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Best approximation problems relating to Monge–Kantorovich duality

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Abstract. Problems of the best approximation of bounded continuous functions on a topological space $X \times X$ by functions of the form $u(x) - u(y)$ are considered. Formulae for the values of the best approximations are obtained and the equivalence between the existence of precise solutions and the non-emptiness of the constraint set of the auxiliary dual Monge–Kantorovich problem with a special cost function is established. The form of precise solutions is described in terms relating to the Monge–Kantorovich duality, and for several classes of approximated functions the existence of precise solutions with additional properties, such as smoothness and periodicity, is proved.

Bibliography: 20 titles.

§ 1. Introduction

1.1. Statement of problems and results. Let X be a topological space[1] and $C^b(X)$ the Banach space of continuous bounded real functions on X with uniform norm

$$\|u\| = \sup_{x\in X} |u(x)|, \qquad u \in C^b(X).$$

In what follows we consider the following extremal problem.

Problem 1. Find the value of the best approximation

$$m(f;\mathscr{H}_0) := \inf_{h_u\in\mathscr{H}_0} \|f - h_u\| = \inf_{u\in C^b(X)} \sup_{x,y\in X} |f(x,y) - u(x) + u(y)| \qquad (1)$$

of a fixed function $f \in C^b(X \times X)$ by functions h_u in the subspace

$$\mathscr{H}_0 = \bigl\{h_u : h_u(x,y) = u(x) - u(y),\ u \in C^b(X)\bigr\} \subset C^b(X \times X).$$

We can state an abstract version of this problem.

Problem 2. Let X be an arbitrary set, $l^\infty(X)$ the Banach space of bounded real functions on X with uniform norm, and H_0 the subspace of $l^\infty(X \times X)$ consisting of functions of the form $u(x) - u(y)$,

$$H_0 = \{h_u : h_u(x,y) = u(x) - u(y),\ u \in l^\infty(X)\}.$$

[1] Throughout, the spaces are assumed to be Hausdorff and completely regular.

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Find the value of the best approximation

$$m(f;H_0) := \inf_{h_u\in H_0} \|f-h_u\| = \inf_{u\in l^\infty(X)} \sup_{x,y\in X} |f(x,y)-u(x)+u(y)| \qquad (2)$$

of a fixed function $f \in l^\infty(X\times X)$ by functions $h_u \in H_0$.

We associate with each function f on $X\times X$ the *cost function* c on $X\times X$:

$$c(x,y) = \min(f(x,y), -f(y,x)). \qquad (3)$$

Theorem 1 (cf. [1], Theorem 5.1). (1) *For each* $f \in l^\infty(X\times X)$,

$$m(f;H_0) = -\inf\left\{\frac{1}{n}\sum_{i=1}^{n} c(x_{i-1},x_i) : x_i\in X,\ x_n = x_0,\ n=1,2,\dots\right\}. \qquad (4)$$

(2) *If X is a compact space, then for each* $f\in C(X\times X)$,

$$m(f;\mathscr{H}_0) = -\inf\left\{\frac{1}{n}\sum_{i=1}^{n} c(x_{i-1},x_i) : x_i\in X,\ x_n = x_0,\ n=1,2,\dots\right\}. \qquad (5)$$

In other words, in both cases the value of the best approximation is equal to the infimum with minus sign of the mean values of the corresponding cost function over the various cycles $(x_0\to x_1\to\cdots\to x_{n-1}\to x_0)$ in X.

Corollary 1. *If X is a compact space and $f\in C(X\times X)$, then $m(f;H_0)=m(f;\mathscr{H}_0)$.*

Now let X be a non-compact topological space. Let $C^b(X)\overline{\otimes}C^b(X)$ be the closure in $C^b(X\times X)$ of the vector subspaces of finite sums of the form $\sum_1^n a_j(x)b_j(y)$, where $a_j, b_j \in C^b(X)$, $j=1,\dots,n$. It follows easily from the Stone–Weierstrass theorem that $C^b(X)\,\overline{\otimes}\,C^b(X)$ is precisely the subspace of functions on $X\times X$ extending continuously to $\beta X\times\beta X$, where βX is the Stone–Čech compactification of X; see [2]. (One has $C(X\times X)=C^b(X\times X)=C^b(X)\,\overline{\otimes}\,C^b(X)=C(X)\,\overline{\otimes}\,C(X)$ for compact X.)

Corollary 2. *If X is an arbitrary (Hausdorff completely regular) topological space and $f\in C^b(X)\,\overline{\otimes}\,C^b(X)$, then part (2) of Theorem 1 and Corollary 1 still hold.*

If the infimum on the right-hand side of (1) or (2) is attained at a function $u\in C^b(X)$ or $u\in l^\infty(X)$, then this function is called a *precise solution* of the corresponding best approximation problem.

We point out that Problem 2 always has a precise solution since $l^\infty(X)$ is the dual Banach space [3] (and therefore closed balls in it are weak-$*$ compact) and the functional $u\mapsto \sup_{x,y\in X}|f(x,y)-u(x)+u(y)|$ on $l^\infty(X)$ is weak-$*$ lower semicontinuous. On the other hand, the question on the existence of precise solutions of Problem 1 is non-trivial.

Theorem 2. *If $f \in C^b(X) \mathbin{\overline{\otimes}} C^b(X)$, then Problem 1 has a precise solution.*

Theorem 3. *Let $X = \mathbb{R}^n$ and let $f(x,y) = g(x-y)$ for $g \in C^b(\mathbb{R}^n)$. Then there exists a bounded infinitely smooth function u in $\mathbb{R}^n$ that is a precise solution of Problem 1:*

$$m(f; \mathscr{H}_0) = \|f - h_u\| = \sup_{x,y \in \mathbb{R}^n} |f(x,y) - u(x) + u(y)|. \tag{6}$$

If, in addition, $g(x) = g(x_1, \dots, x_n)$ is periodic in the variables $x_1, \dots, x_m$, $m \leqslant n$, with periods $\tau_1, \dots, \tau_m$, then there exists a bounded infinitely smooth function u on $\mathbb{R}^n$ with the same periodicity properties that is a precise solution of Problem 1.

Theorem 4. *Let $X = \mathbb{R}^n$ or $X = \mathbb{R}^n_+$, and let $f(x,y) = g(x, y, x-y)$, where $g \in C^b(X \times X \times \mathbb{R}^n)$ satisfies one of the two conditions:*

(a) *$g(x,y,z) = g(x_1, \dots, x_n, y_1, \dots, y_n, z_1, \dots, z_n)$ is non-negative on $X \times X \times \mathbb{R}^n$ and non-decreasing in all the x_i, y_i, $i = 1, \dots, n$;*

(b) *$g(x,y,z)$ is non-positive on $X \times X \times \mathbb{R}^n$ and non-increasing with respect to all the x_i, y_i, $i = 1, \dots, n$.*

Then for each positive integer r there exists a bounded C^r-smooth function on X solving precisely Problem 1.

1.2. Discussion. The above-stated problem fits into the following general scheme of best-approximation problems (see, for instance, [4]–[7]): for a fixed Banach space E, an element $f \in E$, and a closed linear subspace H of E find the quantity $m(f; H) = \inf_{h \in H} \|f - h\|$.

The following duality theorem is well known:

$$m(f; H) = \sup\{\langle \mu, f \rangle : \mu \in H^\perp,\ \|\mu\| \leqslant 1\},$$

where $H^\perp = \{\mu \in E^* : \langle \mu, h \rangle = 0 \text{ for all } h \in H\}$ is the annihilator of H in E^*. Applying it to Problem 1 on a compact space we obtain the formula

$$m(f; \mathscr{H}_0) = \sup\biggl\{\int_{X \times X} f(x,y)\, \mu(d(x,y)) : \pi_1\mu = \pi_2\mu,\ \|\mu\| \leqslant 1\biggr\}, \tag{7}$$

where μ is a Radon measure on $X \times X$, $\pi_1\mu$ and $\pi_2\mu$ are the projections of the measure onto the first and the second factor, that is, the marginal measures on X defined by the equalities $\pi_1\mu(B) = \mu(B \times X)$, $\pi_2\mu(B) = \mu(X \times B)$ for each Borel subset B of X. Assume for simplicity that $f(x,y) \leqslant 0$ for all $(x,y) \in X \times X$. Then f coincides with the function c defined by formula (3), and we can compare (7) and (5). We consider now the cycle $\zeta = (x_0 \to x_1 \to \dots \to x_{n-1} \to x_n = x_0)$ in X and associate with it a measure μ_ζ on $X \times X$:

$$\mu_\zeta = -\frac{1}{n} \sum_1^n \delta_{(x_{i-1}, x_i)},$$

where $\delta_{(x,y)}$ is the Dirac measure (the delta function) at (x,y), $\delta_{(x,y)}(M) = 1$ for $(x,y) \in M$, $\delta_{(x,y)}(M) = 0$ for $(x,y) \notin M$, $M \subset X \times X$. Obviously, $\|\mu_\zeta\| = 1$ and

$$\pi_1\mu_\zeta = \pi_2\mu_\zeta = -\frac{1}{n} \sum_1^n \delta_{x_i}.$$

Taking account of the equality $f=c$, from (7) we obtain

$$m(f;\mathscr{H}_0)=\sup\biggl\{\int_{X\times X}c(x,y)\,\mu(d(x,y)):\pi_1\mu=\pi_2\mu,\ \|\mu\|\leqslant 1\biggr\}$$
$$\geqslant \sup_{\zeta}\int_{X\times X}c(x,y)\,\mu_\zeta(d(x,y))=-\inf_{\zeta}\frac{1}{n}\sum_{i=1}^{n}c(x_{i-1},x_i).$$

Comparing this with equality (5) we see that for $f\leqslant 0$ the duality theorem is a consequence of Theorem 1.

Starting from Chebyshev's classical studies, precise solutions of best-approximation problems have been considered in the literature mainly for finite-dimensional subspaces H. For infinite-dimensional H little is known. Problems 1 and 2 were considered for the first time in [1], where Theorem 1 was stated (without proof). Before that Khavinson [8] had studied a close problem of the best approximation of a continuous function of two variables $f(x,y)$ by sums of the form $\phi(x)+\psi(y)$.

Our approach to best-approximation problems of the kind of Problem 1 is based on the relation between these problems and the Monge–Kantorovich duality. We shall show that a function u is a precise solution of a best-approximation problem if and only if it belongs to the constraint set of a certain auxiliary infinite-dimensional linear programming problem dual to the Monge–Kantorovich problem with a special cost function determined by the approximated function f and the value of the best approximation $m(f;\mathscr{H}_0)$. This relation enables one to prove the above-stated results and, with the use of the concept of reduced cost function [9], to obtain explicitly some precise solutions.

§ 2. Auxiliary information about Monge–Kantorovich duality

Let X be a topological space, $\varphi\in C^b(X\times X)$, σ_1 and σ_2 positive Radon measures in X, and let $\sigma_1(X)=\sigma_2(X)$. The Monge–Kantorovich problem (MKP) consists in finding the optimal value

$$\mathscr{A}(\varphi;\sigma_1-\sigma_2)=\inf\biggl\{\int_{X\times X}\varphi(x,y)\,\mu(d(x,y)):\mu\geqslant 0,\ \pi_1\mu-\pi_2\mu=\sigma_1-\sigma_2\biggr\}.$$

This is the Monge–Kantorovich problem with fixed difference of marginal measures. Better known is another version of the MKP, with fixed marginal measures, in which one seeks the optimal value

$$\mathscr{C}(\varphi;\sigma_1,\sigma_2)=\inf\biggl\{\int_{X\times X}\varphi(x,y)\,\mu(d(x,y)):\mu\geqslant 0,\ \pi_1\mu=\sigma_1,\ \pi_2\mu=\sigma_2\biggr\}.$$

Both problems were posed by Kantorovich [10]–[12], who studied the case of a metric compact space X with the metric taken for the cost function φ. In this case the two versions of the MKP are equivalent, but this is no longer so in the case of an arbitrary cost function.[2] Both problems relate to infinite-dimensional linear programming: the problem with fixed marginal measures is a continual analogue of the

[2] For a more detailed description of the relation between the two types of MKP in the general case see [13], [2], [14].

classical transportation problem, and the problem with fixed difference of marginal measures can be regarded as a generalization of the transportation problem in the network setting, with transit transportation allowed. The optimal value of the dual MKP with fixed difference of marginal measures is defined by the formula

$$\mathscr{B}(\varphi;\sigma_1-\sigma_2)=\sup\biggl\{\int_X u(x)\,(\sigma_1-\sigma_2)(dx):u\in Q(\varphi)\biggr\},$$

where

$$Q(\varphi)=\bigl\{u\in C^b(X):u(x)-u(y)\leqslant\varphi(x,y)\text{ for all }x,y\in X\bigr\}.$$

By analogy with $Q(\varphi)$ we shall consider the sets $Q_0(\varphi)$ and $Q(\varphi;l^\infty(X))$ defined by the equalities

$$Q_0(\varphi)=\bigl\{u\in\mathbb{R}^X:u(x)-u(y)\leqslant\varphi(x,y)\text{ for all }x,y\in X\bigr\},$$
$$Q(\varphi;l^\infty(X))=\bigl\{u\in l^\infty(X):u(x)-u(y)\leqslant\varphi(x,y)\text{ for all }x,y\in X\bigr\};$$

they are the constraint sets of the dual problems for certain non-topological generalizations of the MKP (with fixed difference of projections), see [1], [3]. In [13], [2], for a broad class of spaces X including compact and Polish spaces we developed the duality theory in the mass setting, which gives one a complete description of all the cost functions φ for which the duality relation $\mathscr{A}(\varphi;\sigma_1-\sigma_2)=\mathscr{B}(\varphi;\sigma_1-\sigma_2)$ holds for arbitrary $\sigma_1\geqslant 0$ and $\sigma_2\geqslant 0$, $\sigma_1(X)=\sigma_2(X)$. One can construct a similar theory also for the MKP with fixed marginal measures [2], and for non-topological versions of both problems [1], [3]. We shall require tests for the non-emptiness of $Q(\varphi)$ (as well as $Q_0(\varphi)$ and $Q(\varphi;l^\infty(X))$) and the related concept of reduced cost function.

Let X be an arbitrary set. We can associate with each cost function $\varphi\colon X\times X\to\mathbb{R}$ the *reduced cost function* $\varphi_*\colon X\times X\to\mathbb{R}\cup\{-\infty\}$,

$$\varphi_*(x,y)=\inf\{\varphi^n(x,y):n\in\{0,1,2,\dots\}\},$$

where $\varphi^0(x,y)=\varphi(x,y)$, and for $n\neq 0$,

$$\varphi^n(x,y)=\inf\bigl\{\varphi(x,z_1)+\varphi(z_1,z_2)+\dots+\varphi(z_n,y):z_1,\dots,z_n\in X\bigr\}.$$

Obviously, φ_* satisfies the triangle inequality

$$\varphi_*(x,y)+\varphi_*(y,z)\geqslant\varphi_*(x,z)$$

for arbitrary $x,y,z\in X$. Hence if the function φ_* is equal to $-\infty$ at some point $(x,y)\in X\times X$, then it is identically equal to $-\infty$. We thus have an alternative: either $\varphi_*(x,y)>-\infty$ for all $(x,y)\in X\times X$, or $\varphi_*\equiv-\infty$. In the first case it follows from the triangle inequality that for each fixed $x_0\in X$ the functions $u(x)=\varphi_*(x,x_0)$ and $v(x)=-\varphi_*(x_0,x)$ belong to $Q_0(\varphi_*)$; in the second case $Q_0(\varphi_*)=\varnothing$. Moreover, by the triangle inequality $\varphi_*>-\infty$ if and only if $\varphi_*(x,x)\geqslant 0$ for all $x\in X$ or equivalently, if $\sum_1^n\varphi(x_{i-1},x_i)\geqslant 0$ for each cycle $(x_0\to x_1\to\dots\to x_{n-1}\to x_n=x_0)$ in X.

Now, if $u\in Q_0(\varphi)$, then fixing a point $(x,y)\in X\times X$, a positive integer n, and a transit transfer $(x=z_0\to z_1\to\dots\to z_n\to z_{n+1}=y)$ and adding together the equalities $u(z_{i-1})-u(z_i) \leqslant \varphi(z_{i-1}, z_i)$, $i = 1, \dots, n+1$, we obtain that $u(x)-u(y) \leqslant \varphi(x, z_1) + \varphi(z_1, z_2) + \dots + \varphi(z_n, y)$. Since this holds for arbitrary $x, y \in X$, each n, and each transit transfer of length $n+1$ from x to y, it follows that $u \in Q_0(\varphi_*)$. Hence $Q_0(\varphi) \subseteq Q_0(\varphi_*)$ and since $\varphi_* \leqslant \varphi$, it follows that $Q_0(\varphi) = Q_0(\varphi_*)$. In a similar way one demonstrates that $Q(\varphi; l^\infty(X)) = Q(\varphi_*; l^\infty(X))$ and (for a topological space X) $Q(\varphi) = Q(\varphi_*)$.

Now, if φ_* is bounded above, then $Q_0(\varphi_*) = Q(\varphi_*; l^\infty(X))$, and therefore $Q_0(\varphi) = Q(\varphi; l^\infty(X))$. We finally point out that if X is a topological space and $\varphi \in C^b(X) \mathbin{\overline{\otimes}} C^b(X)$, then $\varphi_* \equiv -\infty$ or $\varphi_* \in C^b(X) \mathbin{\overline{\otimes}} C^b(X)$,[3] and in the latter case for each $x_0 \in X$ the functions $u(x) = \varphi_*(x, x_0)$ and $v(x) = -\varphi_*(x_0, x)$ belong to $Q(\varphi_*) = Q(\varphi)$.

We summarize the above arguments as follows.

Proposition 1 (cf. [15], Lemma 2, [16], Theorem 2.1, [1], Theorem 4.1). *Let X be an arbitrary set and let $\varphi\colon X \times X \to \mathbb{R}$ be a function. Then $Q_0(\varphi) = Q_0(\varphi_*)$, $Q(\varphi; l^\infty(X)) = Q(\varphi_*; l^\infty(X))$ and (for a topological space X) $Q(\varphi) = Q(\varphi_*)$. Moreover, the following properties are equivalent:*

(a) $Q_0(\varphi) \neq \varnothing$;

(b) $\varphi_*(x, y) > -\infty$ *for all* $x, y \in X$;

(c) $\varphi_*(x, x) \geqslant 0$ *for all* $x \in X$;

(d) $\sum_1^n \varphi(x_{i-1}, x_i) \geqslant 0$ *for each cycle* $(x_0 \to x_1 \to \dots \to x_{n-1} \to x_n = x_0)$ *in* X.

If the function φ_* is bounded above, then these properties are equivalent to either of the following two:

(e) $Q(\varphi; l^\infty(X)) \neq \varnothing$;

(f) $\varphi_* \in l^\infty(X \times X)$.

Finally, if X is a topological space and $\varphi \in C^b(X) \mathbin{\overline{\otimes}} C^b(X)$, then one can complete the list of equivalent properties by two further ones:

(g) $Q(\varphi) \neq \varnothing$;

(h) $\varphi_* \in C^b(X) \mathbin{\overline{\otimes}} C^b(X)$.

§ 3. Proof of Theorems 1 and 2

We associate with Problems 1 and 2 the cost functions

$$\varphi_1(x, y) = c(x, y) + m(f; \mathscr{H}_0), \tag{8}$$

$$\varphi_2(x, y) = c(x, y) + m(f; H_0), \tag{9}$$

where $c(x, y) = \min(f(x, y), -f(y, x))$.

Lemma 1. (1) *The following formula holds for the value of the best approximation in Problem 1:*

$$m(f; \mathscr{H}_0) = \inf\{\alpha \in \mathbb{R}_+ : Q(c + \alpha) \neq \varnothing\}. \tag{10}$$

Moreover, $u \in C^b(X)$ is a precise solution of Problem 1 if and only if it belongs to the set $Q(\varphi_1)$.

[3] For compact X this is a consequence of [13], Lemma 2.4; the general case reduces easily to the compact one by the transition from X to βX.

(2) *The following formula holds for the value of the best approximation in Problem 2*:

$$m(f;H_0)=\inf\{\alpha\in\mathbb{R}_+ : Q(c+\alpha;l^\infty(X))\neq\varnothing\} \tag{11}$$

and a function $u\in l^\infty(X)$ *is a precise solution of Problem 2 if and only if it belongs to the set* $Q(\varphi_2;l^\infty(X))$.

Proof. For each $u\in C^b(X)$,

$$\begin{aligned}&\sup_{x,y\in X}|f(x,y)-u(x)+u(y)|\\ &\qquad=\inf\{\alpha\in\mathbb{R}_+ : f(x,y)-\alpha\leqslant u(x)-u(y)\leqslant f(x,y)+\alpha \text{ for all } x,y\in X\}\\ &\qquad=\inf\{\alpha\in\mathbb{R}_+ : u\in Q(c+\alpha)\}.\end{aligned}$$

This yields (10). This also demonstrates that $u\in C^b(X)$ is a precise solution of Problem 1 if and only if $u\in Q(c+m(f;\mathscr{H}_0))=Q(\varphi_1)$. The proof of the first result of the lemma is complete, and the verification of the second is similar.

Remark 1. We have actually proved part (1) of Lemma 1 for Problem 1 with arbitrary $f\in C^b(X\times X)$.

The next result is a consequence of Lemma 1 and Proposition 1.

Corollary 3. *If* $f\in C^b(X)\,\overline{\otimes}\,C^b(X)$, *then the existence of a precise solution of Problem 1 is equivalent to the condition* $\varphi_{1*}\not\equiv-\infty$. (*If this condition holds, then either of the functions* $u(x)=\varphi_{1*}(x,x_0)$, $v(x)=-\varphi_{1*}(x_0,x)$, $x_0\in X$, *is a precise solution of Problem 1.*)

Proof of Theorem 1. As follows from Lemma 1 (see (11)), for each $\alpha>m(f;H_0)$ there exists a function $u\in Q(c+\alpha;l^\infty)$. Then by Proposition 1 ((e) $\Leftrightarrow$ (d)), for each cycle $\zeta=(x_0\to x_1\to\cdots\to x_{n-1}\to x_n=x_0)$ in X we have the inequality $\sum_1^n\varphi(x_{i-1},x_i)\geqslant 0$ with $\varphi(x,y)=c(x,y)+\alpha$. This yields $\sum_1^n c(x_{i-1},x_i)+n\alpha\geqslant 0$. Consequently,

$$\alpha\geqslant-\inf_\zeta\frac{1}{n}\sum_1^n c(x_{i-1},x_i),$$

and since this holds for each $\alpha>m(f;H_0)$, it follows that

$$m(f;H_0)\geqslant-\inf_\zeta\frac{1}{n}\sum_1^n c(x_{i-1},x_i). \tag{12}$$

Assume now that $\alpha<m(f;H_0)$. Then $Q(c+\alpha;l^\infty)=\varnothing$. Hence $(c+\alpha)_*\equiv-\infty$ and $\sum_1^n c(x_{i-1},x_i)+n\alpha<0$ for some cycle $(x_0\to x_1\to\cdots\to x_{n-1}\to x_n=x_0)$. Hence

$$\alpha<-\frac{1}{n}\sum_{i=1}^n c(x_{i-1},x_i)\leqslant-\inf_\zeta\frac{1}{n}\sum_{i=1}^n c(x_{i-1},x_i),$$

and since the same holds for all $\alpha<m(f;H_0)$, it follows that

$$m(f;H_0)\leqslant-\inf_\zeta\frac{1}{n}\sum_{i=1}^n c(x_{i-1},x_i) \tag{13}$$

and (4) is a consequence of (12), (13). The proof of (5) (and of Corollary 2) is similar, with the obvious replacement of $Q(c+\alpha; l^\infty)$ by $Q(c+\alpha)$.

Proof of Theorem 2. In view of Lemma 1, it is sufficient to verify that $Q(\varphi_1)$ is non-empty. Assume the contrary; then by Proposition 1 there exists a cycle $\zeta = (x_0 \to x_1 \to \dots \to x_{n-1} \to x_n = x_0)$ in X such that

$$\sum_1^n \varphi_1(x_{i-1}, x_i) = \sum_1^n (c(x_{i-1}, x_i) + m(f; \mathscr{H}_0)) < 0.$$

In that case $\sum_1^n (c(x_{i-1}, x_i) + m(f; \mathscr{H}_0) + \alpha) < 0$ for sufficiently small $\alpha > 0$. Then it follows by Proposition 1 ((d) $\Leftrightarrow$ (g)) that $Q(c + m(f; \mathscr{H}_0) + \alpha) = \varnothing$; but this contradicts Lemma 1 and is therefore impossible; see (10).

§ 4. Proof of Theorems 3 and 4

We shall require several concepts and facts of lifting theory [17].[4] Let X be a locally compact metrizable space and σ_0 a positive σ-finite Borel measure in it supported by the entire space; let $\mathscr{L}^\infty = \mathscr{L}^\infty(X, \sigma_0)$ be the Banach space of bounded σ_0-measurable real functions on X (we do not identify σ_0-equivalent functions) equipped with the uniform norm $\|u\| = \sup_{x \in X} |u(x)|$. The space $\mathscr{L}^\infty$ is a real Banach algebra with respect to natural (pointwise) multiplication. (It is also a Banach lattice with respect to taking the pointwise supremum and infimum.) A Banach algebra homomorphism, that is, a multiplicative linear operator $\rho\colon \mathscr{L}^\infty \to \mathscr{L}^\infty$ is called a *strong lifting* on $\mathscr{L}^\infty = \mathscr{L}^\infty(X, \sigma_0)$ if the following four conditions hold:

(1) ρ is a projection, that is, $\rho^2 = \rho$;

(2) for each $u \in \mathscr{L}^\infty$ the set $\{x \in X : \rho(u)(x) \neq u(x)\}$ is σ_0-negligible, that is, $\rho(u) = u$ σ_0-a.s.;

(3) for each $u \in \mathscr{L}^\infty$, $u = 0$ a.s. $\Rightarrow \rho(u) \equiv 0$;

(4) $\rho(u) = u$ for all $u \in C^b(X)$.

In combination with the linearity and the multiplicativity of ρ these conditions mean that ρ is also a homomorphism of Banach lattices, so that $\rho(u \vee v) = \rho(u) \vee \rho(v)$ and $\rho(u \wedge v) = \rho(u) \wedge \rho(v)$ for arbitrary $u, v \in \mathscr{L}^\infty$. Moreover, if $u \geqslant v$ σ_0-a.s., then $\rho(u)(x) \geqslant \rho(v)(x)$ for all $x \in X$. (Indeed, since $w = u - v \geqslant 0$ σ_0-a.s., there exists a function $w_1 \in \mathscr{L}^\infty$ such that $w = w_1^2$ σ_0-a.s., and by the multiplicativity of ρ we obtain that $\rho(u)(x) - \rho(v)(x) = \rho(w)(x) = (\rho(w_1)(x))^2 \geqslant 0$ for all $x \in X$.)

The main result that we require is the existence of a strong lifting on the space $\mathscr{L}^\infty = \mathscr{L}^\infty(X, \sigma_0)$ [17] (see also [5], Corollary 1 to Theorem 3.8).

We point out that the standard Lebesgue space $L^\infty = L^\infty(X, \sigma_0)$ is a Banach algebra and a Banach lattice, and the linear operator $\pi\colon \mathscr{L}^\infty \to L^\infty$ taking each function $u \in \mathscr{L}^\infty$ to the class of functions σ_0-equivalent to u is a homomorphism of Banach algebras and Banach lattices. Thus, π is the canonical map of $\mathscr{L}^\infty$ onto the quotient space $L^\infty = \mathscr{L}^\infty / \mathscr{N}_0$, where $\mathscr{N}_0$ is the subspace of σ_0-negligible functions in $\mathscr{L}^\infty$. The standard norm in L^∞ is precisely the quotient norm with respect to π. Since $\rho(u) = \rho(v)$ for $u - v \in \mathscr{N}_0$, ρ induces a homomorphism of Banach algebras

[4] See also [5], Chapter 3.

(and Banach lattices) $\rho'\colon L^\infty \to \mathscr{L}^\infty$ (a strong lifting on L^∞) such that $\rho' \circ \pi = \rho$ and $\pi \circ \rho' = \mathrm{id}_{L^\infty}$.

Lemma 2. *Let X be a locally compact metrizable space, σ_0 a positive σ-finite Borel measure on it supported by the entire space, and let $f \in C^b(X \times X)$. Then the set $Q(\varphi_1; \mathscr{L}^\infty) = Q(\varphi_1; \mathscr{L}^\infty(X, \sigma_0))$, where*

$$Q(\varphi_1; \mathscr{L}^\infty(X, \sigma_0)) := \bigl\{u \in \mathscr{L}^\infty(X, \sigma_0) : u(x) - u(y) \leqslant \varphi_1(x, y) \text{ for all } x, y \in X\bigr\},$$

is non-empty.

Proof. As follows from Lemma 1, for each n there exists a function u_n in $Q(\varphi_1 + 1/n)$. We fix a point $x_0 \in X$ and assume without loss of generality that $u_n(x_0) = 0$, $n = 1, 2, \dots$. For each $x \in X$ we have $-\varphi_1(x_0, x) - 1 \leqslant u_n(x) \leqslant \varphi_1(x, x_0) + 1$, therefore $\{u_n\}$ is a bounded (in norm) subset of $C^b(X)$. Since σ_0 is supported by the entire space X, $C^b(X)$ is naturally linearly isometric to a closed subspace of $L^\infty = L^\infty(X, \sigma_0)$. Hence the sequence $\{u_n\}$ is weak-$*$ bounded and therefore weak-$*$ precompact in $L^\infty = L^{1*}$ and one can select a weak-$*$ convergent subsequence $\{u_{n_k}\}$.[5] To avoid complicated notation we shall assume that the sequence $\{u_n\}$ itself weak-$*$ converges to an element of L^∞. Then there exists a function $v \in \mathscr{L}^\infty$ such that $\{u_n\}$ weak-$*$ converges to $\pi(v)$, and therefore the sequence $\{u_n(x) - u_n(y)\}$ of elements of $C^b(X \times X) \subset L^\infty(X \times X, \sigma_0 \times \sigma_0)$ weak-$*$ converges to an element of $L^\infty(X \times X, \sigma_0 \times \sigma_0)$, which is the equivalence class of a function $v(x) - v(y)$. Now, since $u_n(x) - u_n(y) \leqslant \varphi_1(x, y) + 1/n$ and the positive cone $L^\infty_+(X \times X, \sigma_0 \times \sigma_0)$ is weak-$*$ closed (and therefore one can pass to the limit in the inequalities), it follows that

$$v(x) - v(y) \leqslant \varphi_1(x, y) \quad \text{for} \quad (\sigma_0 \times \sigma_0)\text{-a.s. } (x, y) \in X \times X. \tag{14}$$

We now set

$$N(y) := \{x \in X : v(x) - v(y) > \varphi_1(x, y)\}, \qquad y \in X.$$

It follows from (14) that the set

$$N := \{y \in X : N(y) \text{ is not } \sigma_0\text{-negligible}\}$$

is σ_0-negligible. We regard y as a parameter and observe that for each $y \notin N$ the inequality

$$v(x) - v(y) \leqslant \varphi_1(x, y)$$

holds for σ_0-a.e. $x \in X$. Applying the strong lifting ρ to both sides of this inequality we obtain

$$\rho(v)(x) - v(y) \leqslant \varphi_1(x, y) \tag{15}$$

[5] The space $L^1(X, \sigma_0)$ is separable (as follows, for instance, from [18], Chapter VII, § 1, Theorem 3). Hence, by [19], Theorem V.5.1, the weak-$*$ topology is metrizable on bounded subsets of L^∞ and therefore we can select a weak-$*$ convergent subsequence convergent in the ordinary, rather than in the generalized sense.

for all $x \in X$ and all $y \notin N$. Now, treating x as a parameter and applying ρ to both sides of (15) regarded as functions of y we obtain

$$\rho(v)(x) - \rho(v)(y) \leqslant \varphi_1(x, y) \quad \text{for all } x, y \in X, \tag{16}$$

that is, $\rho(v) \in Q(\varphi_1; \mathscr{L}^\infty)$.

We now proceed directly to the proof of Theorems 3 and 4.

Proof of Theorem 3. By Lemma 2 there exists a function $u_1 \in Q(\varphi_1; \mathscr{L}^\infty)$, where $\mathscr{L}^\infty = \mathscr{L}^\infty(\mathbb{R}^n; \sigma_0)$ and σ_0 is Lebesgue measure on $\mathbb{R}^n$. Let η be a non-negative smooth function on $\mathbb{R}^n$ such that $\int_{\mathbb{R}^n} \eta(x)\, \sigma_0(dx) = 1$ and $\dfrac{\partial^{i_1+\cdots+i_n}\eta(x)}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}} \in L^1(\mathbb{R}^n, \sigma_0)$ for arbitrary $i_1, \dots, i_n$. (For instance, we can take $\eta(x) = \pi^{-n/2} e^{-(x_1^2+\cdots+x_n^2)}$, $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, or an arbitrary non-negative infinitely smooth function with compact support such that the integral of it is 1.) Let u be the convolution of u_1 and η:

$$u(x) = (u_1 * \eta)(x) = \int_{\mathbb{R}^n} u_1(z)\eta(x - z)\, \sigma_0(dz) = \int_{\mathbb{R}^n} u_1(x - z)\eta(z)\, \sigma_0(dz).$$

Obviously, u is a bounded infinitely smooth function in $\mathbb{R}^n$.

Since $f(x, y) = g(x-y)$, it follows that $c(x, y) = \min(g(x-y), -g(y-x))$. Hence $c(x - z, y - z) = c(x, y)$ and $\varphi_1(x - z, y - z) = \varphi_1(x, y)$, so that

$$u_1(x - z) - u_1(y - z) \leqslant \varphi_1(x, y) \quad \text{for all } x, y, z \in \mathbb{R}^n. \tag{17}$$

Multiplying both sides of this inequality by $\eta(z)$ and integrating with respect to $\sigma_0(dz)$ while taking account of the equality $\int_{\mathbb{R}^n} \eta(z)\, \sigma_0(dz) = 1$ we obtain that $u(x) - u(y) \leqslant \varphi_1(x, y)$ for all $x, y \in \mathbb{R}^n$. Hence $u \in Q(\varphi_1)$, and the application of Lemma 1 completes the proof of the first assertion of the Theorem.

For the proof of the second assertion, let X_1 be the topological product of m circles of lengths $\tau_1, \dots, \tau_m$ and the space $\mathbb{R}^{n-m}$. It is convenient to view these circles as the intervals $[0, \tau_i]$ with identified end-points; then the position of a point on the ith circle is defined by a quantity x_i, $0 \leqslant x_i < \tau_i$. The space X_1 is a smooth manifold [6] and, at the same time, a locally compact Abelian group with respect to addition defined by the following agreement: $x + y := z = (z_1, \dots, z_n)$, where $z_i = x_i + y_i \pmod{\tau_i}$ for $i = 1, \dots, m$ and $z_i = x_i + y_i$ for $i = m + 1, \dots, n$. Obviously, one can regard each (smooth) function on X_1 as a (smooth) function on $\mathbb{R}^n$ periodic in the first m variables in accordance with the above description.

By Lemma 2 there exists a function $u_1 \in Q(\varphi_1; \mathscr{L}^\infty)$, where $\mathscr{L}^\infty = \mathscr{L}^\infty(X_1; \sigma_0)$ and σ_0 is Haar measure [7] in X_1, that is, in our case, the product of m linear Lebesgue measures on the corresponding circles and $(n - m)$-dimensional Lebesgue measure on $\mathbb{R}^{n-m}$. (If X_1 is a compact group, then Theorem 2 demonstrates the existence of a function $u_1 \in Q(\varphi_1)$.) The rest of the proof is similar to the proof of the first result of the theorem. Using the special form of the function f we obtain inequality (17)

[6] For $n = 2$ X_1 is either a torus (if $m = 2$) or a cylinder (if $m = 1$).

[7] For more information about Haar measure, see, for instance, [20].

for all $x, y, z \in X_1$. Multiplying it by a non-negative infinitely smooth function with compact support

$$\eta\colon X_1 \to \mathbb{R}, \qquad \int_{X_1} \eta(z)\,\sigma_0(dz) = 1,$$

and integrating after that with respect to Haar measure $\sigma_0(dz)$ we see that the convolution of u_1 with η is infinitely smooth and belongs to $Q(\varphi_1)$. The application of Lemma 1 completes the proof.

Proof of Theorem 4. By Lemma 2 there exists a function $u_1 \in Q(\varphi_1; \mathscr{L}^\infty)$, where $\mathscr{L}^\infty = \mathscr{L}^\infty(X; \sigma_0)$ and σ_0 is Lebesgue measure on X. Using the special form of f we obtain $c(x,y) = -g(y,x,y-x)$, provided that condition (a) holds, and $c(x,y) = g(x,y,x-y)$, provided that condition (b) holds. In either case the function $\varphi_1(x,y) = c(x,y) + m(f; \mathscr{H}_0)$ has the form $\varphi_1(x,y) = h(x,y,x-y)$ with h satisfying (b). Now, for each $z \in \mathbb{R}^n_+$ we obtain

$$u_1(x+z) - u_1(y+z) \leqslant h(x+z, y+z, x-y) \leqslant h(x,y,x-y), \qquad x, y \in X.$$

Multiplying both sides of this inequality by $\eta(z) = e^{-(z_1+\dots+z_n)}$ and integrating after that with respect to Lebesgue measure on $\mathbb{R}^n_+$ while taking account of the equality

$$\int_{\mathbb{R}^n_+} \eta(z)\,\sigma_0(dz) = 1$$

we obtain

$$u_2(x) - u_2(y) \leqslant h(x,y,x-y) = \varphi_1(x,y), \qquad x, y \in X,$$

where

$$u_2(x) := \int_{\mathbb{R}^n_+} u_1(x+z)\eta(z)\,\sigma_0(dz), \qquad x \in X.$$

Setting $x + z = t$ we obtain

$$u_2(x) = u_2(x_1, \dots, x_n) = e^{x_1+\dots+x_n} \int_{x_1}^\infty \cdots \int_{x_n}^\infty u_1(t_1, \dots, t_n) e^{-(t_1+\dots+t_n)}\,dt_1 \cdots dt_n,$$

which yields the differentiability of u_2. In addition, u_2 is bounded because u_1 is bounded. Applying the above argument to u_2 in place of u_1 we obtain the function

$$u_3(x) := \int_{\mathbb{R}^n_+} u_2(x+z)\eta(z)\,\sigma_0(dz), \qquad x \in X,$$

which belongs to $Q(\varphi_1)$ and has second derivatives. Repeating the same procedure $r+1$ times we obtain a function $u_{r+2} \in Q(\varphi_1)$ possessing all the partial derivatives of orders up to $r+1$ and therefore C^r-smooth. The use of Lemma 1 completes the proof.

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