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BEST TESTS FOR TESTING HYPOTHESES ABOUT A RANDOM PARAMETER WITH UNKNOWN DISTRIBUTION[1]

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1. Introduction and summary. Let X be a random variable with a family of possible distributions for X indexed by $\lambda \in \Omega$. λ is the realization of a random variable Λ taking values in the space Ω. For each λ, let f_λ denote the conditional density of X given $\Lambda = \lambda$ with respect to some σ-finite measure μ. Let $\mathscr{G}$ be a family of possible *a priori* distributions G for Λ. After observing X, we wish to test H: $\lambda \in \omega$ against K: $\lambda \in \omega'$ where ω is a subset of Ω and ω' its complement. To determine good tests for this problem, we use an analysis similar to the one of the Neyman–Pearson theory of hypothesis testing. Analogous to the type I and type II errors of the Neyman–Pearson theory are:

type (i) error: $\Lambda \in \omega'$ decided and $\Lambda \in \omega$ occurs,
type (ii) error: $\Lambda \in \omega$ decided and $\Lambda \in \omega'$ occurs.

Analogous to the problem of finding uniformly most powerful level α tests is the problem:

subject to: P_G(type (i) error) $\leqq \alpha$ for all $G \in \mathscr{G}$
minimize P_G(type (ii) error) uniformly for $G \in \mathscr{G}$.

A test which achieves this is called a uniformly most powerful (UMP) level α test relative to $\mathscr{G}$.

The existence of such UMP level α tests is proved for this hypotheses testing problem for various choices of the family of *a priori* distributions $\mathscr{G}$. As might be expected these results are closely related to the Neyman–Pearson theory of hypotheses testing. The second section gives four simple situations where the problem of finding UMP level α tests relative to a family of *a priori* distributions $\mathscr{G}$ reduces to an ordinary testing problem. In the third section, Theorem 1 gives for this testing problem an analogue of the concept of a least favorable distribution from the classical theory of hypotheses testing. Theorem 1 is used to prove Theorem 2 which gives the existence of a UMP level α test when X is real-valued, Ω is a subset of the real numbers, the family of distributions indexed by $\lambda \in \Omega$ has a monotone likelihood ratio in x, and the family $\mathscr{G}$ satisfies a certain condition. The two theorems are applied to several examples.

In the following, as always, a test (randomized) is a function δ defined on the range of X which takes on values in the interval $[0, 1]$. If $X = x$ is observed, K is

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decided to be true with probability $\delta(x)$ and H with probability $1-\delta(x)$. For any test δ and $G\in\mathscr{G}$ we have

(1) $$P_G(\text{type (i) error of } \delta) = \int\int_\omega \delta(x) f_\lambda(x)\, dG(\lambda)\, d\mu \quad \text{and}$$

(2) $$P_G(\text{type (ii) error of } \delta) = \int\int_{\omega'} (1-\delta(x)) f_\lambda(x)\, dG(\lambda)\, d\mu$$

where the integral involving X is over the entire space of X.

It will often be convenient to think of λ as a fixed but unknown parameter and the test δ as a test for the classical testing problem H: $\lambda\in\omega$ against K: $\lambda\in\omega'$. Changing the order of integration in (1) by Fubini's theorem, we have for the test δ the following relationship between the type I error of δ, considered as a test for the classical problem, and the type (i) error of δ, considered as a test for the problem of this paper:

(3) $$P_G(\text{type (i) error of } \delta) = \int_\omega P_\lambda(\text{type I error of } \delta)\, dG(\lambda).$$

In the same way, we have

(4) $$P_G(\text{type (ii) error of } \delta) = \int_{\omega'} P_\lambda(\text{type II error of } \delta)\, dG(\lambda).$$

We will now prove the existence of UMP level α tests for various families of *a priori* distributions.

2. Some simple examples. If $\mathscr{G}$ contains only one distribution G, the simplest case, then the testing problem considered here is a Bayesian hypotheses testing problem with known *a priori* distribution G. Since

$$P_G(\text{type (i) error of } \delta) = P_G(\Lambda\in\omega)\int \delta(x) f_{\omega,G}(x)\, d\mu \quad \text{and}$$
$$P_G(\text{type (ii) error of } \delta) = P_G(\Lambda\in\omega')\int (1-\delta(x)) f_{\omega',G}(x)\, d\mu$$

where $f_{\omega,G}$ and $f_{\omega',G}$ denote the conditional densities of X given $\Lambda\in\omega$ and $\Lambda\in\omega'$ respectively, the problem of finding a most powerful test relative to G is mathematically equivalent to the problem of finding a most powerful test when testing a simple hypothesis against a simple alternate. Therefore, there exists a constant k, depending on α, such that a most powerful test, δ, relative to G is given by:

$$\begin{aligned} \delta(x) &= 1 \quad \text{when} \quad f_{\omega',G}(x) > k f_{\omega,G}(x) \\ &= 0 \quad \text{when} \quad f_{\omega',G}(x) < k f_{\omega,G}(x). \end{aligned}$$

Note that if G is such that $P_G(\Lambda\in\omega) \leqq \alpha$ then the test which is identically one is a most powerful level α test relative to G.

Suppose now that $\mathscr{G}$ is the set of all possible *a priori* distributions of Λ. Assuming that all singletons of Ω are measurable, it follows from (3) and (4) that δ' is a UMP level α test relative to $\mathscr{G}$ if and only if δ' is a UMP level α test for the classical testing problem. This is still true if $\mathscr{G}$ is any class of *a priori* distributions which includes all the distributions which put probability one on some point. If $\mathscr{G}$ is just the set of all one point distributions then the testing problem considered here is just the classical hypotheses testing problem. If for all tests δ, P_λ(type (i) error of δ)

and P_λ(type (ii) error of δ) are continuous functions of λ, then the preceding remark is true if $\mathcal{G}$ contains distributions which put arbitrarily high probability in arbitrarily small neighborhoods of every point of Ω.

These two examples are the two extreme situations in hypotheses testing when the parameter λ is assumed to be a random variable. In the preceding one, λ is a random variable with known distribution. In the latter, λ is a random variable with nothing known about its distribution. The rest of this paper deals with examples where the family of possible distributions falls between these two extremes.

The case where $P_G(\Lambda \in \omega) = \gamma$ for all $G \in \mathcal{G}$ where γ is a known constant corresponds to the situation where the amount of probability assigned to ω and ω' is known but the distribution in ω and ω' is not known. This problem is very similar to the preceding one. For if $\mathcal{G}$ is such that for each $\lambda \in \omega$ there exists a $G \in \mathcal{G}$ with $P_G(\Lambda = \lambda) = \gamma$ then δ is a UMP level α test relative to $\mathcal{G}$ if and only if δ is a UMP level α' test for the classical testing problem where $\alpha' =$ minimum (α/γ and 1).

The next case, in contrast to the previous one, corresponds to the situation where the distribution in ω and ω' is known but the probabilities assigned to ω and ω' are not known. If H and H' are *a priori* distributions of Λ such that $P_H(\Lambda \in \omega) = P_{H'}(\Lambda \in \omega') = 1$ then $\mathcal{G} = (G\colon G = \gamma H + (1-\gamma)H'$ for $0 \leqq \gamma \leqq 1)$. For any test δ and $G = \gamma H + (1-\gamma)H'$ we have

$$P_G(\text{type (i) error of } \delta) = \gamma \int \delta(x) f_H(x)\, d\mu \qquad \text{and}$$

$$P_G(\text{type (ii) error of } \delta) = (1-\gamma) \int (1-\delta(x)) f_{H'}(x)\, d\mu$$

where f_H and $f_{H'}$ are the marginal densities of X when Λ has distributions H and H' respectively. Hence δ' is a UMP level α test relative to $\mathcal{G}$ if and only if it is a most powerful level α test for the simple problem of testing f_H against $f_{H'}$.

3. $\mathcal{G}$ is a parametric family of *a priori* distributions. We consider in this section several examples where the class of possible *a priori* distributions is indexed by a parameter in Euclidian space. Two theorems are proved which give the existence of a UMP level α test in the examples considered. The first theorem shows that an analogue of the least favorable *a priori* distribution concept from the Neyman-Pearson theory holds for the problem considered here.

THEOREM 1. *Let $\mathcal{G}$ be a family of possible a priori distributions for Λ. For each G' let $\phi_{G'}$ be a solution, if it exists, for the problem*

(5) (a) *subject to*: $P_{G'}$(*type* (i) *error*) $\leqq \alpha$

(b) *minimize* P_G(*type* (ii) *error*) *uniformly for* $G \in \mathcal{G}$.

If G^* is a distribution such that ϕ_{G^*} exists and

(6) $$P_G(\textit{type (i) error of } \phi_{G^*}) \leqq \alpha \quad \textit{for} \quad G \in \mathcal{G}$$

then ϕ_{G^*} is a UMP *level α test relative to $\mathcal{G}$. If, in addition, $\phi_{G'}$ exists for each $G' \in \mathcal{G}$ then G^* is a least favorable distribution, i.e., for each $G' \in \mathcal{G}$.*

$$P_G\ (\textit{type (ii) error of } \phi_{G^*}) \geqq P_G\ (\textit{type (ii) error of } \phi_{G'})\ \textit{for}\ G \in \mathcal{G}.$$

Note that Theorem 1 is just a statement of the usual theorem about a least favorable distribution (e.g. Theorem 7 on page 91 of Lehmann [1]) in this new context.

In the following example, we will use Theorem 1 to find a UMP level α test. In all the examples that follow, we assume that α is a fixed number between zero and one since the cases $\alpha = 0$ and $\alpha = 1$ are trivial.

EXAMPLE 1. Let the distribution of X given λ be uniform on $(0, \lambda]$ where $\lambda \in (0, +\infty) = \Omega$. Let $\mathcal{G}$ be the class of uniform distributions on $(0, \theta]$, $\theta > 0$ and let $\omega = (0, \lambda_0]$. We will show that $\theta = \lambda_0$ is a least favorable distribution for this problem.

If δ is the test which is one for $x > c$ and zero otherwise where c is chosen so that $\alpha = P_{\lambda_0}$ (type (i) error of δ) then δ is a solution of (5) with G' taken as λ_0. It is enough to show that δ is best among all tests which reject when x is too large. (Since if δ_0, with power function $\beta_0(\lambda)$, is any test not of this form then there exists a test δ_1, with power function $\beta_1(\lambda)$, of this form such that $\beta_1(\lambda) \leqq \beta_0(\lambda)$ for $\lambda \leqq \lambda_0$ and $\beta_1(\lambda) \geqq \beta_0(\lambda)$ for $\lambda > \lambda_0$ and by (3) and (4), we have that δ_1 is as good as δ_0 for each θ.) Let δ' be a test which rejects when $x > c'$. If $c' < c$ then δ' does not satisfy (5a). If $c' > c$ then $\delta'(x) \leqq \delta(x)$ for all x and it follows by (4) that δ is better then δ' for all θ. Finally, it is easily seen that δ satisfies (6) by calculating the derivative with respect to θ of P_θ (type (ii) error of δ) and δ is a UMP level α test relative to $\mathcal{G}$.

THEOREM 2. *Let X be a real-valued random variable with a family of possible probability distributions indexed by $\lambda \in \Omega$, which is a set of real numbers. Let the family of probability densities f_λ have monotone likelihood ratio in x. λ is the realization of a random variable Λ with a fan ily of possible a priori distributions $\mathcal{G}$. Let $\omega = (\lambda\colon \lambda \leqq \lambda_0)$ and $\omega' = (\lambda\colon \lambda > \lambda_0)$ where λ_0 is a fixed number of Ω.*

Then for each $G \in \mathcal{G}$, if G is the known a priori distribution, there exist constants γ and c and a function δ_G which is of the form

$$\begin{aligned} \delta_G(x) &= 1 \quad \text{for} \quad x > c \\ &= \gamma \quad \text{for} \quad x = c \\ &= 0 \quad \text{for} \quad x < c \end{aligned} \tag{7}$$

such that δ_G is a most powerful level α test relative to G.

If there exists a member G^* of $\mathcal{G}$ such that

$$\delta_{G^*}(x) = \inf_{G \in \mathcal{G}} \delta_G(x) \quad \text{for all} \quad x \tag{8}$$

then δ_{G^*} is a UMP *level α test relative to $\mathcal{G}$. If for each $G' \in \mathcal{G}$ there exists a solution for the problem*

$$\begin{aligned} &\textit{subject to: } P_{G'} \ (\textit{type (i) error}) \leqq \alpha \\ &\textit{minimize} \quad P_G \ (\textit{type (ii) error}) \textit{ uniformly for } G \in \mathcal{G} \end{aligned}$$

then G^* is a least favorable distribution.

PROOF. Let δ be any test with power function $\beta_\delta(\lambda)$. If $\alpha_0 = \beta_\delta(\lambda_0)$ then by Theorem 2 on page 68 of Lehmann [1], we have that there exists a UMP level α_0 test, δ_0, of H: $\lambda \in \omega$ against K: $\lambda \in \omega'$ which has the form

$$\begin{aligned} \delta_0(x) &= 1 \quad \text{for} \quad x > c \\ &= \gamma \quad \text{for} \quad x = c \\ &= 0 \quad \text{for} \quad x < c \end{aligned} \tag{9}$$

and which satisfies $\beta_0(\lambda_0) = \alpha_0$, $\beta_0(\lambda) \leqq \beta_\delta(\lambda)$ for $\lambda \leqq \lambda_0$, and $\beta_0(\lambda) \geqq \beta_\delta(\lambda)$ for $\lambda > \lambda_0$ where $\beta_0(\lambda)$ is the power function of δ_0. By (3) and (4), it follows that for each G P_G (type (i) error of δ_0) $\leqq P_G$ (type (i) error of δ) and P_G (type (ii) error of δ_0) $\leqq P_G$ (type (ii) error of δ) and δ_0 is as good a test as δ relative to G.

If G is the known *a priori* distribution for the testing problem then by Section 2 there exists a most powerful level α test, δ_G relative to G. By the previous remark, we can assume that δ_G is given by (7) for some c and γ and the first part of the theorem is proved. In addition, we can assume that if δ' is any test of form (9) with $\delta'(x) \geqq \delta_G(x)$ for all x and $\mu(x : \delta'(x) \neq \delta_G(x)) > 0$ then P_G (type (i) error of δ') $> \alpha$.

Let G^* satisfy (8) and δ_{G^*} be given by (7). To prove that δ_{G^*} is a UMP level α test relative to $\mathscr{G}$, it is enough to show that δ_{G^*} is a solution for the problem given by (5) with $G' = G^*$ and that δ_{G^*} satisfies (6). Clearly δ_{G^*} satisfies (5a). Let δ_1 be any other test which satisfies (5a). By the first remark in the proof, we can assume that δ_1 is of form (9). Since δ_1 satisfies (5a), it follows that $\mu(x : \delta_1(x) > \delta_{G^*}(x)) = 0$. Therefore, P_G (type (ii) error of δ_{G^*}) $\leqq P_G$ (type (ii) error of δ_1) for $G \in \mathscr{G}$ and δ_{G^*} is a solution of (5). For each G, we have that $\delta_{G^*}(x) \leqq \delta_G(x)$ for all x and (6) holds and by Theorem 1, the second part of the theorem is proved.

Note that the existence of a G^* satisfying (8) is not necessary. If there exists a δ^* which is of the form (7) and satisfies

$$\sup_{G \in \mathscr{G}} P_G(\text{type (i) error of } \delta^*) = \alpha$$

then δ^* is a UMP level α test relative to $\mathscr{G}$.

Next four examples will be given where Theorem 2 gives the existence of a UMP level α test relative to a family of *a priori* distributions $\mathscr{G}$. In these examples, only hypothesis (8) of Theorem 2 will be verified since the others are easily checked.

EXAMPLE 2. Let X given $\Lambda = \lambda$ have the hypergeometric (n, λ, m) distribution where $\lambda \in (0, 1, \cdots, n) = \Omega$ and m and n are known positive integers such that $0 < m < n$. G is the class of binomial (n,p) distributions for $p \in M$ where M is a closed subset of $[0, 1]$.

For each p when the known *a priori* distribution is binomial (n,p) there exist constants c_p and γ_p such that a most powerful level α test δ_p is of form (7). In particular, we will take $\delta_0(x) = 1$ for $x > 0$ and $\delta_0(0) = \alpha$ and $\delta_1(x) = 1$ for all x since then for each x $\delta_p(x)$ is a continuous function of p on the interval $[0, 1]$. Let x_0 be the smallest x such that $\inf_{p \in M} \delta_p(x) > 0$. By the continuity of $\delta_p(x_0)$, there exists a $p^* \in M$ such that $\delta_{p^*}(x_0) = \inf_{p \in M} \delta_p(x_0)$ and δ_{p^*} is a UMP level α test relative to $\mathscr{G}$ since δ_{p^*} satisfies (8).

It is possible to find δ_{p^*} approximately without too much difficulty. For a fixed p it is easy to find δ_p exactly. Since for $0 \leqq x \leqq \lambda \leqq n$

$$\begin{aligned} P_p(X = x, \Lambda = \lambda) &= P_p(\Lambda = \lambda \mid X = x) P_p(X = x) \\ &= \tbinom{n-m}{\lambda - x} p^{\lambda - x}(1-p)^{n-m-(\lambda - x)} \tbinom{m}{x} p^x (1-p)^{m-x} \end{aligned}$$

all the probabilities needed for the computation of δ_p can be found in a binomial table. By calculating δ_p for various values of p in M δ_{p^*} can be found approximately.

EXAMPLE 3. Let the distribution of X given $\Lambda = \lambda$ be normal (λ, σ_1^2) and the distribution of Λ be normal (θ, σ_2^2) where σ_1^2 and σ_2^2 are known and $\theta \in M$, a closed set of real numbers, is unknown.

For each $\theta \in M$ when the known *a priori* distribution is normal (θ, σ_2^2), there exists a constant c_θ such that a most powerful level α test δ_θ is given by $\delta_\theta(x) = 1$ for $x \geqq c_\theta$ and $\delta_\theta(x) = 0$ otherwise. Since c_θ is a continuous function of θ, the existence of a $\theta^* \in M$ such that $\delta_{\theta^*}(x) = \inf_{\theta \in M} \delta_\theta(x)$ for all x follows from (i) $c_\theta \to -\infty$ as $\theta \to +\infty$ and (ii) $c_\theta \to -\infty$ as $\theta \to -\infty$. Since $\lim_{\theta \to +\infty} P_\theta(\Lambda \in \omega) = 0$, (i) is true. To prove (ii), it is enough to show that for each real number a, $\lim_{\theta \to -\infty} P_\theta(X \in [a, +\infty)$ and $\Lambda \in \omega) = 0$. This follows from the fact that $\lim_{\theta \to -\infty} P_\theta(X \in [a, +\infty)) = 0$ since for each θ the marginal density of X is normal $(\theta, \sigma_1^2 + \sigma_2^2)$.

EXAMPLE 4. Let the distribution of X given $\Lambda = \lambda$ be Poisson with parameter λ. The family of distributions for Λ is given by the density functions $(1/t)\exp(-\lambda/t)$ for $\lambda > 0$ where t is not known and $t \in M$, a set of positive real numbers which contains all its limit points except possibly zero.

Let δ_t denote the best test when t is known. The only difficulty in showing the existence of a $t^* \in M$ such that δ_{t^*} satisfies (8) is to check that the inf does not occur as t approaches zero. This follows by considering the test δ which is given by $\delta(x) = 1$ for $x > 0$ and $\delta(0) = \alpha$ and verifying that $\lim_{t \to 0^+} a(t) = \alpha$ and $a'(t) > 0$ for t sufficiently close to zero where $a(t) = P_t$ (type (i) error of δ).

EXAMPLE 5. Let the distribution of X given $\Lambda = \lambda$ be binomial (n, λ) where n is a known positive integer. The family of distributions for Λ is a family of beta distributions with parameters r and s where $(r, s) \in M$, a set in the first quadrant of the plane.

If M is the entire first quadrant, then a UMP level α test of $H: \lambda \leqq \lambda_0$ against $K: \lambda > \lambda_0$ is a UMP level α test relative to M since the family M contains distributions which put arbitrarily high probability in arbitrarily small neighborhoods of every point in $(0, 1)$.

If M is compact then by the continuity of $\delta_{r,s}(x)$ for each x there exists a point $(r^*, s^*) \in M$ such that δ_{r^*, s^*} satisfies (8). Suppose now $M = ((r, s): r = s$ and $0 < r \leqq r')$ where r' is a fixed positive number and assume we are testing $H: \lambda \leqq \frac{1}{2}$ against $K: \lambda > \frac{1}{2}$. $\delta_{r', r'}$ is a UMP level α test relative to M since for every test δ of form (9) $P_{r,r}$ (type (i) error of δ) is a nondecreasing function of r. The preceding is obvious if $\alpha \geqq \frac{1}{2}$ because then $\delta_{r,r}(x) = 1$ for all x since $P_{r,r}(\Lambda \in [0, \frac{1}{2}]) = \frac{1}{2}$.

If we assume that $\alpha < \frac{1}{2}$ and let $M = ((r, s): r = s$ and $0 < r < +\infty)$ then it is easily seen that if δ^* is a UMP level α test of $H: \lambda \leqq \frac{1}{2}$ against $K: \lambda > \frac{1}{2}$ then δ^* is a UMP level α test relative to M. Note that δ^* does not correspond to a least favorable distribution in M.

4. Concluding remark. The testing problem in this paper can be considered as a special case of the following two-set prediction problem. Let X and Y be random variables with a family of possible joint probability distributions indexed by $\theta \in \Theta$ and consider the problem of predicting from X whether or not Y lies in a specified subset ω of the space of values of Y. (In this paper, Λ corresponds to Y, G to θ, and $\mathscr{G}$ to Θ.) If no uniformly most powerful level α predictors exist for this problem then unbiased predictors, invariant predictors, and most stringent predictors defined as in hypotheses testing can be considered.

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REFERENCE

[1] LEHMANN, E. L. (1959). *Testing Statistical Hypotheses.* Wiley, New York.