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BI-INFINITARY CODES (*)

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Abstract. – *The notion of bi-infinitary codes is introduced. For this purpose, the monoid* $^{\infty}A^{\infty}$ *of finite, infinite and bi-infinite words over an alphabet A is defined. A necessary and sufficient condition for a set of words to be a bi-infinitary code is formulated. Conditions for a submonoid of* $^{\infty}A^{\infty}$ *to have a minimal generator set are established. Using a specific kind of Thue system, the notion of bi-quasi free sub-monoids is introduced. An "algebraic" characterization of the submonoids generated by bi-infinitary codes is obtained. Finally, a "combinatorial" characterization of bi-quasi free submonoids is studied.*

Résumé. – *On introduit la notion de code biinfini. On définit d'abord le monoïde* $^{\infty}A^{\infty}$ *des mots finis, infinis ou biinfinis sur un alphabet A. On énonce une condition nécessaire et suffisante pour qu'un ensemble de mots soit un code biinfini. On donne également des conditions pour qu'un sous-monoïde de* $^{\infty}A^{\infty}$ *ait un ensemble minimal de générateurs. En utilisant un système de Thue spécifique, on introduit la notion de sous-monoïde bi-quasi libre. Une caractérisation « algébrique » des sous-monoïdes engendrés par des codes bi-infinis est alors obtenue. Finalement, on étudie une caractérisation « combinatoire » des sous-monoïdes bi-quasi libres.*

INTRODUCTION

There has been a systematic study of codes consisting of finite words, initiated by M. P. Schützenberger [16] and developed by many others taking motivation from information theory (*see* [11-13]).

Recently, infinitary languages consisting of finite and infinite words have served as an adequate tool for studying behaviours of processes. This is the approach of M. Nivat and A. Arnold [14] in some problems of synchronization which stimulated the study of infinite words including bi-infinite words [15].

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Motivated by the theory of codes [1] and the theory of infinitary languages, the notion of infinitary codes has been introduced and examined in [3-10].

This paper is devoted to a study of bi-infinitary codes which are a natural generalization of infinitary codes to bi-infinitary languages *i. e.*, languages of finite, left-infinite, right-infinite and bi-infinite words.

SECTION 1

MONOID ${}^{\infty}A^{\infty}$ AND BI-INFINITARY CODES

Let A be an alphabet. We denote by A^*, the free monoid generated by A. Elements of A^* are called finite words. The length of a word x in A^* is denoted by $|x|$, the empty word by ε and $A^+ = A^* - \{\varepsilon\}$.

We denote by A^N, the set of all right-infinite words, by A^{-N}, the set of all left-infinite words and by A^Z, the set of all bi-infinite words over A. Every (bi) infinite word u has a countable length $|u| = \omega$. For any $X \subseteq A^*$, we denote by X^ω (${}^\omega X$, ${}^\omega X^\omega$), the set of all right-infinite (left-infinite, bi-infinite) words of the form $x_1 x_2 \ldots$ ($\ldots x_2 x_1$, $\ldots x_1 x_2 x_3 \ldots$) for $x_i \in X$. In particular, if $x \in A^*$, then $x^\omega = xxx\ldots$, ${}^\omega x = \ldots xxx$ and ${}^\omega x^\omega = \ldots xxx\ldots$. We write $A^\infty = A^* \cup A^N$, ${}^\infty A = A^* \cup A^{-N}$ and ${}^\infty A^\infty = A^* \cup A^N \cup A^{-N} \cup A^Z$.

We define a product on elements of ${}^\infty A^\infty$ as follows:

$$\alpha . \beta = \begin{cases} \alpha, & \text{if} \quad \alpha \in A^N \cup A^Z \\ \alpha\beta, & \text{if} \quad \alpha \in A^* \cup A^{-N}, \quad \beta \in A^* \cup A^N \\ \beta, & \text{if} \quad \alpha \in A^* \cup A^{-N}, \quad \beta \in A^{-N} \cup A^Z. \end{cases}$$

It is not difficult to verify that the product is associative and therefore ${}^\infty A^\infty$ is a monoid. This monoid has A^*, A^∞ and ${}^\infty A$ as its submonoids. For simplicity, instead of $\alpha . \beta$, we write $\alpha\beta$. For any $X \subseteq {}^\infty A^\infty$, we denote by X^*, the submonoid of ${}^\infty A^\infty$ generated by X and write $X^+ = X^* - \{\varepsilon\}$. If α is a word, instead of $\{\alpha\}^*$, we write α^*.

For any $X \subseteq {}^\infty A^\infty$, we write $X_{\text{fin}} = X \cap A^*$,

$$X_{\text{inf}} = X \cap A^N, \qquad X_{-\text{inf}} = X \cap A^{-N}, \qquad X_{\text{biinf}} = X \cap A^Z,$$

$$X^\infty = X_{\text{fin}} \cup X_{\text{inf}}, \qquad {}^\infty X = X_{\text{fin}} \cup X_{-\text{inf}},$$

$$\bar{X}^{(0)} = X^{(0)} = \{\varepsilon\},$$

$$\bar{X}^{(1)} = X^{(1)} = X,$$

$$X^{(\vec{n})} = \{(x_1, x_2, \ldots, x_n)/x_1, x_2, \ldots, x_{n-1} \in X_{\text{fin}}, x_n \in X^\infty\} \qquad \text{for} \quad n \geq 2,$$

$$X^{(\overleftarrow{n})}=\{(x_1, x_2, \ldots, x_n)/x_1 \in {}^{\infty}X, x_2, x_3, \ldots, x_n \in X_{\text{fin}}\} \quad \text{for} \quad n \geqq 2,$$

$$X^{(\overleftrightarrow{n})}=\{(x_1, x_2, \ldots, x_n)/$$
$$x_1 \in X_{-\text{inf}},\ x_n \in X_{\text{inf}},\ x_2, x_3, \ldots, x_{n-1} \in X_{\text{fin}}\} \quad \text{for} \quad n \geqq 2$$

$$X^{(n)}=X^{(\overrightarrow{n})} \cup X^{(\overleftarrow{n})} \cup X^{(\overleftrightarrow{n})} \quad \text{for} \quad n \geqq 2,$$

$$X^{(*)}=\bigcup_{n \geqq 0} X^{(n)}$$

$$\bar{X}^{(\overrightarrow{n})}=\{x_1 x_2 \ldots x_n/(x_1, x_2, \ldots, x_n) \in X^{(\overrightarrow{n})}\} \quad \text{for} \quad n \geqq 2$$

$$\bar{X}^{(\overleftarrow{n})}=\{x_1 x_2 \ldots x_n/(x_1, x_2, \ldots, x_n) \in X^{(\overleftarrow{n})}\} \quad \text{for} \quad n \geqq 2,$$

$$\bar{X}^{(\overleftrightarrow{n})}=\{x_1 x_2 \ldots x_n/(x_1, x_2, \ldots, x_n) \in X^{(\overleftrightarrow{n})}\} \quad \text{for} \quad n \geqq 2$$

and

$$\bar{X}^{(n)}=\bar{X}^{(\overrightarrow{n})} \cup \bar{X}^{(\overleftarrow{n})} \cup \bar{X}^{(\overleftrightarrow{n})} \quad \text{for} \quad n \geqq 2.$$

We say that a word $\alpha \in {}^{\infty}A^{\infty}$ has a factorization on elements of X if $\alpha = x_1 x_2 \ldots x_n$ for some $(x_1, x_2, \ldots, x_n) \in X^{(*)}$.

DEFINITION 1.1: A subset X of ${}^{\infty}A^{\infty}$ is called a bi-infinitary code if every word $\alpha \in {}^{\infty}A^{\infty}$ has atmost one factorization on elements of X. More precisely, X is a bi-infinitary code if for any n, $m \geqq 1$ and for any $(x_1, x_2, \ldots, x_n) \in X^{(n)}$, $(x'_1, x'_2, \ldots, x'_m) \in X^{(m)}$, the equality $x_1 x_2 \ldots x_n = x'_1 x'_2 \ldots x'_m$ implies $n=m$ and $x_i = x'_i\ (i=1, 2, \ldots, n)$.

Unless otherwise stated, from now on code means bi-infinitary code.

Example 1.1 : If $A=\{a, b\}$, the subset

$$X=\{{}^{\omega}(ab)^{\omega}, {}^{\omega}a, b^{\omega}, ba\}$$

is a code whereas the subset

$$Y=\{{}^{\omega}(ab)^{\omega}, {}^{\omega}a, b^{\omega}, ab\}$$

is not a code, since we have,

$$\begin{aligned} {}^{\omega}ab^{\omega} &= {}^{\omega}a \,.\, ab \,.\, b^{\omega} \\ &= {}^{\omega}a \,.\, b^{\omega}. \end{aligned}$$

SECTION 2

A CHARACTERIZATION OF BI-INFINITARY CODES

In this section, we establish a characterization of codes. We first introduce certain concepts and formulate a fundamental formula.

Let X and Y be two subsets of ${}^{\infty}A^{\infty}$. Define the sets

$$Y^{-1}X = \{\alpha \in {}^{\infty}A^{\infty} \mid \exists\, \beta \in Y : \beta\alpha \in X, \\ (\beta \in Y_{\text{inf}} \cup Y_{\text{biinf}} \Rightarrow \alpha = \varepsilon), \\ (\beta \in {}^{\infty}Y \text{ and } \alpha \in A^{-N} \cup A^{Z} \Rightarrow \beta = \varepsilon)\},$$

$$XY^{-1} = \{\alpha \in {}^{\infty}A^{\infty} \mid \exists\, \beta \in Y : \alpha\beta \in X, (\alpha \in A^{N} \cup A^{Z} \Rightarrow \beta = \varepsilon), \\ (\alpha \in {}^{\infty}A \text{ and } \beta \in Y_{-\text{inf}} \cup Y_{\text{biinf}} \Rightarrow \alpha = \varepsilon)\}.$$

We note that if $u, v \in A^{-N}$ and $u \leqq v$, then $u^{-1}v$ is a subset of A^*. For example, if $u = {}^{\omega}a$ and $v = {}^{\omega}a = {}^{\omega}a . a^*$, then $u^{-1}v = a^*$.

We associate with every subset $X \subseteq {}^{\infty}A^{\infty}$, a sequence of subsets, denoted by $U_n(X)$ or simply by U_n, defined recursively by

$$U_1 = X^{-1}X - \{\varepsilon\}$$
$$U_{n+1} = X^{-1}U_n \cup U_n^{-1}X, \qquad n \geqq 1.$$

LEMMA 2.1: *For any subset X of ${}^{\infty}A^{\infty} - \{\varepsilon\}$, (i) if n is the smallest natural number such that $\varepsilon \in U_n$, then $\forall\, k \in \{1, 2, \ldots, n\}$, $\exists\, u \in U_k$, $\exists\, i, j \geqq 0$:*

$$u(\bar{X}^{(i)} - (A^{-N} \cup A^{Z})) \cap \bar{X}^{(j)} \neq \Phi, \qquad i + j + k = n, \\ u \in A^{N} \cup A^{Z} \Rightarrow i = 0 \tag{2.1}$$

(ii) $\forall\, n \geqq 1$, $\forall\, k \in \{1, 2, \ldots, n\}$:

$$(\exists\, u \in U_k, \exists\, i, j \geqq 0 : u(\bar{X}^{(i)} - (A^{-N} \cup A^{Z})) \cap \bar{X}^{(j)} \neq \Phi, \\ i + j + k = n, \\ u \in A^{N} \cup A^{Z} \Rightarrow i = 0) \Rightarrow \varepsilon \in U_n.$$

Proof: We prove by recurrence on k.

(i) Let n be the smallest natural number such that $\varepsilon \in U_n$. If $k = n$, then (2.1) holds obviously with $u = \varepsilon$, $i = j = 0$. Let $n > k \geqq 1$ and suppose the statement is true for $n, n-1, \ldots, k+1$. We prove for k. Since the statement is true for $k+1$, there exist $v \in U_{k+1}$ and integers i', j' such that

$$v(\bar{X}^{(i')} - (A^{-N} \cup A^{Z})) \cap \bar{X}^{(j')} \neq \Phi, \qquad i'+j'+k+1=n,$$

$v \in A^{N} \cup A^{Z} \Rightarrow i'=0$. Thus we have $x \in \bar{X}^{(i')} - (A^{-N} \cup A^{Z})$ and $y \in \bar{X}^{(j')}$ such that $vx=y$. The fact that $v \in U_{k+1}$ gives rise to two cases.

Case (*a*): $v \in X^{-1} U_k$. Then, there exists $z \in X$, $u \in U_k$ such that

$$zv=u, (z \in X_{\text{inf}} \cup X_{\text{biinf}} \Rightarrow v=\varepsilon)$$

and

$$(z \in {}^{\infty}X \text{ and } v \in A^{-N} \cup A^{Z} \Rightarrow z=\varepsilon).$$

If $v \in A^{N}$, then $i'=0$, $x=\varepsilon$, $z \in {}^{\infty}X$ and $u=zy$. Hence $u(\bar{X}^{(0)} - (A^{-N} \cup A^{Z})) \cap \bar{X}^{(j'+1)} \neq \Phi$. Thus (2.1) holds with $i=0, j=j'+1$.

If $v \in A^{-N} \cup A^{Z}$, then $z \in {}^{\infty}X$ and so $z=\varepsilon$. Thus $\varepsilon \in X$ which contradicts the hypothesis that $X \subseteqq {}^{\infty}A^{\infty} - \{\varepsilon\}$.

If $v \in A^{*}$ and $z \in X_{\text{inf}} \cup X_{\text{biinf}}$, then $v=\varepsilon$ and $u=z$. Hence $u(\bar{X}^{(0)} - (A^{-N} \cup A^{Z})) \cap \bar{X}^{(1)} \neq \Phi$ and therefore (2.1) holds with $i=0, j=1$.

If $v \in A^{*}$ and $z \in {}^{\infty}X$, then $ux=zy$ and so

$$u(\bar{X}^{(i')} - (A^{-N} \cup A^{Z})) \cap \bar{X}^{(j'+1)} \neq \Phi.$$

Thus (2.1) holds with $i=i', j=j'+1$.

Case (*b*): $v \in U_k^{-1} X$. Then, there exist $u \in U_k$ and $z \in X$ such that $uv=z$, $(u \in A^{N} \cup A^{Z} \Rightarrow v=\varepsilon)$ and $(u \in {}^{\infty}A,\ v \in A^{-N} \cup A^{Z} \Rightarrow u=\varepsilon)$.

If $v \in A^{N}$, then $i'=0$, $x=\varepsilon$, $v=y$, $u \in {}^{\infty}A$ and $uy=z$. Hence $u(\bar{X}^{(j')} - (A^{-N} \cup A^{Z})) \cap \bar{X}^{(1)} \neq \Phi$. So, (2.1) holds with $i=j', j=1$.

If $v \in A^{-N} \cup A^{Z}$, then $u \in {}^{\infty}A$ and therefore $u=\varepsilon$. Thus $\varepsilon = u \in U_k$ with $k<n$, which is contrary to the hypothesis that n is the smallest natural number such that $\varepsilon \in U_n$.

If $v \in A^{*}$ and $z \in X_{\text{inf}} \cup X_{\text{biinf}}$, then $v=\varepsilon$, $u=z$ and $y=x$. If $i'=j'=0$, then $k+1=n$ and the equality $u=z$ implies $u(\bar{X}^{(0)} - (A^{-N} \cup A^{Z})) \cap \bar{X}^{(1)} \neq \Phi$. That is, (2.1) holds with $i=0, j=1$. Otherwise we have $k+1<n$ and $v=\varepsilon \in U_{k+1}$ which gives a contradiction.

If $v \in A^{*}$ and $z \in {}^{\infty}X$ then $u \in {}^{\infty}A$. The equation $uy=zx$ gives $u(\bar{X}^{(j')} - (A^{-N} \cup A^{Z})) \cap \bar{X}^{(i'+1)} \neq \Phi$. Thus (2.1) holds with $i=j', j=i'+1$.

(ii) Suppose there exist $u \in U_k$ and two integers $i, j \geqq 0$ such that $u(\bar{X}^{(i)} - (A^{-N} \cup A^{Z})) \cap \bar{X}^{(j)} \neq \Phi$, $i+j+k=n$, $u \in A^{N} \cup A^{Z} \Rightarrow i=0$. We have to prove that $\varepsilon \in U_n$. If $k=n$, then $i=j=0$ and so $u=\varepsilon$. Hence $\varepsilon \in U_n$. Let now

$n>k\geqq 1$ and suppose the statement is true for n, $n-1, \ldots, k+1$. We prove for k. Suppose $x_1 x_2 \ldots x_i \in \bar{X}^{(i)}-(A^{-N} \cup A^Z)$ and $x'_1 x'_2 \ldots x'_j \in \bar{X}^{(j)}$ such that $ux_1 x_2 \ldots x_i = x'_1 x'_2 \ldots x'_j$. We discuss the following cases:

Case (a): Suppose $u \in A^N \cup A^Z$. Then $i=0$, $j+k=n$, $j\geqq 1$ and $u=x'_1 x'_2 \ldots x'_j$. Let $u'=x'_2 x'_3 \ldots x'_j$. Clearly $u' \in U_{k+1}$ and $u'(\bar{X}^{(0)}-(A^{-N} \cup A^Z)) \cap \bar{X}^{(j-1)} \neq \Phi$, $0+j-1+k+1=n$. By recurrence hypothesis $\varepsilon \in U_n$.

Case (b): Suppose $u \in A^*$. If $j=0$, then $i=0$, $u=\varepsilon$ and $k=n$. Thus we have $\varepsilon \in U_n$. Let $j\geqq 1$. If $|u| \geqq |x'_1|$, that is, $u=x'_1 u'$ for some u', then $u' \in U_{k+1}$ and

$$u' x_1 x_2 \ldots x_i = x'_2 x'_3 \ldots x'_j.$$

So $u'(\bar{X}^{(i)}-(A^{-N} \cup A^Z)) \cap \bar{X}^{(j-1)} \neq \Phi$, $i+j-1+k+1=n$. By recurrence hypothesis, $\varepsilon \in U_n$. If $|u| < |x'_1|$, that is, $x'_1 = uu''$ for some u'', then $u'' \in U_{k+1}$ and $u'' x'_2 x'_3 \ldots x'_j = x_1 x_2 \ldots x_i$. Hence

$$u''(\bar{X}^{(j-1)}-(A^{-N} \cup A^Z)) \cap \bar{X}^{(i)} \neq \Phi, \qquad j-1+i+k+1=n.$$

This implies $\varepsilon \in U_n$.

Case (c): Suppose $u \in A^{-N}$. Then $j\geqq 1$. If $j=1$, then $ux_1 x_2 \ldots x_i = x'_1$ which implies $x_1 x_2 \ldots x_i \in u^{-1} x'_1$. Let $u'=x_1 x_2 \ldots x_i$. We have $u' \in U_{k+1}$ and

$$u'(\bar{X}^{(0)}-(A^{-N} \cup A^Z)) \cap \bar{X}^{(i)} \neq \Phi, \qquad 0+i+k+1=n.$$

By recurrence hypothesis $\varepsilon \in U_n$. If $j>1$, there are two subcases.

If u is a left factor of x'_1, we have

$$x_1 x_2 \ldots x_i = u' x'_2 x'_3 \ldots x'_j$$

with $u' \in u^{-1} x'_1$. So, we have $u' \in U_{k+1}$ and

$$u'(\bar{X}^{(j-1)}-(A^{-N} \cup A^Z)) \cap \bar{X}^{(i)} \neq \Phi, \qquad j-1+i+k+1=n.$$

By recurrence hypothesis $\varepsilon \in U_n$.

If x'_1 is a left factor of u, we have

$$x'_2 x'_3 \ldots x'_j = u'' x_1 x_2 \ldots x_i$$

with $u'' \in (x'_1)^{-1} u$. Then $u'' \in U_{k+1}$ and

$$u''(\bar{X}^{(i)}-(A^{-N} \cup A^Z)) \cap \bar{X}^{(j-1)} \neq \Phi, \qquad i+j-1+k+1=n.$$

By recurrence hypothesis $\varepsilon \in U_n$. This proves lemma 2.1.

We are now in a position to formulate the main result of this section which is a generalization of the result proved by Do Long Van in [5, 10]. The latter is a generalization of Sardinas-Patterson theorem. This in many cases gives us a procedure to check whether or not a given set is a bi-infinitary code.

THEOREM 2.1: *A subset X of* $^{\infty}A^{\infty}-\{\varepsilon\}$ *is a code iff for all* $n \geqq 1$, $U_n(X)$ *does not contain the empty word* ε.

Proof: Suppose $\varepsilon \notin U_n(X)$, $n \geqq 1$. Assume that X is not a code. Then there exists a word $\alpha \in {}^{\infty}A^{\infty}$ having two different factorizations on elements of X:

$$\alpha = x_1 x_2 \ldots x_i = x'_1 x'_2 \ldots x'_j \qquad \text{where} \quad (x_1, x_2, \ldots, x_i) \in X^{(i)}$$

and $(x'_1, x'_2, \ldots, x'_j) \in X^{(j)}$.

Case (*a*): Suppose $\alpha \in A^* \cup A^N$. We may assume that $x_1 \neq x'_1$ and $|x_1| > |x'_1|$. Let $x_1 = x'_1 u$ for some $u \neq \varepsilon$. Clearly $u \in U_1$.

If $x_1 \in X_{\text{fin}}$, then $x'_1 \in X_{\text{fin}}$ and $u \in A^+$. So we have

$$u x_2 x_3 \ldots x_i = x'_2 x'_3 \ldots x'_j, \qquad j \geqq 2.$$

Hence

$$u(\bar{X}^{(i-1)} - (A^{-N} \cup A^Z)) \cap \bar{X}^{(j-1)} \neq \Phi.$$

By lemma 2.1 (ii), $\varepsilon \in U_{i+j-1}$ which is a contradiction.

If $x_1 \in X_{\text{inf}}$, then $i=1$, $x'_1 \in X_{\text{fin}}$ and $u \in A^N$. Therefore we have $u = x'_2 x'_3 \ldots x'_j$, j $\geqq 2$. This implies

$$u(\bar{X}^{(0)} - (A^{-N} \cup A^Z)) \cap \bar{X}^{(j-1)} \neq \Phi.$$

Again by lemma 2.1 (ii), $\varepsilon \in U_j$ which is a contradiction.

Case (*b*): Suppose $\alpha \in A^{-N}$. Clearly x_1, $x'_1 \in X_{-\text{inf}}$. Since the case $i=j=1$ is impossible, we may assume that $i \geqq 2$. There are two possibilities.

(i) If $x_1 \neq x'_1$ we can assume that $x_1 = x'_1 u$ with $u \in A^+$ such that $u x_2 x_3 \ldots x_i = x'_2 x'_3 \ldots x'_j$, $j \geqq 2$. Then clearly $u \in U_1$ and

$$u(\bar{X}^{(i-1)} - (A^{-N} \cup A^Z)) \cap \bar{X}^{(j-1)} \neq \Phi.$$

Again by lemma 2.1 (ii), $\varepsilon \in U_{i+j-1}$. This is a contradiction.

(ii) Suppose $x_1 = x'_1$. Here, if $j=1$, then $x_1 x_2 \ldots x_i = x'_1$ and so $x_1 = {}^{\omega}(x_2 x_3 \ldots x_i)$. Let $x_2 x_3 \ldots x_i = u$. Clearly $u \in x_1^{-1} x'_1 \subseteqq U_1$. Hence $u(\bar{X}^{(0)} - (A^{-N} \cup A^Z)) \cap \bar{X}^{(i-1)} \neq \Phi$. This implies $\varepsilon \in U_i$ which is a contradiction.

If $j \geqq 2$, then $x_1 x_2 \ldots x_i = x'_1 x'_2 \ldots x'_j$ with

$$x_2 x_3 \ldots x_i,\ x'_2 x'_3 \ldots x'_j \in A^*.$$

If $x_2 x_3 \ldots x_i = x'_2 x'_3 \ldots x'_j$, we may assume that $x_2 \neq x'_2$, and as in case (*a*), get a contradiction. If $x_2 x_3 \ldots x_i \neq x'_2 x'_3 \ldots x'_j$, then we have either $x_1 = x'_1 u$ and $u x_2 x_3 \ldots x_i = x'_2 x'_3 \ldots x'_j$ or $x'_1 = x_1 u$ and $x_2 x_3 \ldots x_i = u x'_2 x'_3 \ldots x'_j$ for some $u \in A^+$. By symmetry, we shall discuss one of the two possibilities.

Consider $x_1 = x'_1 u$ and $u x_2 x_3 \ldots x_i = x'_2 x'_3 \ldots x'_j$. Now $x_1 = x'_1$ and $x_1 = x'_1 u$ imply $x_1 = x'_1 = {}^{\omega}u$ and $u \in (x'_1)^{-1} x_1 \subseteqq U_1$. Thus $u x_2 x_3 \ldots x_i = x'_2 x'_3 \ldots x'_j$ gives $u(\bar{X}^{(i-1)} - (A^{-N} \cup A^Z)) \cap \bar{X}^{(j-1)} \neq \Phi$ and so $\varepsilon \in U_{i+j-1}$ which is a contradiction.

Case (*c*): Suppose $\alpha \in A^Z$. The case $i = j = 1$ is impossible. We assume $j \geqq 2$. If $i = 1$ then $x_1 = x'_1 x'_2 \ldots x'_j$ and so we have $u = (x'_1)^{-1} x_1 \in U_1$ with $u = x'_2 x'_3 \ldots x'_j$. Hence $u(\bar{X}^{(0)} - (A^{-N} \cup A^Z)) \cap \bar{X}^{(j-1)} \neq \Phi$ which gives a contradiction.

If $i \geqq 2$, then $x_1 x_2 \ldots x_i = x'_1 x'_2 \ldots x'_j$. Now $x_1, x'_1 \in X_{-\text{inf}}$. There are two possibilities.

(i) If $x_1 \neq x'_1$, as in case (*a*), we obtain a contradiction.

(ii) If $x_1 = x'_1$, then we have either

$$x_2 x_3 \ldots x_i = x'_2 x'_3 \ldots x'_j \quad \text{or} \quad x_2 x_3 \ldots x_i \neq x'_2 x'_3 \ldots x'_j.$$

If $x_2 x_3 \ldots x_i = x'_2 x'_3 \ldots x'_j$, then we can assume $x_2 \neq x'_2$ and as in case (*a*), get a contradiction since $x_2 x_3 \ldots x_i$, $x'_2 x'_3 \ldots x'_j \in A^N$. If $x_2 x_3 \ldots x_i \neq x'_2 x'_3 \ldots x'_j$, we can obtain a contradiction as in the last part of Case *b* (ii). Thus X is a code.

We shall prove the converse. Suppose X is a code. Assume that there are some sets $U_i(X)$ containing ε. Let $U_n(X)$ be one among these, with the smallest index. By lemma 2.1 (i), there exists a word $u \in U_1$ with two integers $i, j \geqq 0$ such that

$$u(\bar{X}^{(i)} - (A^{-N} \cup A^Z)) \cap \bar{X}^{(j)} \neq \Phi, \qquad i + j + 1 = n,$$

$u \in A^N \cup A^Z \Rightarrow i = 0$. So, we have $u x_1 x_2 \ldots x_i = x'_1 x'_2 \ldots x'_j$ for some $x_1 x_2 \ldots x_i \in \bar{X}^{(i)} - (A^{-N} \cup A^Z)$ and $x'_1 x'_2 \ldots x'_j \in \bar{X}^{(j)}$. Since $u \in U_1$, there exist words $x, x' \in X$ with either $x \neq x'$ and $x = x' u$ or $x = x'$ and $x = x' u$.

If $u \in A^+$, then both x, x' are either in X_{fin} or in $X_{-\text{inf}}$. Let $x, x' \in X_{\text{fin}}$. Then we have $x \neq x'$ and $x = x' u$. So $x x_1 x_2 \ldots x_i = x' x'_1 \ldots x'_j$ and therefore

X is not a code, a contradiction. Let $x, x' \in X_{-\text{inf}}$. If $x \neq x'$ and $x = x' u$, then as before, we get a contradiction. If $x = x'$ and $x = x' u$, then $x = x' = {}^{\omega}u$ and either

$$x_1 x_2 \ldots x_i = x'_1 x'_2 \ldots x'_j \qquad \text{or} \qquad x_1 x_2 \ldots x_i \neq x'_1 x'_2 \ldots x'_j.$$

If $x_1 x_2 \ldots x_i = x'_1 x'_2 \ldots x'_j$, then $i = j$ and $x_k = x'_k$ $(k = 1, 2, \ldots, i)$ since X is a code. Then the equation $u x_1 x_2 \ldots x_i = x'_1 x'_2 \ldots x'_j$ implies $u = \varepsilon$, a contradiction. If $x_1 x_2 \ldots x_i \neq x'_1 x'_2 \ldots x'_j$ then the equation $x' x_1 x_2 \ldots x_i = x' x'_1 x'_2 \ldots x'_j$ shows that X is not a code, a contradiction.

If $u \in A^N$, then $i = 0$ and either $x \in X_{\text{inf}}$, $x' \in X_{\text{fin}}$ or $x \in X_{\text{biinf}}$, $x' \in X_{-\text{inf}}$. In both cases, we have $x = x' x'_1 \ldots x'_j$ which shows X is not a code, a contradiction.

If $u \in A^Z$, then $i = 0$, $x \in X_{\text{biinf}}$ and $x' = \varepsilon$. Since $x' \in X \subseteqq {}^{\infty}A^{\infty} - \{\varepsilon\}$, this case is not possible.

If $u \in A^{-N}$, then $x = u$ and $x' = \varepsilon$. As before, this case is also not possible. Thus $\varepsilon \notin U_n(X)$, $\forall n \geqq 1$.

Example 2.1: (i) Let $X = \{{}^{\omega}(ab)^{\omega}, {}^{\omega}a, b^{\omega}, ab\}$. $U_1(X) = \{a^+\}$, $U_2(X) = \{b\}$, $U_3(X) = \{b^{\omega}\}$ and $U_4(X) = \{\varepsilon\}$. So X is not a code.

(ii) Let $X = \{{}^{\omega}(ab)^{\omega}, {}^{\omega}a, b^{\omega}, ba\}$. $U_1(X) = \{a^+\}$, $U_2(X) = \Phi$. So, X is a code.

SECTION 3

MINIMAL GENERATOR SET OF A SUBMONOID OF ${}^{\infty}A^{\infty}$.

We recall that a generator set X of a monoid M is minimal if X is contained in any generator set of M. Such a set, if it exists, is unique and called the base of M, denoted as BASE(M). Every submonoid of A^* has a minimal generator set whereas there are submonoids of ${}^{\infty}A^{\infty}$ which have no minimal generator sets. We illustrate this in the following example.

Example 3.1: Let $A = \{a, b\}$ and let M be the submonoid of ${}^{\infty}A^{\infty}$ given by $M = \{\alpha \in {}^{\infty}A^{\infty} \mid |\alpha|_a = |\alpha|_b\}$ where $|\alpha|_a$ stands for the number of occurrences of a in α. This monoid has no minimal generator set.

DEFINITION 3.1: Let M be a submonoid of ${}^{\infty}A^{\infty}$ and u, v, two elements of M_{inf}. We say that u precedes v, denoted by $u \prec v$, if there exists $f \in M_{\text{fin}} - \varepsilon$ such that $u = fv$. An element $u \in M_{\text{inf}}$ is called stable if $\forall v \in M_{\text{inf}}$: $(u \prec v) \Rightarrow (u = v)$. The set of all stable elements of M_{inf} is denoted by STAB(M_{inf}).

Let x, y be two elements of $M_{-\text{inf}}$. Here also we say that x precedes y, denoted by $x \prec y$ if there exists $g \in M_{\text{fin}} - \varepsilon$ such that $x = yg$. As before, $x \in M_{-\text{inf}}$ is called stable if $\forall y \in M_{-\text{inf}} : (x \prec y) \Rightarrow (x = y)$. The set of all stable elements of $M_{-\text{inf}}$ is denoted by $\text{STAB}(M_{-\text{inf}})$.

We say that a submonoid M satisfies the stability condition if every unstable element of M_{inf} (resp. $M_{-\text{inf}}$) precedes a stable element of M_{inf} (resp. $M_{-\text{inf}}$). We introduce the following two sets:

$$\text{BASE}(M_{\text{fin}}) = (M_{\text{fin}} - \varepsilon) - (M_{\text{fin}} - \varepsilon)^2$$
$$\text{UNFAC}(M_{\text{biinf}}) = M_{\text{biinf}} - (M_{-\text{inf}} M_{\text{fin}} M_{\text{inf}}).$$

THEOREM 3.1: *A submonoid M of ${}^{\infty}A^{\infty}$ has a minimal generator set iff M satisfies the stability condition and in that case, the minimal generator set of M is*

$$X = \text{BASE}(M)$$
$$= \text{BASE}(M_{\text{fin}}) \cup \text{STAB}(M_{\text{inf}}) \cup \text{STAB}(M_{-\text{inf}}) \cup \text{UNFAC}(M_{\text{biinf}}).$$

Proof: Assume X satisfies the stability condition. Let

$$X_{\text{fin}} = \text{BASE}(M_{\text{fin}}), \qquad X_{\text{inf}} = \text{STAB}(M_{\text{inf}}), \qquad X_{-\text{inf}} = \text{STAB}(M_{-\text{inf}}),$$
$$X_{\text{biinf}} = \text{UNFAC}(M_{\text{biinf}}) \qquad \text{and} \qquad X = X_{\text{fin}} \cup X_{\text{inf}} \cup X_{-\text{inf}} \cup X_{\text{biinf}}.$$

Since

$$X_{\text{fin}}^* = M_{\text{fin}}, \; M_{\text{inf}} = \text{STAB}(M_{\text{inf}}) \cup (M_{\text{fin}} - \varepsilon)\,\text{STAB}(M_{\text{inf}})$$
$$= M_{\text{fin}}\,\text{STAB}(M_{\text{inf}}) = X_{\text{fin}}^* X_{\text{inf}}.$$

Similarly,

$$M_{-\text{inf}} = X_{-\text{inf}} X_{\text{fin}}^* \qquad \text{and} \qquad M_{\text{biinf}} = \text{UNFAC}(M_{\text{biinf}})$$
$$\cup M_{-\text{inf}} M_{\text{fin}} M_{\text{inf}} = X_{\text{biinf}} \cup X_{-\text{inf}} X_{\text{fin}}^* X_{\text{inf}}.$$

Therefore,

$$M = M_{\text{fin}} \cup M_{\text{inf}} \cup M_{-\text{inf}} \cup M_{\text{biinf}}$$
$$= X_{\text{fin}}^* \cup X_{\text{fin}}^* X_{\text{inf}} \cup X_{-\text{inf}} X_{\text{fin}}^* \cup X_{\text{biinf}}$$
$$\cup X_{-\text{inf}} X_{\text{fin}}^* X_{\text{inf}} = X^*.$$

Thus X is a generator set of M. We shall prove that X is minimal. Let Y be an arbitrary generator set of M. We can assume that $\varepsilon \notin Y$. It is enough if we prove that

$$X_{\text{fin}} \subseteqq Y_{\text{fin}}, \qquad X_{\text{inf}} \subseteqq Y_{\text{inf}},$$
$$X_{-\text{inf}} \subseteqq Y_{-\text{inf}} \quad \text{and} \quad X_{\text{biinf}} \subseteqq Y_{\text{biinf}}.$$

As $Y^*_{\text{fin}} = M_{\text{fin}}$ and X_{fin} is the minimal generator set of M_{fin}, we have $X_{\text{fin}} \subseteqq Y_{\text{fin}}$. Let $u \in X_{\text{inf}}$. Then $u = y_1 y_2 \ldots y_n$ for some $(y_1, y_2, \ldots, y_n) \in Y^{(n)}$, $n \geqq 1$. If $n = 1$, then $u = y_n \in Y_{\text{inf}}$. If $n > 1$, we have $u = f y_n$ with $f = y_1 y_2 \ldots y_{n-1} \in M_{\text{fin}} - \varepsilon$ *i.e.*, $u \prec y_n$. Since u is stable $u = y_n \in Y_{\text{inf}}$. Thus $X_{\text{inf}} \subseteqq Y_{\text{inf}}$. Similarly we can show that $X_{-\text{inf}} \subseteqq Y_{-\text{inf}}$. Let $u \in X_{\text{biinf}}$. Then $u = w_1 w_2 \ldots w_n$ for some $(w_1, w_2, \ldots, w_n) \in Y^{(n)}$, $n \geqq 1$. If $n = 1$, $u = w_1$ where $w_1 \in Y_{\text{biinf}}$. If $n \geqq 2$, $u = w_1 w_2 \ldots w_n$ is an element of $M_{-\text{inf}} M_{\text{fin}} M_{\text{inf}}$ since $w_1 \in Y_{-\text{inf}} Y^*_{\text{fin}} = M_{-\text{inf}}$, $w_n \in Y^*_{\text{fin}} Y_{\text{inf}} = M_{\text{inf}}$ and $w_2 w_3 \ldots w_{n-1} \in Y^*_{\text{fin}}$. This contradicts the choice of u since $u \in \text{UNFAC}(M_{\text{biinf}})$. Hence $u \in Y_{\text{biinf}}$ and so $X_{\text{biinf}} \subseteqq Y_{\text{biinf}}$.

We prove the converse part now. Let Y be a minimal generator set of M. Suppose M does not satisfy the stability condition. Then, there exists an unstable element of M_{inf} (resp. $M_{-\text{inf}}$), which does not precede any stable element of M_{inf} (resp. $M_{-\text{inf}}$). Let u be an unstable element of M_{inf} and v any element of M_{inf} such that $v \neq u$ and $u \prec v$. If $u \in Y_{\text{inf}}$, then since $Y^*_{\text{fin}} = M_{\text{fin}}$, the set $Y' = (Y - \{u\}) \cup \{v\}$ is a generator set of M. Since Y' does not contain Y, we get a contradiction to the minimality of Y. If $u \notin Y_{\text{inf}}$, then $u = y_1 y_2 \ldots y_n$ for some $(y_1, y_2, \ldots, y_n) \in Y^{(n)}$ with $n > 1$. Therefore $u \prec y_n$. By hypothesis, y_n is unstable. Therefore there exists $w \in M_{\text{inf}}$ such that $w \neq y_n$ and $y_n \prec w$. Thus, the set $Y'' = (Y - \{y_n\}) \cup \{w\}$ is a generator set of M. Since Y'' does not contain Y, we have a contradiction. Hence M satisfies the stability condition.

Example 3.2: Let $A = \{a, b\}$. Let M be the submonoid of ${}^\infty A^\infty$ given by

$$M = \{{}^\omega a (ab)^\omega\} \cup A^* \cup {}^\omega b A^* \cup A^* a^\omega \cup {}^\omega b A^* a^\omega.$$

Every element of M_{inf} precedes the unique stable element a^ω. Every element of $M_{-\text{inf}}$ precedes the unique stable element ${}^\omega b$. M satisfies the stability condition. By theorem 3.1, M has a minimal generator set which is $A \cup \{a^\omega, {}^\omega b, {}^\omega a (ab)^\omega\}$.

DEFINITION 3.2: Let M be a submonoid of ${}^\infty A^\infty$. Any increasing sequence $u_1 \prec u_2 \prec \ldots$ of elements of M_{inf} or $M_{-\text{inf}}$ is called a chain. An infinite chain is called stationary if there exists $n \geqq 1$, such that $u_m = u_n$, for all $m \geqq n$. We say that M satisfies the stationary chain condition if every infinite chain of M_{inf} as well as $M_{-\text{inf}}$ is stationary.

We note that stationary chain condition implies the stability condition but the converse is not true.

DEFINITION 3.3: A submonoid M of $^{\infty}A^{\infty}$ is freeable if $M^{-1}M \cap MM^{-1} \subseteq M$.

The next theorem explains the existence of the minimal generator set for a freeable monoid M.

THEOREM 3.2: *For any freeable submonoid M, the following conditions are equivalent.*

(i) *M has a minimal generator set.*

(ii) *M satisfies the stationary chain condition.*

(iii) *M satisfies the stability condition.*

Proof is similar to that of theorem 2.4 of Chapter II in [10] and is therefore omitted. The main difference is to consider infinite chains of elements of $M_{-\mathrm{inf}}$.

DEFINITION 3.4: Let M be a submonoid of $^{\infty}A^{\infty}$. An element u of M_{inf} (resp. $M_{-\mathrm{inf}}$) is maximal if there is no element v of M_{inf} (resp. $M_{-\mathrm{inf}}$) such that $u \prec v$. The set of all maximal elements of M_{inf} (resp. $M_{-\mathrm{inf}}$) is denoted by $\mathrm{MAX}(M_{\mathrm{inf}})$ [resp. $\mathrm{MAX}(M_{-\mathrm{inf}})$]. It is evident that $\mathrm{MAX}(M_{\mathrm{inf}}) \subseteq \mathrm{STAB}(M_{\mathrm{inf}})$ and $\mathrm{MAX}(M_{-\mathrm{inf}}) \subseteq \mathrm{STAB}(M_{-\mathrm{inf}})$. We say that M satisfies the maximality condition if every non maximal element of M_{inf} (resp. $M_{-\mathrm{inf}}$) precedes a maximal element of M_{inf} (resp. $M_{-\mathrm{inf}}$). Clearly, maximality condition implies stability condition but not the converse.

DEFINITION 3.5: Any subset X of $^{\infty}A^{\infty}$ is called distinguished if $X_{\mathrm{inf}} \cap X_{\mathrm{fin}}^{+} X_{\mathrm{inf}} = \Phi$, $X_{-\mathrm{inf}} \cap X_{-\mathrm{inf}} X_{\mathrm{fin}}^{+} = \Phi$ and $X_{\mathrm{biinf}} \cap X_{-\mathrm{inf}} X_{\mathrm{fin}}^{*} X_{\mathrm{inf}} = \Phi$.

The following theorem gives the connection between maximality condition and the distinguished minimal generator set of a monoid M.

THEOREM 3.3: *For any submonoid M, the following conditions are equivalent.*

(i) *M has a distinguished minimal generator set which is*

$$\begin{aligned}\mathrm{BASE}(M_{\mathrm{fin}}) \cup \mathrm{MAX}(M_{\mathrm{inf}}) \cup \mathrm{MAX}(M_{-\mathrm{inf}}) \cup \mathrm{UNFAC}(M_{\mathrm{biinf}}) \\ = (M-\varepsilon) - [(M_{\mathrm{fin}}-\varepsilon)^2 \cup (M_{\mathrm{fin}}-\varepsilon)\, M_{\mathrm{inf}} \cup M_{-\mathrm{inf}}(M_{\mathrm{fin}}-\varepsilon) \\ \cup M_{-\mathrm{inf}} M_{\mathrm{fin}} M_{\mathrm{inf}}]\end{aligned}$$

(ii) M has a distinguished generator set

(iii) M satisfies the maximality condition

Proof: It is clear that (i) implies (ii). We show that (ii) implies (iii). Let Y be a distinguished generator set of M. Since Y is a generator set, it is easy to see that every element of $M_{\mathrm{inf}}-Y_{\mathrm{inf}}$ (resp. $M_{-\mathrm{inf}}-Y_{-\mathrm{inf}}$) precedes an element of Y_{inf} (resp. $Y_{-\mathrm{inf}}$) and so it is enough to prove that

$$Y_{\mathrm{inf}} \subseteqq \mathrm{MAX}(M_{\mathrm{inf}}) \text{ [resp. } Y_{-\mathrm{inf}} \subseteqq \mathrm{MAX}(M_{-\mathrm{inf}})].$$

We shall prove that $Y_{\mathrm{inf}} \subseteqq \mathrm{MAX}(M_{\mathrm{inf}})$. Suppose this is not true. Then, there exists $y \in Y_{\mathrm{inf}}$ which is not maximal. So, for some $v \in M_{\mathrm{inf}}$, we have $y \prec v$. Let $y=gv$ where $g \in M_{\mathrm{fin}}-\varepsilon$ and $v=y_1 y_2 \ldots y_n$ for some $(y_1, y_2, \ldots, y_n) \in Y^{(n)}$, $n \geqq 1$. Since $g y_1 y_2 \ldots y_{n-1} \in Y_{\mathrm{fin}}^{+}$, we have $y \in Y_{\mathrm{inf}} \cap Y_{\mathrm{fin}}^{+} Y_{\mathrm{inf}}$. This is a contradiction since Y is distinguished. Hence (ii) implies (iii).

We now prove (iii) $\Rightarrow$ (i). Let M satisfy the maximality condition. This means M satisfies the stability condition. By theorem 3.1, M has a minimal generator set X, namely,

$$X=\mathrm{BASE}(M_{\mathrm{fin}}) \cup \mathrm{STAB}(M_{\mathrm{inf}}) \cup \mathrm{STAB}(M_{-\mathrm{inf}}) \cup \mathrm{UNFAC}(M_{\mathrm{biinf}}).$$

Since a non maximal stable element cannot precede a maximal element,

$$\mathrm{STAB}(M_{\mathrm{inf}})=\mathrm{MAX}(M_{\mathrm{inf}})=M_{\mathrm{inf}}-(M_{\mathrm{fin}}-\varepsilon) M_{\mathrm{inf}}$$

and

$$\mathrm{STAB}(M_{-\mathrm{inf}})=\mathrm{MAX}(M_{-\mathrm{inf}})=M_{-\mathrm{inf}}-M_{-\mathrm{inf}}(M_{\mathrm{fin}}-\varepsilon).$$

Since

$$\mathrm{UNFAC}(M_{\mathrm{biinf}})=M_{\mathrm{biinf}}-(M_{-\mathrm{inf}} M_{\mathrm{fin}} M_{\mathrm{inf}})$$

and

$$\mathrm{BASE}(M_{\mathrm{fin}})=(M_{\mathrm{fin}}-\varepsilon)-(M_{\mathrm{fin}}-\varepsilon)^2,$$

we have

$$X=(M-\varepsilon)-[(M_{\mathrm{fin}}-\varepsilon)^2 \cup (M_{\mathrm{fin}}-\varepsilon) M_{\mathrm{inf}} \cup M_{-\mathrm{inf}}(M_{\mathrm{fin}}-\varepsilon) \cup M_{-\mathrm{inf}} M_{\mathrm{fin}} M_{\mathrm{inf}}].$$

Since $X=X_{\mathrm{fin}} \cup X_{\mathrm{inf}} \cup X_{-\mathrm{inf}} \cup X_{\mathrm{biinf}}$, let $X_{\mathrm{fin}}=\mathrm{BASE}(M_{\mathrm{fin}})$, $X_{\mathrm{inf}}=\mathrm{MAX}(M_{\mathrm{inf}})$, $X_{-\mathrm{inf}}=\mathrm{MAX}(M_{-\mathrm{inf}})$ and $X_{\mathrm{biinf}}=\mathrm{UNFAC}(M_{\mathrm{biinf}})$. Thus $X_{\mathrm{inf}} \cap X_{\mathrm{fin}}^{+} X_{\mathrm{inf}}=\Phi$, $X_{-\mathrm{inf}} \cap X_{-\mathrm{inf}} X_{\mathrm{fin}}^{+}=\Phi$ and

$$X_{\mathrm{biinf}} \cap X_{-\mathrm{inf}} X_{\mathrm{fin}}^{*} X_{\mathrm{inf}}=\Phi.$$

Hence X is distinguished.

SECTION 4

SUBMONOID GENERATED BY CODES AND A THUE SYSTEM

In this section we introduce a bi-quasi free monoid whose underlying set is the set of all normal forms with respect to a specific Church-Rosser Thue system. We establish a characterisation of codes in terms of morphisms of monoids. We show the relation between bi-quasi free monoids, minimal generator sets and codes.

Let B be any finite alphabet. Let R be a binary relation on B^*. Elements of R are written as equations, i.e., $R=\{(u=v)\mid u, v\in B^*\}$. Let $T(B)=\langle B; R\rangle$. We call $T(B)$ as a Thue system associated with B. We say $(u=v)$ is in $T(B)$ Iff $(u=v)$ is in R.

Define the relation $=_{T(B)}$ on elements of B^* as follows: For any $(u=v)$ in $T(B)$ and any $x, y\in B^*$, we write $xuy=_{T(B)}xvy$. The reflexive transitive closure of the symmetric relation $=_{T(B)}$ is denoted as $\equiv_{T(B)}$. Clearly $\equiv_{T(B)}$ is a congruence relation on B^*. If $x\equiv_{T(B)}y$, for any $x, y\in B^*$, we say that x is congruent to y. The congruence class of x is denoted by $[x]$.

If $(u=v)$ is in $T(B)$, we write $u\rightarrow_{T(B)}v$ if the length of u is greater than the length of v. $\overset{*}{\rightarrow}_{T(B)}$ is the reflexive, transitive closure of the relation $\rightarrow_{T(B)}$. When $x\overset{*}{\rightarrow}_{T(B)}y$, we say that x is an ancestor of y and y is a descendant of x. x is said to be irreducible if it has no descendant except itself. For any $x, y\in B^*$, if $x\equiv_{T(B)}y$ and y is irreducible, then y is called a normal form of x.

$T(B)$ is Church-Rosser if for all $x, y\in B^*$, if $x\equiv_{T(B)}y$, then for some $z\in B^*$, $x\overset{*}{\rightarrow}_{T(B)}z$ and $y\overset{*}{\rightarrow}_{T(B)}z$. This means that every two congruent words have a common descendant. It is known that if $T(B)$ is Church-Rosser, then every congruence class has a unique normal form [2]. We make use of this result in the following discussion.

We partition B into four mutually disjoint subsets B_1, B_2, $\bar{B}_2$, B_3 and call B as a quadruple alphabet $(B_1, B_2, \bar{B}_2, B_3)$. With B, we associate a Thue system defined by $T(B)=\langle B; R\rangle$ where

$$R=\{(bb'=b)\mid b\in B_2\cup B_3,\ b'\in B\}\cup\{(bb'=b')\mid b\in B_1\cup \bar{B}_2,\ b'\in \bar{B}_2\cup B_3\}.$$

Now, $\equiv_{T(B)}$ is a congruence relation on B^*. Consider the quotient monoid $B^*/\equiv_{T(B)}$ and denote this by $B^{[*]}$. It is easy to see that $T(B)$ is Church-Rosser. Hence every congruence class has a unique normal form. It is interesting to note that the set of all normal forms of elements of B^* is

$$B_1^* \cup B_1^* B_2 \cup \bar{B}_2 B_1^* \cup \bar{B}_2 B_1^* B_2 \cup B_3.$$

By a mild abuse of language, we write

$$B^{[*]} = B_1^* \cup B_1^* B_2 \cup \bar{B}_2 B_1^* \cup \bar{B}_2 B_1^* B_2 \cup B_3.$$

Define a product on $B^{[*]}$ as follows: For $x, y \in B^{[*]}$,

$$x . y = \begin{cases} xy & \text{if } x \in B_1^*, \quad y \in B_1^* B_2 \cup B_1^* \\ & \text{or} \\ & x \in \bar{B}_1 B_1^*, \quad y \in B_1^* \\ x & \text{if } x \in B_1^* B_2 \cup B_3 \cup \bar{B}_2 B_1^* B_2 \\ y & \text{if } x \in B_1^* \cup \bar{B}_2 B_1^*, \\ & y \in \bar{B}_2 B_1^* \cup B_3 \cup \bar{B}_2 B_1^* B_2. \end{cases}$$

Clearly $B^{[*]}$ is a monoid which we shall call as a bi-quasi free monoid generated by B.

LEMMA 4.1: *If* $\varphi : B^{[*]} \to {}^\infty A^\infty$ *is an injective morphism and* $\varphi(B) = X$, *then* $\varphi(B_1) = X_{\text{fin}}$, $\varphi(B_2) = X_{\text{inf}}$, $\varphi(\bar{B}_2) = X_{-\text{inf}}$ *and* $\varphi(B_3) = X_{\text{biinf}}$.

Proof: We first show that $\varphi(B_1) \subseteq X_{\text{fin}}$. Suppose it is not true. Then there exists $b \in B_1$ such that

$$\varphi(b) \in X_{\text{inf}} \cup X_{-\text{inf}} \cup X_{\text{biinf}}.$$

If $\varphi(b) \in X_{\text{inf}} \cup X_{\text{biinf}}$, for $b' \in B$,

$$\varphi(bb') = \varphi(b)\varphi(b') = \varphi(b).$$

Since φ is injective, $bb' = b$ which is impossible. If $\varphi(b) \in X_{-\text{inf}}$, for $b' \in \bar{B}_2 \cup B_3$, $bb' = b'$. So, $\varphi(b)\varphi(b') = \varphi(b')$ which is impossible since $\varphi(b) \neq \varepsilon$. Hence $\varphi(B_1) \subseteq X_{\text{fin}}$.

To prove that $\varphi(B_2) \subseteq X_{\text{inf}}$, we suppose that it is not true. Then there exists $b \in B_2$ such that $\varphi(b) \in X_{\text{fin}} \cup X_{-\text{inf}} \cup X_{\text{biinf}}$. For $b' \in B$, $bb' = b$. So, $\varphi(b)\varphi(b') = \varphi(b)$ and this is not possible since $\varphi(b')$ need not be ε. Hence $\varphi(B_2) \subseteq X_{\text{inf}}$.

We now show that $\varphi(\bar{B}_2) \subseteq X_{-\text{inf}}$. If it were not so, there would exist $b \in \bar{B}_2$ such that $\varphi(b) \in X_{\text{fin}} \cup X_{\text{inf}} \cup X_{\text{biinf}}$.Now, for $b' \in \bar{B}_2 \cup B_3$, $bb' = b'$ and so $\varphi(b)\varphi(b') = \varphi(b')$. This is not possible since $\varphi(b) \neq \varepsilon$.

Finally, in order to prove that $\varphi(B_3) \subseteq X_{\text{biinf}}$, assume that it is not true. Then there exists $b \in B_3$ such that $\varphi(b) \in X_{\text{fin}} \cup X_{\text{inf}} \cup X_{-\text{inf}}$. For $b' \in B$, $bb' = b$. Therefore $\varphi(b)\varphi(b') = \varphi(b)$ which is not possible since $\varphi(b') \neq \varepsilon$.

Since $\varphi(B) = X$, we have, $\varphi(B_1) = X_{\text{fin}}$, $\varphi(B_2) = X_{\text{inf}}$, $\varphi(\bar{B}_2) = X_{-\text{inf}}$ and $\varphi(B_3) = X_{\text{biinf}}$. This proves the lemma.

Given a quadruple alphabet $B = (B_1, B_2, \bar{B}_2, B_3)$ we denote $B^{(1)} = B$ and

$$B^{(n)} = \{(b_1, b_2, \ldots, b_n)/b_1, b_2, \ldots, b_{n-1} \in B_1, b_n \in B_1 \cup B_2 \text{ or } b_1 \in B_1 \cup \bar{B}_2, \quad b_2, b_3, \ldots, b_n \in B_1 \text{ or } b_1 \in \bar{B}_2, b_n \in B_2, b_2, b_3, \ldots, b_{n-1} \in B_1\}$$

LEMMA 4.2: (*i*) *If a subset* X *of* ${}^{\infty}A^{\infty}$ *is a code, then every morphism* $\varphi : B^{[*]} \to {}^{\infty}A^{\infty}$ *which induces a bijection from* B *onto* X *with* $\varphi(B_1) \subseteq X_{\text{fin}}$, $\varphi(B_2) \subseteq X_{\text{inf}}$ *and* $\varphi(\bar{B}_2) \subseteq X_{-\text{inf}}$ *is injective.*

(ii) *If* $\varphi : B^{[*]} \to {}^{\infty}A^{\infty}$ *is an injective morphism, then* $X = \varphi(B)$ *is a code.*

Proof is on lines close to that of lemma 1.3 of Chapter III in [10] and is therefore omitted.

We now give a necessary and sufficient condition for a subset of ${}^{\infty}A^{\infty}$ to be a code.

THEOREM 4.1: *A subset* X *of* ${}^{\infty}A^{\infty}$ *is a code iff there exists a bi-quasi free monoid* $B^{[*]}$ *and an injective morphism* $\varphi : B^{[*]} \to {}^{\infty}A^{\infty}$ *such that* $\varphi(B) = X$.

Proof: Let X be a code. Let $B = (B_1, B_2, \bar{B}_2, B_3)$ be a quadruple alphabet chosen so that B_1, B_2, $\bar{B}_2$ and B_3 are in one to one correspondence with X_{fin}, X_{inf}, $X_{-\text{inf}}$ and X_{biinf} respectively. This correspondence shows the existence of an isomorphism

$$\varphi : B^{[*]} \to X^* \quad \text{with} \quad \varphi(B_1) = X_{\text{fin}}, \quad \varphi(B_2) = X_{\text{inf}}, \quad \varphi(\bar{B}_2) = X_{-\text{inf}} \quad \text{and} \quad \varphi(B_3) = X_{\text{biinf}}.$$

By lemma 4.2, the theorem holds.

DEFINITION 4.1: A submonoid M of ${}^{\infty}A^{\infty}$ is said to be bi-quasi free if it is isomorphic to a bi-quasi free monoid $B^{[*]}$.

The following theorem exhibits that the class of submonoids generated by codes coincides with the class of biquasi free submonoids.

THEOREM 4.2: (*i*) *Every bi-quasi free submonoid M has a minimal generator set X which is a code.*

(ii) *If X is a code, then X^* is a bi-quasi free submonoid having X as its minimal generator set.*

Proof: (i) Suppose M is a bi-quasi free submonoid. Then there is an isomorphism $\varphi : B^{[*]} \to M$ from a bi-quasi free monoid onto M. By theorem 4.1, $X=\varphi(B)$ is a code. By lemma 4.2, $\varphi(B_1)=X_{\text{fin}}$, $\varphi(B_2)=X_{\text{inf}}$, $\varphi(\bar{B}_2)=X_{-\text{inf}}$ and $\varphi(B_3)=X_{\text{biinf}}$. We have

$$\begin{aligned} M=\varphi(B^{[*]}). &=\varphi(B_1^* \cup B_1^* B_2 \cup \bar{B}_2 B_1^* \cup \bar{B}_2 B_1^* B_2 \cup B_3) \\ &=[\varphi(B_1)]^* \cup [\varphi(B_1)]^* \varphi(B_2) \cup \varphi(\bar{B}_2)[\varphi(B_1)]^* \\ &\quad \cup \varphi(\bar{B}_2)[\varphi(B_1)]^* \varphi(B_2) \cup \varphi(B_3). \\ &=X_{\text{fin}}^* \cup X_{\text{fin}}^* X_{\text{inf}} \cup X_{-\text{inf}} X_{\text{fin}}^* \cup X_{-\text{inf}} X_{\text{fin}}^* X_{\text{inf}} \cup X_{\text{biinf}}=X^*. \end{aligned}$$

Hence X generates M. To prove the minimality of X, let Y be any generator set of M and $x \in X$. Then $x=y_1 y_2 \ldots y_n$ for some $(y_1, y_2, \ldots, y_n) \in Y^{(n)}$, $n \geqq 0$. Since $x \neq \varepsilon$, $n \geqq 1$. Since X is a code, $n=1$ and so $x=y_1$. Hence $X \subseteq Y$. Thus X is minimal.

(ii) Suppose X is a code. By theorem 4.1, there exists a bi-quasi free monoid $B^{[*]}$ and an injective morphism $\varphi : B^{[*]} \to {}^{\infty}A^{\infty}$ such that $\varphi(B)=X$. Now φ is indeed an isomorphism from $B^{[*]}$ onto $\varphi(B^{[*]})=X^*$. Thus X^* is a bi-quasi free submonoid. By the similar argument as in (i), X is a minimal generator set of X^*.

SECTION 5

A COMBINATORIAL CHARACTERIZATION OF BI-QUASI FREE SUBMONOIDS

LEMMA 5.1: *Every bi-quasi free submonoid is freeable.*

Proof: Let M be a bi-quasi free submonoid with the minimal generator set X. By theorem 4.2, X is a code. Let $\alpha \in M^{-1} M \cap MM^{-1}$. Since $\alpha \in M^{-1} M$, there exists $\beta \in M$ such that $\beta\alpha \in M$, $(\beta \in M_{\text{inf}} \cup M_{\text{biinf}} \Rightarrow \alpha=\varepsilon)$ and $(\beta \in {}^{\infty}M$ and $\alpha \in A^{-\mathbb{N}} \cup A^{\mathbb{Z}} \Rightarrow \beta=\varepsilon)$. Since $\alpha \in MM^{-1}$, there exists $\mathscr{V} \in M$ such that $\alpha \mathscr{V} \in M$,

$$(\alpha \in {}^{\infty}A \text{ and } \mathscr{V} \in M_{-\text{inf}} \cup M_{\text{biinf}} \Rightarrow \alpha=\varepsilon)$$

and

$$(\alpha \in A^N \cup A^Z \Rightarrow \mathscr{V} = \varepsilon).$$

Let

$$\begin{aligned} \beta &= x_1 x_2 \ldots x_k && \text{with } (x_1, x_2, \ldots, x_k) \in X^{(k)}; \\ \alpha \mathscr{V} &= x_{k+1} \ldots x_n && \text{with } (x_{k+1}, x_{k+2}, \ldots, x_n) \in X^{(n-k)}; \\ \beta\alpha &= x_1' x_2' \ldots x_l' && \text{with } (x_1', x_2', \ldots, x_l') \in X^{(l)}; \\ \mathscr{V} &= x_{l+1}' x_{l+2}' \ldots x_m' && \text{with } (x_{l+1}', \ldots, x_m') \in X^{(m-l)}. \end{aligned}$$

If $\beta \in M_{\text{inf}} \cup M_{\text{biinf}}$, then $\alpha = \varepsilon \in M$. If $\beta \in {}^{\infty}M$ and $\alpha \in A^{-N} \cup A^Z$, then $\beta = \varepsilon$. Therefore $\beta\alpha \in M$ implies $\alpha \in M$. If $\alpha \in A^N$, then we have $\mathscr{V} = \varepsilon$ and so $\alpha \mathscr{V} \in M$ implies $\alpha \in M$. When $\alpha \in {}^{\infty}A$ and $\mathscr{V} \in M_{-\text{inf}} \cup M_{\text{biinf}}$, then $\alpha = \varepsilon \in M$. We have to consider the only case when $\beta \in {}^{\infty}M$, $\alpha \in A^*$ and $\mathscr{V} \in M^{\infty}$. Since $\beta(\alpha \mathscr{V}) = (\beta\alpha)\mathscr{V}$, we get

$$x_1 x_2 \ldots x_k x_{k+1} \ldots x_n = x_1' x_2' \ldots x_l' x_{l+1}' \ldots x_m'.$$

Since X is a code, $n = m$ and $x_i = x_i'$, $i = 1, 2, \ldots, n$. Since β is a left factor of $\beta\alpha$, we have $l \geqq k$. Therefore

$$\beta\alpha = x_1' x_2' \ldots x_l' = x_1 x_2 \ldots x_k x_{k+1} \ldots x_l = \beta x_{k+1} \ldots x_l.$$

This implies $\alpha = x_{k+1} \ldots x_l \in M$. Thus $M^{-1} M \cap M M^{-1} \subseteqq M$ and so M is freeable.

DEFINITION 5.1: We say that a submonoid M satisfies finite chain condition if all the chains in M_{inf} and $M_{-\text{inf}}$ are finite.

The finite chain condition implies the maximality condition.

LEMMA 5.2: *Every bi-quasi free submonoid satisfies the finite chain condition.*

Proof is similar to that of proposition 3.3 of Chapter III in [10] and is omitted. The difference is to consider infinite chains in $M_{-\text{inf}}$.

THEOREM 5.1: *For any submonoid M, the following conditions are equivalent.*

(i) M is bi-quasi free *i. e.*, generated by a code.

(ii) M is freeable and satisfies the finite chain condition.

(iii) M is freeable and satisfies the maximality condition

(iv) M is freeable and has a distinguished (minimal) generator set.

Proof: It is clear that (iii) $\Leftrightarrow$ (iv) by theorem 3.3. (i) $\Rightarrow$ (ii) is by lemmas 5.1 and 5.2. (ii) $\Rightarrow$ (iii) is evident. We have to show that (iii) $\Rightarrow$ (i).

Suppose M is freeable and satisfies the maximality condition. By theorem 3.3, M has a distinguished minimal generator set X which is $\mathrm{BASE}(M_{\mathrm{fin}}) \cup \mathrm{MAX}(M_{\mathrm{inf}}) \cup \mathrm{MAX}(M_{-\mathrm{inf}}) \cup \mathrm{UNFAC}(\mathrm{M}_{\mathrm{biinf}})$.
By theorem 4.2, it is enough if we prove that X is a code. Suppose X is not a code. Then there exists a word α such that it has two different factorizations on elements of X. *i.e.*,

$$\alpha = x_1 x_2 \ldots x_n = x'_1 x'_2 \ldots x'_m$$

where $n,\ m \geqq 1$, $(x_1, x_2, \ldots, x_n) \in X^{(n)}$ and $(x'_1, x'_2, \ldots, x'_m) \in X^{(m)}$. Clearly either n, or, m should be greater than 1. Let $m > 1$.

Case (*a*): Suppose $\alpha \in A^*$. We may assume that $x_1 \neq x'_1$. Let $|x_1| > |x'_1|$. Then there exists a word $f \neq \varepsilon$ such that $x_1 = x'_1 f$ and $f x_2 x_3 \ldots x_n = x'_2 x'_3 \ldots x'_m$. From the freeability of M, it follows that $f \in M_{\mathrm{fin}} - \varepsilon$. This contradicts the hypothesis that $x_1 \in \mathrm{BASE}(M_{\mathrm{fin}})$.

Case (*b*): Suppose $\alpha \in A^N$. Then $x_n,\ x'_m \in \mathrm{MAX}(M_{\mathrm{inf}})$. If $n = 1$, then $x_n = x'_1 x'_2 \ldots x'_m$ and so $x_n \prec x'_m$ which is a contradiction to the maximality of x_n. Suppose $x \geqq 2$. If

$$|x_1 x_2 \ldots x_{n-1}| = |x'_1 x'_2 \ldots x'_{m-1}|,$$

then

$$x_1 x_2 \ldots x_{n-1} = x'_1 x'_2 \ldots x'_{m-1} \in A^*.$$

As in case (*a*), we get a contradiction. If not, we assume that $|x_1 x_2 \ldots x_{n-1}| > |x'_1 x'_2 \ldots x'_{m-1}|$. This implies that there exists $f \neq \varepsilon$ with

$$x_1 x_2 \ldots x_{n-1} = x'_1 x'_2 \ldots x'_{m-1} f \qquad \text{and} \quad f x_n = x'_m.$$

Again, by freeability of M, $f \in M_{\mathrm{fin}} - \varepsilon$ and so we have $x'_m \prec x_n$ which contradicts the maximality of x'_m.

Case (*c*): Suppose $\alpha \in A^{-N}$. Then $x_1,\ x'_1 \in \mathrm{MAX}(M_{-\mathrm{inf}})$. We can discuss as in case (*b*) and obtain a contradiction.

Case (*d*): Suppose $\alpha \in A^Z$. If $n = 1$, then $x_1 = x'_1 x'_2 \ldots x'_m$ which is a contradiction since X is a distinguished minimal generator set. Suppose $n \geqq 2$. Then $x_1,\ x'_1 \in \mathrm{MAX}(M_{-\mathrm{inf}})$. There are two possibilities.

(i) If $x_1 \neq x'_1$, we assume $x_1 = x'_1 f$ and so we have

$$f x_2 x_3 \ldots x_n = x'_2 x'_3 \ldots x'_m,\ m \geqq 2.$$

This implies $f \in M_{\text{fin}} - \varepsilon$ since M is freeable. Hence $x_1 \prec x'_1$ which contradicts the maximality of x_1.

(ii) Suppose $x_1 = x'_1$. We have either

$$x_2 x_3 \ldots x_n = x'_2 x'_3 \ldots x'_m$$

or

$$x_2 x_3 \ldots x_n \neq x'_2 x'_3 \ldots x'_m.$$

If $x_2 x_3 \ldots x_n = x'_2 x'_3 \ldots x'_m$, then we assume $x_2 \neq x'_2$ and proceed as in case (*b*) and get a contradiction. If $x_2 x_3 \ldots x_n \neq x'_2 x'_3 \ldots x'_m$ since α has two factorizations, we have either $x_1 = x'_1 f$ and

$$f x_2 x_3 \ldots x_n = x'_2 x'_3 \ldots x'_m \qquad \text{or} \qquad x'_1 = x_1 f$$

and

$$x_2 x_3 \ldots x_n = f x'_2 x'_3 \ldots x'_m.$$

Since the two cases are similar, it is enough to consider any one of the possibilities, say $x_1 = x'_1 f$ and $f x_2 x_3 \ldots x_n = x'_2 x'_3 \ldots x'_m$. Clearly $f \in M_{\text{fin}} - \varepsilon$ as M is freeable and hence $x_1 \prec x'_1$ which contradicts the maximality of x_1.

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