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BEST CONSTANT FOR THE EMBEDDING OF THE SPACE $H^2 \cap H_0^1(\Omega)$ INTO $L^{2N/(N-4)}(\Omega)$

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1. Introduction. In the study of fourth order elliptic equations in bounded domains [16], it is important to know the constant of the continuous, noncompact embedding of the space $H^2 \cap H_0^1$ into $L^{2N/(N-4)}$,

$$H^2 \cap H_0^1(\Omega) \to L^{\frac{2N}{N-4}}(\Omega).$$

This specific embedding plays a role when fourth order problems with boundary conditions $u = \Delta u = 0$ on $\partial\Omega$ are considered. Here Ω is a smooth, bounded domain in $\mathbb{R}^N$. The main objective of this paper is to solve the following problem.

Problem (I). *Find the largest constant K_1 for which the inequality*

$$\|u\|_{2N/(N-4)} \leq K_1^{-1/2} \|u\|_{2,2}, \quad \forall u \in H^2 \cap H_0^1(\Omega), \tag{1.1}$$

is valid.

The norms used here are defined by

$$\|u\|_{2N/(N-4)} = \left(\int_\Omega |u|^{\frac{2N}{N-4}} \, dx \right)^{\frac{N-4}{2N}}, \quad \|u\|_{2,2} = \left(\int_\Omega |\Delta u|^2 \, dx \right)^{1/2}.$$

The analog of Problem (I) for the case $\Omega = \mathbb{R}^N$ can be answered and the largest constant possible is K_0 which is given by

$$K_0 = \min \left\{ \int_{\mathbb{R}^N} |\Delta u|^2 \, dx : \; u \in \mathcal{D}^{2,2}(\mathbb{R}^N), \; \int_{\mathbb{R}^n} |u|^{\frac{2N}{N-4}} \, dx = 1 \right\}, \tag{1.2}$$

where the space $\mathcal{D}^{2,2}(\mathbb{R}^N)$ is the completion of $\mathcal{D}(\mathbb{R}^N)$ in the norm $\| \cdot \|_{2,2}$. This was studied by Lions [8]. He also proved that there exists a minimizer for (1.2) which is uniquely determined up to translations and dilations. The explicit form of this minimizer can be found in [5], [7] and is given by

$$U_{\epsilon,x_0}(x) = C_N \frac{\epsilon^{\frac{N-4}{2}}}{(|x - x_0|^2 + \epsilon^2)^{\frac{N-4}{2}}}, \tag{1.3}$$

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where

$$C_N = ((N-4)(N-2)N(N+2))^{\frac{N-4}{8}}.$$

With this expression, the constant K_0 can be evaluated explicitly;

$$K_0 = \pi^2 N(N-4)(N^2-4)\left\{\frac{\Gamma(\frac{N}{2})}{\Gamma(N)}\right\}^{\frac{4}{N}}. \tag{1.4}$$

It is now interesting to ask whether $K_1 = K_0$.

About the corresponding question for the embedding

$$H_0^1(\Omega) \to L^{\frac{2N}{N-2}}(\Omega),$$

we know that the constants S_0 and S_1, which play the role of K_0 and K_1 here, indeed satisfy

$$S_1 = S_0.$$

This result was proved in [2], [4], [12]. The argument they used to prove that $S_1 = S_0$ is based on the fact that one can extend H_0^1-functions with zero in $\mathbb{R}^N$ outside Ω. A consequence which follows from the work of [4] is that on a bounded domain there is never a function u such that the constant S_1 is achieved.

For the embedding

$$H_0^2(\Omega) \to L^{\frac{2N}{N-4}}(\Omega),$$

a similar result was proved in [5] using an analogous method.

Returning to Problem (I), we shall show that

$$K_1 = K_0. \tag{1.5}$$

Specifically, we shall prove the following theorem.

Theorem 1. *Let $\Omega \subset \mathbb{R}^N$, $N > 4$, be a bounded domain with a smooth boundary $\partial\Omega$. Consider the space $H^2 \cap H_0^1(\Omega)$. Then*

$$H^2 \cap H_0^1(\Omega) \to L^{\frac{2N}{N-4}}(\Omega)$$

is a continuous embedding (not compact) with

$$\|u\|_{2N/(N-4)} \le K_0^{-1/2}\|u\|_{2,2}, \quad \forall u \in H^2 \cap H_0^1(\Omega), \tag{1.1}$$

where

$$K_0 = \min\left\{\int_{\mathbb{R}^N} |\Delta u|^2\,dx :\ u \in \mathcal{D}^{2,2}(\mathbb{R}^N),\ \int_{\mathbb{R}^N} |u|^{\frac{2N}{N-4}}\,dx = 1\right\}. \tag{1.2}$$

We note that Problem (I) is equivalent to the minimization problem

$$\text{(II)} \qquad K_1 = \inf\left\{\int_{\Omega} |\Delta u|^2\,dx :\ u \in H^2 \cap H_0^1(\Omega),\ \int_{\Omega} |u|^{\frac{2N}{N-4}}\,dx = 1\right\}.$$

For this minimization problem, we proved the following result.

Theorem 2. *The infimum K_1 is never achieved by a function $u \in H^2 \cap H_0^1(\Omega)$ when Ω is bounded.*

The arguments based on extending functions by zero employed in [4], [5] fail for Problem (I) because, in general, functions in $H^2 \cap H_0^1(\Omega)$ can not be extended by zero in $\mathbb{R}^N$ outside Ω. We shall, however, use more involved techniques. The most important methods which will be used are arguments involving a concentration compactness principle [8], a symmetrization comparison principle [13] and nonexistence results employing variational identities [9], [11], and [14].

The organization of the paper is as follows. In Section 2, we shall prove Theorem 1. The proof of Theorem 2 will follow directly from the proof of Theorem 1. In this section we shall use some technical lemmas which will be proved in the appendix.

2. The Proof of Theorem 1. In order to prove Theorem 1, we shall introduce some notation and cite the basic lemmata which are required in the proof. These lemmata will be proved in the appendix. As mentioned in Section 1, Problem (I) is equivalent to the evaluation of

$$\text{(II)} \qquad K_1 = \inf\left\{ \int_\Omega |\Delta u|^2\,dx : \ u \in H^2 \cap H_0^1(\Omega),\ \int_\Omega |u|^{\frac{2N}{N-4}}\,dx = 1 \right\}. \tag{2.1}$$

We formulate some results from the concentration-compactness principle using the Hilbert space $H^2 \cap H_0^1(\Omega)$. Two related minimization problems are crucial during the entire proof. They are

$$K_0 = \min\left\{ \int_{\mathbb{R}^N} |\Delta u|^2\,dx : \ u \in \mathcal{D}^{2,2}(\mathbb{R}^N),\ \int_{\mathbb{R}^N} |u|^{\frac{2N}{N-4}}\,dx = 1 \right\}. \tag{2.2}$$

$$K_2 = \inf\left\{ \int_{x_1>0} |\Delta u|^2 dx : \ u \in \mathcal{D}^{2,2}(\mathbb{R}^N), u\,\big|_{x_1=0} = 0,\ \int_{x_1>0} |u|^{\frac{2N}{N-4}} dx = 1 \right\}. \tag{2.3}$$

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a smooth boundary $\partial\Omega$ and $N > 4$. Then if*

$$K_1 < \min\{K_0,\, K_2\}, \tag{2.4}$$

every minimizing sequence (u_n) of Problem (II) *is relatively compact in $H^2 \cap H_0^1(\Omega)$ and there exists a minimizer for Problem* (II).

For a more complete version of Lemma 2.1, we refer to Lemma A1 in the appendix. The proof of Lemma A1 is essentially due to Lions [8]. If condition (2.4) of Lemma 2.1 is satisfied, a minimizer $u(x)$ of Problem (II) satisfies the biharmonic equation

$$\Delta^2 u = \mu u^{\frac{N+4}{N-4}} \quad \text{in} \quad \mathcal{D}'(\Omega), \tag{2.5}$$

with $\mu = K_1 > 0$, in the distributional sense because $K_1 = 0$ implies $u \equiv 0$ a.e. in Ω, which is a contradiction. From Lemma B3 of the appendix, we recall that a solution of (2.5) is indeed a $C^4(\Omega) \cap C^3(\bar{\Omega})$-function which satisfies the boundary conditions

$$u = 0,\ \ \Delta u = 0 \quad \text{on} \quad \partial\Omega.$$

Another observation we can make about the minimization Problem (II) is that, if condition (2.4) is satisfied, there is a positive minimizer of K_1. To see this, we argue as follows. The minimization Problem (II) can also be written as

$$\text{(III)} \qquad K_1 = \inf\left\{ \frac{\int_\Omega |\Delta u|^2\,dx}{\left(\int_\Omega |u|^{\frac{2N}{N-4}}\,dx\right)^{\frac{N-4}{N}}} \;:\; u \in H^2 \cap H_0^1(\Omega) \right\}.$$

Now solve the following problem in $H^2 \cap H_0^1(\Omega)$:

$$-\Delta v = |\Delta u|. \tag{2.6}$$

By the maximum principle, one observes that $v > 0$ in Ω. If we add Δu to both sides of (2.6), we obtain

$$-\Delta(v-u) = |\Delta u| + \Delta u \geq 0.$$

Of course we have zero data on the boundary. Applying the maximum principle, we find that $v \geq u$ pointwise in $\bar{\Omega}$. Similarly, if we add $-\Delta u$ to both sides of (2.6), we obtain $v \geq -u$, which together with the previous estimate gives that $v \geq |u|$. For the numerator and the denominator in the quotient in Problem (III) this yields

$$\int_\Omega |\Delta v|^2\,dx = \int_\Omega |\Delta u|^2\,dx \quad \text{and} \quad \int_\Omega |v|^{\frac{2N}{N-4}}\,dx \geq \int_\Omega |u|^{\frac{2N}{N-4}}\,dx.$$

Consequently, the quotient in (III) is minimized by the positive functions, and so there are positive minimizers. The previous list of observations makes it possible to write down a partial differential equation which a minimizer of K_1 has to satisfy if condition (2.4) of Lemma 2.1 is fulfilled:

$$\text{(IV)} \quad \begin{cases} \Delta^2 u = K_1 u^{\frac{N+4}{N-4}}, \quad u > 0 \quad \text{in } \Omega, & (2.7) \\ u = 0,\ \Delta u = 0 \quad \text{on } \partial\Omega. & (2.8) \end{cases}$$

In the next lemma, we prove a crucial observation about the right hand side of condition (2.4).

Lemma 2.2. *We have*

$$K_2 = K_0. \tag{2.9}$$

Proof. First we shall formulate an equivalent expression for K_2. Let

$$X = \left\{ u \in \mathcal{D}^{2,2}(\mathbb{R}^N) : u\,\big|_{x_1=0} = 0 \right\}.$$

Then we have

$$K_2 = \inf\left\{ \frac{\int_{x_1>0} |\Delta u|^2\,dx}{\left(\int_{x_1>0} |u|^{\frac{2N}{N-4}}\,dx\right)^{\frac{N-4}{N}}} \;:\; u \in X \right\}. \tag{2.10}$$

The proof is based on the symmetrization principle due to Talenti [13]. The next step then shall be to apply a symmetrization comparison principle of Talenti. In order to apply the Talenti comparison principle, we first modify (2.10) and then pass to the limit.

Let $u \in X$ and set $-\Delta u = f$; then $f \in L^2(x_1 > 0)$. By a straightforward density argument, it is possible to construct a sequence of functions $(f_n) \in C_0^\infty(x_1 > 0)$ which is an approximating sequence for f in the L^2-norm. These functions f_n have compact support for every n. We have $f_n \to f$ as $n \to \infty$ in the sense that

$$\|f_n - f\|_2 \to 0 \quad \text{as} \quad n \to \infty.$$

This construction also implies pointwise convergence in $(x_1 > 0)$; i.e.,

$$f_n(x) \to f(x) \text{ as } n \to \infty, \text{ a.e. on } (x_1 > 0).$$

We shall start by considering the problem

$$(*) \qquad \begin{cases} -\Delta u_n = f_n & \text{in } (x_1 > 0), \qquad (2.11) \\ u_n \big|_{x_1=0} = 0. & \qquad (2.12) \end{cases}$$

Write supp $f_n = \mathcal{A}_n$. Then, by construction, meas $(\mathcal{A}_n) < \infty$ and

$$\|f_n\|_{2N/(N+2)} \leq \text{meas}\,(\mathcal{A}_n)^{1/N} \|f_n\|_2 < \infty,$$

where the integrals are taken with respect to $\mathcal{A}_n$. This implies that for any n, $f_n \in L^{2N/(N+2)}$ $(x_1 > 0)$ and hence that Problem $(*)$ has a weak solution $u_n \in \mathcal{D}^{1,2}$ $(x_1 > 0)$.

Because we want to apply the Talenti comparison principle, we recall that the Schwarz symmetrization of a function ϕ, which we denote by ϕ^*, is defined as

$$\phi^*(x) = \inf\{y \geq 0 \colon \mu(y) < \omega_N |x|^N\}, \tag{2.13}$$

where $\mu(y) = \text{meas}\,\{x \in \Omega \colon |\phi(x)| > y\}$ and ω_N is the measure of the N-dimensional unit ball. Consequently, the Schwarz symmetrization is defined on the ball Ω^* with the additional property that meas $(\Omega) =$ meas (Ω^*). The Talenti comparison principle states the following.

Proposition 1. (Talenti [13]). *Suppose Ω is a regular domain in $\mathbb{R}^N$ and let u be a weak solution of the problem*

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

in which $f \in L^{2N/(N+2)}(\Omega)$. Then

$$u^* \leq v \quad \text{a.e in } \Omega^*$$

and v is the weak solution of

$$\begin{cases} -\Delta v = f^* & \text{in } \Omega^*, \\ v = 0 & \text{on } \partial\Omega^*. \end{cases}$$

Consider the family of problems

$$(**)\qquad \begin{cases} -\Delta v_n = f_n^* \quad \text{in} \quad \mathbb{R}^N, & (2.14)\\ v_n \to 0 \quad \text{as} \quad |x| \to \infty, & (2.15)\end{cases}$$

where the v_n are weak $\mathcal{D}^{1,2}(\mathbb{R}^N)$-solutions because $f_n^* \in L^2 \cap L^{2N/(N+2)}(\mathbb{R}^N)$ for all n, a property of the Schwarz symmetrization. We can apply Proposition 1 in the domain $(x_1 > 0)$ and obtain the pointwise estimate

$$u_n^* \le v_n \quad \text{a.e. on } \mathbb{R}^N. \tag{2.16}$$

The final step consists of establishing an estimate for v and u^* by passing to the limit. We proceed as follows. From the appendix, we recall the contraction property of the map $f \to f^*$. In particular, we have that the map $\mathcal{S}\colon \phi \to \phi^*$ is a contraction map; i.e.,

$$\|\phi_n^* - \phi^*\|_p \le \|\phi_n - \phi\|_p, \quad 1 \le p \le \infty. \tag{2.17}$$

Now we use (2.17) with f and $p = 2$;

$$\|f_n^* - f^*\|_2 \le \|f_n - f\|_2. \tag{2.18}$$

Since the right hand side of (2.18) tends to zero as n tends to infinity, we conclude that

$$\|f_n^* - f^*\|_2 \to 0 \quad \text{as} \quad n \to \infty.$$

This implies that there exists a subsequence f_{n_k} with the property

$$f_{n_k}^*(x) \to f^*(x) \quad \text{a.e. on } \mathbb{R}^N. \tag{2.19}$$

Because $\{f_n^*\}$ is a Cauchy sequence in $L^2(\mathbb{R}^N)$, it follows that $\{v_n\}$ is a Cauchy sequence in $\mathcal{D}^{2,2}(\mathbb{R}^N)$. This can be seen by using $-\Delta v_n + \Delta v_m = f_n^* - f_m^*$† and the relation

$$\|v_n - v_m\|_{2,2} = \|f_n^* - f_m^*\|_2 \to 0 \quad \text{as} \quad n, m \to \infty.$$

Consequently, because $\mathcal{D}^{2,2}(\mathbb{R}^N)$ is a Hilbert space, there exists a function $v \in \mathcal{D}^{2,2}(\mathbb{R}^N)$ such that

$$v_n \to v \text{ in } \mathcal{D}^{2,2}(\mathbb{R}^N) \text{ as } n \to \infty.$$

By the Sobolev inequality for $D^{2,2}(\mathbb{R}^N)$, we obtain

$$v_n \to v \text{ in } L^{2N/(N-4)}(\mathbb{R}^N) \text{ as } n \to \infty, \tag{2.20}$$

$$v_{n_k} \to v \text{ a.e. on } \mathbb{R}^N \text{ as } n_k \to \infty. \tag{2.21}$$

As a direct consequence, it follows that

$$-\Delta v = f^* \quad \text{a.e. on } \mathbb{R}^N. \tag{2.22}$$

†Using the Green function of $-\Delta$ on $\mathbb{R}^N$, one obtains $v_n = G * f_n$, which is well defined by the Hardy-Littlewood-Sobolev inequality. Clearly, $-\Delta v_n \in L^2(\mathbb{R}^N)$. Similarly, u_n exists and $-\Delta u_n \in L^2(x_1 > 0)$.

Next, we turn to the sequence $\{u_n\}$. We do this by considering $-\Delta(u_n - u) = f_n - f$. We have

$$\int_{x_1>0} |\Delta(u_n - u)|^2\, dx = \|f_n - f\|_2^2. \tag{2.23}$$

Observe that $u_n \to u$ in X as n tends to infinity. Extending u_n and u oddly in x_1 to $\mathbb{R}^N$, we can apply the Sobolev inequality for the Hilbert space $\mathcal{D}^{2,2}(\mathbb{R}^N)$ to deduce from (2.23) that

$$\left(\int_{x_1>0} |u_n - u|^{\frac{2N}{N-4}}\, dx\right)^{\frac{N-4}{2N}} \leq C\|f_n - f\|_2 \to 0. \tag{2.24}$$

From this we conclude that

$$u_n \to u \text{ in } L^{2N/(N-4)}(x_1 > 0) \text{ as } n \to \infty, \tag{2.25}$$

$$u_{n_k} \to u \text{ a.e. on } (x_1 > 0) \text{ as } n_k \to \infty. \tag{2.26}$$

Since the L^p-norm is invariant under Schwarz symmetrization, it follows from (2.16) that

$$\|u_n\|_{2N/(N-4)} = \|u_n^*\|_{2N/(N-4)} \leq \|v_n\|_{2N/(N-4)}.$$

Combining (2.20), (2.21), (2.25) and (2.26), we arrive at the inequality

$$\|u\|_{2N/(N-4)} \leq \|v\|_{2N/(N-4)}, \tag{2.27}$$

where we used

$$\big|\|u\|_{2N/(N-4)} - \|u_n\|_{2N/(N-4)}\big| \leq \|u - u_n\|_{2N/(N-4)} \to 0$$

and similarly for v and v_n.

Remark. Again using the contraction property of the Schwarz symmetrization, we can even obtain the inequality

$$u^* \leq v \quad \text{a.e. on } \mathbb{R}^N. \tag{2.28}$$

To see this, we use (2.17) with $p = 2N/(N-4)$. Then, using (2.24), we obtain

$$\|u_n^* - u^*\|_{2N/(N-4)} \leq \|u_n - u\|_{2N/(N-4)} \leq C\|f_n - f\|_2 \to 0, \tag{2.29}$$

from which we conclude that

$$u_n^* \to u^* \text{ in } L^{2N/(N-4)}(\mathbb{R}^N) \text{ as } n \to \infty, \tag{2.30}$$

$$u_{n_k}^* \to u^* \text{ a.e. on } \mathbb{R}^N \text{ as } n_k \to \infty. \tag{2.31}$$

Hence, again using (2.16), we obtain (2.28).

At this point in the proof, we remark that we have now constructed for each function $u \in X$, a positive radially symmetric decreasing function $v \in \mathcal{D}^{2,2}(\mathbb{R}^N)$ which, according to (2.27) or (2.28), has the additional property

$$\frac{\int_{x_1>0} |\Delta u|^2\,dx}{\left(\int_{x_1>0} |u|^{\frac{2N}{N-4}}dx\right)^{\frac{N-4}{N}}} = \frac{\int_{x_1>0} |\Delta u|^2\,dx}{\left(\int_{\mathbb{R}^N} |u^*|^{\frac{2N}{N-4}}dx\right)^{\frac{N-4}{N}}} \geq \frac{\int_{\mathbb{R}^N} |(\Delta u)^*|^2\,dx}{\left(\int_{\mathbb{R}^N} |v|^{\frac{2N}{N-4}}dx\right)^{\frac{N-4}{N}}}.$$

Employing (2.22), we see that $(-\Delta u)^* = f^* = -\Delta v$; so, for the infimum K_2, given in (2.10), we obtain

$$K_2 = \inf \frac{\int_{x_1>0} |\Delta u|^2\,dx}{\left(\int_{x_1>0} |u|^{\frac{2N}{N-4}}dx\right)^{\frac{N-4}{N}}} \geq \inf \frac{\int_{\mathbb{R}^N} |\Delta v|^2\,dx}{\left(\int_{\mathbb{R}^N} |v|^{\frac{2N}{N-4}}dx\right)^{\frac{N-4}{N}}} = K_0. \tag{2.32}$$

By (2.32) we have $\min\{K_2, K_0\} = K_0$, and so we have proved the first part of the lemma.

To establish the reverse inequality, we use a test function of the type

$$u_\lambda = U(x + \lambda e_1) - U(x - \lambda e_1), \tag{2.33}$$

where U is the minimizer of K_0, given in (1.3), and $e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^N$. For convenience, we write

$$u_1 = U(x + \lambda e_1), \quad u_2 = U(x - \lambda e_1), \quad u_\lambda = u_1 + u_2.$$

Calculating the integrals over $\mathbb{R}^N$ in (2.32), we obtain for $\lambda \to \infty$,

$$\int_{\mathbb{R}^N} |\Delta u_\lambda|^2\,dx = \int_{\mathbb{R}^N} |\Delta u_1|^2\,dx + \int_{\mathbb{R}^N} |\Delta u_2|^2\,dx + o(1),$$
$$\int_{\mathbb{R}^N} |u_\lambda|^{\frac{2N}{N-4}}\,dx = \int_{\mathbb{R}^N} |u_1|^{\frac{2N}{N-4}}\,dx + \int_{\mathbb{R}^N} |u_2|^{\frac{2N}{N-4}}\,dx + o(1).$$

This gives, simply by the translation invariance of the integrals,

$$\int_{\mathbb{R}^N} |\Delta u_\lambda|^2\,dx = 2\int_{\mathbb{R}^N} |\Delta U|^2\,dx + o(1), \tag{2.34}$$

$$\int_{\mathbb{R}^N} |u_\lambda|^{\frac{2N}{N-4}}\,dx = 2\int_{\mathbb{R}^N} |U|^{\frac{2N}{N-4}}\,dx + o(1). \tag{2.35}$$

Now recall from (2.3), after odd extension in x_1, that

$$K_2 = \frac{1}{2}\inf\left\{\int_{\mathbb{R}^N} |\Delta u|^2 dx : \; u \in \mathcal{D}^{2,2}(\mathbb{R}^N),\, u\,\big|_{x_1=0} = 0,\, \int_{\mathbb{R}^N} |u|^{\frac{2N}{N-4}} dx = 2\right\}.$$

Furthermore, if we write

$$K(u) = 2^{-4/N}\frac{\int_{\mathbb{R}^N} |\Delta u|^2\,dx}{\left(\int_{\mathbb{R}^N} |u|^{\frac{2N}{N-4}}\,dx\right)^{\frac{N-4}{N}}}, \tag{2.36}$$

then

$$K_2 = \{\inf K(u) : \ u \in \mathcal{D}^{2,2}(\mathbb{R}^N), \ \ u \text{ odd in } x_1\}.$$

Combing (2.34) and (2.35) in (2.36), we obtain

$$K(u_\lambda) = \frac{\int_{\mathbb{R}^N} |\Delta U|^2 \, dx}{\left(\int_{\mathbb{R}^N} |U|^{\frac{2N}{N-4}} \, dx \right)^{\frac{N-4}{N}}} + o(1) \quad \text{as} \quad \lambda \to \infty.$$

For the infimum, this implies that

$$K_2 \leq K_0. \tag{2.37}$$

This proves the reverse inequality and consequently the equality $K_2 = K_0$.

Remark. The question whether or not the infimum K_2 is achieved does not seem to be straightforward. However, we believe that the infimum K_2 is not achieved. From the first part of this section, we know that if the infimum K_2 is achieved, then there must be positive minimizers. These minimizers also satisfy the equations of Problem (IV) in the case $\Omega = (x_1 > 0)$. A formal calculation with variational identities [9], [11], [14] leads to an integral identity which proves that Problem (IV) cannot have any positive solutions on $(x_1 > 0)$. In order to use integral identities, one needs additional integrability of the minimizers; for example, $u \in H^2(x_1 > 0)$. These considerations are similar to those used in [6]. The precise evaluation of the terms in the integral identity involved needs a further careful study of the asymptotic behaviour of these minimizers.

Our next step in the proof of Theorem 1 is to recall a nonexistence result for Problem (IV) from [9], [14] in the case $\Omega = B_R$.

Lemma 2.3. *Consider Problem* (IV) *with* Ω *strictly star shaped. Then Problem* (IV) *has no (positive) solution* $u \in C^4(\Omega) \cap C^3(\bar{\Omega})$.

Employing Lemmas 2.1, 2.2 and 2.3 now enables us to complete the proof of Theorem 1. Once more we shall use the Talenti comparison principle in order to transform Problem (II) to the ball Ω^*. We should remark that from this point onward the proof is straightforward if Ω is star shaped, as will follow from [9], [14]. However, in the general case, a more involved line of argument has to be followed. The constant K_1 given by (2.1) can equivalently be calculated as

$$K_1 = \inf_{H^2 \cap H_0^1} \frac{\int_\Omega |\Delta u|^2 \, dx}{\left(\int_\Omega |u|^{\frac{2N}{N-4}} \, dx \right)^{\frac{N-4}{N}}}. \tag{2.38}$$

Exactly as before we write

$$-\Delta u = f \quad \text{in } \Omega, \quad u \in H^2 \cap H_0^1(\Omega). \tag{2.39}$$

Because $f \in L^2(\Omega)$, it follows that $f^* \in L^2(\Omega^*)$ and that $f \in L^{2N/(N+2)}(\Omega)$, using the inequality

$$\|f\|_{2N/(N+2)} \leq \text{meas}\,(\Omega)^{1/N} \|f\|_2 < \infty.$$

Next we construct a radially symmetric function v as before, using Talenti's principle. Let v be the H_0^1-solution of the problem

$$\begin{cases} -\Delta v = f^* \quad \text{in } \Omega, & (2.14) \\ v = 0 \quad \text{on } \partial\Omega. & (2.15) \end{cases}$$

Because $f^* \in L^2(\Omega^*)$, standard regularity yields $v \in H^2 \cap H_0^1(\Omega^*)$. The Talenti comparison principle now immediately gives

$$u^* < v \quad \text{a.e. in } \Omega^*. \tag{2.40}$$

For the integrals, we obtain using (2.40)

$$\int_{\Omega^*} |v|^{\frac{2N}{N-4}}\, dx \geq \int_{\Omega^*} |u^*|^{\frac{2N}{N-4}}\, dx = \int_{\Omega} |u|^{\frac{2N}{N-4}}\, dx, \tag{2.41}$$

$$\int_{\Omega^*} |\Delta v|^2\, dx = \int_{\Omega^*} (f^*)^2\, dx = \int_{\Omega} f^2\, dx = \int_{\Omega} |\Delta u|^2\, dx. \tag{2.42}$$

Using (2.38), the expressions (2.41) and (2.42) yield for the infimum K_1,

$$K_1(\Omega) = \inf \frac{\int_\Omega |\Delta u|^2\, dx}{\left(\int_\Omega |u|^{\frac{2N}{N-4}}\, dx\right)^{\frac{N-4}{N}}} \geq \inf \frac{\int_{\Omega^*} |\Delta v|^2\, dx}{\left(\int_{\Omega^*} |v|^{\frac{2N}{N-4}}\, dx\right)^{\frac{N-4}{N}}} = K_1(\Omega^*). \tag{2.43}$$

We proceed using a concentration-compactness argument. Assume

$$K_1(\Omega^*) < K_0. \tag{2.44}$$

By Lemmas 2.1 and 2.2 and symmetrization, we can conclude in this case that $K_1(\Omega^*)$ has a minimizer $\mathcal{U} \in H^2 \cap H_0^1(\Omega^*)$ which is positive and radially symmetric decreasing in $|x|$. As before, by regularity (see the appendix), this solution has to satisfy equation (2.7) and boundary conditions (2.8). We obtain Problem (IV) with the domain Ω^*. We recall

$$\text{(IVa)} \quad \begin{cases} \Delta^2 \mathcal{U} = K_1 \mathcal{U}^{\frac{N+4}{N-4}}, \quad \mathcal{U} > 0 \text{ in } \Omega^*, & (2.7a) \\ \mathcal{U} = 0, \quad \Delta\mathcal{U} = 0 \text{ on } \partial\Omega^*. & (2.8a) \end{cases}$$

From Lemma 2.3, we know that Problem (IVa) has no positive solutions; so, we have a contradiction. We conclude that the opposite of (2.44) holds; i.e.,

$$K_1(\Omega^*) \geq K_0.$$

Together with (2.43), this leads to the chain of inequalities

$$K_1(\Omega) \geq K_1(\Omega^*) \geq K_0. \tag{2.45}$$

A simple scaling argument shows that the reverse inequality holds (see [16]),

$$K_1(\Omega) \leq K_0. \tag{2.46}$$

Combining (2.45) and (2.46), we conclude that

$$K_1(\Omega) = K_0. \tag{2.47}$$

From (2.47) it is automatically clear that $K_1(\Omega)$ is domain independent. Because the infimum $K_1(\Omega)$ is now known, we have that for all $u \in H^2 \cap H_0^1(\Omega)$,

$$\frac{\int_\Omega |\Delta u|^2 \, dx}{\left(\int_\Omega |u|^{\frac{2N}{N-4}} \, dx \right)^{\frac{N-4}{N}}} \geq K_0,$$

which yields

$$\|u\|_{2N/(N-4)} \leq K_0^{-1/2} \|u\|_{2,2}, \tag{1.1}$$

and the theorem is proved.

Finally, to prove Theorem 2, we assume that $K_1(\Omega)$ has a minimizer $u \in H^2 \cap H_0^1(\Omega)$. Using symmetrization techniques as before, we construct the map

$$\mathcal{T} : u \to v,$$

where v is positive, radially symmetric and decreasing in Ω^*. We observe that v is a minimizer of $K_1(\Omega^*)$. As before, we show that $K_1(\Omega^*)$ has no minimizer, which contradicts the assertion. Consequently, the infimum $K_1(\Omega)$ is not achieved.

From the explicit form of the only minimizers of K_0, we can easily calculate the explicit form of K_0. The minimizers (1.3) can be found in [5], [7] and [15]. In [5] and [15], a uniqueness proof is also given. For a straight forward computation of K_0, we refer to these papers. In [15], higher order embedding constants are also computed.

Appendices. In these appendices, a number of lemmas will be given which were crucially used in the proof of this paper. The concentration compactness lemma is a virtual transcription of the principle due to Lions [8]. Though in [8] it is not explicitly used for fourth order equations, it was mentioned that it also applies to higher order equations. For completeness, we shall give the details for the case of fourth order equations.

A. A concentration compactness principle. Following [8] we define

$$\mathcal{E}(u) = \int_\Omega |\Delta u|^2 \, dx, \quad \mathcal{J}(u) = \int_\Omega |u|^{\frac{2N}{N-4}} \, dx$$

and consider the minimization problems (compare with [8], page 172)

$$K_1 = \inf\{\mathcal{E}(u) : \quad u \in H^2 \cap H_0^1(\Omega), \quad \mathcal{J}(u) = 1\}, \tag{A1}$$

$$K_{1,\lambda} = \inf\{\mathcal{E}(u) : \quad u \in H^2 \cap H_0^1(\Omega), \quad \mathcal{J}(u) = \lambda\}. \tag{A2}$$

Plainly

$$K_{1,\lambda} = \lambda^{\frac{N-4}{N}} K_1.$$

The idea now is to introduce the quantities $\mathcal{E}_y^\infty(u)$ and $\mathcal{J}_y^\infty(u)$ which arise from the transformations

$$u(x) \to \epsilon^{-\frac{N-4}{2}} u((x+y)/\epsilon), \quad y \in \bar{\Omega}, \quad \epsilon \to 0^+. \tag{A3}$$

Let us define

$$\mathcal{E}_y^\infty(u) = \lim_{\epsilon\to 0} \mathcal{E}(\epsilon^{-\frac{N-4}{2}} u((x+y)/\epsilon) = \begin{cases} \int_{\mathbb{R}^N} |\Delta u|^2 \, d\tilde{x} \text{ if } y \in \Omega \\ \int_{x_1>0} |\Delta u|^2 \, d\tilde{x} \text{ if } y \in \partial\Omega, \end{cases}$$

$$\mathcal{J}_y^\infty(u) = \lim_{\epsilon\to 0} \mathcal{J}(\epsilon^{-\frac{N-4}{2}} u((x+y)/\epsilon) = \begin{cases} \int_{\mathbb{R}^N} |u|^{\frac{2N}{N-4}} \, d\tilde{x} \text{ if } y \in \Omega \\ \int_{x_1>0} |u|^{\frac{2N}{N-4}} \, d\tilde{x} \text{ if } y \in \partial\Omega. \end{cases}$$

The reason why the quantities $\mathcal{E}_y^\infty(u)$ and $\mathcal{J}_y^\infty(u)$ are introduced is to take into account the loss of compactness caused by to the transformations (A3). Following [8], page 92–94, we introduce

$$K_y^\infty = \inf\{\mathcal{E}_y^\infty(u): \quad \mathcal{J}_y^\infty(u) = 1\} \tag{A4}$$

and (compare with [8], page 80)

$$K^\infty = \inf_{y\in\bar{\Omega}} K_y^\infty. \tag{A5}$$

In determining the information in (A5), it makes a difference whether $y \in \Omega$ or $y \in \partial\Omega$. Thus, it is convenient to write

$$K^\infty = \min\{\inf_{y\in\Omega} K_y^\infty, \ \inf_{y\in\partial\Omega} K_y^\infty\}. \tag{A6}$$

Write $K_0 = \inf_{y\in\Omega} K_y^\infty$ and $K_2 = \inf_{y\in\partial\Omega} K_y^\infty$. Then

$$K_0 = \inf\left\{ \int_{\mathbb{R}^N} |\Delta u|^2 \, dx : \ u \in \mathcal{D}^{2,2}, \ \int_{\mathbb{R}^N} |u|^{\frac{2N}{N-4}} \, dx = 1 \right\},$$

$$K_2 = \inf\left\{ \int_{x_1>0} |\Delta u|^2 \, dx : \ u \in \mathcal{D}^{2,2}, \ u\,\Big|_{x_1=0} = 0, \ \int_{x_1>0} |u|^{\frac{2N}{N-4}} \, dx = 1 \right\}.$$

Consequently,

$$K^\infty = \min\{K_0, \, K_2\}.$$

In the following lemma, we give a compactness criterion using the quantities introduced above.

Lemma A1. *Consider the minimization problem* (A1). *Suppose*

$$K_1 < K^\infty. \tag{A7}$$

Then every minimizing sequence (u_n) *of* (A1) *is relatively compact in* $H^2 \cap H_0^1(\Omega)$. *Moreover,* K_1 *is attained by a function* $u \in H^2 \cap H_0^1(\Omega)$. *If* (A7) *is not satisfied, i.e.,*

$$K_1 = K^\infty, \tag{A8}$$

then there exists a noncompact minimizing sequence (u_n) *of* (A1) *such that*

(i) $u_n \to 0$ *weakly in* $H^2 \cap H_0^1(\Omega)$ *and* $|\Delta u_n|^2 \to K^\infty \delta_{x_0}$ *as* $n \to \infty$,

(ii) $|u_n|^{\frac{2N}{N-4}} \to \delta_{x_0}$ *for some* $x_0 \in \bar{\Omega}$,

(iii) $\exists \sigma_n \in \mathbb{R}$, $\sigma_n \to \infty$, $\exists y_n \in \mathbb{R}^N$, $\frac{y_n}{\sigma_n} \to x_0$, $\sigma_n^{-\frac{N-4}{2}} u_n((\cdot + y_n)/\sigma_n) \to \bar{u}$, *the minimum of* $K_{x_0}^\infty$.

Proof. The proof of Lemma A1 consists mainly of obtaining a concentration lemma as in [8] (Lemmas 1.1 and 1.4). As in the proof of Lemma 1.4 of [8], one uses $\mathcal{E}_y^\infty$, $\mathcal{J}_y^\infty$ and K_y^∞ to establish the following concentration principle.

Let $u_n \to u$ weakly in $H^2 \cap H_0^1(\Omega)$ and $|u_n|^{\frac{2N}{N-4}}$ converge weakly in the sense of measures to $|u|^{\frac{2N}{N-4}} + \sum_{k\in J} \nu_k \delta_{x_k}$. Then we may assume that

$$|\Delta u_{n_k}|^2 \to \mu$$

for some positive bounded measure μ and

$$\mu \geq |\Delta u|^2 + \sum_{k\in J} \nu_k^{\frac{N-4}{N}} K^\infty \delta_{x_k}.$$

If we now follow the proof in [8] on pages 175–177 and use the above concentration principle, the proof of the lemma follows.

B. Regularity of solutions. Consider the distributional equation

$$\Delta^2 u = a(x)u \quad \text{in} \quad \mathcal{D}'(\Omega), \quad \text{where} \quad a(x) \in L^{N/4}(\Omega). \tag{$*$}$$

The technique of obtaining L^p-estimates for all $p < \infty$, as employed by Brezis and Kato [3] using Moser iteration, does not seem to be straightforward for this equation. Using a completely different method, we establish the following L^p-regularity result.

Lemma B1. *Let $u \in H^2 \cap H_0^1(\Omega)$ be a solution of $(*)$ in the distributional sense. Then*

$$u \in L^p(\Omega) \quad \textit{for all } p, \quad 1 \leq p < \infty.$$

Proof. Step 1. In order to prove the lemma, we shall first establish an auxiliary result in which we show that it is possible to write the product $a(x)u(x)$ as a sum $a(x)u(x) = q_\varepsilon(x)u(x) + f_\varepsilon(x)$, in which q_ε is arbitrary small in $L^{N/4}$ and $f_\varepsilon \in L^\infty(\Omega)$.

Lemma B2. *For every $\varepsilon > 0$, there are functions $q_\varepsilon \in L^{N/4}(\Omega)$, $f_\varepsilon \in L^\infty(\Omega)$ and a constant $K_\varepsilon > 0$ such that*

$$a(x)u = q_\varepsilon(x)u + f_\varepsilon$$

and

$$\|q_\varepsilon\|_{N/4} < \varepsilon, \quad \|f_\varepsilon\|_\infty \leq K_\varepsilon.$$

Proof. Define the sets

$$\Omega_k = \{x \in \Omega \colon |a| < k\} \quad \text{and} \quad \Omega_l = \{x \in \Omega \colon |u| < l\},$$

where k and l are chosen such that

$$\|a\|_{L^{N/4}(\Omega_k^c)} < \frac{1}{4}\varepsilon, \quad \|a\|_{L^{N/4}(\Omega_l^c)} < \frac{1}{4}\varepsilon \tag{B1}$$

and $\Omega_k \cap \Omega_l \neq \emptyset$, $a(x) \not\equiv 0$ on $\Omega_k \cap \Omega_l$. Clearly these conditions can be met provided k, l are large enough, because $a(x)$ and $u(x)$ are L^1-functions. We now define

$$q_\varepsilon(x) = \begin{cases} \frac{1}{n} a(x) & \text{on } \Omega_k \cap \Omega_l \\ a(x) & \text{on } \Omega_k^c \cup \Omega_l^c \end{cases}$$

and

$$f_\varepsilon(x) = \{a(x) - q_\varepsilon(x)\} u(x). \tag{B2}$$

Then clearly

$$f_\varepsilon(x) = 0 \quad \text{for all} \quad x \in (\Omega_k \cap \Omega_l)^c.$$

For the $L^{N/4}$-norm of q_ε we find

$$\begin{aligned} \|q_\varepsilon\|_{N/4}^{N/4} &= \int_\Omega |q_\varepsilon|^{N/4}\,dx = \int_{\Omega_k \cap \Omega_l} |q_\varepsilon|^{N/4}\,dx + \int_{(\Omega_k \cap \Omega_l)^c} |q_\varepsilon|^{N/4}\,dx \\ &\leq \int_{\Omega_k \cap \Omega_l} |q_\varepsilon|^{N/4}\,dx + \int_{\Omega_k^c} |q_\varepsilon|^{N/4}\,dx + \int_{\Omega_l^c} |q_\varepsilon|^{N/4}\,dx \\ &= \Big(\frac{1}{n}\Big)^{N/4} \int_{\Omega_k \cap \Omega_l} |a|^{N/4}\,dx + \int_{\Omega_k^c} |a|^{N/4}\,dx + \int_{\Omega_l^c} |a|^{N/4}\,dx \\ &< \Big(\frac{1}{n}\Big)^{N/4} \int_{\Omega_k \cap \Omega_l} |a|^{N/4}\,dx + \frac{1}{4}\varepsilon + \frac{1}{4}\varepsilon. \end{aligned}$$

So,

$$\|q_\varepsilon\|_{N/4} < \frac{1}{n}\|a\|_{N/4} + \frac{1}{2}\varepsilon. \tag{B3}$$

Hence, for $n = n_\varepsilon > 2\|a\|_{N/4}/\varepsilon$, we have

$$\|q_\varepsilon\|_{N/4} < \varepsilon. \tag{B4}$$

As for f_ε, since it is only nonzero in $\Omega_k \cap \Omega_l$, it follows from the construction that

$$\|f_\varepsilon\|_\infty = \left|1 - \frac{1}{n_\varepsilon}\right| kl < \infty. \tag{B5}$$

Step 2. By Step 1, Problem $(*)$ can be written as

$$\Delta^2 u = q_\varepsilon u + f_\varepsilon, \quad \text{where} \quad \|q_\varepsilon\|_{N/4} < \varepsilon \quad \text{and} \quad f_\varepsilon \in L^\infty(\Omega), \tag{B6}$$

or, equivalently, as

$$u - A^\varepsilon u = h_\varepsilon, \tag{B7}$$

where $A^\varepsilon u = (-\Delta)^{-2}\ (q_\varepsilon u)$‡ and $h_\varepsilon = (-\Delta)^{-2} f_\varepsilon$. We shall show using Lemma B2 that A^ε acts from L^p to L^p and that $q_\varepsilon(x)$ can be chosen such that the operator

‡The operator $(-\Delta)^{-2}$ is defined by using the Green function for the operator $(-\Delta)^2$, with the boundary conditions $u = \Delta u = 0$ on $\partial\Omega$. Thus, $(-\Delta)^{-2} u = G * u$.

norm $\|A^\varepsilon\|_{L^p\to L^p} < 1/2$. We proceed as follows. Let $v \in L^p$. Then by using the Hardy-Littlewood-Sobolev inequality

$$\|A^\varepsilon v\|_p = \|(-\Delta)^{-2}(q_\varepsilon v)\|_p \le C(p)\|q_\varepsilon v\|_r, \tag{B8}$$

where

$$\frac{1}{r} = \frac{1}{p} + \frac{4}{N}. \tag{B9}$$

Employing Hölder's inequality and the fact that $q_\varepsilon \in L^{N/4}(\Omega)$, we derive

$$\|A^\varepsilon v\|_p < C(p)\|q_\varepsilon\|_{N/4}\|v\|_p, \tag{B10}$$

which shows that K_2^ε acts from L^p to L^p for some $p \ge 2N/(N-4)$. According to Lemma B2, we can choose $\varepsilon = \varepsilon^* = \frac{1}{2C(p)}$. This results in

$$\|A^{\varepsilon^*} v\|_p < \frac{1}{2}\|v\|_p,$$

which gives for all $p \ge 2N/(N-4)$,

$$\|A^{\varepsilon^*}\|_{L^p\to L^p} < \frac{1}{2}. \tag{B11}$$

Step 3. To conclude the proof, we consider the operator

$$(I - A^{\varepsilon^*})^{-1} \colon L^p \to L^p. \tag{B12}$$

Note that by (B11),

$$\|(I - A^{\varepsilon^*})^{-1}\|_{L^p\to L^p} < 2. \tag{B13}$$

From equation (B7), we deduce that

$$u = (I - A^{\varepsilon^*})^{-1} h_{\varepsilon^*},$$

and from (B13) that

$$\|(I - A^{\varepsilon^*})^{-1} h_{\varepsilon^*}\|_p < 2\|h_{\varepsilon^*}\|_p = 2\|(-\Delta)^{-2} f_{\varepsilon^*}\|_p \le 2C(p)\|f_{\varepsilon^*}\|_p \le \tilde{C}(p)\|f_{\varepsilon^*}\|_\infty.$$

This yields the estimate

$$\|u\|_p < \tilde{C}(p)\|f_{\varepsilon^*}\|_\infty \quad \text{for every} \quad p \ge 2N/(N-4). \tag{B14}$$

Because $u \in L^{2N/(N-4)}(\Omega)$, we finally derive

$$u \in L^p(\Omega) \quad \text{for all } p, \quad 1 \le p < \infty, \tag{B15}$$

which completes the proof.

From this essential Lemma B1, it is possible to obtain a straightforward regularity result for weak solutions of the problem

$$\int_\Omega \Delta u \Delta \phi \, dx = \int_\Omega g(u) u \phi \, dx \quad \text{for all} \quad \phi \in H^2 \cap H_0^1(\Omega). \tag{**}$$

Lemma B3. *Let $u \in H^2 \cap H_0^1(\Omega)$ be a weak solution of $(**)$, $\partial\Omega$ of class $C^{4,\lambda}$ and $g(s) \in C^{0,\alpha}(\mathbb{R})$ with $0 \le q \le \frac{8}{N-4}$*

$$|g(s)| \le C_1|s|^q + C_2, \quad \textit{for positive constants} \quad C_1,\, C_2.$$

Then

$$u \in C^4(\Omega) \cap C^3(\bar{\Omega})$$

and u satisfies the equation

$$\begin{cases} \Delta^2 u = g(u)u & \textit{in } \Omega, \\ u = \Delta u = 0 & \textit{on } \partial\Omega. \end{cases}$$

If $g(s) \in C^k(\mathbb{R})$, $k \in \mathbb{N}$ and $\partial\Omega$ of class $C^{4+k,\lambda}$, then $u \in C^{4+k}(\Omega) \cap C^3(\bar{\Omega})$.

Remark. If for example $g(s) = |s|^{8/(N-4)}$, the conditions of Lemma B3 are satisfied and consequently $u \in C^4(\Omega) \cap C^3(\bar{\Omega})$. If positive solutions are considered then, since $g(s) \in C^\infty(\mathbb{R}^+)$, Lemma B3 implies that

$$u \in C^\infty(\Omega).$$

Proof. Let $u \in H^2 \cap H_0^1(\Omega)$ be a weak solution of Problem $(**)$ and thus $g(u(x)) \in L^{N/4}(\Omega)$. Then, if we set $a(x) = g(u(x))$, u satisfies

$$\Delta^2 u = a(x)u \quad \text{in} \quad \mathcal{D}'(\Omega).$$

According to Lemma B1, we have $u \in L^p(\Omega)$ for every $p \ge 1$. Consequently,

$$a(x)u \in L^p(\Omega), \quad 1 \le p < \infty. \tag{B16}$$

Now we set $a(x)u(x) = f(x)$. Let $\hat{u} \in H^2 \cap H_0^1(\Omega)$ be the unique solution of

$$\int_\Omega \Delta\hat{u}\Delta\phi\, dx = \int_\Omega f(x)\phi\, dx, \quad \forall\phi \in H^2 \cap H_0^1(\Omega),$$

with the additional L^p-estimates due to Agmon, Douglis and Nirenberg [1] (smoothness $\partial\Omega$); namely, $\|\hat{u}\|_{4,p} \le C_p\|f\|_p$. Furthermore,

$$\Delta^2\hat{u} = f(x) \quad \text{a.e. in } \Omega.$$

Notice that for u and $\hat{u}$ we have

$$\int_\Omega \Delta(\hat{u} - u)\Delta\phi\, dx = 0, \quad \forall\phi \in H^2 \cap H_0^1(\Omega),$$

which implies $\hat{u} = u$ a.e. on Ω. We thus have

$$\begin{aligned} &\|u\|_{4,p} \le C_p\|f\|_p, \quad p > 1, \\ &\Delta^2 u = f(x), \quad \text{a.e. in } \Omega. \end{aligned} \tag{B17}$$

We recall from standard embeddings theorems, taking p large enough,

$$W^{4,p}(\Omega) \to C^{3,\lambda}(\bar{\Omega}), \quad p > N, \quad \lambda \leq 1 - N/p,$$
$$C^{3,\lambda}(\bar{\Omega}) \to C^3(\bar{\Omega}).$$

Using (B17) proves

$$u \in C^3(\bar{\Omega}). \tag{B18}$$

Finally, because $W^{4,p}(\Omega) \to C^{0,\lambda}(\bar{\Omega})$, when $p > N/4$ we have $g(u(x))u(x) \in C^{0,\lambda}(\bar{\Omega})$. By Schauder estimates [1] (using the smoothness of $\partial\Omega$), we conclude that

$$u \in C^4(\Omega). \tag{B19}$$

Combining (B18) and (B19) proves $u \in C^4(\Omega) \cap C^3(\bar{\Omega})$. Because u is a $H^2 \cap H_0^1$-solution, partial integration yields the boundary conditions. The last remark on the C^{k+4}-solutions can be achieved again by Schauder estimates because the left hand side is smoother in this case.

C. A contraction property of the symmetrization map. Let's denote the Schwarz symmetrization of a function ϕ as ϕ^*. The Schwarz symmetrization of ϕ is defined as

$$\phi^*(x) = \inf\{y \geq 0\colon \mu(y) < \omega_N |x|^N\},$$

where $\mu(y) = \text{meas}\{x \in \Omega\colon |\phi(x)| > y\}$ and ω_N is the measure of the N-dimensional unit ball. Consequently, the Schwarz symmetrization is defined on the ball Ω^*, with the additional property meas $(\Omega) =$ meas (Ω^*). Denote the symmetrization map by

$$\begin{cases} \mathcal{S}\colon L^p(\Omega) \to L^p(\Omega), \\ \mathcal{S}\phi = \phi^*. \end{cases}$$

Lemma C1. *Suppose ϕ, $\psi \in L^p(\Omega)$ for a certain $p \in [1, \infty]$. Then*

$$\|\mathcal{S}\phi - \mathcal{S}\psi\|_p \leq \|\phi - \psi\|_p.$$

Proof. See [10].

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