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BETTER BOUNDS FOR PERIODIC SOLUTIONS OF DIFFERENTIAL EQUATIONS IN BANACH SPACES

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ABSTRACT. Let f be Lipschitz with constant L in a Banach space and let $x(t)$ be a P-periodic solution of $x'(t) = f(x(t))$. We show that $P \geq 6/L$. An example is given with $P = 2\pi/L$, so the bound is nearly strict. We also give a short proof that $P \geq 2\pi/L$ in a Hilbert space.

I. Introduction. Suppose f is Lipschitz with constant L in a Banach space and $x(t)$ is a P-periodic solution of $x'(t) = f(x(t))$. How small can LP be?

This question was first addressed when Lasota and Yorke [**3**] proved $LP \geq 4$. Busenberg and Martelli [**2**] showed $LP \geq 4\frac{1}{2}$. We show that $LP \geq 6$.

The lowest known LP is 2π, e.g. for nonzero solutions of $u'(t) = v(t)$ and $v'(t) = -u(t)$. Also, in a Hilbert space, $LP \geq 2\pi$ (Lasota and Yorke [**3**], Yorke [**4**]—A short proof is given in §III). So the Hilbert space bound is strict, but the Banach space bound may not be strict.

II. In Banach spaces, $LP \geq 6$.

LEMMA 1. *Let B be a Banach space and let $y: \mathbf{R} \to B$ be continuous and P-periodic with $\|y'(t)\|$ integrable. Then*

$$\int_0^P \int_0^P \|y(t) - y(s)\| \, ds \, dt \leq \frac{P}{6} \int_0^P \int_0^P \|y'(t) - y'(s)\| \, ds \, dt.$$

PROOF.

$$\begin{aligned} A \equiv \int_0^P \int_0^P \|y(t) - y(s)\| \, ds \, dt &= \int_0^P \int_0^P \|y(s+t) - y(s)\| \, ds \, dt \\ &= \int_0^P \int_0^P \frac{(P-t)t}{P} \left\| \frac{y(s+t) - y(s)}{t} - \frac{y(s) - y(s+t-P)}{P-t} \right\| ds \, dt \\ &= \int_0^P \int_0^P \frac{(P-t)t}{P^2} \left\| \int_0^P \left(y'\left(s + \frac{tr}{P}\right) - y'\left(s + \frac{tr}{P} - r\right) \right) dr \right\| ds \, dt \\ &\leq \int_0^P \int_0^P \frac{(P-t)t}{P^2} \int_0^P \left\| y'\left(s + \frac{tr}{P}\right) - y'\left(s + \frac{tr}{P} - r\right) \right\| dr \, ds \, dt \\ &= \int_0^P \frac{(P-t)t}{P^2} \int_0^P \int_0^P \left\| y'\left(s + \frac{tr}{P}\right) - y'\left(s + \frac{tr}{P} - r\right) \right\| ds \, dr \, dt. \end{aligned}$$

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Since the inner integral is over one period, it can be shifted by $tr/P - r$ to yield

$$A \le \int_0^P \frac{(P-t)t}{P^2}\,dt \int_0^P \int_0^P ||y'(s+r) - y'(s)||\,ds\,dr$$
$$= \frac{P}{6}\int_0^P \int_0^P ||y'(r) - y'(s)||\,ds\,dr.$$

THEOREM 1. *If f is Lipschitz with constant L in a Banach space and $x(t)$ is a nonconstant P-periodic solution of $x'(t) = f(x(t))$, then $LP \ge 6$.*

PROOF.

$$\int_0^P \int_0^P ||x(t) - x(s)||\,ds\,dt \le \frac{P}{6}\int_0^P \int_0^P ||x'(t) - x'(s)||\,ds\,dt$$
$$= \frac{P}{6}\int_0^P \int_0^P ||f(x(t)) - f(x(s))||\,ds\,dt \le \frac{LP}{6}\int_0^P \int_0^P ||x(t) - x(s)||\,ds\,dt.$$

Solving for LP gives the result.

III. A short proof that $LP \ge 2\pi$ in Hilbert spaces.

LEMMA 2 (*a Hilbert space analog of Wirtinger's inequality*). *Let H be a Hilbert space and let $y: \mathbf{R} \to H$ be continuous and P-periodic with $\int_0^P y(t)\,dt = 0$ and $||y'(t)||^2$ integrable. Then*

$$\int_0^P ||y(t)||^2\,dt \le \frac{P^2}{4\pi^2}\int_0^P ||y'(t)||^2\,dt.$$

PROOF. Since the path of $y(t)$ is compact, there is a countable orthonormal set $e_1, e_2, \dots$ with $y(t) = \sum_{i=1}^\infty a_i(t)e_i$. Each $a_i(t)$ is P-periodic and $\int_0^P a_i(t)\,dt = 0$, so Wirtinger's inequality [**1**] gives $\int_0^P a_i(t)^2\,dt \le (P^2/4\pi^2)\int_0^P a_i'(t)^2\,dt$. Then

$$\int_0^P ||y(t)||^2\,dt = \int_0^p \left|\left|\sum_{i=1}^\infty a_i(t)e_i\right|\right|^2 dt = \int_0^P \sum_{i=1}^\infty a_i(t)^2\,dt$$
$$= \sum_{i=1}^\infty \int_0^P a_i(t)^2\,dt \le \frac{P^2}{4\pi^2}\sum_{i=1}^\infty \int_0^P a_i'(t)^2\,dt = \frac{P^2}{4\pi^2}\int_0^P \sum_{i=1}^\infty a_i'(t)^2\,dt$$
$$= \frac{P^2}{4\pi^2}\int_0^P \left|\left|\sum_{i=1}^\infty a_i'(t)e_i\right|\right|^2 dt = \frac{P^2}{4\pi^2}\int_0^P ||y'(t)||^2\,dt.$$

THEOREM 2 [**3**, **4**]. *If f is Lipschitz with constant L in a Hilbert space and $x(t)$ is a nonconstant P-periodic solution of $x'(t) = f(x(t))$, then $LP \ge 2\pi$.*

PROOF. Pick h with $x(h) \ne x(0)$. Since $x(t)$ is P-periodic, $x(t+h) - x(t)$ is P-periodic and $\int_0^P (x(t+h) - x(t))\,dt = 0$. Then from Lemma 2

$$\int_0^P ||x(t+h) - x(t)||^2\,dt \le \frac{P^2}{4\pi^2}\int_0^P ||x'(t+h) - x'(t)||^2\,dt$$
$$= \frac{P^2}{4\pi^2}\int_0^P ||f(x(t+h)) - f(x(t))||^2\,dt \le \frac{L^2P^2}{4\pi^2}\int_0^P ||x(t+h) - x(t)||^2\,dt.$$

Solving for LP gives the result.

NOTE ADDED IN PROOF. The bound in Theorem 1 is now known to be sharp. We have constructed an example in a Banach space with $LP = 6$.

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