

Research Article

Bessel Transform of (k, γ) -Bessel Lipschitz Functions

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Using a generalized translation operator, we obtain an analog of Theorem 5.2 in Younis (1986) for the Bessel transform for functions satisfying the (k, γ) -Bessel Lipschitz condition in $L_{2,\alpha}(\mathbb{R}_+)$.

1. Introduction and Preliminaries

Younis Theorem 5.2 [1] characterized the set of functions in $L^2(\mathbb{R})$ satisfying the Cauchy Lipschitz condition by means of an asymptotic estimate growth of the norm of their Fourier transforms; namely, we have the following.

Theorem 1 (see [1]). *Let $f \in L^2(\mathbb{R})$. Then the followings are equivalent:*

- (1) $\|f(x+h) - f(x)\|_2 = O(h^\alpha / (\log(1/h))^\beta)$ as $h \rightarrow 0, 0 < \alpha < 1, \beta > 0$,
- (2) $\int_{|x| \geq r} |\mathcal{F}(f)(x)|^2 dx = O(r^{-2\alpha} (\log r)^{-2\beta})$ as $r \rightarrow +\infty$,

where \mathcal{F} stands for the Fourier transform of f .

In this paper, we obtain a generalization of Theorem 1 for the Bessel transform. For this purpose, we use a generalized translation operator.

Assume that $L_{2,\alpha}(\mathbb{R}_+)$; $\alpha > -1/2$ is the Hilbert space of measurable functions $f(t)$ on \mathbb{R}_+ with finite norm

$$\|f\|_{2,\alpha} = \left(\int_0^\infty |f(x)|^2 x^{2\alpha+1} dx \right)^{1/2}. \quad (1)$$

Let

$$B = \frac{d^2}{dt^2} + \frac{(2\alpha+1)}{t} \frac{d}{dt} \quad (2)$$

be the Bessel differential operator.

For $\alpha \geq -1/2$, we introduce the Bessel normalized function of the first kind j_α defined by

$$j_\alpha(z) = \Gamma(\alpha+1) \sum_{n=0}^\infty \frac{(-1)^n}{n! \Gamma(n+\alpha+1)} \left(\frac{z}{2}\right)^{2n}, \quad (3)$$

where Γ is the gamma function (see [2]).

The function $y = j_\alpha(x)$ satisfies the differential equation

$$By + y = 0, \quad (4)$$

with the initial conditions $y(0) = 1$ and $y'(0) = 0$. $j_\alpha(z)$ is function infinitely differentiable, even, and, moreover, entirely analytic.

Lemma 2. *For $x \in \mathbb{R}_+$ the following inequality is fulfilled:*

$$|1 - j_\alpha(x)| \geq c, \quad (5)$$

with $x \geq 1$, where $c > 0$ is a certain constant which depends only on α .

Proof. Analog of Lemma 2.9 is in [3]. □

Lemma 3. *The following inequalities are valid for Bessel function j_α :*

- (1) $|j_\alpha(x)| \leq 1$, for all $x \in \mathbb{R}^+$,
- (2) $1 - j_\alpha(x) = O(x^2)$, $0 \leq x \leq 1$.

Proof. See [4]. □

The Bessel transform we call the integral transform from [2, 5, 6]

$$\widehat{f}(\lambda) = \int_0^\infty f(t) j_\alpha(\lambda t) t^{2\alpha+1} dt, \quad \lambda \in \mathbb{R}^+. \quad (6)$$

The inverse Bessel transform is given by the formula

$$f(t) = (2^\alpha \Gamma(\alpha + 1))^{-2} \int_0^\infty \widehat{f}(\lambda) j_\alpha(\lambda t) \lambda^{2\alpha+1} d\lambda. \quad (7)$$

We have the Parseval's identity

$$\|\widehat{f}\|_{2,\alpha} = 2^\alpha \Gamma(\alpha + 1) \|f\|_{2,\alpha}. \quad (8)$$

In $L_{2,\alpha}(\mathbb{R}_+)$, consider the generalized translation operator T_h defined by

$$T_h f(t) = c_\alpha \int_0^\pi f\left(\sqrt{t^2 + h^2 - 2th \cos \varphi}\right) \sin^{2\alpha} \varphi d\varphi, \quad (9)$$

where

$$c_\alpha = \left(\int_0^\pi \sin^{2\alpha} \varphi d\varphi\right)^{-1} = \frac{\Gamma(\alpha + 1)}{\Gamma(1/2) \Gamma(\alpha + (1/2))}. \quad (10)$$

The following relations connect the generalized translation operator and the Bessel transform; in [7] we have

$$(\widehat{T_h f})(\lambda) = j_\alpha(\lambda h) \widehat{f}(\lambda). \quad (11)$$

2. Main Result

In this section we give the main result of this paper. We need first to define (k, γ) -Bessel Lipschitz class.

Definition 4. Let $0 < k < 1$ and $\gamma \geq 0$. A function $f \in L_{2,\alpha}(\mathbb{R}^+)$ is said to be in the (k, γ) -Bessel Lipschitz class, denoted by $\text{Lip}(k, \gamma, 2)$, if

$$\|T_h f(t) - f(t)\|_{2,\alpha} = O\left(\frac{h^k}{(\log(1/h))^\gamma}\right), \quad \text{as } h \rightarrow 0. \quad (12)$$

Our main result is as follows.

Theorem 5. Let $f \in L_{2,\alpha}(\mathbb{R}^+)$. Then the followings are equivalent

- (1) $f \in \text{Lip}(k, \gamma, 2)$.
- (2) $\int_r^\infty |\widehat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda = O(r^{-2k}/(\log r)^{2\gamma})$, as $r \rightarrow +\infty$.

Proof. (1) \Rightarrow (2) Assume that $f \in \text{Lip}(k, \gamma, 2)$. Then we have

$$\begin{aligned} & \|T_h f(t) - f(t)\|_{2,\alpha}^2 \\ &= \frac{1}{(2^\alpha \Gamma(\alpha + 1))^2} \int_0^\infty |1 - j_\alpha(\lambda h)|^2 |\widehat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda. \end{aligned} \quad (13)$$

If $\lambda \in [1/h, 2/h]$ then $\lambda h \geq 1$ and Lemma 2 implies that

$$1 \leq \frac{1}{c^2} |1 - j_\alpha(\lambda h)|. \quad (14)$$

Then

$$\begin{aligned} & \int_{1/h}^{2/h} |\widehat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda \\ & \leq \frac{1}{c^2} \int_{1/h}^{2/h} |1 - j_\alpha(\lambda h)|^2 |\widehat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda \\ & \leq \frac{1}{c^2} \int_0^\infty |1 - j_\alpha(\lambda h)|^2 |\widehat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda \\ & = O\left(\frac{h^{2k}}{(\log(1/h))^{2\gamma}}\right). \end{aligned} \quad (15)$$

We obtain

$$\int_r^{2r} |\widehat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda \leq C \frac{r^{-2k}}{(\log r)^{2\gamma}}, \quad (16)$$

where C is a positive constant.

So that

$$\begin{aligned} & \int_r^\infty |\widehat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda \\ &= \left[\int_r^{2r} + \int_{2r}^{4r} + \int_{4r}^{8r} + \dots \right] |\widehat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda \\ & \leq C \frac{r^{-2k}}{(\log r)^{2\gamma}} + C \frac{(2r)^{-2k}}{(\log 2r)^{2\gamma}} + C \frac{(4r)^{-2k}}{(\log 4r)^{2\gamma}} + \dots \\ & \leq C \frac{r^{-2k}}{(\log r)^{2\gamma}} \left(1 + 2^{-2k} + (2^{-2k})^2 + (2^{-2k})^3 + \dots\right) \\ & \leq CK \frac{r^{-2k}}{(\log r)^{2\gamma}}, \end{aligned} \quad (17)$$

where $K = (1 - 2^{-2k})^{-1}$ since $2^{-2k} < 1$.

This proves that

$$\int_r^\infty |\widehat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda = O\left(\frac{r^{-2k}}{(\log r)^{2\gamma}}\right) \quad \text{as } r \rightarrow +\infty. \quad (18)$$

(2) \Rightarrow (1) Suppose now that

$$\int_r^\infty |\widehat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda = O\left(\frac{r^{-2k}}{(\log r)^{2\gamma}}\right) \quad \text{as } r \rightarrow +\infty. \quad (19)$$

We write

$$\int_0^\infty |1 - j_\alpha(\lambda h)|^2 |\widehat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda = I_1 + I_2, \quad (20)$$

where

$$\begin{aligned} I_1 &= \int_0^{1/h} |1 - j_\alpha(\lambda h)|^2 |\widehat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda, \\ I_2 &= \int_{1/h}^\infty |1 - j_\alpha(\lambda h)|^2 |\widehat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda. \end{aligned} \quad (21)$$

Estimate the summands I_1 and I_2 from above. It follows from the inequality $|j_\alpha(\lambda h)| \leq 1$ that

$$\begin{aligned} I_2 &= \int_{1/h}^\infty |1 - j_\alpha(\lambda h)|^2 |\widehat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda \\ &\leq 4 \int_{1/h}^\infty |\widehat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda = O\left(\frac{h^{2k}}{(\log(1/h))^{2\gamma}}\right). \end{aligned} \quad (22)$$

To estimate I_1 , we use the inequality (2) of Lemma 3. Set

$$\phi(x) = \int_x^\infty |\widehat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda. \quad (23)$$

Using integration by parts, we obtain

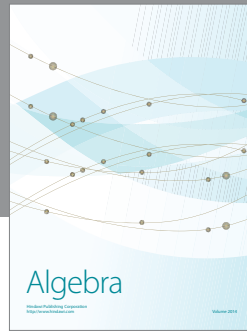
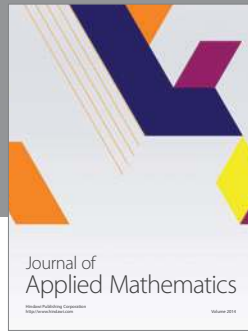
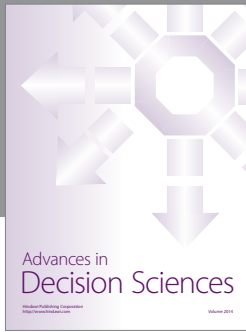
$$\begin{aligned} I_1 &\leq -C_1 h^2 \int_0^{1/h} s^2 \phi'(s) ds \\ &\leq -C_1 \phi\left(\frac{1}{h}\right) + 2C_1 h^2 \int_0^{1/h} s \phi(s) ds \\ &\leq C_2 h^2 \int_0^{1/h} s \phi(s) ds \\ &\leq C_2 h^2 \int_0^{1/h} s s^{-2k} (\log s)^{-2\gamma} ds \\ &\leq C_3 h^{2k} (\log(1/h))^{-2\gamma}, \end{aligned} \quad (24)$$

where C_1, C_2 , and C_3 are positive constants and this ends the proof. \square

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