Beyond Stabilizer Codes I: Nice Error Bases

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Abstract

Nice error bases have been introduced by Knill as a generalization of the Pauli basis. These bases are shown to be projective representations of finite groups. We classify all nice error bases of small degree, and all nice error bases with abelian index groups. We show that in general an index group of a nice error basis is necessarily solvable.

Keywords

Quantum computing, nice error bases, generalizations of the Pauli basis, projective group representations, quantum error correcting codes.

I. Introduction

Errors resulting from imperfect gate operations and decoherence are serious obstructions to quantum computing. Recent progress in quantum error control and in fault tolerant computing gives hope that large scale quantum computers can be built [1], [2], [3], [4], [5]. Although arbitrary error operators E might affect the state of a qubit, it is always possible to keep track of the error amplitudes by expressing E in terms of an error operator basis, such as

$$
\rho(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \rho(0,1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
$$
$$
\rho(1,0) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \rho(1,1) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{1}
$$

This is an example of an error basis for a two-dimensional quantum system. The purpose of this note is to study error bases for higher dimensional quantum systems. There are many possible bases for the algebra of $n \times n$-matrices. A particularly useful class of unitary error bases – called nice error bases – has been introduced by Knill in [6], [7]. The nice error bases are the pillar of quantum error control codes [7], [8], [9] which generalize the stabilizer codes [1], [2].

Note that the error operators in (1) are parametrized by elements of the Klein four-group $\mathbf{Z}_2 \times \mathbf{Z}_2$. In general, nice error bases are also parametrized by a group. The group structure underlying the bases turns out to be important in quantum error control applications and in fault-tolerant quantum computing.

The purpose of this note is to give detailed information about nice error bases of small degree. To save space, we do not list the bases themselves. Instead, we focus on the group structure underlying the bases. We will show that nice error bases are projective representations of finite groups. Thus, the knowledge of the groups is sufficient to construct the corresponding error bases. We provide the programs for this construction at the web site

$$\texttt{http://www.cs.tamu.edu/faculty/klappi/ueb/ueb.html}$$

This site contains a complete list of nice error bases up to degree 11.

II. Error Bases

Let G be a group of order n^2 with identity element 1. A *nice error basis* on $\mathcal{H} = \mathbf{C}^n$ is a set $\mathcal{E} = \{\rho(g) \in \mathcal{U}(n) \,|\, g \in G\}$ of unitary matrices such that

(i) $\rho(1)$ is the identity matrix,

(ii) $\operatorname{tr} \rho(g) = n\, \delta_{g,1}$ for all $g \in G$,

(iii) $\rho(g)\rho(h) = \omega(g, h)\, \rho(gh)$ for all $g, h \in G$,

where $\omega(g, h)$ is a nonzero complex number depending on $(g, h) \in G \times G$; the function $\omega \colon G \times G \to \mathbf{C}^\times$ is called the factor system of ρ. We call G the *index group* of the error basis $\mathcal{E}$.

We discuss some consequences of the definitions. The conditions (i) and (iii) state precisely that ρ is a projective representation of the index group G, see [10, p. 349]. Projective representations have been introduced by Issai Schur at the beginning of the last century. In this note, we will make profitable use of some tools developed by Schur and others. We refer the reader to [10] for a gentle introduction to projective representations of finite groups.

Condition (ii) ensures that the matrices $\rho(g)$ are pairwise orthogonal with respect to the trace inner product $\langle A, B \rangle = \operatorname{tr}(A^\dagger B)/n$, hence they form a basis of the matrix algebra $M_n(\mathbf{C})$. It follows that ρ is an irreducible projective representation. In other words, $\{0\}$ and $\mathcal{H}$ are the only subspaces of the vector space $\mathcal{H}$ that remain invariant under the action of the representing matrices $\rho(g)$.

Furthermore, condition (ii) implies that the projective representation ρ is faithful, which simply means that $\rho(g)$ is not a scalar multiple of the identity matrix unless $g = 1$.

Theorem 1: Let $\mathcal{E} = \{\rho(g) \,|\, g \in G\}$ be a set of unitary matrices parametrized by the elements of a finite group G. The set $\mathcal{E}$ is a nice error basis with index group G if and only if ρ is a unitary irreducible faithful projective representation of G of degree $|G|^{1/2}$.

Proof: If $\mathcal{E}$ is a nice error basis, then the above discussion shows that ρ is indeed a unitary irreducible faithful projective representation of degree $|G|^{1/2}$. Conversely, ρ satisfies conditions (i) and (iii), since it is a projective representation. The assumption on the degree gives condition (ii) for $g = 1$. The representation ρ is supposed to be faithful, which means that

$$\ker \rho = \{g \in G \,|\, \rho(g) = cI \text{ for some } c \in \mathbf{C}\} = \{1\}.$$

Here, I denotes the identity matrix. The extremal degree condition $\operatorname{tr}\rho(1) = |G|^{1/2}$ implies that $\operatorname{tr}\rho(g) = 0$ holds for all $g \in G - \ker \rho$, see Corollary 11.13 in [11, p. 79]. It follows that the representation ρ satisfies condition (ii) for the nonidentity elements as well. $\square$

As a consequence of this theorem, we obtain a complete classification of all nice error bases with abelian index group. Recall that a group G is said to be of symmetric type if $G \cong H \times H$ for some group H.

Theorem 2: If a nice error basis $\mathcal{E}$ has an abelian index group G, then G is of symmetric type. Conversely, any finite abelian group G of symmetric type is index group of a nice error basis.

Proof: Theorem 1 shows that the error basis $\mathcal{E}$ can be understood as a faithful, irreducible projective representation of G. It was already shown by Frucht [12, XIII, p. 24] in 1931 that an abelian group admits a faithful irreducible projective representation only if it is of symmetric type.

On the other hand, if G is a finite abelian group of symmetric type, then it has a faithful irreducible unitary projective representation ρ, again by result XIII in [12, p. 24]. The degree of ρ is given by $\sqrt{|G|}$, see result XII in [12, p. 22]. Therefore, the set of representing matrices $\{\rho(g) \,|\, g \in G\}$ is a nice error basis according to Theorem 1. $\square$

III. Abstract Error Groups

The previous section characterized all nice error bases with abelian index groups. The case of nonabelian index groups is more complicated. We show that an index group of a nice error basis is

a solvable group. We derive this result by studying slightly larger groups known as abstract error groups.

Let G be the index group of a nice error basis $\{\rho(g) \,|\, g \in G\}$. We assume that the factor system ω is of finite order, i. e., that there exists a natural number m such that $\omega(g, h)^m = 1$ for all $g, h \in G$. This can always be achieved by multiplying the representation matrices $\rho(g)$ with a suitable phase factor if necessary [7]. Denote by T the cyclic group generated by the values of ω. Define an operation $\circ$ on the set $H = T \times G$ by

$$(a, g) \circ (b, h) = (ab\,\omega(g, h), gh), \qquad a, b \in T, g, h \in G.$$

It turns out that H is a finite group with respect to this multiplication, the ω-covering group of G [13, p. 134]. A group isomorphic to such an ω-covering group of an index group of a nice error basis was called *abstract error group* by Knill.

Given a nice error basis $\{\rho(g) \,|\, g \in G\}$, then the abstract error group is isomorphic to the group generated by the matrices $\rho(g)$. The assumption that the factor system ω is of finite order ensures that the abstract error group is finite. For instance, if we take the n-fold tensor product of matrices (1), then we obtain a nice error basis for a system of n qubits. The abstract error group H generated by these matrices is isomorphic to a so-called extraspecial 2-group, cf. [1].

Theorem 3: A group H is an abstract error group if and only if H is a group of central type with cyclic center $Z(H)$. In particular, all abstract error groups are solvable groups.

Proof: Recall that a group H is said to be of central type if and only if there exists an irreducible ordinary character χ of H with $\chi(1)^2 = (H : Z(H))$.

Step 1. If H is an abstract error group, then it is (isomorphic to) an ω-covering group of an index group G, which has an irreducible projective representation of degree $|G|^{1/2}$ with factor system ω. In particular, $G \cong H/T$ for some cyclic central subgroup T of H. Each irreducible projective repesentation of G with factor system ω lifts to an ordinary irreducible representation of H of the same degree. Consequently, H has an irreducible ordinary character χ with $\chi(1)^2 = (H : T)$. Each irreducible character χ of a group H satisfies the inequality $\chi(1)^2 \leq (H : Z(H))$, whence $T = Z(H)$. It follows that H is a group of central type with cyclic center.

Step 2. Conversely, suppose that H is a group of central type with cyclic center. It was shown in a seminal work by Pahlings [14] that H has a faithful irreducible unitary representation ϱ of degree

$(H : Z(H))^{1/2}$; this can be obtained by combining the Corollary and Proposition 5 in [14].

Let $G = H/Z(H)$, and denote by $W = \{x_g \mid g \in G\}$, with $1 \in W$, a set of coset representatives for $Z(H)$ in H. Define a projective representation ρ of G by

$$\rho(g) = \varrho(x_g), \qquad g \in G.$$

This projective representation ρ is unitary, irreducible, and faithful. Theorem 1 shows that G is an index group of a nice error basis. Finally, H is by construction isomorphic to an ω-covering group of G, hence an abstract error group.

Step 3. A deep result by Howlett and Isaacs shows that all groups of central type are solvable. The proof of this fact relies on the classification of finite simple groups [15]. □

The classification of all abstract error groups was posed as an open problem by Knill [7]. The previous theorem showed that all abstract error groups are solvable. It is known that all solvable groups can occur as *subgroups* of index groups of nice error bases, see Theorem 1.2 in [16]. On the other hand, it is known that not all solvable groups can occur as index groups [17]. This delicate situation makes it difficult to find a complete characterizaton of index groups of nice error bases and thus of abstract error groups.

Unlike the general case, it is fairly simple to characterize the abstract error groups with abelian index groups:

Theorem 4: A group H is an abstract error group with abelian index group $G \cong H/Z(H)$ if and only if H is nilpotent of class at most 2 and $Z(H)$ is a cyclic group.

Proof: We denote by H' the commutator subgroup

$$H' = \langle [g, h] = g^{-1}h^{-1}gh \mid g, h \in H \rangle$$

of the group H. Recall that a quotient group H/N of H is abelian if and only if the normal subgroup N of H contains the commutator subgroup H'.

Therefore, an abstract error group H has an abelian index group $G \cong H/Z(H)$ if and only if $H' \subseteq Z(H)$. In other words, H is nilpotent of class at most 2. The center of H is cyclic, since there exists a faithful irreducible representation of H.

Conversely, if H is nilpotent of class 2 with cyclic center, then it has a faithful irreducible representation of degree $(H : Z(H))^{1/2}$, cf. the Corollary to Proposition 4 in [14]. This shows that H is an

abstract error group. If H is abelian, hence cyclic, then H is an abstract error group with trivial index group.   □

IV. Small Index Groups

For practical purposes it is desirable to know the index groups of nice error bases of small order. We determine all nonabelian index groups of order 121 or less in this section. Recall that all groups of order p^2, p a prime, are abelian. Therefore, the search for nonabelian index groups of order n^2 can be restricted to composite numbers n. We used two different methods for the computer search. The first method searches the solvable groups of order n^3 to find abstract error groups, whence index groups appear as factor groups. The second method determines the degrees of the irreducible projective representations of potential index groups of order n^2 by calculating the Schur representation group.

Method 1. For each index group G of order n^2 there exists an abstract error group H of order at most n^3 such that $G \cong H/Z(H)$. Thus, we can search the catalogue of solvable groups [18] to find groups of central type with cyclic center. A calculation of the character table of all possible groups would be impractical. Here it is advantageous to use the following criterion by Pahlings: *Let H be a group with cyclic center Z, $G = H/Z$. Denote by $\mathrm{Cl}_G(x)$ the conjugacy class of x in G. The group H is of central type if and only if $|\mathrm{Cl}_H(x)| > |\mathrm{Cl}_G(xZ)|$ for all $x \in H - Z$.* This test allows to determine rapidly if a group is an abstract error group or not.

Method 2. In the second method we need to check whether a potential index group G of order n^2 has an irreducible projective representation of degree n. This can be done by calculating the Schur representation group G^* of G. This is a central extension of G which has the particular feature that *all* irreducible projective representations of G can be derived from the ordinary representations of G^*. In particular, G is an index group if and only if there exists an irreducible ordinary representation of G^* of degree n.

The first method has the advantage that the test is rather quick, but has the drawback that many groups have to be tested. Using this method we were able to determine all nonabelian index groups G of order n^2 with the exception of the case $n = 8$. There exist 10 494 213 groups of order $n^3 = 8^3 = 512$ so that it is not feasible to search for abstract error groups of this size. In contrast, the second method has to search only 256 potential nonabelian index groups of order $n^2 = 64$. The calculation of the Schur representation group and its ordinary representations of each candidate showed that there are

39 nonabelian index groups of order 8^2. The number of nonabelian index groups of order n^2 are summarized in Table I for small values of n. The corresponding index groups are tabulated in the Appendix.

TABLE I

THIS TABLE LISTS THE NUMBER OF NONABELIAN INDEX GROUPS (NAIGS) OF NICE ERROR BASES OF DEGREE n

FOR ALL $n < 12$.

Degree n	1	2	3	4	5	6	7	8	9	10	11
# NAIGs	0	0	0	2	0	2	0	39	3	1	0

V. EXAMPLE

We construct in this section a family of abstract error groups H_n. The example illustrates how to decide whether a group is an abstract error group or not. Let $H_n = \langle \tau, \alpha \rangle$ be the group generated by composition of the maps

$$\tau = (x \mapsto x + 1 \bmod 2^n), \qquad \alpha = (x \mapsto 5x \bmod 2^n).$$

Notice that $H_n = A \rtimes B$ is a semidirect product of the groups $A = \langle \tau \rangle$ and $B = \langle \alpha \rangle$, that is, A is a normal subgroup of H_n and the factor group H_n/A is isomorphic to B.

Theorem 5: The group H_n is an abstract error group of order 2^{2n-2}. The index group $H_n/Z(H_n)$ is nonabelian provided that $n \geq 5$.

Proof: We show in the first part of the proof that the center $Z = Z(H_n)$ of H_n is cyclic of order 4. Then we show that H_n is a group of central type. Here we use the criterion by Pahlings in the following form: the conjugacy class of any noncentral element contains two distinct elements, which are in the same coset modulo the center $Z(H_n)$.

1. The elements of the group H_n can be written in the form

$$H_n = \{\tau^k \alpha^l \mid 0 \leq k < 2^n, 0 \leq l < 2^{n-2}\},$$

since the relation $\alpha\tau = \tau^5\alpha$ holds.

2. An element $\tau^k \alpha^l$ of H_n is in the center if and only if it coincides with all its conjugates. Comparing

$$\tau^{-1}\tau^k \alpha^l \tau = (x \mapsto 5^l x + 5^l + k - 1 \bmod 2^n)$$

with $\tau^k \alpha^l = (x \mapsto 5^l x + k \bmod 2^n)$ shows that $5^l \equiv 1 \bmod 2^n$ needs to holds for elements in Z. This implies $l = 0$, and thus all elements of Z are in A. Notice that

$$\alpha \tau^k \alpha^{-1} = (x \mapsto x + 5k \bmod 2^n)$$

is equal to $\tau^k = (x \mapsto x + k \bmod 2^n)$ if and only if $4k \equiv 0 \bmod 2^n$ holds. Therefore, the center is given by $Z = \langle \tau^{2^{n-2}} \rangle$, a cyclic group of order 4.

3. Consider an element of the form

$$a = \tau^k \alpha^l = (x \mapsto 5^l x + k \bmod 2^n)$$

with $1 \leq l < 2^{n-2}$. Conjugation with τ^m yields

$$b = \tau^{-m}(\tau^k \alpha^l)\tau^m = (x \mapsto 5^l x + (5^l - 1)m + k \bmod 2^n).$$

Choose the smallest m with $0 < m \leq 2^{n-3}$ such that

$$(5^l - 1)m \equiv 0 \bmod 2^{n-2}$$

holds. Then $a \equiv b \bmod Z$, but $a \neq b$.

4. Consider an element of the form

$$a = \tau^k = (x \mapsto x + k \bmod 2^n).$$

Conjugation with α^m yields

$$\alpha^{-m}\tau^k \alpha^m = (x \mapsto x + 5^m k \bmod 2^n).$$

Recall that the subgroup generated by 5 in the group of units $(\mathbf{Z}/2^n\mathbf{Z})^\times$ is of the form $\langle 5 \rangle = \{1 + 4\mathbf{Z} \bmod 2^n\}$, see [19, p. 72]. Write k in the form $k = k_2 k_2'$, where k_2 is a power of 2 and k_2' is not divisible by 2. Consequently, the conjugacy class $\mathrm{Cl}(a)$ of a is given by

$$\mathrm{Cl}(a) = \{x + r \mid r \in k + 4k_2\mathbf{Z} \bmod 2^n\},$$

since $x \mapsto k_2' x \bmod 2^n$ is a permutation of the set $4\mathbf{Z}/2^n\mathbf{Z}$.

If $|\mathrm{Cl}(a)| > 1$, then the elements τ^k and τ^ℓ, with $\ell = k + 2^{n-1} \bmod 2^n$, are both in $\mathrm{Cl}(a)$, and $\tau^k \equiv \tau^\ell \bmod Z$.

The last two steps showed that in the conjugacy class of a noncentral element there are always two distinct elements which are congruent modulo Z. Therefore, H_n is of central type.

5. The factor group H_n/Z is nonabelian for $n \geq 5$, since the maps

$$\tau\alpha = (x \mapsto 5x + 1 \bmod 2^n) \quad \text{and} \quad \alpha\tau = (x \mapsto 5x + 5 \bmod 2^n)$$

are not equivalent modulo Z, hence $\alpha Z \cdot \tau Z \neq \tau Z \cdot \alpha Z$. $\quad \square$

In some sense the groups H_n are getting more and more complicated for larger n. Indeed, it is not difficult to show that the group H_n is nilpotent of class $\lfloor (n+1)/2 \rfloor$ for $n \geq 5$. On the other hand, the construction of an error basis for the index group $H_n/Z(H_n)$ is rather simple.

For a semidirect product $A \rtimes B$ with abelian normal subgroup A it is easy to write down the representations with the help of the method of little groups by Mackey and Wigner [20, p. 146]. Let $\phi \colon \mathbf{Z}/2^n\mathbf{Z} \to \mathbf{C}$ be the function defined by $\phi(x) = \exp(2\pi i\, 5^x/2^n)$, where $i^2 = -1$. Then the diagonal matrix $\rho(\tau) = \mathrm{diag}(\phi(0), \phi(1), \ldots, \phi(2^{n-2} - 1))$ and the shift

$$\rho(\alpha) = \begin{pmatrix} 0 & 1 & & & \\ & \ddots & \ddots & & \\ & & & \ddots & 1 \\ 1 & & & & 0 \end{pmatrix}$$

define a faithful irreducible unitary ordinary representation of H_n of degree 2^{n-2}. We obtain a nice error basis with nonabelian index group by

$$\mathcal{E} = \{\, \rho(\tau)^k \rho(\alpha)^\ell \mid 0 \leq k, \ell \leq 2^{n-2} - 1 \,\}.$$

VI. Conclusions

Nice error bases have found various applications in quantum computing. The teleportation of quantum states [21] is such an instance; the relation of teleportation schemes and unitary error bases is discussed in [22]. Unitary error bases are an essential tool in the construction of quantum error control codes. The knowledge of the group structure is of particular importance here, since Clifford theory provides a natural explanation of stabilizer codes and their higher-dimensional generalizations [1], [2], [5], [7]. Nice error bases are also of interest in the theory of noiseless subsystems [23], [24].

Appendix

List of Nonabelian Index Groups

We list in this appendix all nonabelian index groups of nice error bases which are of order 121 or less. The first column of Table II denotes the degree n of the faithful projective representation of

the index group G. This index group G is then of order n^2. The second column gives the number of the group in the Neubüser catalogue used in MAGMA and GAP, cf. [25], [18]. For instance, `G := SmallGroup(100,15)` gives the nonabelian index group of degree 10 in GAP. The isomorphism type of the group is tabulated in the third column.

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TABLE II

LIST OF ALL NONABELIAN INDEX GROUPS OF NICE ERROR BASES OF DEGREE LESS THAN 12.

Degree	Catalog No.	Isomorphism type
1	—	—
2	—	—
3	—	—
4	3	$Z_4 \ltimes (Z_2 \times Z_2)$
	11	$Z_2 \times D_8$
5	—	—
6	11	$Z_3 \times A_4$
	13	$Z_2 \ltimes (Z_3 \times Z_6)$
7	—	—
8	3	$Z_8 \ltimes Z_8$
	8	$Z_4 \ltimes (Z_2 \times D_8)$
	10	$Z_2 \ltimes (Z_4 \ltimes Z_8)$
	34	$Z_4 \ltimes (Z_4 \times Z_4)$
	35	$Z_4 \ltimes (Z_4 \times Z_4)$
	58	$Z_4 \times (Z_2 \ltimes (Z_2 \times Z_4))$
	60	$Z_4 \ltimes (Z_4 \times (Z_2 \times Z_2 \times Z_2 \times Z_2))$
	62	$Z_2 \ltimes (Z_2 \times Z_2 \times Z_4)$
	67	$Z_2 \ltimes (Z_2 \times Z_2 \times Z_2 \times Z_4)$
	68	$Z_4 \ltimes (Z_4 \ltimes Z_4)$
	69	$Z_2 \ltimes (Z_2 \times Z_4 \times Z_4)$
	71	$Z_2 \ltimes (Z_2 \times Z_4 \times Z_4)$
	72	$Z_4 \ltimes (Z_2 \times Q_8)$
	73	$Z_2 \ltimes (Z_2 \times Z_2 \times D_8)$
	74	$Q_8 \ltimes (Z_2 \times Z_2 \times Z_2)$
	75	$D_8 \ltimes (Z_2 \times Z_4)$
	77	$Z_2 \ltimes (Z_2 \times (Z_4 \ltimes Z_4))$
	78	$Z_2 \ltimes (Z_2 \times (Z_4 \ltimes Z_4))$
	82	not a split extension
	90	$Z_4 \ltimes (Z_2 \times Z_2 \times Z_4)$
	91	$Z_4 \ltimes (Z_2 \times Z_2 \times Z_4)$
	123	$Z_2 \ltimes (Z_4 \times D_8)$
	128	$Z_2 \ltimes (Z_2 \times D_{16})$
	138	$Z_2 \ltimes (D_8 Y D_8)$
	162	$Z_2 \ltimes (Z_2 \times (Z_4 \ltimes Z_4))$
	167	$Z_2 \ltimes (Z_4 \times Z_8)$
	174	$Z_2 \ltimes (Z_4 \times Z_8)$
	179	$Q_8 \ltimes Z_8$
	193	$Z_2 \ltimes (Z_2 \times Z_2 \times Z_2 \times Z_4)$
	195	$Z_2 \times (Z_2 \ltimes (Z_4 \times Z_4))$
	202	$Z_2 \times (Z_2 \ltimes (Z_2 \times Z_2 \times Z_2 \times Z_2))$
	203	$Z_2 \times (Z_2 \ltimes (Z_2 \times Z_2 \times Z_4))$
	207	$Z_2 \times (Z_2 \times Z_4 \times Z_4)$
	211	$Z_2 \times (Z_2 \times Z_4 \times Z_4)$
	216	$Z_2 \ltimes (Z_2 \times Z_2 \times D_8)$
	226	$D_8 \times D_8$
	236	$Z_2 \ltimes (Z_4 \times D_8)$
	242	$Z_2 \ltimes (Z_2 \times D_{16})$
	261	$Z_2 \times Z_2 \times Z_2 \times D_8$
9	4	$Z_9 \ltimes Z_9$
	9	$Z_3 \ltimes (Z_3 \times Z_9)$
	12	$Z_3 \times (Z_3 \ltimes (Z_3 \times Z_3))$
10	15	$Z_2 \times (Z_2 \ltimes (Z_5 \times Z_5))$
11	—	—