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Beta-Normal Distribution: Bimodality Properties and Application

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The beta-normal distribution is characterized by four parameters that jointly describe the location, the scale and the shape properties. The beta-normal distribution can be unimodal or bimodal. This paper studies the bimodality properties of the beta-normal distribution. The region of bimodality in the parameter space is obtained. The beta-normal distribution is applied to fit a numerical bimodal data set. The beta-normal fits are compared with the fits of mixture-normal distribution through simulation.

Key words: Bimodal region, percentiles, curve estimation, egg size distribution

Introduction

Bimodal distributions occur in many areas of science. Withington et al. (2000), in their study of cardiopulmonary bypass in infants showed that plasma vecuronium and vecuronium clearance requirements have bimodal distributions. They concluded that their findings on bimodal distributions for plasma vecuronium and vecuronium clearance requirements highlight the need for individual monitoring of neuromuscular blockade. Espinoza et al. (2001) discussed the importance of bimodal distributions in the study of size distribution of metals in aerosols. Bimodal distributions also occur in the study of genetic diversity (Freeland et al., 2000), in the study of agricultural farm size distribution (Wolf & Sumner, 2001), in the study of atmospheric pressure (Zangvil et al., 2001), and in the study of anabolic steroids on animals (Isaacson, 2000).

Acknowledgement

Let $F(x)$ be the cumulative distribution function (CDF) of a random variable X. The cumulative distribution function for a generalized class of distributions for the random variable X can be defined as the logit of the beta random variable given by

$$G(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^{F(x)} t^{\alpha-1}(1-t)^{\beta-1}dt, \quad 0 < \alpha, \beta < \infty .$$

(1.1)

Eugene et al. (2002) considered $F(x)$ as the CDF of the normal distribution with parameters μ and σ. Thus, the random variable X has the beta-normal distribution with probability density function (pdf)

$$g(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \left[\Phi\left(\frac{x-\mu}{\sigma}\right)\right]^{\alpha-1} \left[1-\Phi\left(\frac{x-\mu}{\sigma}\right)\right]^{\beta-1} \sigma^{-1}\phi\left(\frac{x-\mu}{\sigma}\right)$$

(1.2)

where $\phi\left(\dfrac{x-\mu}{\sigma}\right)$ is the normal pdf and $\Phi\left(\dfrac{x-\mu}{\sigma}\right)$ is the normal CDF. We denote the beta-normal distribution with parameters α, β, μ, and σ as $BN(\alpha, \beta, \mu, \sigma)$.

The distribution in (1.2) may be symmetric, skewed to the left, or skewed to the

right. The distribution may be unimodal or bimodal. Eugene et al. (2002) discussed the shape properties of the unimodal beta-normal distribution. Furthermore, they considered the estimation of its parameters by the method of maximum likelihood.

In the analysis of bimodal data, a mixture of two normal densities is often used as a model (e.g., Cobb et al., 1983). The mixture of normal distribution is used as a model to analyze bimodal data because the mixture of normal densities can take on bimodal shapes depending on the parameters of the distribution. Eisenberger (1964) showed how the parameters of a mixture of normal distributions determine its shape. When a mixture assumption is not required or justified the beta-normal distribution can serve as a model to analyze data since only one distribution has to be used and one less parameter to estimate.

In the rest of the paper, we provide some bimodality properties of the beta-normal distribution. We obtain the region of bimodality in the parameter space. We also illustrate the application of beta-normal distribution to a numerical data set that exhibits two modes and compare the fit with mixture-normal distribution. A simulation study is conducted to compare the performance between beta-normal and mixture-normal distributions in fitting bimodal data.

Bimodality Properties

In this section, some results on the bimodality properties of beta-normal distribution are obtained.

Fact: A mode of the $BN(\alpha, \beta, \mu, \sigma)$ is any point $x_0 = x_0(\alpha, \beta)$ that satisfies

$$x_0 = \frac{\sigma\phi(\frac{x_0-\mu}{\sigma})}{1-\Phi(\frac{x_0-\mu}{\sigma})}\left\{2-\alpha-\beta\right\} + \frac{(\alpha-1)\phi(\frac{x_0-\mu}{\sigma})\sigma}{\Phi(\frac{x_0-\mu}{\sigma})[1-\Phi(\frac{x_0-\mu}{\sigma})]} + \mu. \quad (2.1)$$

Proof: Differentiating $BN(\alpha, \beta, \mu, \sigma)$ in (1.2) with respect to x, setting it equal to zero, and solving for x gives the result in (2.1).

Corollary 1: If $\alpha = \beta$ and one mode of $BN(\alpha, \beta, \mu, \sigma)$ is at x_0, then the other mode is at the point $2\mu - x_0$.

Proof: If $BN(\alpha, \beta, \mu, \sigma)$ is unimodal, then the only mode occurs at the point $x_0 = \mu$. For bimodal case, we need to show that if we replace x_0 with $2\mu - x_0$, then equation (2.1) remains the same. When $\alpha = \beta$, equation (2.1) becomes

$$x_0 = \frac{\sigma\phi(\frac{x_0-\mu}{\sigma})(\alpha-1)}{\Phi(\frac{x_0-\mu}{\sigma})[1-\Phi(\frac{x_0-\mu}{\sigma})]}\left\{1-2\Phi(\frac{x_0-\mu}{\sigma})\right\} + \mu. \quad (2.2)$$

If x_0 in (2.2) is replaced with $2\mu - x_0$, we obtain

$$2\mu - x_0 = \frac{\sigma\phi(\frac{\mu-x_0}{\sigma})(\alpha-1)}{\Phi(\frac{\mu-x_0}{\sigma})[1-\Phi(\frac{\mu-x_0}{\sigma})]}\left\{1-2\Phi(\frac{\mu-x_0}{\sigma})\right\} + \mu \quad (2.3)$$

By using $\phi\left(\frac{x_0-\mu}{\sigma}\right) = \phi\left(\frac{\mu-x_0}{\sigma}\right)$ and $\Phi\left(\frac{x_0-\mu}{\sigma}\right) = 1-\Phi\left(\frac{\mu-x_0}{\sigma}\right)$ in (2.3) and on simplification, we get the result in (2.2).

Corollary 2: If $BN(\alpha, \beta, \mu, \sigma)$ has a mode at x_0, then $BN(\beta, \alpha, \mu, \sigma)$ has a mode at $2\mu - x_0$.

Proof: We need to show that if we replace α with β, and $2\mu - x_0$ with x_0, equation (2.1) remains the same. Equation (2.1) can be written as

$$x_0 = \frac{\sigma\phi(\frac{x_0-\mu}{\sigma})}{\Phi(\frac{x_0-\mu}{\sigma})[1-\Phi(\frac{x_0-\mu}{\sigma})]}\left\{(2-\alpha-\beta)\Phi(\frac{x_0-\mu}{\sigma})+(\alpha-1)\right\} + \mu. \quad (2.4)$$

If x_0 is replaced with $2\mu - x_0$ and α is replaced with β in (2.4), using $\phi\left(\dfrac{x_0 - \mu}{\sigma}\right) = \phi\left(\dfrac{\mu - x_0}{\sigma}\right)$ and $\Phi\left(\dfrac{x_0 - \mu}{\sigma}\right) = 1 - \Phi\left(\dfrac{\mu - x_0}{\sigma}\right)$, and on simplification, we obtain the result in (2.4).

Corollary 3: The modal point $x_0(\alpha,\ \beta)$ is an increasing function of α and a decreasing function of β.

Proof: Differentiating the result in (2.1) with respect to α and β gives

$$\frac{\partial x_0(\alpha,\beta)}{\partial \alpha} = \frac{\sigma\phi(\frac{x_0 - \mu}{\sigma})}{\Phi(\frac{x_0 - \mu}{\sigma})} > 0$$

and

$$\frac{\partial x_0(\alpha,\beta)}{\partial \beta} = \frac{-\sigma\phi(\frac{x_0 - \mu}{\sigma})}{1 - \Phi(\frac{x_0 - \mu}{\sigma})} < 0.$$

Hence $x_0(\alpha,\beta)$ is an increasing function of α and a decreasing function of β.

Eugene et al. (2002) showed that the beta-normal distribution is symmetric about μ when $\alpha = \beta$. From this result and corollary 3, the modal value is greater than μ if $\alpha > \beta$. Also, the modal value is less than μ if $\alpha < \beta$. The beta-normal distribution has a very distinct property in that it can be used to describe both bimodal and unimodal data.

Region of Bimodality

The beta-normal distribution becomes bimodal for certain values of the parameters α and β, and the analytical solution of α and β, where the distribution becomes bimodal, cannot be solved algebraically. A numerical solution is obtained, however, by solving the number of roots of the derivative of $BN(\alpha, \beta, \mu, \sigma)$. Table 1 shows a grid of values where the distribution is bimodal. The "2" in Table 1 indicates that the beta-normal distribution has two turning points which implies that the distribution is bimodal and the "1" indicates that the beta-normal distribution has one turning point which implies that the distribution is unimodal.

Table 1. Number of turning points of $BN(\alpha,\ \beta,\ 0,\ 1)$ for various values of α and β

Alpha \ Beta	.01	.02	.03	.04	.05	.06	.07	.08	.09	.10	.11	.12	.13	.14	.15	.16	.17	.18	.19	.02	.21	.22
.01	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	1	1	1	1	1	1
.02	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	1	1	1	1	1	1
.03	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	1	1	1	1	1	1
.04	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	1	1	1	1	1	1
.05	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	1	1	1	1	1	1
.06	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	1	1	1	1	1	1
.07	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	1	1	1	1	1	1
.08	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	1	1	1	1	1
.09	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	1	1	1	1	1
.1	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	1	1	1	1	1
.11	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	1	1	1	1	1
.12	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	1	1	1	1	1
.13	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	1	1	1	1
.14	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	1	1	1	1
.15	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	1	1	1	1
.16	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	1	1	1	1
.17	1	1	1	1	1	1	1	2	2	2	2	2	2	2	2	2	2	2	2	1	1	1
.18	1	1	1	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2	2	1	1	1
.19	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2	2	2	2	1	1
.2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2	2	1	1
.21	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2	1
.22	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
.23	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
.24	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
.25	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
.26	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

Note: "2" indicates where bimodality occurs and "1" indicates where unimodality occurs

Numerically, the largest value of α or β that gives bimodal property is approximately 0.214. Figure 1 shows a plot of the boundary region of α and β values where $BN(\alpha, \beta, 0,1)$ is bimodal.

Figure 1. Plot of bimodal region for beta-normal distribution $BN\,(\alpha,\,\beta,\,0,\,1)$
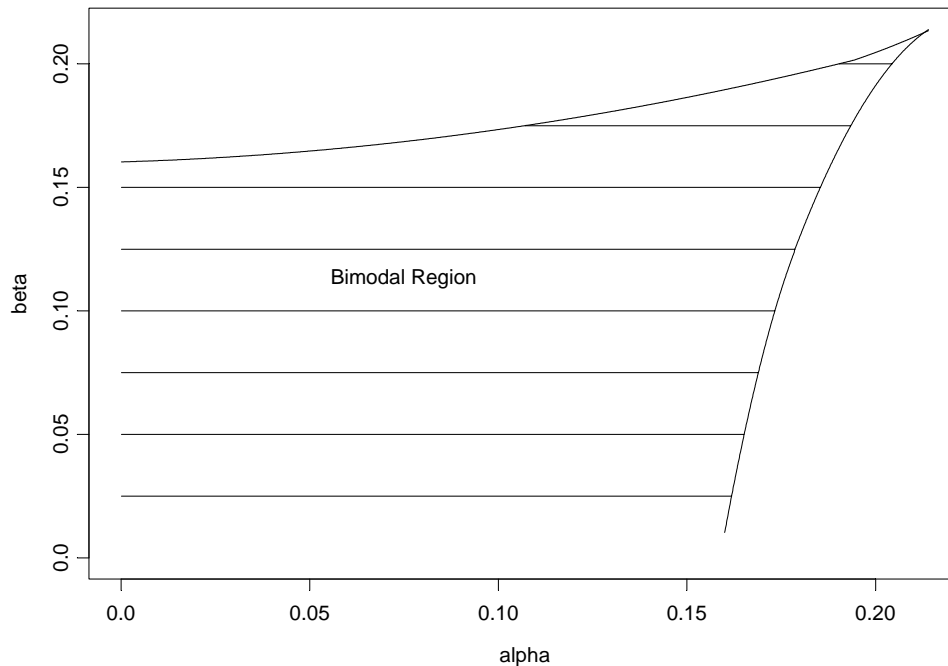
Corollary 4: The bimodal property of BN (α, β, μ, σ) is independent of the parameters μ and σ.

Proof: The mode(s) of $BN(\alpha,\,\beta,\,\mu,\,\sigma)$ is at the point $x_0 = x_0(\alpha,\,\beta)$ given in (2.1). On re-writing (2.1), one obtains (2.4). On taking the μ on the right hand side of (2.4) to the left hand side, dividing through by σ, and replacing $(x_0 - \mu)/\sigma$ by z_0, one obtains

$$z_0 = \frac{\phi(z_0)}{\Phi(z_0)[1-\Phi(z_0)]}\{(2-\alpha-\beta)\Phi(z_0)+(\alpha-1)\}$$

(2.5)

which is independent of parameters μ and σ.

In corollary 4, we showed that the bimodal property of $BN(\alpha,\,\beta,\,\mu,\,\sigma)$ is robust against the parameters μ and σ. In other words, regardless of the values of μ and σ, the α and β range for the bimodality of $BN(\alpha,\,\beta,\,\mu,\,\sigma)$ remains the same. To get more accurate values

of the pairs of (α, β) values that lie on the boundary of the region where the beta-normal distribution becomes bimodal, regression lines were drawn to estimate each boundary. The regression line that traced the boundaries of Figure 1 was approximated using curve estimation. For the values of α in the interval [0.01, 0.1943), the values of β at the upper boundary in Figure 1 were estimated by $\hat{\beta} = 0.8591\alpha^2 + 0.0453\alpha + 0.1603$. For α in the interval [0.1943, 0.214] we estimated β values by $\hat{\beta} = 4.4113\alpha^2 - 1.1966\alpha + 0.2675$. For the values of α in the interval [0.16, 0.1785), the values of β at the lower boundary were estimated by

$$\hat{\beta} = -116.15\alpha^2 + 45.4657\alpha - 4.2908.$$

For α in the interval [0.1785, 0.214] we obtained the equation

$$\hat{\beta} = -41.972\alpha^2 + 18.9913\alpha - 1.9281$$

to estimate the value of β.

If $BN(\alpha,\ \beta,\ \mu,\ \sigma)$ is unimodal, the distribution is skewed to the right whenever $\alpha > \beta$ and it is skewed to the left whenever $\alpha < \beta$. If $BN(\alpha,\ \beta,\ \mu,\ \sigma)$ is bimodal, the distribution is skewed to the right when $\alpha < \beta$ and it is skewed to the left when $\alpha > \beta$. Thus, the beta-normal distribution provides great flexibility in modeling symmetric, skewed and bimodal distributions.

Percentile of beta-normal distribution

Let $CBN(t)$ denote the cumulative probability of the beta-normal distribution up to a point t, which is given by

$$CBN(t) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)}$$
$$\int_{-\infty}^{t} [\Phi(\frac{x-\mu}{\sigma})]^{\alpha-1}[1-\Phi(\frac{x-\mu}{\sigma})]^{\beta-1}\sigma^{-1}\phi(\frac{x-\mu}{\sigma})dx \ . \qquad (2.6)$$

The percentiles in Table 2 are computed by solving (2.6) for t such that $CBN(t)$ takes the values 0.50, 0.75, 0.90, 0.95, and 0.99.

When $\beta = 1$, the result in (2.6) reduces to

$$CBN(t) = \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha)}$$
$$\int_{-\infty}^{t} [\Phi(\frac{x-\mu}{\sigma})]^{\alpha-1}\sigma^{-1}\phi(\frac{x-\mu}{\sigma})dx = [\Phi(\frac{t-\mu}{\sigma})]^{\alpha}.$$

When $\alpha = 1$, (2.6) becomes

$$CBN(t) = \frac{\Gamma(\beta + 1)}{\Gamma(\beta)}$$
$$\int_{-\infty}^{t} [1-\Phi(\frac{x-\mu}{\sigma})]^{\beta-1}\sigma^{-1}\phi(\frac{x-\mu}{\sigma})dx = 1-[1-\Phi(\frac{t-\mu}{\sigma})]^{\beta}.$$

Notice that if we compare the values of the mean of the unimodal beta-normal distribution with its median in Table 2, the mean of the beta-normal distribution is always greater than its median whenever $\alpha > \beta$. When the distribution is bimodal the mean of the beta-normal distribution is less than its median whenever $\alpha > \beta$. The percentiles in Table 2 are clearly increasing functions of α and decreasing functions of β. A graph of α versus the median (50th percentile) is plotted for β = 0.1, 0.5, 1, and 10 in Figure 2(a). Similar graphs for the 75th and 90th percentiles show the same pattern in Figure 2(b) and Figure 2(c) respectively.

Table 2. Mean and percentiles of BN(α, β, 0,1) for different values of α and β

α	β	Mean	0.50	0.75	0.90	0.95	0.99
0.05	0.05	0.0000	0.0000	4.7784	7.6544	9.2705	12.2382
	0.10	-2.2859	-2.7789	1.6446	4.3972	5.7191	8.0155
	0.50	-4.6527	-4.4917	-2.2674	-0.3400	0.7053	2.2795
	1.00	-5.0344	-4.4630	-2.2795	-1.1671	-0.3625	0.9074
	5.00	-5.5403	-5.1608	-3.3464	-2.1275	-1.5810	-0.9765
	10.00	-5.7016	-5.2981	-3.5470	-2.4100	-1.9186	-1.2382
0.10	0.05	2.2859	2.7789	5.8402	8.3706	9.8727	12.7024
	0.10	0.0000	0.0000	3.1398	5.2216	6.3834	8.5077
	0.50	-2.7633	-2.7086	-0.9199	0.6213	1.4295	2.7266
	1.00	-3.2639	-3.0973	-1.5865	-0.3889	0.2501	1.3069
	5.00	-3.9268	-3.6519	2.4222	-1.5832	-1.1778	-0.5432
	10.00	-4.1331	-3.8377	-2.6793	-1.9194	-1.5638	-1.0233
0.50	0.05	4.6527	4.4917	6.8631	9.1215	10.5189	13.2127
	0.10	2.7633	2.7086	4.5059	6.1677	7.1840	9.1287
	0.50	0.0000	0.0000	1.0518	1.9691	2.5031	3.4843
	1.00	-0.7043	-0.6745	0.1573	0.8779	1.2959	2.0558
	5.00	-1.7558	-1.6779	-1.1274	-0.6831	-0.4351	0.0028
	10.00	-2.0809	-1.9945	-1.5095	-1.1289	-0.9204	-0.5588
1.00	0.05	5.0344	4.7630	7.0477	9.2623	10.6416	13.3109
	0.10	3.2639	3.0973	4.7630	6.3613	7.3520	9.2623
	0.50	0.7043	0.6745	1.5341	2.3263	2.8070	3.7190
	1.00	0.0000	0.0000	0.6745	1.2816	1.6449	2.3264
	5.00	-1.1630	-1.1290	-0.6994	-0.3344	-0.1238	0.2582
	10.00	-1.5388	-1.4988	-1.1290	-0.8215	-0.6468	-0.3344
5.00	0.05	5.5403	5.1608	7.3275	9.4786	10.8308	13.4631
	0.10	3.9268	3.6519	5.1547	6.6640	7.6167	9.4751
	0.50	1.7558	1.6779	2.3038	2.9367	3.3438	4.1515
	1.00	1.1630	1.1290	1.5900	2.0365	2.3187	2.8769
	5.00	0.0000	0.0000	0.2742	0.5216	0.6702	0.9503
	10.00	-0.4556	-0.4517	-0.2233	-0.0201	0.1004	0.3251
10.0	0.05	5.7016	5.2981	7.4264	9.5558	10.8986	13.5179
	0.10	4.1331	3.8377	5.2916	6.7719	7.7117	9.5521
	0.50	2.0809	1.9945	2.5625	3.1526	3.5380	4.3130
	1.00	1.5388	1.4988	1.9055	2.3087	2.5679	3.0889
	5.00	0.4556	0.4517	0.6836	0.8960	1.0248	1.2706
	10.00	0.0000	0.0000	0.1915	0.3640	0.4075	0.6621

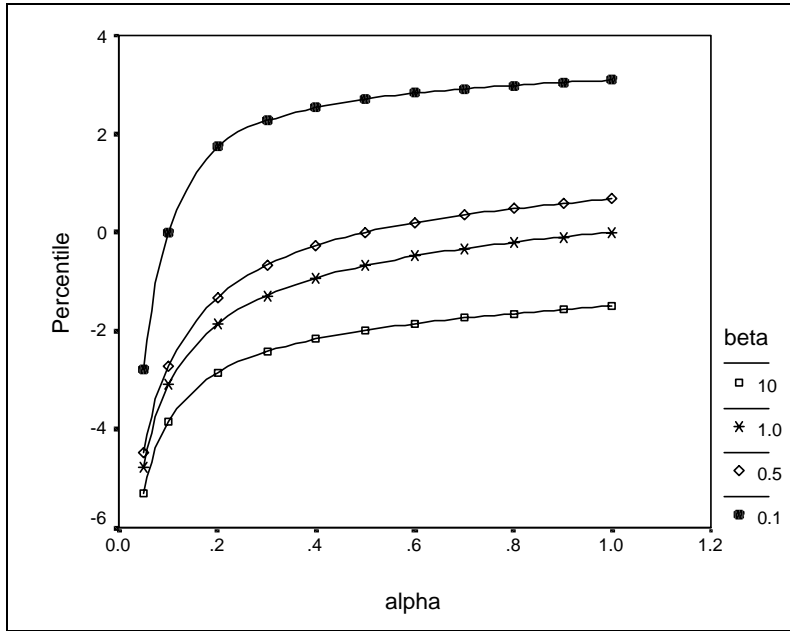Figure 2(a). Plot of 50th percentile versus α for some β values

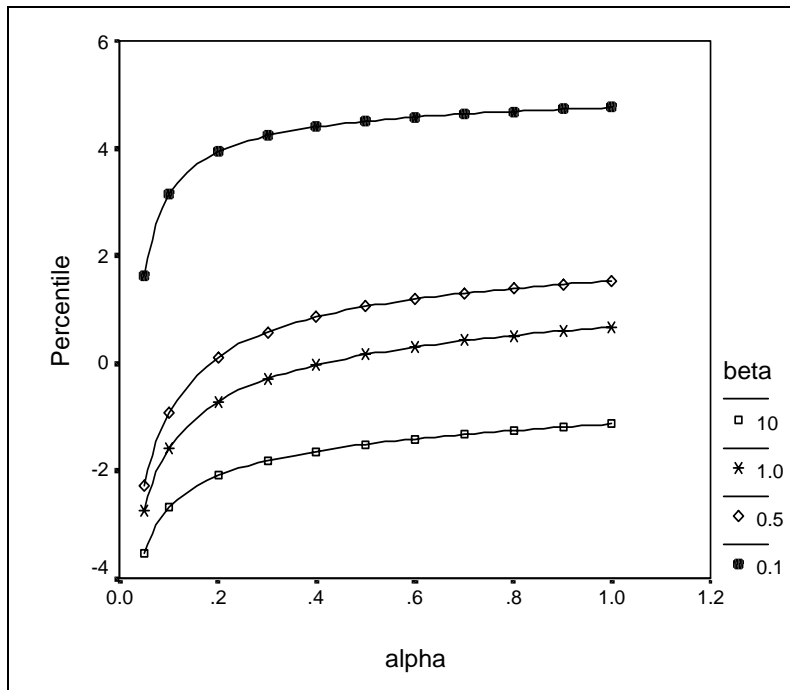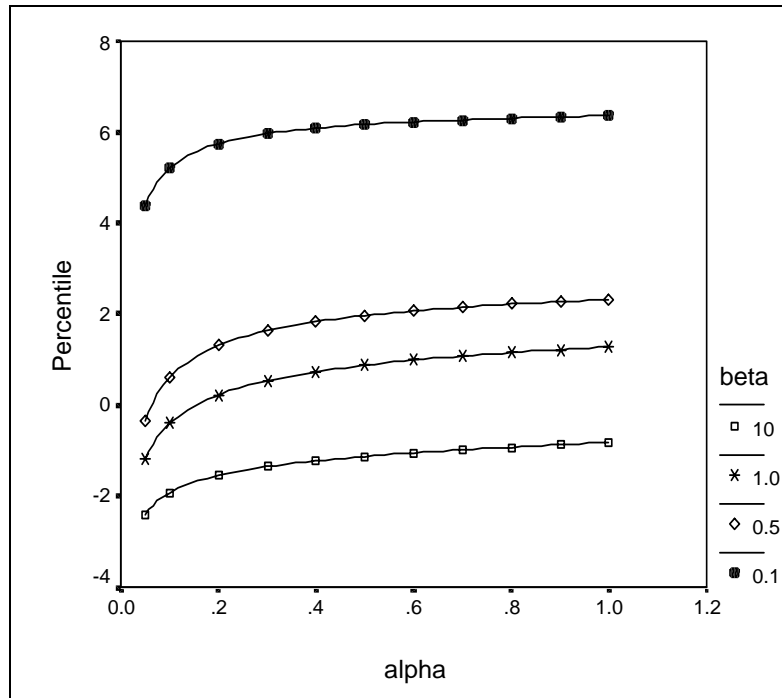Figure 2(b). Plot of 75th percentile versus α for some β values

Figure 2(c). Plot of 90th Percentile versus α for some β values

The percentiles increase very rapidly when α and β are less than 0.2. This rate of increase is due to the fact that the variation of the beta-normal distribution increases when α or β decreases. When α or β gets closer to 0.2, this variation decreases.

Application to Bimodal Data
Egg Size Distribution

Sewell and Young (1997) studied the egg size distributions of echinoderm. In marine invertebrates, a species produces either many small eggs with planktotrophic development or fewer larger eggs with lecithotrophic development, Thorson (1950). The models developed by Vance (1973a, 1973b) viewed planktotrophy and lecithotrophy as extreme forms of larvae development. Subsequent modifications of these models (see references in Sewell and Young, 1997) predict that eggs of marine invertebrates have bimodal distributions. Christiansen and Fenchel (1979) reported a bimodal distribution of egg sizes within prosobranchs. Emlet et al. (1987) described bimodal distributions in asteroid and echinoid echinoderms.

For echinoids and asteroids (see Tables 2 and 7 of Emlet et al., 1987), the egg diameters for species with planktotrophic larvae have less variation than species with lecithotrophic larvae (see Table 3). Because of this variation, the egg diameters appear to have one mode. However, with logarithmic transformation, the effect of large eggs in lecithotrophic species is reduced and the distribution of eggs becomes bimodal for both echinoids and asteroids. The transformation brings the modes nearer to each other and possibly makes their existence easier to detect.

Table 3. Descriptive statistics for asteroids species data

Types	n	Egg Diameter		Log Egg Diameter	
		Mean	SD	Mean	SD
Planktotrophic	35	153.11	34.26	5.01	0.23
Lecithotrophic	36	828.28	304.20	6.64	0.42
Brooding	17	1496.47	1066.58	7.05	0.77
All Types	88	688.83	705.59	6.07	0.99

Sewell and Young (1997) reported that many of the early studies used data sets that were not appropriate for a valid test of the egg size distribution patterns. They defined three criteria for appropriate data sets. The most widely cited example of bimodality in egg sizes is the data set compiled by Emlet et al. (1987). This data set satisfied the three criteria defined by Sewell and Young.

Sewell and Young (1997) reexamined the asteroid and echinoid egg size data in Emlet et al. (1987) with some additional data from more recent study. The additional data used by Sewell and Young were not available in their published article.

In this article, we have applied the beta-normal distribution to fit the logarithm of the egg diameters of the asteroids data in Emlet et al. (1987). The valid data consists of 88 asteroid species divided into three types consisting of 35 planktotrophic larvae, 36 lecithotrophic larvae, and 17 brooding larvae. These species are from a variety of habitats.

The maximum likelihood estimation method is used for parameter estimation. Eugene et al. (2002) gave the detailed discussion of this estimation technique. The parameter estimates for beta-normal distribution are $\hat{\alpha} = 0.0129$, $\hat{\beta} = 0.0070$, $\hat{\mu} = 5.7466$, and $\hat{\sigma} = 0.0675$. The estimates for α and β fall in the bimodal region in Figure 1. The log-likelihood value is –109.48. By using the result in (2.1), the two modes for the beta-normal distribution are at the points (log of egg diameters) 5.16 and 6.55.

A mixture of two normal distributions (Johnson et al. (1994) page 164) with parameters μ_1, μ_2, σ_1, σ_2, and p is fitted to the asteroids data. The maximum likelihood estimates for the parameters are $\hat{\mu}_1 = 5.0014$, $\hat{\mu}_2 = 6.7462$, $\hat{\sigma}_1 = 0.2232$, $\hat{\sigma}_2 = 0.6056$, and $\hat{p} = 0.3875$. The log-likelihood value for the mixture-normal is –101.31. A histogram of the data with the beta-normal and mixture-normal distributions superimposed is presented in Figure 3.

The Kolmogorov-Smirnov test (see DeGroot & Schervish, 2002, p. 568) is used to compare the goodness of fit of beta-normal and mixture-normal distributions to the data. In Figure 4, the empirical CDF, the beta-normal CDF, and the mixture-normal CDF for the data are presented. The absolute maximum difference between the empirical cumulative distribution function and the beta-normal cumulative distribution function is $D_n^* = 0.1233$.

This provides a test statistic $\sqrt{n}D_n^* = 1.1570$ with a significance probability of 0.1370. The corresponding results for the mixture-normal distribution are $D_n^* = 0.0654$, $\sqrt{n}D_n^* = 0.6135$ with a p-value of 0.8459. Thus, both the beta-normal and mixture-normal distributions provide an adequate fit to the data. However, the mixture-normal appears to provide a better fit.

Figure 3. Histogram of asteroids data with beta-normal and mixture-normal superimposed
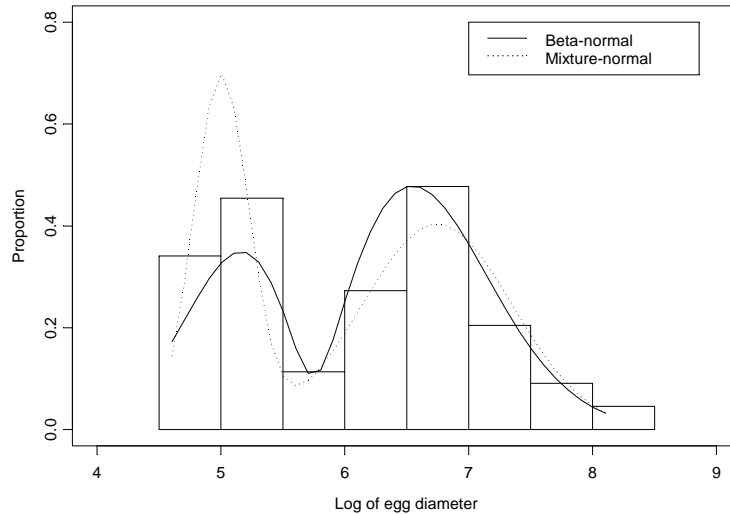
In examining the histogram for the log of egg diameter in Figure 3, both modes appear to have come from two symmetric distributions. This may explain in part why mixture-normal distribution provides a better fit than the beta-normal. Another reason is that mixture-normal has five parameters whereas the beta-normal has four parameters.

Test of Bimodality for the Egg Size Distribution Data

Schilling et al. (2002) derived a condition for the unimodality of mixtures of two normal distributions with unequal variances. If σ_1^2 and σ_2^2 are the variances of two normal distributions with means μ_1 and μ_2, the mixture is unimodal for any mixture proportion p if and only if $|\mu_2 - \mu_1| \le S(r)[\sigma_1 + \sigma_2]$, where $r = \sigma_1^2 / \sigma_2^2$ and

$$S(r) = \left(-2 + 3r + 3r^2 - 2r^3 + 2(1 - r + r^2)\right)^{1/2} / \left(\sqrt{r}(1 + \sqrt{r})\right).$$

From the fit of mixture-normal to the asteroids data, the parameter estimates gave $|\hat{\mu}_2 - \hat{\mu}_1| = 1.7$ and $S(r)[\hat{\sigma}_1 + \hat{\sigma}_2] = 0.56$. Thus, there is evidence that the parameter values do not lie in the region where the mixture is unimodal for any value of p.

Figure 4. Empirical, beta-normal, and mixture-normal CDF for asteroids data
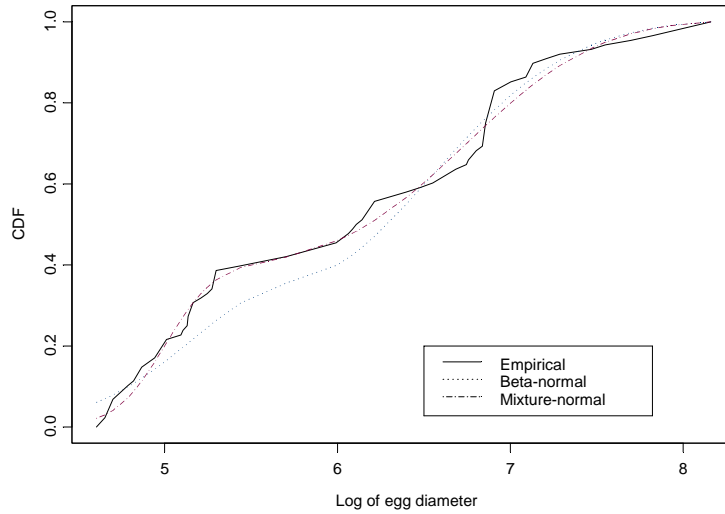
A Comparison Between Beta-Normal and Mixture-Normal Distributions

A simulation study is conducted to compare the performance between beta-normal and mixture-normal for bimodal data. One hundred simulations, each with sample size $n = 400$, are conducted. In each simulation, data are generated from two Weibull distributions, W ($\lambda=2$, $\beta=5$) and W ($\lambda=2$, $\beta=10$), where λ and β are the scale and shape parameters respectively. Bimodal data are obtained from mixing the data from the two Weibull distributions in the form

$$\left[p_1 W(2,\ 5) + 10 \right] + \left[(1 - p_1) W(2,\ 10) \right]. \quad (5.1)$$

The value 10 that is added to the first quantity in (5.1) is used to adjust the location of the modes. The different mixing proportions p_1 considered in the simulation study are 0.2, 0.3, 0.4, and 0.5. A variety of other types of mixtures are also considered and the results are similar. Some of the simulations failed due to numerical difficulty in estimating the beta-normal parameters using S-PLUS on personal computer. The main difficulty is that the optimization algorithm in S-PLUS failed to converge. There is a need for better algorithms to solve this numerical difficulty and this will be taken up in future research.

We wish to compare the mixture-normal (MN) density $f(x)$ and the beta-normal (BN) density $g(x)$. Given these two densities, we test the null hypothesis

H_0 : MN and BN are equivalent

against the alternative hypothesis (5.2)

H_f : MN is better than BN, or H_g : BN is better than MN. (5.3)

To test the null hypothesis in (5.2), we use the likelihood ratio test proposed by Vuong (1989).

Vuong's Likelihood Ratio Statistic

The likelihood ratio statistic for testing H_0 in (5.2) is

$$L_* = \sum_{i=1}^{n} \log\left(\frac{f(x_i)}{g(x_i)}\right). \qquad (5.4)$$

Because the mixture-normal and the beta-normal densities are non-nested, the statistic in (5.4) is not chi-square distributed. Vuong (1989) used the Kullback-Liebler Information Criterion to discriminate between two non-nested models and proposed an unadjusted test statistic

$$T_* = \frac{L_*}{\hat{\omega}\sqrt{n}}, \qquad (5.5)$$

where

$$\omega^2 = \frac{1}{n}\sum_{i=1}^{n}\left[\log\left(\frac{f(x_i)}{g(x_i)}\right)\right]^2 - \left[\frac{1}{n}\sum_{i=1}^{n}\log\left(\frac{f(x_i)}{g(x_i)}\right)\right]^2,$$

is an estimate of the variance of $L_*/\sqrt{n}$.

When comparing the goodness of fit for two non-nested parametric distributions, the number of parameters may not be the same. To account for the different number of parameters, Vuong proposed two adjusted test statistics:

$$T_1 = \frac{L_* - K_1(f,g)}{\hat{\omega}\sqrt{n}}, \text{ and } T_2 = \frac{L_* - K_2(f,g)}{\hat{\omega}\sqrt{n}},$$

where $K_1(f,g) = p - q$ is a correction factor for only the number of parameters and $K_2(f,g) = \ln(n)[(p-q)/2]$ is a correction factor for the number of parameters and the sample size n. In the test statistic, p is the number of parameters in $f(x)$ and q is the number of parameters in $g(x)$. In this case $p = 5$, $q = 4$, and $n = 400$. We apply both adjusted statistics T_1 and T_2 in our comparison. T_i ($i = 1$, 2) is approximately standard normal distributed under the null hypothesis that the two densities are equivalent (Vuong, 1989).

At significant level α, one compares T_i with $z_{\alpha/2}$. If $T_i < -z_{\alpha/2}$, H_0 is rejected in favor of H_g, BN is better than MN. If $T_i > z_{\alpha/2}$, H_0 is rejected in favor of H_f, MN is better than BN. However, if $|T_i| \leq z_{\alpha/2}$, H_0 is not rejected.

Thus, we do not have sufficient evidence to say that both densities are not equivalent. For each generated data, the test statistics T_1 and T_2 are computed for testing H_0. From the 100 simulations, we record the number of times the BN density is better than the MN density, the number of times the MN density is better than the BN density and the number of times both densities are equivalent.

Simulation Results and Discussion

Table 4 summarizes the Vuong's T_1 and T_2 goodness of fit statistics from 100 simulated data sets. The comparison is conducted at 10% and 5% level of significance. From Table 4, both beta-normal and mixture-normal distributions fit the mixtures of Weibull distribution data equally well for most cases. In general, beta-normal fits better than the mixture-normal, especially when using the adjusted statistic T_2.

Figures 5 (A – D) give the histograms and the empirical CDF's of some simulated data sets with $n = 400$ and the corresponding fitted distributions of beta-normal and mixture-normal. The fitted distributions shown in Figure 5 indicate that both BN and MN fit these Weibull mixtures well. The histogram in Figure 5 (A) looks less like a bimodal distribution. The beta-normal distribution fits the data as unimodal distribution, while the mixture-normal distribution fits this data as bimodal distribution.

As mentioned before, we encountered some numerical difficulties when using the S-PLUS optimization routines to estimate the parameters of beta-normal distributions. Further research to develop better estimation algorithms will be needed to address this numerical problem in the estimation of BN parameters.

Table 4. Comparison between Beta-Normal and Mixture-Normal densities for Fitting 100 Simulated Mixtures of Weibull Distributions, $\left[p_1 W(2,\ 5)+10\right]+\left[(1-p_1)W(2,\ 10)\right]$.

| α		$p_1 = 0.2$		$p_1 = 0.3$		$p_1 = 0.4$		$p_1 = 0.5$	
		T_1	T_2	T_1	T_2	T_1	T_2	T_1	T_2
0.10	BN is better	0	7	9	52	16	55	1	9
	MN is better	6	2	0	0	0	0	1	0
	Both equivalent	94	91	91	48	84	45	98	91
0.05	BN is better	0	4	4	41	9	50	1	6
	MN is better	3	0	0	0	0	0	1	0
	Both equivalent	97	96	96	59	91	50	98	94

Figure 5. Histogram of Weibull mixture data with BN and MN superimposed; Empirical, BN and MN CDF for Weibull Mixture data

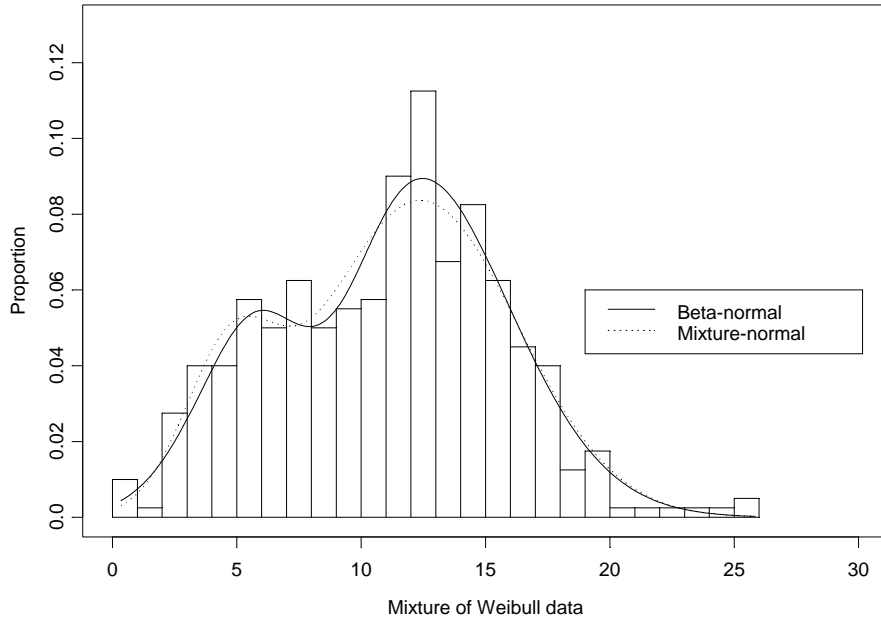(A) Mixing proportion $p_1 = 0.2$

Histogram of simulated data
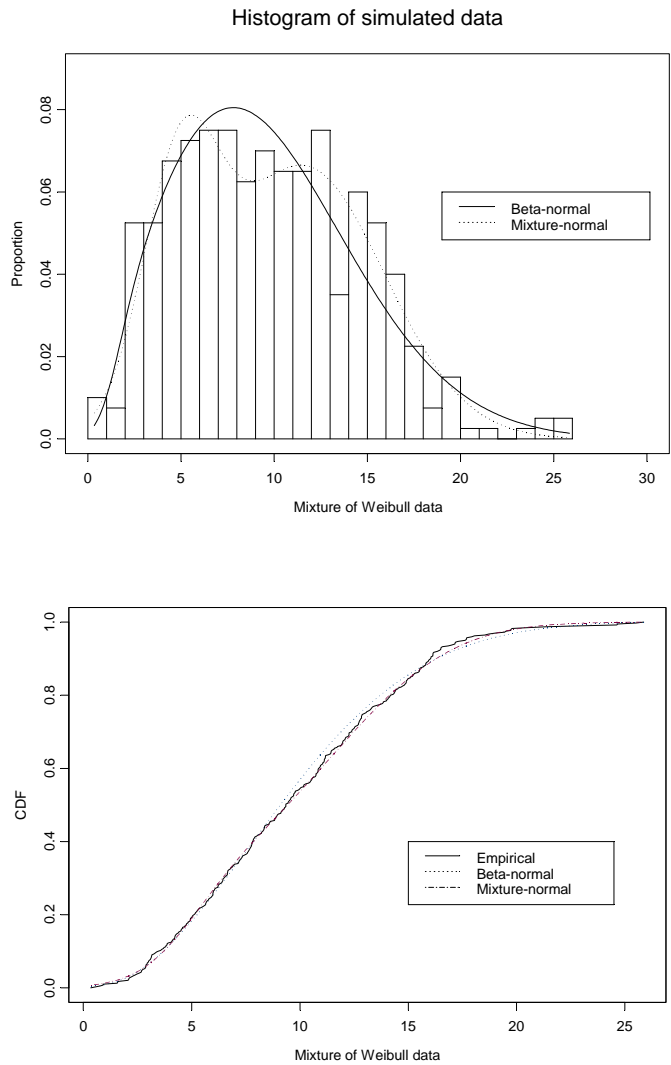
(B) Mixing proportion $p_1 = 0.3$
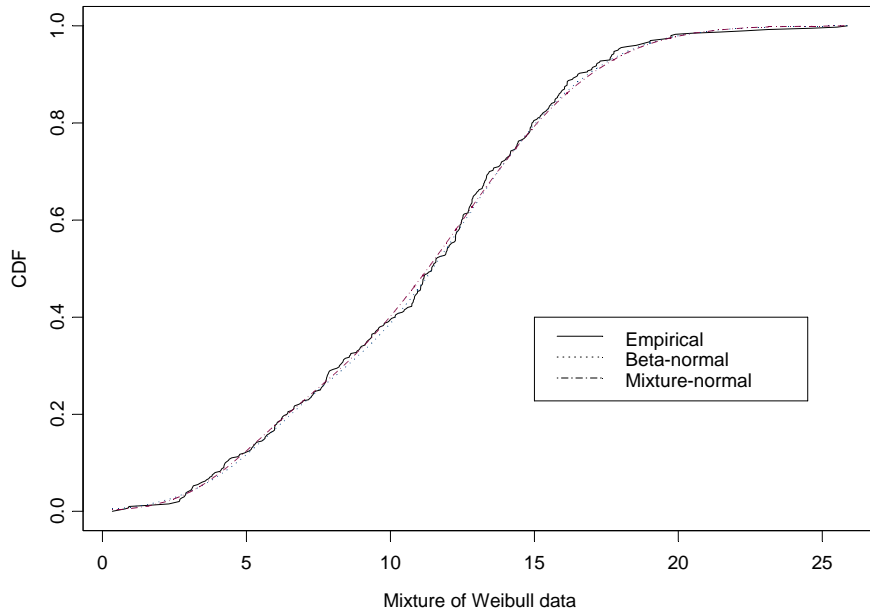
Histogram of simulated data
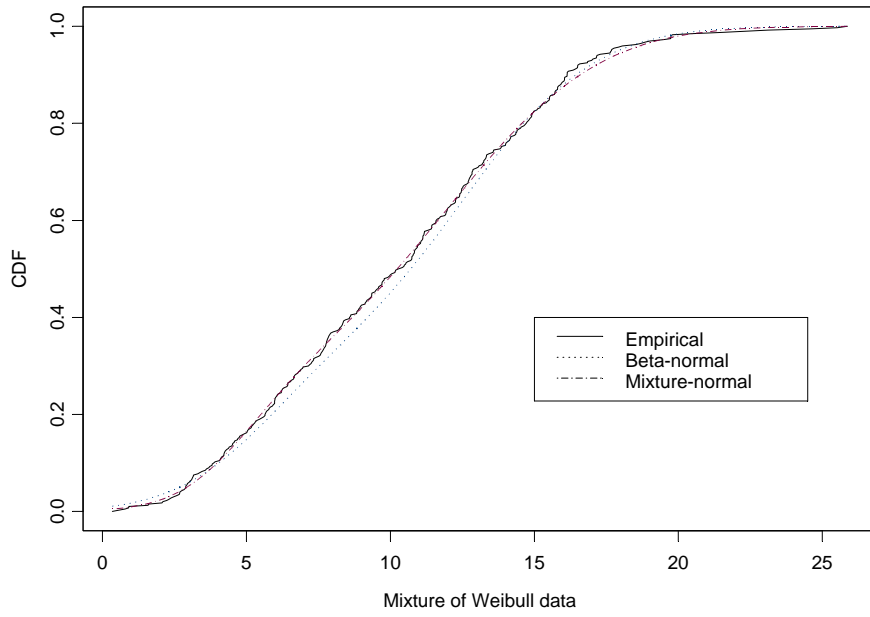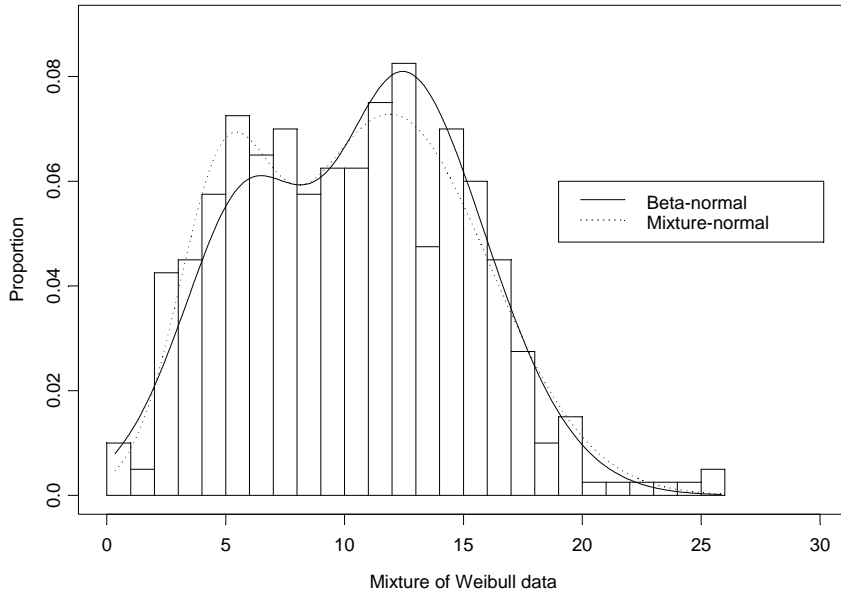
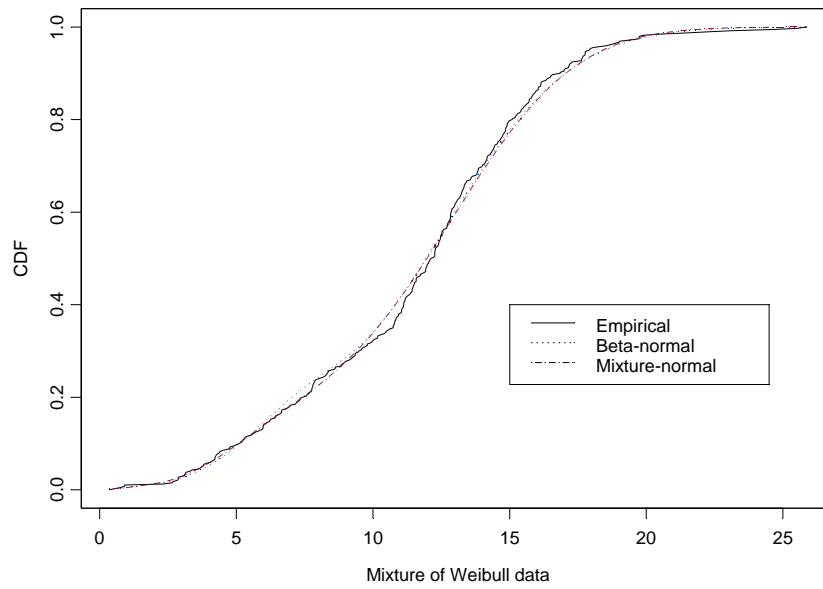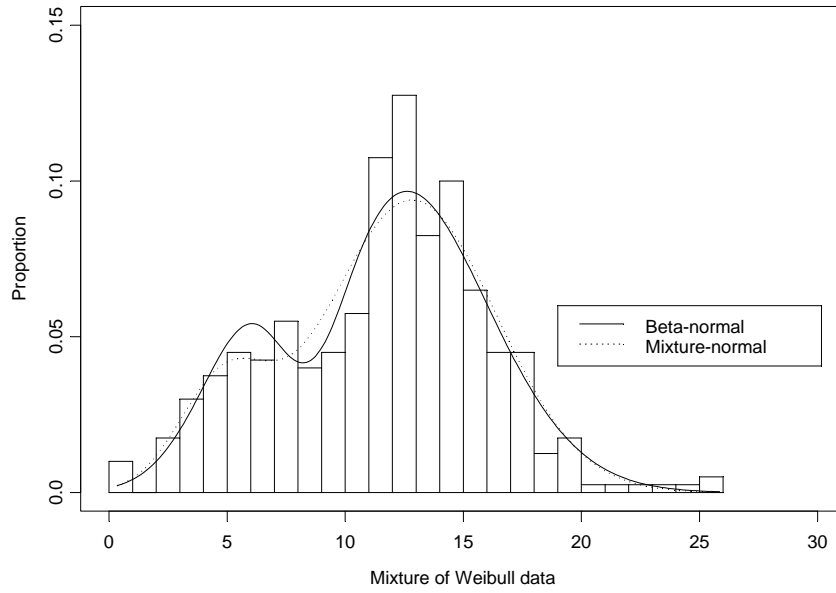(C) Mixing proportion $p_1 =$

Histogram of simulated data

0.4

(D) Mixing proportion $p_1 = 0.5$

Histogram of simulated data

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