(t)f\|_{L_2} = 0,$$

implies $f = 0$.

Taking $\lambda = h^{-\nu}$ we get in particular

$$B_2^{\eta s,\infty} \subseteq \Lambda_{\eta(s-\nu)}, \qquad 0 < \eta \leq 1,$$

and conversely

$$\Lambda_{\eta(s-\nu(1-q_0))} \subseteq B_2^{\eta s,\infty}, \qquad 0 < \eta \leq 1.$$

In our example we have $s = 2$ and $q_0 = 0$ if $\theta > 1/2$, $q_0 = 1$ if $\theta = 1/2$ and thus for $0 < \eta \leq 1$, $0 < \nu \leq 2$

$$\Lambda_{\eta(2-\nu)} = B_2^{2\eta,\infty}, \qquad \theta > 1/2,$$

$$\Lambda_{2\eta} \subseteq B_2^{2\eta,\infty} \subseteq \Lambda_{\eta(2-\nu)}, \qquad \theta = 1/2,$$

8. **Stability of function of $P_h(D)$ and $P(D)$.**

As an introduction we discuss the following d-dimensional analogue of example 7.1. Consider the equation

$$\frac{\partial u}{\partial t} = -P(D)u = \sum_{j=1}^{d} \frac{\partial^2 u}{\partial x_j^2}, \qquad t > 0,$$

$$u = f, \qquad t = 0,$$

where $x \in R^d$. Then $P(\xi) = |\xi|^2$. Define the difference operator $E_h(k)$ by the relation

$$\frac{1}{k}(E_h(k) - 1)u(x, t) = \sum_{j=1}^{d} (\theta E_h(k) + (1-\theta))\frac{1}{h^2}\delta_j u(x, t),$$

where $0 \leq \theta \leq 1$ and

$$\delta_j g(x) = g(x + he_j) - 2g(x) + g(x - he_j).$$

Here e_j is the $j: th$ unit vector

$$e_j = (\delta_{1j}, \ldots, \delta_{dj}).$$

Then, with the notation of section 7

$$\frac{1}{k}(e_h(k, \xi) - 1) = \sum_{j=1}^{d} (\theta e_h(k, \xi) + (1-\theta))\frac{4}{h^2}\sin^2 h\xi_j/2,$$

and hence

$$e_h(k; \xi) = \frac{1 - 4\lambda(1-\theta)\sum_{j=1}^{d} \sin^2 h\xi_j/2}{1 + 4\lambda \sum_{j=1}^{d} \sin^2 h\xi_j/2}, \qquad k = \lambda h^2.$$

Consequently we can write

$$e_h(k; \xi) = f\left(\frac{1}{d}\sum_{j=1}^{d} \sin^2 h\xi_j/2\right),$$

with

$$f(u) = \frac{1 - 4d\lambda(1-\theta)u}{1 + 4d\lambda\theta u}, \qquad 0 \leq \theta \leq 1.$$

Writing

$$H(\eta) = \frac{1}{d} P(\eta) = \frac{1}{d} |\eta|^2$$

and

$$\sin h\xi/2 = (\sin h\xi_1/2, \ldots, \sin h\xi_d/2),$$

we get

$$P_h(\xi) = -k^{-1} \log e_h(k; \xi) = -k^{-1} \log f(H(\sin h\xi/2)).$$

Note that

$$\sup H(\sin x/2) = 1. \tag{8.1}$$

This is the starting point of this section. We shall assume that $P_h(\xi)$ is a function of the particular form

$$P_h(\xi) = -k^{-1} \log f(H(\sin h\xi/2)), \tag{8.2}$$

where $H(\eta)$, $\eta = (\eta_1, \ldots, \eta_d)$ is homogenous of order $m > 0$, positive and infinitely differentiable for $\eta \neq 0$. We shall suppose that H is normalized by the condition (8.1). We shall study the stability of the operator

$$\exp(-tP_h(D)), \qquad t = Nk, \qquad N = 0, 1, \ldots,$$

defined by

$$(\mathcal{F} \exp(-tP_h(D))g)(\xi) = \exp(-tP_h(\xi))g^\wedge(\xi).$$

This means that we shall give conditions on the function f, which suffices for

$$\| \exp(-tP_h(\xi) \|_{M_p} \leq C, \qquad t = Nk, \qquad 0 < h < 1.$$

However we shall also consider more general functions of $P_h(D)$ than the exponential function. Thus we shall give conditions on f and φ, such that

$$\varphi(tP_h(D)) \tag{8.3}$$

is stable in L_p. The operator (8.3) is defined in the natural way:

$$(\mathcal{F}\varphi(tP_h(D))g)(\xi) = \varphi(tP_h(\xi))g^\wedge(\xi),$$

and the stability of (8.3) in L_p means

$$\| \varphi(tP_h(\xi)) \|_{M_p} \leq C.$$

Before carrying out the program indicated above we note that lemma 1.4 implies the stability of the operator

$$\varphi(tP(D)),$$

provided that $\varphi(u)$ is infinitely differentiable on $0 < u < \infty$ and satisfies

$$(8.4) \qquad |\varphi(u) - \varphi(0)| \leq C_0 u^{\alpha}, \qquad 0 < u < 1$$

$$(8.5) \qquad |\varphi(u)| \leq C_0 u^{-\beta}, \qquad 1 < u < \infty,$$

$$(8.6) \qquad |D^J \varphi(u)| \leq C_J \min(u^{\alpha-1}, u^{-\beta-J}), \qquad 0 < u < \infty$$

for $J = 1, 2, \ldots$ Here $\alpha, \beta > 0$.

We now proceed to the study of the stability of the operator $\varphi(tP_h(D))$. We shall prove two results. The first one concerns the case

$$\varphi(x) = \exp(-x),$$

and

$$t = Nk, \qquad N = 1, 2, \ldots, k = \lambda h^m.$$

Then

$$\varphi(tP_h(D)) = f(H(\sin hD/2))^N.$$

In case H is a polynomial, this operator is a difference operator and our first result reduces in this case to a stability theorem of the type used in section 7.

In our second theorem, φ is « arbitrary ». Note that φ has to be defined on the set

$$\{z;\ z = -t \log f(u)\}.$$

If we suppose $|f(u)| \leq 1$, which we shall do, this is a subset of $\{z;\ \mathrm{Re}\, z \geq 0\}$. But if we assume that f is non-negative, the function $-\log f(u)$ is non-negative. Thus it suffices to define φ on the positive real axis. We shall assume that f is positive in our second theorem. However we shall have no restriction on t; t runs through the entire positive axis.

We now present our two results.

THEOREM 8.1. - Let $P_h(\xi)$ be defined by (8.2), where H is normalized by (8.1), and assume that f is infinitely differentiable on $0 < u < 1$ and has bounded derivatives there, that $f(0) = 1$ and

$$(8.7) \qquad |f(u)| \leq \exp(-\delta u), \qquad 0 < u < 1,$$

where $\delta > 0$. Then

$$\| \exp(-NkP_h(\xi)) \|_{M_p} \leq C. \tag{8.8}$$

for $N = 0, 1, 2, \ldots$ and $0 < h < 1$.

THEOREM 8.2. - Suppose that $P_h(\xi)$ is defined by (8.2) and H, f satisfy the assumptions of theorem 8.1. Assume in addition that $f(u) > 0$, $0 \leq u \leq 1$, and that

$$| \varphi(u) - \varphi(0) | \leq C_0 u^\alpha$$

$$| D^J \varphi(u) | \leq C_J u^{\alpha - J}$$

for some $\alpha > 0$. Then

$$\| \varphi(tP_h(\xi)) \|_{M_p} \leq C, \qquad 0 < t < \infty, \qquad 0 < h < 1. \tag{8.9}$$

Theorem 8.1 is similar to the (one dimensional) stability theorem by STRANG [30]. For difference operators the result follows from the general stability theory developed by WIDLUND [34] and others. We shall present an independent proof, based on the theory of interpolation spaces. This technique will also be used in the proof of theorem 8.2, which does not follow from the ordinary stability theory.

Before we prove our theorems let us however return to the introductory example of this section. Then

$$f(u) = \frac{1 - 4d\lambda(1-\theta)u}{1 + 4d\lambda\theta u} \, 0 \leq u \leq 1, \qquad (0 \leq \theta \leq 1).$$

Clearly

$$f(u) = 1 - \frac{4d\lambda u}{1 + 4d\theta u} \leq 1 - \delta u \leq e^{-\delta u}, \qquad 0 \leq u \leq 1$$

with

$$\delta = \frac{4d\lambda}{1 + 4d\lambda\theta}.$$

Moreover

$$- f(u) \leq e^{-\delta u}$$

if and only if $e^{-\delta} + f(1) \geq 0$ i.e.

$$\frac{4d\lambda}{1 + 4d\lambda\theta} \leq 1 + e^{-\delta} \quad \text{for some} \quad \delta > 0$$

i.e.

$$\frac{4d\lambda}{1 + 4d\lambda\theta} < 2.$$

This gives

$$2d\lambda(1-2\theta) < 1. \tag{8.10}$$

It follows that

$$|f(u)| \leq \exp(-\delta u), \qquad 0 \leq u \leq 1$$

if and only if (8.10) holds. Thus (8.10) is a sufficient condition for stability of the operator

$$f(H(\sin hD/2))^N, \qquad 0 < h < 1, \qquad N = 1, 2, \ldots .$$

We also see that $f(u) > 0$, $0 \leq u \leq 1$ if and only if

$$\frac{4d\lambda u}{1 + 4d\lambda\theta u} < 1, \qquad 0 \leq u \leq 1$$

i.e.

$$\frac{4d\lambda}{1 + 4d\lambda\theta} < 1.$$

This gives

$$4d\lambda(1-\theta) < 1 \tag{8.11}$$

and we get from theorem 8.2, that (8.10) and (8.11) are sufficient for the stability of the operator

$$\varphi(-tP_h(D)), \qquad 0 < h < 1, \qquad 0 < t < \infty,$$

where φ satisfies the assumptions of theorem 8.2. and

$$-kP_h(D) = \log f(H(\sin hD/2)).$$

REMARK 8.1. - It can be proved that we have stability in the extremal case

$$2d\lambda(1-2\theta) = 1, \tag{8.12}$$

(c.f. example 7.1). This does not follow directly from theorem 8.1, because $(f1) = 1$, when (8.12) holds. Note however that in this example

$$H(\sin \xi/2) = \frac{1}{d}\sum_{j=1}^{d} \sin^2 \xi_j/2 = 1 - H(\cos \xi/2).$$

Note also that the translation invariance of M_p gives

$$\| f(H(\sin \xi/2))^N \|_{M_p} = \| f(H(\cos \xi/2))^N \|_{M_p} = \| f(1 - H(\sin \xi/2))^N \|_{M_p}. \tag{8.13}$$

If (8.12) holds, $|f(1-u)| \leq \exp(-\delta u)$, $0 < u < 1/2$ and $|f(u)| \leq \exp(-\delta u)$, $0 < u < 1/2$. Thus we can write $f(u) = f_0(u) + f_1(u)$, where $f_0(u)$ and $f_1(1-u)$ satisfies the assumptions of theorem 8.1. From (8.13) it follows then easily that (8.8) holds also in the extremal case (8.12), (c.f. THOMÉE [32], STRANG [30]).

For the proof of our theorems we shall need some auxiliary discussions and lemmata. The technique in the proof resembles the one used in PEETRE [20] and LÖFSTRÖM [16].

Clearly it suffices to prove (8.8) and (8.9) for $p = 1$. In view of the invariance for homotheties it is also enough to consider $h = 1$. We shall therefore consider

$$G_t(\xi) = \varphi(tP_1(\xi)) = \varphi(-t\lambda^{-1}\log f(H(\sin \xi/2))).$$

The G_t, being a periodic function, is the FOURIER transform of a measure of the form

$$\sum_{\alpha \in Z^d} G_t^\alpha \delta_\alpha, \qquad \delta_\alpha(f) = f(\alpha).$$

Now (1.11) shows that

$$\| G_t \|_{M_1} = \sum_\alpha | G_t^\alpha |$$

Let Φ_k be the standard functions in the definition of the BESOV spaces, and write

$$G_{t,k}^\alpha = \Phi_k(\alpha) G_t^\alpha .$$

Put

$$G_{t,k}(\xi) = \sum_\alpha G_{t,k}^\alpha \exp i < \alpha, \xi > .$$

According to remark 2.3 we write

$$\|G_t\|_{\dot{B}_p^{s,q}[T^d]} = \Big(\sum_{-\infty}^{+\infty} (2^{sk} \| G_{t,k} \|_{L_p[T^d]})^q\Big)^{1/q}.$$

We have

LEMMA 8.1. - For any $t > 0$ we have

$$\sum_\alpha | G_t^\alpha | \leq C_d \| G_t \|_{\dot{B}_2^{d/2,1}[T^d]} \tag{8.14}$$

and consequently

$$\| G_t \|_{M_p} \leq C_d \| G_t \|_{\dot{B}_2^{d/2,1}[T^d]}, \qquad 1 \leq p \leq \infty.$$

PROOF. - We shall prove (8.14). The technique is very similar to the one used in lemma 1.1. We have

$$\sum_{2^{j-1}\leq|\alpha|<2^{j+1}} | G^{\alpha}_{t,k} | \leq C_d 2^{jd/2} \Big(\sum_{\alpha\in Z^d} | G^{\alpha}_{t,k} |^2 \Big)^{1/2} \leq$$

$$\leq C_d 2^{jd/2} \| G_{t,k} \|_{L_2[T^d]}.$$

Since $G^{\alpha}_{t,k} \neq 0$ only if $2^{-1}|\alpha| < 2^k < 2|\alpha|$ and

$$\sum_{-\infty}^{+\infty} G^{\alpha}_{t,k} = G^{\alpha}_t,$$

we conclude

$$\sum_{2^{j-1}\leq|\alpha|<2^j} | G^{\alpha}_t | \leq C_d \sum_{2^{j-1}\leq|\alpha|<2^{j+1}} \sum_{2^{-1}|\alpha|<2^k<2|\alpha|} | G^{\alpha}_{r,k} | \leq$$

$$\leq C_d 2^{jd/2} \sum_{j-2<k<j+2} \| G_{t,k} \|_{L_2[T^d]}.$$

The conclusion now follows since

$$\sum_{\alpha} | G^{\alpha}_t | \leq C_d \sum_{j=-\infty}^{+\infty} 2^{jd/2} \sum_{j-2<k<j+2} \| G_{t,k} \|_{L_2[T^d]} \leq$$

$$\leq 3 \cdot 2^d C_d \sum_{-\infty}^{+\infty} 2^{kd/2} \| G_{t,k} \|_{L_2[T^d]}.$$

We now introduce two auxiliary spaces X and X^L, defined by the norm

$$\| g \|_X = \Big(\int_0^1 | g(u) |^2 w(u) du \Big)^{1/2},$$

and the semi-norm

$$\| g \|_{X^L} = \sum_{M=1}^{L} \| u^{M-L/m} g^{(M)}(u) \|_X,$$

respectively. Here

$$w(u) = \int \frac{dS_u}{| \operatorname{grad} H(\sin \xi/2) |} = \frac{d}{du} \int_{H(\sin \xi/2)\leq u} d\xi. \tag{8.15}$$

The first integral means integration over the surface

$$\Sigma_u : H(\sin \xi/2) = u, \qquad \xi \in T^d.$$

It is clear that

$$\int_{T^d} | g(H(\sin \xi/2)) |^2 d\xi = \int_0^1 | g(u) |^2 w(u) du.$$

Using lemma 1.5 and the fact that

$$|D^J H(\sin \xi/2)| \leq C_J H(\sin \xi/2)^{(m-J)/m},$$

we conclude

$$|D^L g(H(\sin \xi/2))| \leq C_L \sum_{K=1}^{L} (H(\sin \xi/2))^{K-L/m} \cdot g^{(K)}(H(\sin \xi/2)).$$

Therefore

$$\| D^L g(H(\sin \xi/2)) \|_{L_2[T^d]} \leq$$

$$\leq C_L \sum_{K=1}^{L} \| (H(\sin \xi/2))^{K-L/m} g^{(K)} \cdot (H(\sin \xi/2)) \|_{L_2[T^d]} =$$

$$= C_L \sum_{K=1}^{L} \| u^{K-L/m} g^{(K)}(u) \|_K = C_L \| g \|_{X^L}.$$

Let us write

$$\| g(H(\sin \xi/2)) \|_{\dot{H}_2^L[T^d]} = \max \| D^L g(H(\sin \xi/2)) \|_{L_2[T^d]}$$

where the maximum is taken over all derivatives D^L of order L. Then we have showed

LEMMA 8.2. - The linear operator S, defined by

$$(Sg)(\xi) = g(H(\sin \xi/2)),$$

maps X into $L_2[T^d]$ and X^L into $\dot{H}_2^L[T^d]$, continuously.

By means of the two space X and X^L, we shall now construct a new space $X^{s,q}$. This will be done by means of the following general device.

Let A_0 and A_1 be two (semi-)normed spaces, continuously imbedded in a topological vector space. Put

$$K(r, g; A_0, A_1) = \inf_{g=g_0+g_1} (\| g_0 \|_{A_0} + r \| g_1 \|_{A_1}), \qquad g \in A_0 + A_1,$$

$$J(r, g; A_0, A_1) = \max (\| g \|_{A_0}, r \| g \|_{A_1}), \qquad g \in A_0 \cap A_1.$$

Then we construct a new (semi-)normed space $(A_0, A_1)_{\theta, q}$ $(0 < \theta < 1,\ 1 \leq q \leq \infty)$ defined by each one of the equivalent (semi-)norms

$$\left(\int_0^\infty (r^{-\theta} K(r, g; A_0, A_1))^q \frac{dr}{r} \right)^{1/q},$$

$$\left(\sum_{k=-\infty}^{+\infty} (2^{\theta L k} K(2^{-Lk}, g; A_0, A_1))^q \right)^{1/q},$$

$$\inf\left(\int_0^\infty (r^{-\theta}J(r,\ g(r);\ A_0,\ A_1))^q \frac{dr}{r}\right)^{1/q}.$$

$$\left(\text{infimum over all } g(r) \text{ such that } g = \int_0^\infty g(r)\frac{dr}{r} \text{ in } A_0 + A_1\right),$$

$$\inf\left(\sum_{k=-\infty}^{+\infty} (2^{\theta Lk},\ g_k;\ A_0,\ A_1))^q\right)^{1/q},$$

$$(\text{infimum over all } g_k \text{ such that } g = \sum_{-\infty}^{+\infty} g_k \text{ in } A_0 + A_1).$$

The spaces $(A_0,\ A_1)_{\theta,q}$ have the following interpolation property: If T is a bounded linear operator from A_0 into B_0 with norm M_0 and from A_1 into B_1 with norm M_1, then T maps $(A_0,\ A_1)_{\theta,q}$ into $(B_0,\ B_1)_{\theta,q}$ with norm M_θ and

$$M_\theta \leq M_0^{1-\theta} M_1^\theta,$$

(see PEETRE [19], [20]).

Now we define $X^{s,q}$ by

$$X^{s,q} = (X,\ X^L)_{s/L,\,q}, \qquad 0 < s < L, \qquad 1 \leq q \leq \infty.$$

One can show by means of the so called stability theorem for interpolation spaces (see PEETRE [17], [18]) that $X^{s,q}$ is independent of L.

From the interpolation property we now get

COROLLARY 8.1. - Let S be the linear operator defined in lemma 8.2. Then S is a bounded operator from $X^{s,q}$ into $\dot{B}_s^{s,q}[T^d]$. In particular we have

$$\| G_t \|_{M_1} \leq C \| g_t \|_{X^{d/2,1}},$$

if

$$g_t(u) = \varphi(-t\lambda^{-1} \log f(u)),$$

$$G_t(\xi) = \varphi(tP_1(\xi)),$$

$$P_1(\xi) = -t\lambda^{-1} \log f(H(\sin \xi/2)).$$

PROOF. - The corollary follows immediately if we can show

$$\dot{B}_p^{s,q}[T^d] = (L_p[T^d], \qquad \dot{H}_p^L[T^d])_{s/L,\,q}.$$

In the R^d-case this is proved in PEETRE [20], but the proof carries over to our situation. Since we return to related questions in section 10 we prefer to postpone the proof to that section. The reader is thus referred to section 10.1.

Before we can start proving our theorems we have to estimate the weight $w(u)$, given by (8.15). We shall only need an estimate when u is small. Note however that

$$\int_0^1 w(u)du = \int_{T^d} d\xi < \infty. \tag{8.16}$$

LEMMA 8.3. - Suppose that $w(u)$ is given by (8.15). Then

$$w(u) \leq Cu^{\frac{d}{m}-1}, \qquad 0 < u < 1/2. \tag{8.17}$$

PROOF. - According to (8.15) we have

$$w(u) = \int \frac{dS_u}{|\operatorname{grad} H(\sin \xi/2)|}.$$

Since H is homogenous we have

$$|\sin \xi/2|^m \leq CH(\sin \xi/2) = Cu$$

on the surface $\Sigma_u : H(\sin \xi/2) = u$. Using that the derivatives

$$H_j(\eta) = \frac{\partial}{\partial \eta_j} H(\eta)$$

are homogenous of order $m-1$ and never vanishes simultanously, we get

$$|\operatorname{grad} H(\sin \xi/2)|^2 = \frac{1}{4}\sum_{j=1}^{d} (\cos \xi_j/2)^2 (H_j(\sin \xi/2))^2 \geq CH(\sin \xi/2)^{2\frac{m-1}{m}}.$$

Consequently

$$|\operatorname{grad} H(\sin \xi/2)| \geq Cu^{\frac{m-1}{m}}, \qquad \xi \in \Sigma_u,$$

provided $u < 1/2$. Therefore we get

$$w(u) \leq Cu^{-\frac{m-1}{m}} \int dS_u.$$

However

$$\int dS_u \leq Cu^{\frac{d-1}{m}}, \qquad 0 < u < 1/2.$$

The reader will easily supply the details in the proof of this fact. Then the estimate (8.17) follows.

For conveniance we shall write

$$w(u) = u^{\frac{d}{m}-1} w_0(u),$$

where according to

$$w_0(u) \leq C, \qquad 0 < u < 1/2.$$

Clearly $w_0(u)$ is integrable over the interval $0 < u < 1$, (see (8.16)).

PROOF OF THEOREM 8.1. - We write $t = Nk = N\lambda$, since $k = \lambda h^m$, $h = 1$. We shall use corollary 8.1. We write

$$g_N(u) = f(u)^N$$

and our proof will consist of the decomposition of g_N into a sum of two functions g_{N0} and g_{N1}, depending on u and r, such that

$$N^{d/2m} \| g_{N0} \|_X \leq C \min(1,\ r^{m+d/2}), \tag{8.18}$$

$$r^L N^{(d/2-L)/m} \| g_{N1} \|_{X^L} \leq C \min(1,\ r^{m+d/2}), \tag{8.19}$$

with $L > m + \dfrac{d}{2}\cdot$

As soon as these inequalities are proved, theorem 8.1 follows easilv. Write $K(r,\ g) = K(r,\ g;\ X,\ X^L)$. Then (8.18) and (8.20) implies

$$N^{d/2m} K(N^{-L/m} r^L,\ g_N) \leq C \min(1,\ r^{m+d/2}).$$

According to the definition of the space $X^{d/2,1}$ we get

$$\| g_N \|_{X^{d/2,\,2}} = \int_0^\infty \frac{K(s,\ g_N)}{s^{d/2L}} \frac{ds}{s} = (s = N^{-L/m} r^L)$$

$$= L \int_0^\infty \frac{N^{d/2m} K(N^{-L/m} r^L,\ g_N)}{r^{d/2}} \frac{dr}{r} \leq$$

$$\leq CL \int_0^\infty \frac{\min(1,\ r^{m+d/2})}{r^{d/2}} \frac{dr}{r} \leq C' < \infty .$$

Using corollary 8.1 we now get the conclusion of theorem 8.1.

It remains to define g_{N0} and g_{N1} and to prove (8.18) and (8.19). Let χ be infinitely differentiable and such that

$$\chi(u) = 1, \qquad u < 1/4,$$
$$\chi(u) = 0, \qquad u > 1/2$$
$$0 \leq \chi(u) \leq 1.$$

We now define

$$g_{N0}(u) = \chi\left(\frac{Nu}{r^m}\right)(g_N(u) - 1),$$

$$g_{N1}(u) = \left(1 - \chi\left(\frac{Nu}{r^m}\right)\right)(g_N(u) - 1) + 1,$$

for $0 < r < 1$. For $r \geq 1$ we write

$$g_{N0}(u) = g_N(u),$$

$$g_{N1}(u) = 0,$$

We leave to the reader to verify that (8.18) holds in case $r \geq 1$, and concentrate on the case $0 < r < 1$. The argument needed in the case $r \geq 1$ is similar to the one used below.

We shall prove (3.18) for $0 < r < 1$. By condition (8.7) we have

(8.20) $$|g_N(u) - 1| = |f(u)^N - 1| \leq 1 - \exp(-N\delta u) \leq C_\delta(Nu).$$

Therefore, using (8.20),

$$\|g_{N0}\|_X \leq C_\delta \left\{ \int\limits_{0<u<r^m/2N} (Nu)^2 \cdot u^{d/m-1} w_0(u) du \right\}^{1/2} \leq C N^{-d/2m} r^{m+d/2},$$

which gives (8.18).

To prove (8.19) we have to estimate the derivatives $D^M g_{N1}$, $M = 1, \ldots, L$. By LEIBNITZ' rule we get

(8.21) $$D^M g_{N1} = \psi_0 - \psi_M - \sum_{K=1}^{M-1} C_{M,K} \psi_K,$$

where

(8.22) $$\psi_0(u) = \left(1 - \chi\left(\frac{Nu}{r^m}\right)\right) g_N^{(M)}(u),$$

(8.23) $$\psi_M(u) = \left(\frac{N}{r^m}\right)^M \chi^{(M)}\left(\frac{Nu}{r^m}\right)(g_N(u) - 1),$$

(8.24) $$\psi_K(u) = \left(\frac{N}{r^m}\right)^K \chi^{(K)}\left(\frac{Nu}{r^m}\right) g_N^{M-K}(u).$$

Now condition (8.7) gives

(8.25) $$| D^J g_N(u) | = | D^J f(u)^N | \leq C_J N^J \exp(-\delta N u).$$

Thus we have

$$\| u^{M-L/m}\psi_0(u) \|_X \leq$$

$$\leq CN^M \left\{ \int_{r^m/4N\leq u\leq 1} \exp(-2\delta N u) u^{2(mM+d/2-L)/m-1} w_0(u) du \right\}^{1/2} \leq$$

$$\leq CN^M \left\{ \int_{r^m/4N\leq u\leq 1/2} u^{2(mM+d/2-L)/m-1} du + \exp(-N\delta) \right\}^{1/2}$$

Consequently

$$\| u^{M-L/m}\psi_0(u) \|_X \leq C \{ N^{(L-d/2)/m} r^{m+d/2-L} + 1 \}^{1/2} \leq$$

$$\leq CN^{(L-d/2)/m} \cdot r^{m+d/2-L},$$

provided that $L > m + d/2$, $r < 1$.

In the same way we get

$$\| u^{M-L/m}\psi_M(u) \|_X \leq CN^M r^{-Mm} \left\{ \int_{r^m/4N\leq u\leq r^m/2N} (| g_N(u) - 1 | u^{M-L/m})^2 w(u) du \right\}^{1/2}$$

$$\leq CN^N r^{-Mm} \left\{ \int_{r^m/N\leq u\leq r^m/2N} (Nu \cdot u^{M-L/m})^2 u^{d/m-1} du \right\}^{1/2} \leq$$

$$\leq CN^{(L-d/2)/m} \cdot r^{m+d/2-L}$$

and similarly

$$\| u^{M-L/m}\psi_K(u) \|_X \leq CN^{(L-d/2)/m} r^{m+d/2-L}.$$

This gives (8.19) and theorem 8.1 is proved.

REMARK 8.2. - The construction of g_{N0} and g_{N1} and the proof of (8.18) and (8.19) is analogous to the proof of theorem 4.4 in LÖFSTRÖM [16] and theorem 2.2 in PEETRE [20].

PROOF OF THEOREM 8.2. - Since $f(u)$ is positive for $0 \leq u \leq 1$, we have by (8.7)

(8.26) $$\delta u \leq -\log f(u) \leq \rho u, \qquad 0 \leq u \leq 1,$$

for some ρ. Moreover, $\log f(u)$ is infinitely differentiable on $0 < u < 1$, and has bounded derivatives there.

Put

$$g_t(u) = \varphi(-t \log f(u)).$$

Without loss of generality we can assume that $\varphi(0) = 0$, and

$$| D^J \varphi(u) | \leq C u^{\alpha - J} \tag{8.27}$$

Then, using lemma 1.5, (8.26) and (8.27)

$$| D^J g_t(u) | \leq C \sum_{K=1}^{J} t^K | \varphi^{(K)}(-t \log f(u)) | \leq C \sum_{K=1}^{J} (tu)^\alpha u^{-K}$$

and therefore

$$| D^J g_t(u) | \leq C (tu)^\alpha u^{-J}.$$

Let χ be the function used in the proof of theorem 8.1 and put

$$g_{t0}(u) = \chi\left(\frac{tu}{r^m}\right) g_t(u),$$

$$g_{t1}(u) = \left(1 - \chi\left(\frac{tu}{r^m}\right)\right) g_t(u),$$

for $0 < r^m < \min(1;\ t)$. Then

$$t^{d/2m} \| g_{t0} \|_X \leq C t^{d/2m} \left\{ \int\limits_{0<u<r^m/2t} (tu)^{2\alpha} u^{d/m-1} w_0(u) du \right\}^{1/2} \leq C r^{\alpha m + d/2}.$$

As in the proof of theorem 8.1 we write

$$D^M g_{t1} = \psi_0 - \psi_M - \sum_{K=1}^{M-1} C_{M,K} \psi_K$$

where the functions ψ_J, $J = 0, 1, \ldots, M$ are defined by formulas (8.22), (8.23) and (8.24). However we replace N by t and $g_N(u) - 1$ by $g_t(u)$. We have

$$t^{(d/2-L)/m} \| u^{M-L/m} \psi_0(u) \|_X \leq$$

$$\leq C t^{(d/2-L)/m} \left\{ \int\limits_{r^m/4t \leq u \leq 1} (tu)^{2\alpha} u^{-2L/m + d/m - 1} w_0(u) du \right\}^{1/2} \leq$$

$$\leq C \left\{ t^{(d-2L)/m} \int\limits_{r^m 4t \leq u \leq 1/2} (tu)^{2\alpha} u^{-2L/m + d/m - 1} du + t^{(2\alpha m + d - 2L)/m} \right\}^{1/2} \leq$$

$$\leq C r^{\alpha m + d/2 - L}$$

since $0 < r^m < t$ and $L > \frac{d}{2} + \alpha m$. Similarly

$$t^{(d/2-L)/m} \| u^{M-L/m} \psi_K(u) \|_X \leq C r^{\alpha m + d/2 - L},$$

for $1 \leq K \leq M$ and therefore

$$r^L t^{(d/2-L)/m} \| g_t(u) \|_{X^L} \leq C r^{\alpha m + d/2},$$

for $0 < r^m < \min(1, t)$.

In the case $t < 1$, $t < r^m < \infty$ we take

$$g_{t0}(u) = g_t(u),$$

$$g_{t1}(u) = 0,$$

and then we easily get

$$t^{d/2m} \| g_{t0} \|_X \leq C \min(1, r^{\alpha m + d/2}),$$

It remains to define g_{t0} and g_{t1} for $r > 1$, when $t > 1$. We put

$$g_{t0}(u) = \chi(tu) g_t(u),$$

$$g_{t1}(u) = (1 - \chi(tu)) g_t(u).$$

Then the calculations above shows that

$$t^{d/2m} \| g_{t1} \|_X \leq C$$

and

$$t^{(d/2-L)/m} \| g_{t1} \|_{X^L} \leq C.$$

We have proved that

$$t^{d/2m} \| g_{t0} \|_X \leq C \min(1, r^{\alpha m + d/1}),$$

$$r^L t^{(d/2-L)/m} \| g_{t1} \|_{X^N} \leq C \min(1, r^{\alpha m + d/2}).$$

Using the same argument as in the proof of theorem 8.1 we see that theorem 8.2 follows.

9. - Rate of convergence of functions of $P_h(D)$.

In this section we shall study the rate of convergence of the operator

$$E_h(t) = \varphi(tP_h(D)),$$

to the operator

$$E(t) = \varphi(tP(D)).$$

We shall assume that $E_h(t)$ and $E(t)$ are stable, in the sense that

$$\|\varphi(tP_h(\xi))\|_{M_p} \leq C,\ 0 < h < 1,\ t \in I, \tag{9.1}$$

$$\|\varphi(tP(\xi))\|_{M_p} \leq C,\ t \in I, \tag{9.2}$$

where

$$I = \{t;\ 0 < t < \infty\}.$$

We assume that $P(t)$ is homogenous of order $m > 0$ positive and infinitely differentiable for $\xi \neq 0$. We suppose that $P_h(\xi)$ is positive for $\xi \neq 0$ and that $P_h(D)$ approximates $P(D)$ of order exactly s, in the sense of definition 7.1.

We shall prove the following analogue of theorem 7.1.

THEOREM 9.1. - Assume that $E_h(t)$ and $E(t)$ are stable and that $P_h(D)$ approximates $P(D)$ with degree exactly s. Suppose moreover that φ satisfies

$$|D^J\varphi(x)| \leq C_J \min(x^{\alpha-J},\ x^{-\beta-J}) \tag{9.3}$$

for $J = 1, 2, \dots Nt$, $N > \frac{d}{2}$, and $\alpha, \beta > 0$, and that

$$|\varphi'(0)| > 0. \tag{9.4}$$

Then $E_h(t)$ is a saturated approximation of $E(t)$ with order s and the corresponding LIPSCHITZ spaces Λ_σ satisfy

$$\Lambda_\sigma = B_p^{\sigma,\infty},\ 0 < \sigma \leq s.$$

PROOF. - The proof is parallel to the proof of theorem 7.1, so we shall only indicate the main steps in the proof.

First we prove

$$\Lambda_\sigma \supseteq B_p^{\sigma,\infty},\ 0 < \sigma \leq s,$$

using theorem 5.1 and corollary 5.1. Thus we shall prove

$$\sum_{h2^n \leq \varepsilon} 2^{-ns} h^{-s} \| \varphi(tP_h(\xi)) - \varphi(tP(\xi)) \|_n \leq C, \tag{9.5}$$

for $0 < t < \infty$, $0 < h < 1$. We recall that

$$P_h(\xi) - P(\xi) = h^s |\xi|^{m+s} Q(h\xi),$$

where $Q(\eta)$ is infinitely differentiable and bounded for $0 < |\eta| < \varepsilon_0$. Consequently we have

$$\varphi(tP_h(\xi)) - \varphi(tP(\xi)) = t(P_h(\xi) - P(\xi)) \int_0^1 \varphi'(th^{-m} S_r(h\xi)) dr$$

where

$$S_r(h\xi) = P(h\xi) + rh^m(P_h(\xi) - P(\xi)) = P(h\xi) + r(h|\xi|)^{m+s} Q(h\xi).$$

As in the proof of theorem 7.1 we get for $h2^n < \varepsilon$, ε small

$$\| \varphi(tP_h(\xi)) - \varphi(tP(\xi)) \|_n \leq Ct2^{mn}(h2^n)^s \int_0^1 \| \varphi'(th^{-m} S_r(h\xi)) \|_n \, dr.$$

By lemma 1.2 we see that

$$\| \varphi'(th^{-m} S_r(h(\xi)) \|_n \leq C \sup_{2^{n-2} \leq |\xi| \leq 2^{n+2}} \sup_{0 \leq J \leq N} | |\xi|^J D^J \varphi'(th^{-m} S_r(h\xi)) | .$$

since

$$|\xi|^M | D^M S_r(h\xi) | \leq C_M h^m |\xi|^m \leq C_M \varepsilon^m . \tag{9.6}$$

Now by (9.3)

$$| \varphi^{(K+1)}(th^{-m} S_r(h\xi)) | \leq C_K \min \{ (th^{-m} S_r(h\xi))^{\alpha-K-1} ; (th^{-m} S_r(h\xi))^{-\beta-K-1} \} .$$

If ε is small enough

$$| S_r(h\xi) | \geq C(h|\xi|)^m (1 - D\varepsilon^s) \geq C'(h2^n)^m$$

for $\xi \in U_{2^n}$. Using this estimate and (9.6) (for $M = 0$), we get

$$| \varphi^{(K+1)}(th^{-m} S_r(h\xi)) | \leq C_K \min \{ (t2^{nm})^{\alpha-1} ; (t2^{nm})^{-\beta-1} \} (t2^{nm})^{-K} . \tag{9.7}$$

It follows that for $\xi \in U_{2^n}$, $h2^n < \varepsilon$

$$|\xi|^J | D^J \varphi'(th^{-m} S_r(h\xi)) | \leq C_J \min \{ (t2^{nm})^{\alpha-1} ; (t2^{nm})^{-\beta-1} \} ,$$

and therefore

$$\|\varphi(tP_h(\xi))\|_n \leq C(t2^{nm})(h2^n)^s \min\{(t2^{nm})^{\alpha-1};\ (t2^{nm})^{-\beta-1}\}$$

for $h2^n \leq \varepsilon$, ε small. This gives

$$\sum_{h2^n \leq \varepsilon} (h2^n)^{-s}\|\varphi(tP_h(\xi)) - \varphi(tP(\xi))\|_n \leq$$

$$\leq C \sum_{-\infty}^{-\infty} (t2^{nm}) \min((t2^{nm})^{\alpha-1};\ (t2^{nm})^{-\beta-1}) \leq$$

$$\leq C\int_0^\infty (tx^m \min((tx^m)^{\alpha-1};\ (tx^m)^{-\beta-1})\frac{dx}{x} = C\int_0^\infty \min(x^\alpha;\ x^{-\beta})\frac{dx}{y}.$$

The integral is finite if $\alpha, \beta > 0$.

Next we prove that lor $t = \tau^m 2^{-nm}$, $h = l2^{-n}$ (τ, l small), we have

$$\|\{\varphi(tP_h(\xi)) - \varphi(tP(\xi))\}^{-1}\|_n \leq C\tau^{-m}l^{-s},\ n = 1, 2, \ldots . \tag{9.8}$$

From this inequality follows

$$\Lambda_\sigma \subseteq B_p^{\sigma,\infty},\ 0 < \sigma,$$

and that the approximation is saturated of order s.

Now for $t = \tau^m 2^{-nm}$, $h = l2^{-n}$ we have

$$tP_h(\xi) = tP(\xi) + th^s|\xi|^{m+s}Q(h\xi) = P(\tau 2^{-n}\xi) + \tau^m l^s(2^{-n}|\xi|)^{m+s}Q(l2^{-n}\xi).$$

Thus we have (using (1.17)),

$$\|\{\varphi(tP_h(\xi)) - \varphi(tP(\xi))\}^{-1}\|_n =$$

$$= \|\{\varphi(P(\tau\xi) + \tau^m l^s|\xi|^{m+s}Q(l\xi)) - \varphi(P(\tau\xi))\}^{-1}\|_1$$

However, the conditions on Q shows that for 1 sufficiently small

$$\|(|\xi|^{m+s}Q(l\xi))^{-1}\|_1 \leq C.$$

Thus

$$\|\{\varphi(tP_h(\xi)) - \varphi(tP(\xi))\}^{-1}\|_n \leq$$

$$\leq C\tau^{-m}l^{-s}\left\|\frac{\tau^m 1^s|\xi|^{m+s}Q(1\xi)}{\varphi(P(\tau\xi) + \tau^m 1^s|\xi|^{m+s}Q(1\xi)) - \varphi(P(\tau\xi))}\right\|_1.$$

Now for $\xi \in U_1$

$$\left| \frac{\tau^m l^s |\xi|^{m+s} Q(l\xi)}{\varphi(P(\tau\xi) + \tau^m l^s |\xi|^{m+s} Q(l\xi)) - \varphi(P(\tau\xi))} \right| \leq C$$

if τ and l sufficiently small, since

$$|\varphi'(0)| \geq C > 0.$$

By induction il is also easy to show that

$$\left| D^J \frac{\tau^m l^s |\xi|^{m-s} Q(l\xi)}{\varphi(P(\tau\xi) + \tau^m l^s |\xi|^{m+s} Q(l\xi)) - \varphi(P(\tau\xi))} \right| \leq C, \ J = 0,\ 1, \dots, N,\ \xi \in U_1.$$

if τ and l are sufficiently small. Thus, by lemma 1.2,

$$\left\| \frac{\tau^m l^s |\xi|^{m+s} Q(l\xi)}{\varphi(P(\tau\xi) + \tau^m l^s |\xi|^{m+s} Q(l\xi)) - \varphi(P(\tau\xi))} \right\|_1 \leq C.$$

This gives

$$\| \{ \varphi(tP_h(\xi)) - \varphi(tP(\xi)) \}^{-1} \|_n \leq C\tau^{-m} l^{-s}$$

if $t = \tau^m l^{-nm}$, $h = l2^{-n}$ and τ and l are small enough. This is what we wanted to prove.

As an application we give

COROLLARY 9.1. - Suppose that $P_h(D)$ is the infinitesimal generator of a strongly continuous semi-group $E_h(t)$ on L_p, which is uniformly bounded.

Suppose moreover that $P_h(D)$ approximates $P(D)$ with order exactly s. Let $R_h(\mu)$ and $R(\mu)$ be the resolvents of $P_h(D)$ and $P(D)$, respectivety, i.e.

$$R_h(\mu) = (\mu + P_h(D))^{-1}$$

$$R(\mu) = (\mu + P(D))^{-1}.$$

Write

$$F_h(t) = \frac{1}{t} R_h\left(\frac{1}{t}\right) = (1 + tP_h(D))^{-1}$$

$$F(t) = \frac{1}{t} R\left(\frac{1}{t}\right) = (1 + tP(D))^{-1}.$$

Then $F_h(t)$ is a saturated approximation of $F(t)$ with order s and the corresponding LIPSCHITZ spaces Λ_σ satisfy

$$\Lambda_\sigma = B_p^{\sigma, \infty},\ 0 < \sigma \leq s.$$

PROOF. - From our assumptions we deduce that

$$\|R_h(\mu)f\|_{L_p} \leq C_0\mu^{-1}\|f\|_{L_p}, \ \mu > 0,$$

$$\|R(\mu)f\|_{L_p} \leq C_1\mu^{-1}\|f\|_{L_p}, \ \mu > 0.$$

This follows immediately from the formulas

$$R_h(\mu)f = \int_0^\infty e^{-\mu t}E_h(t)fdt,$$

$$R(\mu)f = \int_0^\infty e^{-\mu t}E(t)fdt, \ E(t) = \exp\,(-tP(D)),$$

(See i.g. BUTZER-BERENS [7]). Thus

$$\|F_h(t)f\|_{L_p} \leq C_0\|f\|_{L_p}, \ 0 < t < \infty,$$

$$\|F(t)f\|_{L_p} \leq C_1\|f\|_{L_p}, \ 0 < t < \infty.$$

Now

$$F_h(t) = \varphi(tP_h)), \ F(t) = \varphi(tP(D))$$

where

$$\varphi(x) = (1 + x)^{-1}.$$

This function clearly satisfies the assumptions of theorem 9.1. Therefore the conclusion follows.

10. - Convergence of summation methods for Fourier series.

Until now we have considered $L_p[R^d]$. In this section we shall discuss operator on $L_p[T^d]$, where T^d is the d-dimensional torus. In doing so we illustrate how the methods used above carry over to Fourier series, (c.f. remark 1.1 and 2.1). We shall discuss operators E_h on $L_p = L_p[T^d]$, defined by

$$E_hf(x) = (2\pi)^{-d} \sum_{\xi \in Z^d} \exp\,(i < x, \ \xi >)\, e_h(\xi)f^\wedge(\xi),$$

where

$$f^\wedge(\xi) = \int_{T^d} \exp\,(i, < x, \ \xi >)f(x)dx.$$

The function $e_h(\xi)$ will be of the particular form

$$e_h(\xi) = \varphi(hP(\xi)),$$

where φ is a given function on the positive real line and $P(\xi)$ is again a homogeneous function of order $m > 0$. We assume that $P(\xi)$ is defined for $\xi \in R^d$, $\xi \neq 0$ and that $P(\xi)$ is infinitely differentiable and positive for $\xi \neq 0$.

We shall cousider two questions; the first one is the question of stability i.e.

$$\|E_h f\|_{L_p} \leq C\|f\|_{L_p}, \ 0 < h.$$

The second one is the question of the rate of convergence of $E_h f$ to f. We want to characterize the space of all function $f \in L_p$, such that

$$\|F_h f - f\|_{L_p} = 0(h^s), \ k \to 0.$$

These two questions are closely related, as we have shown above. The object is to carry out an analysis similar to the one developed in LÖFSTRÖM [16], PEETRE [20] and in section 6 in this work. This section has two parts. In the first one we consider the question of stability, in the second one we deal with the rate of convergence.

10.1. - Stability theorems.

We shall denote by m_p the space of Fourier multipliers on $L_p = L_p[T^d]$. Thus m_p is defined by

$$\|\psi * f\|_{L_p} \leq \|\psi^\wedge\|_{m_p}\|f\|_{L_p},$$

and $\psi^\wedge(\xi)$, $\xi \in Z^d$ are the Fourier coefficients of ψ. The stability in L_p of an operator $E_h f$ given by $(E_h f)^\wedge(\xi) = e_h(\xi) f^\wedge(\xi)$ is clearly equivalent to

$$\|e_h\|_{m_p} \leq C.$$

Most of the facts about multipliers on $L_p[R^d]$ carry over to the m_p-spaces. In particular m_p is a BANACH algebra under pointwise multiplication. We define the space l_p by means of the norm

$$\|g\|_{l_p} = \Big(\sum_{\xi \in Z^d} |g(\xi)|^p \Big)^{1/p}, \ (1 \leq p \leq \infty).$$

Then $m_2 = l_\infty$ and $m_1 \subseteq m_p \subseteq m_2$. Clearly

$$\|\Psi^\wedge\|_{m_1} \leq \|\Psi\|_{L_1} \leq \|\Psi^\wedge\|_{l_1} \tag{10.1}$$

and lemma 8.1 shows

$$\|\Psi^\wedge\|_{m_1} \leq C_d \|\Psi\|_{\dot{B}_2^{d/2,1}[T^d]}. \tag{10.2}$$

We define the local multiplier spaces $m_p(V)$ in analogy with definition of $M_p(V)$. Thus

$$\|\Psi^\wedge\|_{m_{p(V)}} = \inf \|\chi^\wedge\|_{m_p}, \tag{10.3}$$

where the infimum is taken over all $\chi^\vee \in m_p$, such that $\Psi^\vee = \chi^\wedge$ on the finite subset V of Z^d.

We shall now establish the analogue of lemma 1.2 for the local multiplier spaces $m_p(U_r)$, where U_r is the annulus $2^{-1}r \leq |\xi| \leq 2r$, $\xi \in Z^d$. The role of the differential operator D in lemma 1.2 will be played by the difference operator $\Delta^\alpha = \Delta_1^{\alpha_1} \ldots \Delta_d^{\alpha_d}$, $\alpha = (\alpha_1, \ldots, \alpha_d)$, where

$$\Delta_j g(\xi) = g(\xi + e_j) - g(\xi).$$

Here e_j is ths unit vector $(\delta_{1j}, \ldots, \delta_{dj})$. If $g(\xi)$ are the Eourier coefficients of the function $g^v(x)$, then $\Delta^\alpha g(\xi)$ are the Fouries coefficients of

$$\prod_{j=1}^{d} (\exp(-ix_j) - 1)^{\alpha_j} g^v(x).$$

Motivated by this formula we define the operator $|\Delta|$ by saying that $|\Delta| g(\xi)$ are the Fourier coefficients of the function

$$\left(\sum_{j=1}^{d} |\exp(-ix_j) - 1|^2\right)^{1/2} g^v(x) = 2\left|\sin\frac{x}{2}\right| g^v(x),$$

where we write

$$\left|\sin\frac{x}{2}\right|^2 = \sum_{j=1}^{d} \left(\sin\frac{x_j}{2}\right)^2.$$

Note that

$$|\Delta|^2 = \Delta_1^2 + \ldots + \Delta_d^2. \tag{10.4}$$

If g is a function defined on R^d we shall write $g \in l_p$ or $g \in m_p$, if the restriction of g to Z^d, belongs to l_p or m_p. If g is sufficiently differentiable, then $\Delta^L g$ can be estimated by the corresponding derivative $D^L g$. In fact, we have

$$|\Delta^L g(\xi)| \leq C_L \sup_{\substack{|\eta| \leq L \\ \eta \in R^d}} |D^L g(\xi + \eta)|, \quad \xi \in Z^d. \tag{10.5}$$

It is clearly sufficient to prove this for $L = 1$ in the one-dimensional case. But then (10.5) is obvious.

Before we formulate our basic result on local multipliers, let us also write down the analogue of LEIBNITZ' formula for differentialion of a product. We define the translation operator τ^α by $\tau^\alpha = \tau_1^{\alpha_1} \dots \tau_d^{\alpha_d}$ and

$$\tau_j g(\xi) = g(\xi + e_j).$$

If Δ^L denotes any difference operator of order L, we then have

$$\Delta^L fg = \sum_{M \le L} C_{L,M}(\Delta^{L-M}\tau^M f)\Delta^M g. \tag{10.6}$$

LEMMA 10.1. - Let L be an even integer larger than $d/2$ and let U_r be the annulus $2^{-1}r \le |\xi| \le 2r$, $\xi \in Z^d$. Then

$$\|g\|_{m_1(U_r)} \le C_d(\sup_{4^{-1}r \le |\xi| \le 4r} |g(\xi)|)^{1-d/2L} \max_{1 \le M < L} \sup_{4^{-1}r \le |\xi| \le 4r} |\xi|^M |\Delta^M g(\xi)|^{d/2L} \tag{10.7}$$

In particular

$$\|g\|_{m_1(U_r)} \le C_d \max_{0 \le M \le L} \sup_{4^{-1}r \le |\xi| \le 4r} |\xi|^M |\Delta^M g(\xi)|. \tag{10.8}$$

PROOF. - By means of CAUCHY-SCHWARZ' inequality and PARSEVAL'S relation we get, just as in the proof of lemma 1.2.

$$\int_{|\sin \frac{x}{2}| \ge r^{-1}} |g^v(x)|\, dx \le C_d r^{L-d/2} \|\Delta^L g\|_{l_2}, \quad L > d/2, \tag{10.9}$$

$$\int_{|\sin \frac{x}{2}| \le r^{-1}} |g^v(x)|\, dx \le C_d r^{-d/2} \|g\|_{l_2} \tag{10.10}$$

Now write

$$\|g\|_{\dot{h}_2^L} = \max_{|\alpha|=L} \|\Delta^\alpha g\|_{l_2}.$$

Then if L is even we get from (10.4)

$$\| |\Delta|^L g\|_{l_2} \le C \|g\|_{\dot{h}_2^L}.$$

Therefore (10.9) and (10.10) give, combined with (10.1)

$$\|g\|_{m_1} \le C_d r^{-d/2} (\|g\|_{l_2} + r^L \|g\|_{\dot{h}_2^L}),$$

for all $r > 0$. Taking the infimum over all $r > 0$ we conclude

$$\|g\|_{m_1} \leq C_d \|g\|_{l_2}^{1-d/2L} \|g\|_{\dot{h}_2^L}^{d/2L}. \tag{10.11}$$

It remain to localize (10.11). Let χ be an infinitely differentiable function on the real line, which is identically 1 on $2^{-1} \leq u \leq 2$ and vanishes outside $4^{-1} \leq u \leq 4$. Then

$$\|g\|_{m_1(U_r)} \leq \|\Psi\|_{m_1},$$

if

$$\Psi(\xi) = \chi(r^{-1}|\xi|)g(\xi).$$

Now LEIBNITZ' formula and (10.5) gives

$$|\Delta^L \Psi(\xi)| \leq C_L \sum_{1 \leq M \leq L} \sup_{\eta \in R^d} |D^{L-M}\Psi(r^{-1}|\eta|)| \, |\Delta^M g(\xi)| \leq$$

$$\leq C'_L \max_{1 \leq M \geq L} r^{M-L} |\Delta^M g(\xi)|, \; 4^{-1}r \leq |\xi| \leq 4r.$$

But it easily seen that

$$\sum_{4^{-1}r \leq |\xi| \leq 4r} 1 \leq C_d r^d.$$

In fact, for small values of r there is nothing to prove. But if r is large, then the sum is smaller than the volume of the annulus $8^{-1}r \leq |\eta| \leq 8r$, $\eta \in R^d$, which is of the order r^d. Thus

$$\|\Psi\|_{\dot{h}_2^L} \leq C_d r^{d/2-L} \max_{1 \leq M \leq L} \Big(\sup_{4^{-1}r \leq |\xi| \leq 4r} |\xi|^M |\Delta^M g(\xi)| \Big).$$

In a similar way

$$\|\Psi\|_{l_2} \leq C_d r^{d/2} \Big(\sup_{4^{-1}r \leq |\xi| \leq 4r} |g(\xi)| \Big).$$

Now (10.7) follows immediately from (10.11), and it is clear that (10.8) follows from (10.7).

REMARK 10.1. - Let $\varphi_k(\eta)$ be the standard functions in the definition of the BESOV spaces $\dot{B}_p^{s,q}[T^d]$, i.e. suppose that $\varphi_k(\eta) = \varphi(2^{-k}\eta)$, where $\varphi(\xi)$ is infinitely differentiable, supported by $2^{-1} \leq |\xi| \leq 2$ and positive on $2^{-1} < |\xi| < 2$. Then we get from the proof of lemma 10.1

$$\|\overset{v}{\varphi}_k\|_{L_1[T^d]} \leq C. \tag{10.12}$$

By means of (10.12) we can now complete the proof of corollary 8.1. We can show

$$\dot{B}_p^{s,q}[T^d] = (L_p[T^d], \dot{H}_p^L[T^d])_{s/L,q}. \tag{10.13}$$

For the convenience of the reader we shall briefly sketch the proof. From (10.12) we get

$$\|D^L\varphi_k\|_{L_1[T^d]} \leq C_L 2^{Lk}, \tag{10.14}$$

$$\| |D|^{-L}\varphi_k^v\|_{L_1[T^d]} \leq C_L 2^{-Lk}. \tag{10.15}$$

Suppose now that $g = g_0 + g_1$. Then (10.12) and (10.15) give

$$\|\varphi_k^v * g_0\|_{L_p[T^d]} \leq C\|g_0\|_{L_p[T^d]},$$

$$\|\varphi_k^v * g_1\|_{L_p[T^d]} \leq C2^{-Lk}\|g_1\|_{\dot{H}_p^L[L^d]}.$$

Thus

$$\|\varphi_k^v * g\|_{L_p[T^d]} \leq CK(2^{-Lk}, g; L_p[T^d], \dot{H}_p^L[T^d]),$$

and therefore

$$(L_p[T^d], \dot{H}_p^L[T^d])_{s/L,q} \subseteq \dot{B}_p^{s,q}[T^d].$$

The converse inclusion follows from (10.14), since

$$\|\varphi_k^v * g\|_{\dot{H}_p^L[T^d]} \leq C2^{Lk} \sum_{j=-1}^{1} \|\varphi_{k+j}^v * g\|_{L_p[T^d]},$$

which implies

$$J(2^{-Lk}, g; L_p[T^d], \dot{H}_p^L[T^d]) \leq C \sum_{j=-1}^{1} \|\varphi_{k+j}^v * g\|_{L_p[T^d]}.$$

Now we are ready to prove two theorems on the stability of the operator E_h, defined by

$$E_h f(x) = (2\pi)^{-d} \sum_{\xi \in Z^d} \varphi(hP(\xi)) f^\wedge(\xi) \exp(i < x, \xi >),$$

where $P(\xi)$ is homogenous of order $m > 0$, infinitely differentiable and positive on $R^d -/\{0\}$. Our first theorem is analogous to lemma 1·4.

THEOREM 10.1. - Let φ be infinitely differentiable and suppose that

$$|\varphi(u) - \varphi(0)| \leq C_0 u^\alpha, \quad 0 < u \leq 1,$$

$$|\varphi(u)| \leq C_0 u^{-\beta}, \quad 1 \leq u < \infty,$$

$$|D^J\varphi(u)| \leq C_J u^{-J} \min(u^\alpha, u^{-\beta}), \quad 0 < u < \infty,$$

for $J = 1, 2, \ldots$. Here α and β are positive numbers. Then E_h is stable in L_p, $1 \leq p \leq \infty$, i.e.

$$\|\varphi(hP(\xi))\|_{m_1} \leq C, \quad 0 < h < \infty.$$

PROOF. - It is no restriction to assume that $\varphi(0)=0$. Let b be a large number. Then (10.1) gives

$$\|\varphi(hP(\xi))\|_{m_1(|\xi|\leq 2b)} \leq \sum_{|\xi|\leq 2b} |\varphi(hP(\xi))|.$$

Now the sum on the right hand side is bounded by

$$C_0 \sum_{|\xi|\leq 2b} \min\left((hP(\xi))^{\alpha},\ (hP(\xi))^{-\beta}\right),$$

which is clearly uniformly bounded in $0<h<\infty$. Thus it remains to show

$$\|\varphi(hP(\xi))\|_{m_1(|\xi|\geq 2)} \leq C,\ 0<h<\infty.$$

But this follows if we prove

$$\sum_{2^k\geq b} \|\varphi(hP(\xi))\|_{m_1(U_{2^k})} \leq C. \tag{10.16}$$

Now the estimate (10.8) of lemma 10.1 and (10.5) give

$$\|\varphi(hP(\xi)\|_{m_1(U_{2^k})} \leq C_d \max_{0\leq M\leq L} \sup_{\substack{2^{k-2}\leq|\xi|\leq 2^{k+2}\\ |\eta|\leq L}} |\xi|^M |D^M\varphi(hP(\xi+\eta))|.$$

By lemma 1.5 we have

$$|D^M\varphi(hP(\xi+\eta))| \leq C_M \sum_{1\leq J\leq M} P(\xi+\eta)^{J-M/m}h^J|\varphi^{(J)}(hP(\xi+\eta))| \leq$$

$$\leq C'_M P(\xi+\eta)^{-M/m} \min\left((hP(\xi+\eta))^{\alpha},\ (hP(\xi+\eta))^{-\beta}\right),$$

(c.f. section 8 and the proof of lemma 1.4). If $|\xi|$ is large and $|\eta|\leq L$, then there is a constant $A>0$ so that

$$A^{-1}|\xi|^m \leq P(\xi+\eta) \leq A|\xi|^m.$$

Thus we get

$$\|\varphi(hP(\xi))\|_{m_1(U_{2^k})} \leq C \min\left((h2^{mk})^{\alpha},\ (h2^{mk})^{-\beta}\right),$$

for $2^k\geq b$, if b sufficiently large. Now (10.16) follows and the proof is complete.

The proof of theorem 10.1 also gives

COROLLARY 10.1. - Suppose that φ is infinitely differentiable on the positive real line and

$$|D^J\varphi(u)| \leq C_J u^{-J},\ J=0,\ 1,\ \dots\,.$$

Then for all $r > 0$

$$\|\varphi(hP(\xi))\|_{m_1(U_r)} \leq C.$$

Simple functions satisfying the assumptions of theorem 10.1 are

$$\varphi(u) = u^\alpha \ \exp\ (-zu),\ \mathrm{Re} z > 0,\ \alpha \geq 0,$$

$$\varphi(u) = u^{-\beta}(\exp\ (-zu) - 1),\ \mathrm{Re} z > 0,\ 0 < \beta \leq 1,$$

$$\varphi(u) = u^\alpha(1 + u)^{-\gamma},\ 0 \leq \alpha < \gamma.$$

Clearly any infinitety differentiable function on the real line, which has compact support, also satisfies the assumptions of theorem 10.1.

In our next theorem we shall weaken the condition of φ at infinity.

THEOREM 10.2. - Let φ be infinitely differentiable on the real line and let φ vanish in a neighbourhood of the origin. Assume

$$|D^J\varphi(u)| \leq C_J u^{-\beta},\ J = 0,\ 1,\ 2,\ \dots,$$

for some β, such that

$$\beta > d|p^{-1} - 2^{-1}|.$$

Then

$$\|\varphi(hP(\xi))\|_{m_p} \leq C,\ 0 < h < \infty.$$

PROOF. - Suppose that $\varphi(u)$ vanishes for $u < 1$. We shall first prove the result for $p = 1$, using the estimate (10.7) of lemma 10.1.

As in the proof of theorem 10.1 we see that it suffices to show

$$\sum_{2^k \geq b} \|\varphi(hP(\xi))\|_{m_1(U_{2^k})} \leq C,$$

where b is a large fixed number. Now it is clear that

$$\max_{2^{k-2} \leq |\xi| \leq 2^{k+2}} |\varphi(hP(\xi))| \leq C(h2^{km})^{-\beta}.$$

To estimate

$$\sup_{2^{k-2} \leq |\xi| \leq 2^{k+2}} |\xi|^M |\Delta^M\varphi(hP(\xi))|,\ 1 \leq M \leq L,$$

we use again (10.5). Thus we see that it suffices to estimate

$$\sup_{\substack{2^{k-2} \leq |\xi| \leq 2^{k+2} \\ |\eta| \leq L}} |\xi|^M |D^M\varphi(hP(\xi + \eta))|.$$

But clearly

$$|D^M\varphi(hP(\xi+\eta))| \leq C_M \sum_{1\leq J\leq M} P(\xi+\eta)^{J-M/m} h^J |\varphi^{(J)}(hP(\xi+\eta))| \leq$$

$$\leq C_L h^{M/m}(hP(\xi+\eta))^{L-M/m}(hP(\xi+\eta))^{-\beta},$$

since $hP(\xi+\eta)\geq 1$. Thus we conclude that if $2^k \geq b$, b sufficiently large, then

$$\sup_{2^{k-2}\leq|\xi|\leq 2^{k+2}} |\xi|^M |\Delta^M\varphi(hP(\xi))| \leq C_L(h2^{mk})^{L-\beta},$$

and thus lemma 10.1 gives

$$\|\varphi(hP(\xi))\|_{m_1(U_{2_k})} \leq C(h2^{mk})^{-(\beta-d/2)}, \; 2^k \geq b. \tag{10.17}$$

Noting that we must have $h2^{mk} \geq c > 0$, we get

$$\sum_{2^k\geq b} \|\varphi(hP(\xi))\|_{m_1(U_{2_k})} \leq C \sum_{2h^{mk}\geq C} (h2^{mk})^{-(\beta-d/2)}.$$

The sum on the right hand side is bounded if $\beta > d/2$ and thus the conclusion follows in the case $p=1$.

To prove the theorem in its general form we note that RIESZ' interpolation theorem gives

$$\|g\|_{m_p} \leq \|g\|_{m_1}^{\eta} \|g\|_{m_2}^{1-\eta}, \qquad p^{-1} = 1+2^{-1}\eta, \qquad 0\leq\eta\leq 1. \tag{10.18}$$

Thus

$$\|g\|_{m_p(U_{2^k})} \leq \|g\|_{m_1(U_{2^k})}^{\eta} \|g\|_{m2(U_{2^k})}^{1-\eta}. \tag{10.18'}$$

However $m_2 = l_\infty$ and so

$$\|\varphi(hP(\xi))\|_{m_2(U_{2^k})} \leq C(h2^{mk})^{-\beta}.$$

Consequently (10.17) and (10.18') show

$$\|\varphi(hP(\xi))\|_{m_p(U_{2^k})} \leq C(h2^{mk})^{-(\beta - d(p^{-1}2^{-1}))}.$$

The gives the conclusion for $1\leq p\leq 2$, and since $m_{p'} = m_p$ for $p'^{-1} = 1-p^{-1}$ the conclusion follows in this full generality.

The result in theorem 10.2 was proved by WAINGER [33] for the special function $\varphi(u) = u^{-\beta}\exp(iu)$, $u>1$, and with $P(\xi) = |\xi|^m$, $0<m<1$. WAINGER also proved that in this case the lower bound for β is the best possible. The analogue of theorem 10.2 does also hold in $L_p[R^d]$. This has been proved by SJÖSTRAND [27], by the same technique as we have used here. By taking $\varphi(u) = u^{-\beta} \exp(iu)$, $P(\xi) = |\xi|^m$, SJÖSTRAND proved, that on $L_p[R^d]$ it is not possible to improve the lower bound for β if $m \neq 1$.

The theorems 10.1 and 10.2 we have considered a function φ, which is «irregular» at 0 and ∞. Now we shall consides the case when φ is irregular in a certain sense in a point u_1, $0 < u_1 < \infty$. Since we want to consider the RIESZ' mean operator R_h^α, given by

$$(10.19) \qquad (\mathcal{R}_h^\alpha f)^\wedge(\xi) = (1 - hP(\xi))_+^\alpha f^\wedge(\xi),$$

we want to include the function

$$\varphi(u) = (1 - u)_+^\alpha .$$

THEOREM 10.3. - Suppose lhat φ is infinitely differentiable on $u \neq u_1$, $(0 < u_1 < \infty)$ and has compact support on $0 < u < \infty$. Assume

$$|D^J\varphi(u)| \leq C_J |u - u_1|^{\alpha - J}, \; u \neq u_1,$$

for $J = 0, 1, 2, \ldots$, for some α, such that

$$(10.20) \qquad \alpha > (d-1)|p^{-1} - 2^{-1}|.$$

Then

$$\|\varphi(hP(\xi))\|_{m_p} \leq C, \; 0 < h < \infty.$$

Clearly we can write $(1-u)_+^\alpha$ as the sum of two functions φ_0 and φ_1, where φ_0 is infinitely differentiable and has compact support on the real line and φ_1 satisfies the assumptious of theorem 10.3. Thus we get from theorem 10.1 and theorem 10.3.

COROLLARY 10.2. - The RIESZ' mean operator $\mathcal{R}_h^\alpha$ given by (10.19) is stable in L_p if (10.20) holds.

The corollary was proved by STEIN [29] for $P(\xi) = |\xi|^2$. STEIN also proved that the lower bound for α can not be improved if $P(\xi) = |\xi|^2$ and $p = 1$.

We also note that the corollary in its general form also follows from the work of SPANNE [28], who used the results of HÖRMANDER [13] and [13'].

In the proof of theorem 10.3 we can not use lemma 10.1, but we shall another (more refined) consequence of the estimates (10.9) and (10.10).

Let χ be infinitely differentiable on the real line and let χ vanish outside the interval $2^{-1} \leq u \leq 2$. Put

$$\Psi_k^*(x) = \chi\left(2^k \left|\sin\frac{x}{2}\right|\right), \; x \in T^d.$$

We can assume that

$$\Sigma\ \Psi_k^*(x) = 1,\ x \neq 0,\ y \in T^d.$$

Put

$$\Psi_k(x) = \Psi_{k+k_0}^*(x),\ k > 0,$$

$$\Psi_0(x) = 1 - \sum_{k>0} \Psi_k(x).$$

where k_0 is a large integer, which we shall choose in the proof of theorem 10.3. It is easy to see that

$$\|(\Delta^L \Psi_k^\wedge)^\vee\|_{M_1[T^d]} \leq C2^{-Lk}, \tag{10.21}$$

and

$$\|(|\Delta|^- \Psi_k^\wedge)^\vee\|_{M_1[T^d]} \leq C2^{Lk}. \tag{10.22}$$

We now define BESOV space $\dot{b}_p^{s,q}$ by

$$\|g\|_{\dot{b}_p^{s,q}} = \left(\sum_{k=0}^{\infty} (2^{-sk} \|\Psi_k^\wedge * g\|_{l_p})^q\right)^{1/q}.$$

We also define the space $\dot{h}_p^L$ by

$$\|g\|_{\dot{h}_p^L} = \max_{|\alpha|=L} \|\Delta^\alpha g\|_{l_p}.$$

Then $\dot{b}_p^{s,q}$ can be given as an interpolation space:

$$\dot{b}_p^{s,q} = (l_p,\ \dot{h}_p^L)_{s/L,\, q},\ 0 < s < L,\ 1 \leq q \leq \infty, \tag{10.23}$$

This follows from (10.21) and (10.22) just as in the proof of (10.13) in remark 10.1. In particular, it is easy to see that

$$\|g\|_{\dot{b}_p^{s,q}} \leq C \int_{2^{k_0}}^{\infty} r^{-s} K r^L,\ g;\ 1_q,\ \dot{h}_q^L) \frac{dr}{r} = \tag{10.24}$$

$$= C \int_{h2^{k_0}}^{\infty} r^{-s} h^s K(r^L h^{-L},\ g;\ 1_q,\ \dot{h}_q^L) \frac{dr}{r}.$$

The substitute for lemma 10.1. which we shall need in the proof of theorem 10.3 is the following lemma.

LEMMA 10.2. - We have

(10.25) $$\|g\|_{m_1} \leq C\|g\|_{\dot{b}_2^{d/2,\,1}}$$

and if $q^{-1} > p^{-1} - 2^{-1} \geq 0$, $2 < q < \infty$, then

(10.26) $$\|g\|_{m_p} \leq C\|g\|_{\dot{b}_q^{d/d,\,1}}.$$

(c.f. the theorems of BERNSTEIN and HIRSCHMAN in PEETRE [20]).

PROOF OF LEMMA 10.2. - Let $g = g_0 + g_1$ and apply (10.9) and (10.10) to the functions $\Psi_k^\wedge * g_0$ and $\Psi_k^\wedge * g_1$. With $r = 2^{k+k_0}$ we get

$$\|\Psi_k^\wedge * g\|_{m_1} \leq C2^{-(k+k_0)d/2} K(2^{L(k+k_0)},\ g;\ l_2,\ \dot{h}_2^L)$$

and thus (10.25) follows from (10.23). Now the interpolation property (see section 8) gives

(10.27) $$(l_\infty,\ \dot{b}_2^{d/2,\,1})_{\theta,\,q} \subseteq (m_2,\ m_1)_{\theta,\,q},\ \theta = 2/q.$$

We shall prove

(10.28) $$(m_2,\ m_1)_{\theta,\,q} \subseteq m_p,$$

and

(10.29) $$\dot{b}_q^{d/q,\,1} \subseteq (l_\infty,\ \dot{b}_q^{d/2,\,1})_{\theta,\,q}.$$

This clearly gives the conclusion.
To prove (10.28) suppose

$$g = \Sigma\ g_k.$$

Then (10.18) gives

$$\|g\|_{m_p} \leq \Sigma\ 2^{-k\eta}\|g_k\|_{m_2}^{1-\eta}(2^k\|g_k\|_{m_1})^\eta \leq$$

$$\leq \Sigma\ 2^{-k\eta} J(2^k,\ g_k;\ m_2,\ m_1),\ p^{-1} = 1 + \eta/2,$$

where

$$J(2^k,\ g_k;\ m_2,\ m_1) = \max\ (\|g_k\|_{m_2},\ 2^k\|g_k\|_{m_1}).$$

(c.f. section 8). Thus $(m_2,\ m_1)_{\eta,\,1} \subseteq m_p$. But it is easily see that $m_1 \subseteq m_2$ implies

$$(m_2,\ m_1)_{\theta,\,p} \subseteq (m_2,\ m_1)_{\eta,\,1},\ \eta < \theta.$$

Thus (10.28) follows.

To prove (10.29) we write $g_k = \Psi_k^\wedge * g$. Then

$$\|g_k\|_{\dot{b}_2^{d/2,1}} \leq C\, 2^{kd/2} \sum_{j=-1}^{1} \|\Psi_{k+j}^\wedge * g\|_{l_2}$$

and thus, by the interpolation property

$$\|g_k\|_{(l_\infty,\, \dot{b}_2^{d/2,1})_{\theta,q}} \leq C 2^{k\theta\, d/2} \sum_{j=-1}^{1} \|\Psi_{k+j}^\wedge * g\|_{l_q},$$

for $\theta = 2/q$. This follows from the well-known fact that

$$(l_\infty,\, l_2)_{\theta,q} = l_p, \ \theta = 2/q.$$

But now (10.29) follows at once and the proof is complete.

(The proof of lemma 10.2 is a word by word transcription of the corresponding proof for multipliers on $L_p[R^d]$ in PEETRE [20]).

We are now ready for the proof of theorem 10.3.

PROOF OF THEOREM 10.3. - For simplicity we take $u_1 = 1$. It suffices to prove the theorem for $m = 1$. In fact, suppose that the theorem is true in this case and let φ satisfy the assumptions. Write $\varphi_m(u) = \varphi(u^m)$ and $P_0(\xi) = P(\xi)^{1/m}$, where P is a given homogenous function of order m. Then P_0 has order 1 and $\varphi(hP(\xi) = \varphi_m(h^{1/m}P_0(\xi))$. But is quite easy to see that if φ has compact support on $0 < u < \infty$ and satisfies the assumptions of the theorem, then θ_m has the same properties. Thus the theorem for $m = 1$ gives the conclusion for $\varphi(hP(\xi))$.

Take $m = u_1 = 1$. By lemma 10.2 and (10.24) it suffices to show

$$\int_{Ah} r^{-d/q} h^{d/q} K(r^L h^{-L}) \frac{dr}{r} \leq C, \qquad 0 < h < \infty, \tag{10.30}$$

where $A = 2^{k_0}$ is a large costant, and

$$K(s) = K(s,\ \varphi(hP(\xi));\ l_q,\ \dot{h}_q^L).$$

If we want to show the conclusion of the theorem for $p = 1$ we shall take $2\alpha > d - 1$ and $q = 2$, thus using (10.25). If we want to get the result for $1 < p \leq 2$ we take $q \cdot \alpha > d - 1$, $q^{-1} > p^{-1} - 2^{-1}$ and use (10.26).

Note that $K(s)$ is defined by

$$K(r^L h^{-L}) = \inf\ (\|\varphi_0\|_{l_q} + r^L h^{-L} \|\varphi_1\|_{\dot{h}_q^L}),$$

where we take infimum over all φ_0 and φ_1 such that

$$\varphi(hP(\xi)) = \varphi_0(\xi) + \varphi_1(\xi). \tag{10.31}$$

Want we have to do is therefore to define a decomposition (10.31), which is near the optimal one and then try to show (10.30).

First take $\varphi_1(\xi) = 0$. Let φ be supported by the set $a \leq u \leq b$, $(0 < a < 1 < b < \infty)$. Then we get

$$K(r^L h^{-L}) \leq C \sum_{a \leq hP(\xi) \leq b} 1)^{1/q} \leq C(\sum_{a' \leq h|\xi| \leq b'} 1)^{1/q}$$

where a' and b' are suitable constants. Thus we get (as in the proof of lemma 10.1).

$$K(r^L h^{-L}) \leq C h^{-d/q}$$

and so

$$r^{-d/q} h^{d/q} K(r^L h^{-L}) \leq C r^{-d/q}.$$

Consequently we get (10.30) if h is large, and in any case we see that it suffices to let h and r be small, but still $r \geq Ah$.

Write

$$\varphi_0(\xi) = \chi\left(\frac{hP(\xi) - 1}{r}\right) \varphi(hP(\xi)),$$

$$\varphi_1(\xi) = \left(1 - \chi\left(\frac{hP(\xi) - 1}{r}\right)\right) \varphi(hP(\xi)),$$

where χ is infinitely differentiable and

$$\chi(u) = 1, \ |u| < 1/4,$$

$$\chi(u) = 0, \ |u| < 1/2.$$

Then

$$\|\varphi_0'\|_{l_q} \leq Cr^{\alpha}(\sum_{|hP(\xi)-1|<r/2} 1)^{1/q} \leq C r^{\alpha + 1/q} h^{d/q},$$

since the sum is bounded by the volume of $\{\eta \mid \eta \in R^d, \ |P(\eta) - h^{-1}| \leq 2r/h\}$ if $r/h \geq A$, and A is sufficiently large.

To estimate $\|\varphi_1\|_{l^L_q}$ we write

$$\omega(u) = \left(1 - \chi\left(\frac{hu - 1}{r}\right)\right) \varphi(hu).$$

Then

$$\omega^{(M)}(u) = \left(1 - \chi\left(\frac{hu-1}{r}\right)\right) h^M \varphi^{(M)}(hu) -$$

$$- \sum_{1 \leq J \leq M} C_{M,J} r^{-J} h^J \chi^{(J)} \left(\frac{hu - 1}{r}\right) h^{M-J} \varphi^{(M-J)}(hu) = \omega_M - \sum_{J=1}^{M} \omega_{M-J}.$$

Clearly $\omega(P(\xi) = \varphi_1(\xi)$. Now (10.5) gives

$$|\Delta^L \omega(P(\xi))| \leq C_L \sup_{|\eta| \leq L} |D^L \omega(P(\xi + \eta))|$$

and by lemma 1.5

$$|D^L \omega(P(\xi + \eta))| \leq C_L \sum_{1 \leq M \leq L} P(\xi + \eta)^{M-L} |\omega^{(M)}(P(\xi + \eta))| \leq$$

$$\leq C'_L \sum_{1 \leq M \leq L} P(\xi + \eta)^{M-L} \sum_{0 \leq J \leq M} |\omega_{M-J}(P(\xi + \eta))|.$$

Thus we see that it suffices to estimate the norms

$$N_J = \{ \sum_{a \leq hP(\xi+\eta) \leq b} |P(\xi + \eta)^{M-L} |\omega_{M-J}(P(\xi + \eta))||^q \}^{1/q}$$

uniformly in $|\eta| \leq L$, $0 \leq J \leq M \leq L$.

Clearly

$$N_0 \leq C \{ \Sigma \{ P(\xi + \eta)^{M-L} h^M |hP(\xi + \eta) - 1|^{\alpha - M} \}^q \}^{1/q} \leq$$

$$\leq C' h^L \{ \Sigma |hP(\xi + \eta) - 1|^{-q(L-\alpha)} \}^{1/q},$$

summation over the set $Ar \leq |hP(\xi + \eta) - 1|$, $a \leq hP(\xi + \eta) \leq b$, $\xi \in Z^d$. This set has a large distance to the origin, i.e. $|\xi|$ is large, because h^{-1} and rh^{-1} are large number. But it is easy to see that

$$|hP(\tau) - 1| \geq C |hP(\theta) - 1| \tag{10.32}$$

if $|\tau|$ is large, $|\theta - \tau| \leq D$ and $|P(\tau) - 1| \geq Ah/4$. If we apply (10.32) to $\theta = \sigma$, $\tau = (\xi + \eta)$, where $|\sigma - \xi| \leq 1/2$ and $|\xi|$ is sufficiently large, we get

$$|hP(\xi + \eta) - 1| \geq C |hP(\sigma) - 1|,$$

if $|hP(\xi + \eta) - 1| \geq r/4$. Thus

$$|hP(\xi + \eta) - 1|^{-q(L-\alpha)} \leq C \int_{|\sigma - \xi| < 1/2} |hP(\sigma) - 1|^{-q(L-\alpha)} d\sigma.$$

It is clear that there is a number $C_0 > 0$ so that the union of the spheres $|\sigma - \xi| < 1/2$, where $\xi \in Z^d$ and $|hP(\xi + \eta) - 1| \geq Ar/4$, is contained in the set $C_0 r \leq |hP(\sigma) - 1|$, $\sigma \in R^d$. Thus easily

$$N_0 \leq C' h^L \Big(\int_{C_0 r \leq |hP(\sigma) - 1|} |hP(\sigma) - 1|^{-q(L-\alpha)} d\sigma \Big)^{1/q} \leq$$

$$\leq C'' h^{L - d/q} r^{\alpha - (d-1)/q + d/q - L},$$

if $q(L - \alpha) > d$.

The norms N_J, $1 \leq J \leq M$ are estimated in a similar way. We have

$$N_J \leq Cr^{-J} \{ \sum_{\substack{r/4 \leq |hP(\xi+\eta)-1| \leq r/2 \\ a \leq hP(\xi+\eta) \leq b}} \{ P(\xi+\eta)^{-L} |hP(\xi+\eta)-1|^{\alpha-L+J} \}^q \}^{1/q} \leq$$

$$\leq Ch^L r^{\alpha-L} (\sum_{r/4 \leq |hP(\xi+\eta)-1| \leq r/2} 1)^{1/q}.$$

The number of points $\xi \in Z^d$, such that $r/4h \leq |P(\xi+\eta) - h^{-1}| \leq r/2h$ is bounded by the volume of a set of the form $C^{-1}r/h \leq |P(\xi) - h^{-1}| \leq Cr/h$, $\sigma \in R^d$ if $r/h \geq A$, A sufficiently large. Thus

$$N_J \leq Ch^{L-d/q} r^{\alpha-(d-1)/p+d/q-L}.$$

We conclude

$$\| \varphi_1 \|_{l_q^L} \leq Ch^{L-d/q} r^{\alpha-(d-1)/q+d/q-L},$$

and so

$$r^{-d/q} h^{d/q} K(r^L h^{-L}) \leq Cr^{\alpha-(d-1)/q}.$$

This gives the desired bound for the remaining part of the integral (10.30). The proof is complete.

10.2. - On the rate of convergence.

We shall denote by $D_p(P)$ the domain in $L_p = L_p[T^d]$ of the operator $P(D)$ given by

$$(P(D)f)^\wedge(\xi) = P(\xi) f^\wedge(\xi).$$

The space $H_p(P)$ is a BANACH space with the graph norm

$$\|f\|_{D_p(P)} = \|f\|_{L_p} + \|P(D)f\|_{L_p}.$$

We denote by $H_p^*(P)$ the interpolation space

$$H_p^*(P) = (L_p, D_p(P))_{1,\infty}.$$

The norm on $H_p^*(P)$ is equivalent to

$$|f\|_{H_p^*(P)} = \|f\|_{L_p} + \sup_{0<h<1} h^{-1} \| G_p(h)f - f \|_{L_p},$$

where the operator $G_p(h)$ is given by

$$(G_p(h)f)^\wedge(\xi) = \exp(-hP(\xi)) f^\wedge(\xi).$$

By theorem 10.1 we see that $G_p(h)$ is a uniformly bounded, strongly continuuus semi-group on L_p. For $1 < p < \infty$ we have

$$H_p^*(P) = D_p(P).$$

(These facts follows from general theorems on semi-groups of operators on a BANACH space, see BUTZER-BERENS [7], LÖFSTRÖM [16]).

By means of the so called stability theorem in interpolation theory (see PEETRE [19]), we get

$$(L_p,\ H_p^*(P))_{s/m,\,q} = B_p^{s,\,q} = B_p^{s,\,q}[T^d].$$

for $0 < s < m$, $1 \leq q \leq \infty$. We also note that $B_p^{s,\,q}$ can be defined by means of moduli of continuity, as in section 2. Put

$$\omega_p^r(t,\ f) = \sup_{|h|\leq t} \|(T_h - 1)^r f\|_{L_p},$$

$$T_h f(x) = f(x + h).$$

For $s = L + \alpha$, $0 < \alpha < 1$ we have $f \in B_p^{s,\,q}$ if and only if

$$D^J f \in L_p,\ J \leq L \text{ and } \left(\int_0^1 \left(\frac{\omega_p^1(t,\ D^L f)}{t^\alpha}\right)^q \frac{dt}{t}\right)^{1/q} < \infty,$$

and for $s = L + 1$ we have $f \in B_p^{s,\,q}$ if and only if

$$D^J f \in L_p,\ J \leq L \text{ and } \left(\int_0^1 \left(\frac{\omega_p^2(t,\ D^L f)}{t^\alpha}\right)^q \frac{dt}{t}\right)^{1/q} < \infty.$$

From general theorems on semi-groups of operators (see BUTZER-BERENS [7], LÖFSTRÖM [16] and PEETRE [19]), we also get that the norm on $B_p^{\sigma,\infty}$ is equivalent to each one of the norms

$$\|f\|_{L_p} + \sup_{0<h<1} h^{-\sigma} \|G_p(h)f - f\|_{L_p},\ 0 < \sigma < 1, \tag{10.26}$$

$$\|f\|_{L_p} + \sup_{0<h<1} h^{1-\sigma} \|P(D)G_p(h)f\|_{L_p},\ 0 < \sigma < 1. \tag{10.27}$$

After these preliminary remarks we are ready to discuss the rate of convergence of the operator E_h to the identity operator. Here E_h is given by

$$(E_h f)^\wedge(\xi) = \varphi(hP(\xi))f^\wedge(\xi).$$

We shall assume throughout this section that E_h in stable in L_p, i.e.

$$\|E_h f\|_{L_p} \leq C\|f\|_{L_p}.$$

As in the previous sections we write

$$\|f\|_{\Lambda_s} = \|f\|_{L_p} + \sup_{0<h<1} h^{-s}\|E_h f - f\|_{L_p}.$$

THEOREM 10.4. - Suppose that E_h is stable in L_p. Assume $\varphi(0) = 1$ and let $u^{-1}(\varphi(u) - 1)$ satisfy the assumptions of corollary 10.1 in a neighborhood of the origin. Then

$$B_p^{m,1} \subseteq \Lambda_1, \tag{10.28}$$

and

$$B_p^{\sigma m,\infty} \subseteq \Lambda_\sigma,\ 0 < \sigma < 1. \tag{10.29}$$

If in addition $\varphi'(0) \neq 0$, then

$$\Lambda_\sigma \subseteq B_p^{\sigma m,\infty},\ 0 < \sigma. \tag{10.30}$$

PROOF. - It is easy to see that

$$\|P(\xi)\|_{m_p(U_{2^n})} \leq C2^{nm},\ U_{2^n} = \{\xi \,|\, 2^{n-1} \leq |\xi| \leq 2^{n+1}\}.$$

Thus corollary 10.1 gives

$$h^{-1}2^{-nm}\|\varphi(hP(\xi)) - 1\|_{m_p(U_{2^n})} \leq C\left\|\frac{\varphi(hP(\xi)) - 1}{hP(\xi)}\right\|_{m_p(U_{2^n})} \leq C',$$

if $h\,2^{nm} < \varepsilon$, ε sufficiently small. But now theorem 5.1 gives (10.28) and (10.29) follows by interpolation.

If $\varphi'(0) \neq 0$ then $u(\varphi(u) - 1)^{-1}$ satisfies the assumptions of corollary 10.1 in a neighbourhood of the origin. Thus, if Φ_n are the standard functions in the definition of the BESOV spaces,

$$\|\Phi_n(\xi)(\varphi(2^{-nm}lP(\xi)) - 1)^{-1}\|_{m_p} \leq Cl^{-1}$$

for l sufficiently small. Thus, by theorem 5.3

$$2^{\sigma nm}\|\Phi_n^v * f\|_{L_p} \leq C \sup_{0<h<1} h^{-\sigma}\|E_h f - f\|_{L_p},$$

which gives the result.

THEOREM 10.5. - Suppose that $u^{-1}(\varphi(u)-1)$ satisfies the assumptions of theorem 10.1 in a neighborhood of the origin and let φ be a sum of three functions satisfying the assumptions of theorem 10.1, 10.2 and 10.3, respectively. Then

$$H_p^*(P) \subseteq \Lambda_1$$

$$B_p^{\sigma,\infty} \subseteq \Lambda_\sigma, \; 0<\sigma<1.$$

If, in addition $\varphi'(0) \neq 0$ and $\varphi(u)<1$, $u>0$, then

$$H_p^*(P) = \Lambda_1, \tag{10.33}$$

$$B_p^{\sigma m,\infty} = \Lambda_\sigma, \; 0<\sigma<1. \tag{10.34}$$

PROOF. - In a neighbourhood of $u=0$ we write

$$\Psi(u) = (\varphi(u)-1)(\exp(-u)-1)^{-1} = u^{-1}(\varphi(u)-1)\cdot u(\exp(-\exp u)-1)^{-1}.$$

Thus we see easily that Ψ satisfies the assumptions of theorem 10.1 in a neighbourhood of $u=0$.

Clearly $\Psi(u)$ satisfies the assumptions of theorem 10.3 in a neighbourhood of $u=u_1$. For large values of u we write

$$\Psi(u) = 1-(1-\exp u)^{-1} - \varphi(u) + \varphi(u)(1-\exp u)^{-1},$$

which satisfies the assumptions of theorems 10.1 and / or 10.2. Thus

$$\|\Psi(hP(\xi))\|_{m_p} \leq C_0$$

and so

$$\|E_h f - f\|_{L_p} \leq C_0 \|G_p(h)f - f\|_{L_p}.$$

This gives the conclusion in view of (10.26).

To prove the second statement we note that

$$ue^{-u}(\varphi(u)-1)^{-1},$$

satisfies the assumptions of theorem 10.1 and 10.3 at $0, \infty$ and u_1, respectively. Thus

$$\|hP(D)G_p(h)f\|_{L_p} \leq C\|E_h f - f\|_{L_p},$$

which gives the result, in view of (10.27).

As applications we consides the cases

(i) $\varphi(u) = \exp(-zu)$, $\mathrm{Re} z > 0$,

(ii) $\varphi(u) = (1 + u)^{-\beta}, \ \beta > 0,$

(iii) $\varphi(u) = (1 - u)_+^{\alpha}, \ \alpha > (d-1)\,|p^{-1} - 2^{-1}|.$

The corresponding operators are

$$G_{p,z}(h)f = (2\pi)^{-d} \sum_{\xi \in Z^d} \exp(-zhP(\xi)) f^\wedge(\xi) \exp(i < x, \xi >),$$

$$\mathfrak{F}^\beta(h)f = (2\pi)^{-d} \sum_{\xi \in Z^d} (1 + hP(\xi))^{-\beta} f^\wedge(\xi) \exp(i < y, \xi >),$$

$$\mathfrak{R}^\alpha(h)f = (2h)^{-d} \sum_{\xi \in Z^d} (1 + hP(\xi))_+^{\alpha} f^\wedge(\xi) \exp(i < x, \xi >),$$

(c.f. LÖFSTRÖM [16]). Then theorem 10.5 shows that the following conditions are equivalent:

$$f \in H_p^*(P)$$

$$\| G_{p,z}(h)f - f \|_{L_p} = 0(h).$$

$$\| \mathfrak{F}^\beta(h)f - f \|_{L_p} = 0(h),$$

$$\| \mathfrak{R}^\beta(h)f - f \|_{L_p} = 0(h).$$

In the same way we see that the following statements are equivalent for each σ, $0 < \sigma < 1$

$$f \in B_p^{\sigma m, \infty}$$

$$\| G_{p,z}(h)f - f \|_{L_p} = 0(h^\sigma).$$

$$\| \mathfrak{F}^\beta(h)f - f \|_{L_p} = 0(h^\sigma),$$

$$\| \mathfrak{R}^\beta(h)f - f \|_{L_p} = 0(h^\sigma.$$

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