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Research Article

Best Possible Bounds for Yang Mean Using Generalized Logarithmic Mean

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We prove that the double inequality $L_p(a,b) < U(a,b) < L_q(a,b)$ holds for all $a,b > 0$ with $a \neq b$ if and only if $p \leq p_0$ and $q \geq 2$ and find several sharp inequalities involving the trigonometric, hyperbolic, and inverse trigonometric functions, where $p_0 = 0.5451\cdots$ is the unique solution of the equation $(p+1)^{1/p} = \sqrt{2}\pi/2$ on the interval $(0,\infty)$, $U(a,b) = (a-b)/[\sqrt{2}\arctan((a-b)/\sqrt{2ab})]$, and $L_p(a,b) = [(a^{p+1} - b^{p+1})/((p+1)(a-b))]^{1/p}$ $(p \neq -1, 0)$, $L_{-1}(a,b) = (a-b)/(\log a - \log b)$ and $L_0(a,b) = (a^a/b^b)^{1/(a-b)}/e$ are the Yang, and pth generalized logarithmic means of a and b, respectively.

1. Introduction

For $p \in \mathbb{R}$ and $a,b > 0$ with $a \neq b$, the pth generalized logarithmic mean $L_p(a,b)$ is defined by

$$L_p(a,b) = \begin{cases} \left[\dfrac{a^{p+1} - b^{p+1}}{(p+1)(a-b)}\right]^{1/p}, & p \neq 0, -1, \\ \dfrac{1}{e}\left(\dfrac{a^a}{b^b}\right)^{1/(a-b)}, & p = 0, \\ \dfrac{a-b}{\log a - \log b}, & p = -1. \end{cases} \quad (1)$$

It is well known that $L_p(a,b)$ is continuous and strictly increasing with respect to $p \in \mathbb{R}$ for fixed $a,b > 0$ with $a \neq b$. Many classical bivariate means are the special case of the generalized logarithmic mean. For example, $G(a,b) = \sqrt{ab} = L_{-2}(a,b)$ is the geometric mean, $L(a,b) = (a-b)/(\log a - \log b) = L_{-1}(a,b)$ is the logarithmic mean, $I(a,b) = (a^a/b^b)^{1/(a-b)}/e = L_0(a,b)$ is the identric mean, and $A(a,b) = (a+b)/2 = L_1(a,b)$ is the arithmetic mean. Recently, the generalized logarithmic mean has been the subject of intensive research.

Stolarsky [1] proved that the inequality

$$L_p(a,b) < M_{(2+p)/3}(a,b) \quad (2)$$

holds for all $a,b > 0$ with $a \neq b$ and $p \in (-2,-1/2) \cup (1,\infty)$, and inequality (2) is reversed for $p \in (-\infty,-2) \cup (-1/2,1)$, where $M_r(a,b) = [(a^r + b^r)/2]^{1/r}$ $(r \neq 0)$ and $M_0(a,b) = \sqrt{ab}$ is the rth power mean of a and b.

Yang [2] proved that the double inequality

$$A(a,b) < L_p(a,b) < M_p(a,b) \quad (3)$$

holds for all $a,b > 0$ with $a \neq b$ if $p > 1$, and inequality (3) is reversed if $p < 0$.

In [3], the authors proved that the inequality

$$L_p(a,b) < \frac{a+b}{(p+1)^{1/p}} \quad (4)$$

holds for all $a,b > 0$ with $a \neq b$ and $p > 1$.

Li et al. [4] proved that the function $p \mapsto L_p(a,b)/L_p(1-a,1-b)$ is strictly increasing (decreasing) on $\mathbb{R}$ if $0 < a < b \leq 1/2$ $(1/2 \leq a < b < 1)$. In [5, 6], the authors proved that the function $q \mapsto L_q(a,b)/L_q(a,c)$ is strictly decreasing on $\mathbb{R}$ if $0 < a < b < c$ and the function $r \mapsto L_r(d,d+\varepsilon)/L_r(d+\delta,d+\varepsilon+\delta)$ is strictly increasing on $\mathbb{R}$ for all $d,\varepsilon,\delta > 0$.

Shi and Wu [7] proved that the double inequality

$$\left[\frac{(\lambda b+((1-\lambda)/2)(a+b))^{p+1}-(\lambda a+((1-\lambda)/2)(a+b))^{p+1}}{\lambda(p+1)(b-a)}\right]^{1/p}<L_p(a,b)<\left[\frac{(a+b-c)^{p+1}-c^{p+1}}{(p+1)(a+b-2c)}\right]^{1/p} \quad (5)$$

for all $b>a>c>0$ and $0<\lambda<1$ if $p>1$, and inequality (5) is reversed if $p\in(-1,0)\cup(0,1)$.

Long and Chu [8] and Matejíčka [9] presented the best possible parameters $p=p(\alpha)$ and $q=q(\alpha)$ such that the double inequality

$$L_p(a,b)<\alpha A(a,b)+(1-\alpha)G(a,b)<L_q(a,b) \quad (6)$$

holds for all $a,b>0$ with $a\neq b$ and $\alpha\in(0,1/2)\cup(1/2,1)$.

In [10], Qian and Long answered the question: what are the greatest value p and the least value q such that the double inequality

$$L_p(a,b)<G^{\alpha}(a,b)H^{1-\alpha}(a,b)<L_q(a,b) \quad (7)$$

holds for all $a,b>0$ with $a\neq b$ and $\alpha\in(0,1)$, where $H(a,b)=2ab/(a+b)$ is the harmonic mean of a and b.

In [11, 12], the authors proved that the double inequalities

$$\begin{gathered}L_{p_1}(a,b)<M(a,b)<L_{q_1}(a,b),\\ L_{p_2}(a,b)<T(a,b)<L_{q_2}(a,b)\end{gathered} \quad (8)$$

hold for all $a,b>0$ with $a\neq b$ if and only if $p_1\leq p_1^*$, $q_1\geq 2$, $p_2\leq 3$, $q_2\geq q_2^*$, where $p_1^*=1.843\cdots$ is the unique solution of the equation $(p+1)^{1/p}=2\log(1+\sqrt{2})$ on the interval $(0,\infty)$, $q_2^*=3.152\cdots$ is the unique solution of the equation $(q+1)^{1/q}=\pi/2$ on the interval $(0,\infty)$, $M(a,b)=(a-b)/[2\sinh^{-1}((a-b)/(a+b))]$ is the Neuman-Sándor mean, and $T(a,b)=(a-b)/[2\arctan((a-b)/(a+b))]$ is the second Seiffert mean.

In [13, 14], the authors presented the best possible parameters $p_1=p_1(q)$, $p_2=p_2(q)$, $\lambda=\lambda(\alpha)$, and $\mu=\mu(\alpha)$ such that the double inequalities

$$\begin{gathered}L_{p_1}(a,b)<[L(a^q,b^q)]^{1/q}<L_{p_2}(a,b),\\ L_{\lambda}(a,b)<G^{\alpha}(a,b)\left[\frac{A(a,b)+G(a,b)}{2}\right]^{1-\alpha}\\ <L_{\mu}(a,b)\end{gathered} \quad (9)$$

hold for all $a,b>0$ with $a\neq b$, $q>0$ with $q\neq 1$ and $\alpha\in(0,2/3)\cup(2/3,1)$.

Gao et al. [15] provided the greatest value α and the least value β such that the double inequality

$$L_{\alpha}(a,b)<P(a,b)<L_{\beta}(a,b) \quad (10)$$

holds for all $a,b>0$ with $a\neq b$, where $P(a,b)=(a-b)/[2\arcsin((a-b)/(a+b))]$ is the first Seiffert mean of a and b.

Very recently, Yang [16] introduced the Yang mean

$$U(a,b)=\frac{a-b}{\sqrt{2}\arctan\left((a-b)/\sqrt{2ab}\right)} \quad (11)$$

of two distinct positive real numbers a and b and proved that the inequalities

$$\begin{gathered}P(a,b)<U(a,b)<T(a,b),\\ \frac{G(a,b)T(a,b)}{A(a,b)}<U(a,b)<\frac{P(a,b)Q(a,b)}{A(a,b)},\\ Q^{1/2}(a,b)\left[\frac{2G(a,b)+Q(a,b)}{3}\right]^{1/2}<U(a,b)\\ <Q^{2/3}(a,b)\left[\frac{G(a,b)+Q(a,b)}{2}\right]^{1/3},\\ \frac{G(a,b)+Q(a,b)}{2}<U(a,b)\\ <\left[\frac{2}{3}\left(\frac{G(a,b)+Q(a,b)}{2}\right)^{1/2}+\frac{1}{3}Q^{1/2}(a,b)\right]^2\end{gathered} \quad (12)$$

hold for all $a,b>0$ with $a\neq b$, where $Q(a,b)=\sqrt{(a^2+b^2)/2}$ is the quadratic mean of a and b.

The Yang mean $U(a,b)$ is the special case of the Seiffert type mean $T_{M,q}(a,b)=(a-b)/[q\arctan((a-b)/(qM(a,b)))]$ defined by Toader in [17], where $M(a,b)$ is a bivariate mean and q is a positive real number. Indeed, $U(a,b)=T_{G,\sqrt{2}}(a,b)$.

In [18, 19], the authors proved that the double inequalities

$$\begin{gathered}\left[\frac{2}{3}\left(\frac{G(a,b)+Q(a,b)}{2}\right)^{p}+\frac{1}{3}Q^{p}(a,b)\right]^{1/p}<U(a,b)\\ <\left[\frac{2}{3}\left(\frac{G(a,b)+Q(a,b)}{2}\right)^{q}+\frac{1}{3}Q^{q}(a,b)\right]^{1/q},\\ \frac{2^{1-\lambda}(G(a,b)+Q(a,b))^{\lambda}Q(a,b)+G(a,b)Q^{\lambda}(a,b)}{2^{1-\lambda}(G(a,b)+Q(a,b))^{\lambda}+Q^{\lambda}(a,b)}\\ <U(a,b)\\ <\frac{2^{1-\mu}(G(a,b)+Q(a,b))^{\mu}Q(a,b)+G(a,b)Q^{\mu}(a,b)}{2^{1-\mu}(G(a,b)+Q(a,b))^{\mu}+Q^{\mu}(a,b)},\\ M_{\alpha}(a,b)<U(a,b)<M_{\beta}(a,b)\end{gathered} \quad (13)$$

hold for all $a,b>0$ with $a\neq b$ if and only if $p\leq p_0$, $q\geq 1/5$, $\lambda\geq 1/5$, $\mu\leq p_1$, $\alpha\leq 2\log 2/(2\log\pi-\log 2)$, and $\beta\geq 4/3$, where $p_0=0.1941\cdots$ is the unique solution of the equation

$p\log(2/\pi)-\log(1+2^{1-p})+\log 3=0$ on the interval $(1/10,\infty)$, and $p_1=\log(\pi-2)/\log 2=0.1910\cdots$.

Zhou et al. [20] proved that $\alpha=1/2$ and $\beta=\log 3/(1+\log 2)=0.6488\cdots$ are the best possible parameters such that the double inequality

$$\left[\frac{a^{\alpha}+(ab)^{\alpha/2}+b^{\alpha}}{3}\right]^{1/\alpha}<U(a,b)$$
$$<\left[\frac{a^{\beta}+(ab)^{\beta/2}+b^{\beta}}{3}\right]^{1/\beta} \quad (14)$$

holds for all $a,b>0$ with $a\neq b$.

The main purpose of this paper is to present the best possible parameters p and q such that the double inequality $L_p(a,b)<U(a,b)<L_q(a,b)$ holds for all $a,b>0$ with $a\neq b$. As application, we derive several sharp inequalities involving the trigonometric, hyperbolic, and inverse trigonometric functions. Some complicated computations are carried out using Mathematica computer algebra system.

2. Lemmas

In order to prove our main result we need two lemmas, which we present in this section.

Lemma 1. *Let $n\geq 4$, $-\infty\leq a<b\leq +\infty$, $f_0, f_i, f_i^*:(a,b)\to\mathbb{R}$ be n-times differentiable functions such that $f_i(x)=f_i^*(x)f_{i-1}'(x)$ and $f_i^*(x)>0$ for $1\leq i\leq n$ and $x\in(a,b)$. If*

$$f_{n-2}(a^-)<0,$$
$$f_{n-2}(b^+)>0, \quad (15)$$
$$f_{n-1}(a^-)\geq 0,$$

$$f_i(a^-)\leq 0,$$
$$f_i(b^+)>0 \quad (16)$$

for $0\leq i\leq n-3$ and

$$f_n(x)>0 \quad (17)$$

for $x\in(a,b)$, then there exists $x_0\in(a,b)$ such that $f_0(x)<0$ for $x\in(a,x_0)$ and $f_0(x)>0$ for $x\in(x_0,b)$.

Proof. From (15) and (17) we clearly see that there exists $x_{n-2}\in(a,b)$ such that $f_{n-2}(x)<0$ for $x\in(a,x_{n-2})$ and $f_{n-2}(x)>0$ for $x\in(x_{n-2},b)$, which implies that $f_{n-3}(x)$ is strictly decreasing on $(a,x_{n-2}]$ and strictly increasing on $[x_{n-2},b)$. Then (16) leads to the conclusion that there exists $x_{n-3}\in(a,b)$ such that $f_{n-3}(x)<0$ for $x\in(a,x_{n-3})$ and $f_{n-3}(x)>0$ for $x\in(x_{n-3},b)$.

Making use of (16) and the same method as above we know that for $0\leq i\leq n-4$ there exists $x_i\in(a,b)$ such that $f_i(x)<0$ for $x\in(a,x_i)$ and $f_i(x)>0$ for $x\in(x_i,b)$. □

Lemma 2. *Let $p\in\mathbb{R}$, and*

$$\begin{aligned} f(x,p) = {} & px^{4p+10}-2(p+1)x^{4p+8} \\ & +2(3p-1)x^{4p+6}-2(p+1)x^{4p+4} \\ & -(3p+2)x^{4p+2}-p(2p+1)x^{2p+10} \\ & +(2p^2+5p+4)x^{2p+8}+4(1-p)x^{2p+6} \\ & +4(1-p)x^{2p+4}+(2p^2+5p+4)x^{2p+2} \\ & -p(2p+1)x^{2p}-(3p+2)x^8 \\ & -2(p+1)x^6+2(3p-1)x^4 \\ & -2(p+1)x^2+p. \end{aligned} \quad (18)$$

Then the following statements are true:

(1) *if $p=2$, then $f(x,p)>0$ for all $x\in(1,\infty)$;*

(2) *if $p_0=0.5451\cdots$ is the unique solution of the equation $(p+1)^{1/p}=\sqrt{2}\pi/2$ on the interval $(0,\infty)$ and $p=p_0$, then there exists $\lambda\in(1,\infty)$ such that $f(x,p)<0$ for $x\in(1,\lambda)$ and $f(x,p)>0$ for $x\in(\lambda,\infty)$.*

Proof. For part (1), if $p=2$, then (18) becomes

$$f(x,p)=2(x-1)^6(x+1)^6(x^2+1)^3. \quad (19)$$

Therefore, part (1) follows from (19).

For part (2), let $p=p_0=0.5451\cdots$ be the unique solution of the equation $(p+1)^{1/p}=\sqrt{2}\pi/2$ on the interval $(0,\infty)$, $f_1(x,p)=(1/2x)\partial f(x,p)/\partial x$, $f_2(x,p)=(1/2x)\partial f_1(x,p)/\partial x$, $f_3(x,p)=(1/2x)\partial f_2(x,p)/\partial x$, $f_4(x,p)=(1/2x)\partial f_3(x,p)/\partial x$, $f_5(x,p)=(x^{9-2p}/2p)\partial f_4(x,p)/\partial x$, $f_6(x,p)=(1/2(p+1)x)\partial f_5(x,p)/\partial x$, $f_7(x,p)=(1/8x)\partial f_6(x,p)/\partial x$, $f_8(x,p)=(1/2x)\partial f_7(x,p)/\partial x$, and $f_9(x,p)=(1/2x)\partial f_8(x,p)/\partial x$. Then elaborated computations lead to

$$f(1^-,p)=0,$$
$$\lim_{x\to+\infty}f(x,p)=+\infty,$$
$$f_1(1^-,p)=0,$$
$$\lim_{x\to+\infty}f_1(x,p)=+\infty,$$
$$f_2(1^-,p)=0,$$
$$\lim_{x\to+\infty}f_2(x,p)=+\infty,$$
$$f_3(1^-,p)=0,$$
$$\lim_{x\to+\infty}f_3(x,p)=+\infty,$$
$$f_4(1^-,p)=-48p^2(2-p)(p+1)<0,$$

$$\lim_{x\to+\infty} f_4(x,p) = +\infty,$$

$$f_5(1^-,p) = -120p(p+1)(2-p)(2p+1) < 0,$$

$$\lim_{x\to+\infty} f_5(x,p) = +\infty,$$

$$f_6(1^-,p) = -80p(p+1)\left(-7p^2+8p+8\right) < 0,$$

$$\lim_{x\to+\infty} f_6(x,p) = +\infty,$$

$$f_7(1^-,p) = 5p(p+1)\left(44p^3+16p^2-47p-58\right) < 0,$$

$$\lim_{x\to+\infty} f_7(x,p) = +\infty,$$

$$f_8(1^-,p) = p\left(288p^5+606p^4+1009p^3+1676p^2+1571p+250\right) > 0,$$

$$\begin{aligned}
f_9(x,p) &= (p+2)^2(p+3)(p+4)(p+5)(2p+1)\cdot(2p+3)(2p+5)x^{2p+2} - 4(p+1)^2(p+2)^2(p+3)(p+4)(2p+1)(2p+3)x^{2p} \\
&\quad + 2p(p+1)(p+2)(p+3)\left(4p^2-1\right)(2p+3)(3p-1)x^{2p-2} - 4p\left(p^2-1\right)^2(p+2)\left(4p^2-1\right)x^{2p-4} \\
&\quad - p(p-1)^2(p-2)(2p-3)\left(4p^2-1\right)(3p+2)x^{2p-6} - 30(p+2)(p+3)(p+4)(p+5)(2p+1)x^2 \\
&\quad + 6(p+2)(p+3)(p+4)\left(2p^2+5p+4\right) > (p+2)(p+3)(p+4)(2p+1)\left[(p+2)(p+5)\cdot(2p+3)(2p+5) - 4(p+1)^2(p+2)(2p+3) - 30(p+5)\right]\times x^{2p+2} \\
&\quad + \left[2p(p+1)(p+2)\cdot(p+3)\left(4p^2-1\right)(2p+3)(3p-1) - 4p\left(p^2-1\right)^2(p+2)\left(4p^2-1\right) - p(p-1)^2\cdot(p-2)(2p-3)\left(4p^2-1\right)(3p+2) + 6(p+2)\cdot(p+3)(p+4)\left(2p^2+5p+4\right)\right]x^{2p-4} \\
&= (p+2)\cdot(p+3)(p+4)(2p+1)\left[-4p^4+79p^2+159p-24\right]x^{2p+2} + \left(8p^8+428p^7+694p^6+781p^5+100p^4+163p^3+1250p^2+1400p+576\right)x^{2p-4} > 0
\end{aligned} \tag{20}$$

for $x \in (1,\infty)$.

Therefore, part (2) follows easily from Lemma 1 and (20). □

3. Main Result

Theorem 3. *The double inequality*

$$L_p(a,b) < U(a,b) < L_q(a,b) \tag{21}$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $p \leq p_0$ and $q \geq 2$, where $p_0 = 0.5451\cdots$ is the unique solution of the equation $(p+1)^{1/p} = \sqrt{2}\pi/2$ on the interval $(0,\infty)$.

Proof. Since $U(a,b)$ and $L_p(a,b)$ are symmetric and homogeneous of degree one, without loss of generality, we assume that $a = x^2 > 1$ and $b = 1$. Let $p \in \mathbb{R}$ and $p \neq 0, -1$. Then (1) and (11) lead to

$$\begin{aligned}
\log[U(a,b)] - \log\left[L_p(a,b)\right] &= \log\left[U\left(x^2,1\right)\right] - \log\left[L_p\left(x^2,1\right)\right] \\
&= \log\left[\frac{x^2-1}{\sqrt{2}\arctan\left(\left(x^2-1\right)/\sqrt{2}x\right)}\right] - \frac{1}{p}\cdot\log\left[\frac{x^{2p+2}-1}{(p+1)\left(x^2-1\right)}\right] =: F(x,p),
\end{aligned} \tag{22}$$

$$\lim_{x\to1} F(x,p) = 0, \tag{23}$$

$$\lim_{x\to\infty} F(x,p) = \frac{1}{p}\log(p+1) + \frac{1}{2}\log 2 - \log\pi, \tag{24}$$

$$\frac{\partial F(x,p)}{\partial x} = \frac{2(p+1)x\left(x^{2p}-1\right)}{p\left(x^2-1\right)\left(x^{2p+2}-1\right)\arctan\left(\left(x^2-1\right)/\sqrt{2}x\right)}\cdot F_1(x,p), \tag{25}$$

where

$$F_1(x,p) = \arctan\left(\frac{x^2-1}{\sqrt{2}x}\right) - \frac{p\left(x^4-1\right)\left(x^{2p+2}-1\right)}{\sqrt{2}(p+1)x\left(x^4+1\right)\left(x^{2p}-1\right)}, \tag{26}$$

$$\lim_{x\to1} F_1(x,p) = 0, \tag{27}$$

$$\lim_{x\to\infty} F_1(x,p) = -\infty, \tag{28}$$

$$\frac{\partial F_1(x,p)}{\partial x} = -\frac{\sqrt{2}}{2(p+1)x^2\left(x^4+1\right)^2\left(x^{2p}-1\right)^2}f(x,p), \tag{29}$$

where $f(x,p)$ is defined by (15).

We divide the proof into four cases.

Case 1 ($p = 2$). Then from Lemma 2(1) and (29) we clearly see that the function $x \to F_1(x,p)$ is strictly decreasing on $(1,\infty)$. Then (27) leads to the conclusion that

$$F_1(x,p) < 0 \tag{30}$$

for all $x \in (1,\infty)$. Therefore,

$$U(a,b) < L_2(a,b) \tag{31}$$

follows easily from (22), (23), (25), and (30).

Case 2 ($p = p_0$). Then from Lemma 2(2) and (29) we know that there exists $\lambda \in (1,\infty)$ such that the function $x \to F_1(x,p)$ is strictly increasing on $(1,\lambda]$ and strictly decreasing on $[\lambda,\infty)$.

It follows from (25)–(28) and the piecewise monotonicity of the function $x \to F_1(x,p)$ that there exists $\lambda^* \in (1,\infty)$ such that the function $x \to F(x,p)$ is strictly increasing on $(1,\lambda^*]$ and strictly decreasing on $[\lambda^*,\infty)$.

Note that (24) becomes

$$\lim_{x\to\infty} F(x,p) = 0. \tag{32}$$

Therefore,

$$U(a,b) > L_{p_0}(a,b) \tag{33}$$

follows easily from (22), (23), and (32) together with the piecewise monotonicity of the function $x \to F(x,p)$.

Case 3 ($p < 2$). Let $x > 0$ and $x \to 0$; then making use of Taylor expansion we get

$$\begin{aligned} U(1+x,1) &- L_p(1+x,1) \\ &= \frac{x}{\sqrt{2}\arctan\left(x/\sqrt{2(1+x)}\right)} \\ &\quad - \left[\frac{(1+x)^{p+1}-1}{(p+1)x}\right]^{1/p} = \frac{2-p}{24}x^2 + o\left(x^2\right). \end{aligned} \tag{34}$$

Equation (34) implies that there exists small enough $\delta \in (0,1)$ such that

$$U(1+x,1) > L_p(1+x,1) \tag{35}$$

for all $x \in (0,\delta)$.

Case 4 ($p > p_0$). Then from (24) and the fact that the function $p \to \log(p+1)/p$ is strictly decreasing on $(0,\infty)$ we get

$$\lim_{x\to\infty} F(x,p) < \frac{1}{p_0}\log(1+p_0) + \frac{1}{2}\log 2 - \log\pi = 0. \tag{36}$$

Equation (22) and inequality (36) imply that there exists large enough $X > 1$ such that

$$U\left(x^2,1\right) < L_p\left(x^2,1\right) \tag{37}$$

for all $x \in (X,\infty)$. □

4. Applications

As applications of Theorem 3 in engineering problems, we present several sharp inequalities involving the trigonometric, hyperbolic, and inverse trigonometric functions in this section.

From (1) and (11) together with Theorem 3 we get Theorem 4 immediately.

Theorem 4. *Let $p_0 = 0.5451\cdots$ be the unique solution of the equation $(p+1)^{1/p} = \sqrt{2}\pi/2$ on the interval $(0,\infty)$. Then the double inequality*

$$\begin{aligned} \frac{\sqrt{6}(a-b)}{2\sqrt{a^2+ab+b^2}} &< \arctan\left(\frac{a-b}{\sqrt{2ab}}\right) \\ &< \frac{\pi(a-b)^{1+1/p_0}}{2\left(a^{p_0+1}-b^{p_0+1}\right)^{1/p_0}} \end{aligned} \tag{38}$$

holds for all $a, b > 0$ with $a \neq b$.

Let $t > 0$, $b = 1$, and $a = t^2 + t\sqrt{t^2+2} + 1$. Then Theorem 4 leads to the following.

Theorem 5. *Let $p_0 = 0.5451\cdots$ be the unique solution of the equation $(p+1)^{1/p} = \sqrt{2}\pi/2$ on the interval $(0,\infty)$. Then the double inequality*

$$\begin{aligned} \frac{\sqrt{6}\left(t^2+t\sqrt{t^2+2}\right)}{2\sqrt{2t^4+5t^2+\left(2t^3+3t\right)\sqrt{t^2+2}+3}} &< \arctan(t) \\ &< \frac{\pi\left(t^2+t\sqrt{t^2+2}\right)^{1+1/p_0}}{2\left[\left(t^2+t\sqrt{t^2+2}\right)^{1+p_0}-1\right]^{1/p_0}} \end{aligned} \tag{39}$$

holds for all $t > 0$.

Let $a > b > 0$, $x = \log\sqrt{a/b} \in (0,\infty)$. Then (1) and (11) lead to

$$\begin{aligned} \frac{L_p(a,b)}{\sqrt{ab}} &= \left[\frac{\sinh(p+1)x}{(p+1)\sinh x}\right]^{1/p}, \\ \frac{U(a,b)}{\sqrt{ab}} &= \frac{\sqrt{2}\sinh x}{\arctan\left(\sqrt{2}\sinh x\right)}. \end{aligned} \tag{40}$$

It follows from Theorem 3 and (40) that one has the following theorem.

Theorem 6. *Let $p_0 = 0.5451\cdots$ be the unique solution of the equation $(p+1)^{1/p} = \sqrt{2}\pi/2$ on the interval $(0,\infty)$. Then the double inequality*

$$\begin{aligned} \frac{\sqrt{6}\sinh x}{\sqrt{3+4\sinh^2 x}} &< \arctan\left(\sqrt{2}\sinh x\right) \\ &< \pi\left(\frac{\sinh x}{\sinh(p_0+1)x}\right)^{1/p_0} \end{aligned} \tag{41}$$

holds for all $x > 0$.

Let $a > b > 0$, $x = \arcsin[(a-b)/(a+b)] \in (0, \pi/2)$. Then (1) and (11) lead to

$$\frac{L_p(a,b)}{\sqrt{ab}} = \left[\frac{((1+\sin x)/\cos x)^{p+1} - (\cos x/(1+\sin x))^{p+1}}{2(p+1)\tan x}\right]^{1/p}, \quad (42)$$

$$\frac{U(a,b)}{\sqrt{ab}} = \frac{\sqrt{2}\tan x}{\arctan\left(\sqrt{2}\tan x\right)}.$$

Theorem 3 and (42) lead to the following.

Theorem 7. *Let $p_0 = 0.5451\cdots$ be the unique solution of the equation $(p+1)^{1/p} = \sqrt{2}\pi/2$ on the interval $(0,\infty)$. Then the double inequality*

$$\frac{\sqrt{6}\sin x}{\sqrt{3+\sin^2 x}} < \arctan\left(\sqrt{2}\tan x\right) < \frac{2^{1/p_0}\pi\sin^{1+1/p_0} x}{\left[(1+\sin x)^{1+p_0} - (1-\sin x)^{1+p_0}\right]^{1/p_0}} \quad (43)$$

holds for all $x \in (0, \pi/2)$.

Let $a > b > 0$, $x = \arctan((a-b)/(a+b)) \in (0, \pi/4)$, $y = \sinh^{-1}((a-b)/(a+b)) \in (0, \log(1+\sqrt{2}))$. Then from (1) and (11) we have

$$\frac{L_p(a,b)}{\sqrt{ab}} = \left(\frac{1}{2(p+1)}\right)^{1/p} \cdot \frac{1}{\sqrt{\cos(2x)}}\left[\frac{(\cos x+\sin x)^{p+1} - (\cos x-\sin x)^{p+1}}{\sin x}\right]^{1/p} = \left(\frac{1}{2(p+1)}\right)^{1/p} \cdot \frac{1}{\sqrt{1-\sinh^2(y)}}\left[\frac{(1+\sinh y)^{p+1} - (1-\sinh y)^{p+1}}{\sinh y}\right]^{1/p}, \quad (44)$$

$$\frac{U(a,b)}{\sqrt{ab}} = \frac{\sqrt{2}\sin x/\sqrt{\cos(2x)}}{\arctan\left(\sqrt{2}\sin x/\sqrt{\cos(2x)}\right)} = \frac{\sqrt{2}\sinh y/\sqrt{1-\sinh^2 y}}{\arctan\left(\sqrt{2}\sinh y/\sqrt{1-\sinh^2 y}\right)}. \quad (45)$$

From (44), (45), and Theorem 3 one has the following.

Theorem 8. *Let $p_0 = 0.5451\cdots$ be the unique solution of the equation $(p+1)^{1/p} = \sqrt{2}\pi/2$ on the interval $(0,\infty)$. Then the double inequalities*

$$\frac{\sqrt{6}\sin x}{\sqrt{2+\cos(2x)}} < \arctan\left(\frac{\sqrt{2}\sin x}{\sqrt{\cos(2x)}}\right) < \frac{2^{1/p_0}\pi\sin^{1+1/p_0} x}{\left[(\cos x+\sin x)^{1+1/p_0} - (\cos x-\sin x)^{1+1/p_0}\right]^{1/p_0}},$$

$$\frac{\sqrt{6}\sinh y}{\sqrt{3+\sinh^2 y}} < \arctan\left(\frac{2\sinh y}{\sqrt{3-\cosh(2y)}}\right) < \frac{2^{1/p_0}\pi\sinh^{1+1/p_0} y}{\left[(1+\sinh y)^{1+1/p_0} - (1-\sinh y)^{1+1/p_0}\right]^{1/p_0}} \quad (46)$$

hold for all $x \in (0, \pi/4)$ and $y \in (0, \log(1+\sqrt{2}))$.

Competing Interests

The authors declare that there is no conflict of interests regarding the publications of this paper.

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