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BI-LIPSCHICITY OF QUASICONFORMAL HARMONIC MAPPINGS IN THE PLANE

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Abstract

We show that quasiconformal harmonic mappings on the proper domains in $\mathbb{R}^2$ are bi-Lipschitz with respect to the quasihyperbolic metric.

1 Introduction

Continuity properties of quasiconformal mappings $f : D \longrightarrow D'$, where D and D' are domains in plane, with respect to various natural metrics have been studied extensively in [AKM], [KM], [KP] and [P].

Since the inverse of a K-quasiconformal mapping is also K-quasiconformal mapping, such results apply at the same time to f and f^{-1}.

In this paper we deal with harmonic quasiconformal mappings $f : D \longrightarrow D'$, note that f^{-1} is not, in general, harmonic.

Our main result is that harmonic K-quasiconformal mapping $f : D \longrightarrow D'$ in plane is bi-Lipschitz with respect to quasihyperbolic metric.

We note that in [M] this result is proved in n-dimensional setting, but only in the case where D and D' are the upper half space in $\mathbb{R}^n$.

In the case $n = 2$, in [M] this result is proved for $D = D' = \mathbb{D} = \{z : |z| < 1\}$, with explicit bounds in terms of K.

2 Result

Theorem 1. *Suppose D and D' are proper domains in $\mathbb{R}^2$. If $f : D \longrightarrow D'$ is K-qc and harmonic, then it is bi-Lipschitz with respect to quasihyperbolic metrics on D and D'.*

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We recall definition from [AG, Definition 1.5]

$$\alpha_f(z) = \exp\left(\frac{1}{n}(\log J_f)_{B_z}\right),$$

where

$$(\log J_f)_{B_z} = \frac{1}{m(B_z)}\int_{B_z} \log J_f \, dm, \quad B_z = B(z, d(z, \partial D)).$$

In the case $n = 2$ we have

$$\frac{1}{\alpha_f(z)} = \exp\left(\frac{1}{2}\frac{1}{m(B_z)}\int_{B_z} \log\frac{1}{J_f(w)}\, dm(w)\right). \tag{1}$$

We are going to use the following result:

Theorem 2. *[AG, Theorem 1.8] Suppose that D and D' are domains in $\mathbb{R}^n$ if $f : D \longrightarrow D'$ is K-qc, then*

$$\frac{1}{c}\frac{d(f(z), \partial D')}{d(z, \partial D)} \le \alpha_f(z) \le c\,\frac{d(f(z), \partial D')}{d(z, \partial D)}$$

for $z \in D$, where c is a constant wich depends only on K and n.

3 Proof of Theorem 1

Our proof is based on the theorem of Astala and Gehring.

Proof. Since f is harmonic we have a local representation

$$f(z) = g(z) + \overline{h(z)},$$

where g and h are analytic functions. Then Jacobian $J_f(z) = |g'(z)|^2 - |h'(z)|^2 > 0$ (note that $g'(z) \neq 0$).

Further,

$$J_f(z) = |g'(z)|^2\left(1 - \frac{|h'(z)|^2}{|g'(z)|^2}\right) = |g'(z)|^2\left(1 - |\omega(z)|^2\right),$$

where $\omega(z) = \frac{h'(z)}{g'(z)}$ is analytic and $|\omega| < 1$. Now we have

$$\log\frac{1}{J_f(z)} = -2\log|g'(z)| - \log(1 - |\omega(z)|^2).$$

The first term is harmonic function (it is well known that logarithm of moduli of analytic function is harmonic everywhere except where that analytic function vanishes, but $g'(z) \neq 0$ everywhere).

The second term can be expanded in series

$$\sum_{k=1}^{\infty} \frac{|\omega(z)|^{2k}}{k},$$

and each term is subharmonic (note that ω is analytic).

So, $-\log(1-|\omega(z)|^2)$ is a continuous function represented as a locally uniform sum of subharmonic functions. Thus it is also subharmonic.

Hence

$$\log \frac{1}{J_f(z)} \text{ is a subharmonic function.} \tag{2}$$

Note that representation $f(z) = g(z) + \overline{h(z)}$ is local, but that suffices for our conclusion (2).

¿From (2) we have

$$\frac{1}{m(B_z)} \int_{B_z} \log \frac{1}{J_f(w)}\, dm(w) \geq \log \frac{1}{J_f(z)}.$$

Combining this with (1) we have

$$\frac{1}{\alpha_f(z)} \geq \exp\left(\frac{1}{2} \log \frac{1}{J_f(z)}\right) = \frac{1}{\sqrt{J_f(z)}}$$

and therefore

$$\sqrt{J_f(z)} \geqslant \alpha_f(z).$$

Applying the first inequality from Theorem 2 we have

$$\sqrt{J_f(z)} \geq \frac{1}{c} \frac{d(f(z), \partial D')}{d(z, \partial D)}. \tag{3}$$

Note that

$$J_f(z) = |g'(z)|^2 - |h'(z)|^2 \leq |g'(z)|^2$$

and by K-quasiconformality of f, $|h'| \leq k|g'|$, $0 \leq k < 1$, where $K = \frac{1+k}{1-k}$.

This gives $J_f \geq (1-k^2)|g'|^2$. Hence,

$$\sqrt{J_f} \asymp |g'| \asymp |g'| + |h'| = L(f, z),$$

where

$$L(f, z) = \max_{|h|=1} |f'(z)h|.$$

Finally (3) and the above asymptotic relation give

$$L(f, z) \geq \frac{1}{c} \frac{d(f(z), \partial D')}{d(z, \partial D)}, \quad c = c(k).$$

For the reverse inequality we again use $J_f(z) \geq (1-k^2)|g'(z)|^2$, i.e.

$$\sqrt{J_f(z)} \geq \sqrt{1-k^2}|g'(z)| \tag{4}$$

Further, we know that for $n = 2$

$$\alpha_f(z) = \exp\left(\frac{1}{m(B_z)}\int_{B_z} \log\sqrt{J_f(x)}\,dm(w)\right).$$

Using (4)

$$\begin{aligned}
\frac{1}{m(B_z)}\int_{B_z} \log\sqrt{J_f(x)}\,dm(w) &\geq \frac{1}{m(B_z)}\int_{B_z} \log\sqrt{1-k^2} + \log|g'(w)|\,dm(w)\\
&= \log\sqrt{1-k^2} + \frac{1}{m(B_z)}\int_{B_z} \log|g'(w)|\,dm(w)\\
&= \log\sqrt{1-k^2} + \log|g'(z)|.
\end{aligned}$$

Since $\log|g'|$ is harmonic, we have

$$\begin{aligned}
\alpha_f(z) &= \exp\left(\frac{1}{m(B_z)}\int_{B_z} \log\sqrt{J_f(x)}\,dm(w)\right)\\
&\geq \exp(\log\sqrt{1-k^2} + \log|g'(z)|)\\
&= \sqrt{1-k^2}|g'(z)|\\
&\geq \frac{1}{2}\sqrt{1-k^2}(|g'(z)| + |h'(z)|)\\
&= \frac{\sqrt{1-k^2}}{2}L(f,z).
\end{aligned}$$

Again using the second inequality in [AG, Theorem 1.8]

$$L(f,z) \leqslant c\sqrt{J_f(z)} \leqslant c\,\alpha_f(z) \leqslant c\,\frac{d(f(z),\partial D')}{d(z,\partial D)}, \quad c = c(k).$$

Therefore, we proved

$$L(f,z) \asymp \frac{d(f(z),\partial D')}{d(z,\partial D)},$$

however, quasiconformality gives

$$L(f,z) \asymp l(f,z),$$

where

$$l(f,z) = \min_{|h|=1} |f'(z)h|.$$

Therefore, we have

$$l(f,z) \asymp \frac{d(f(z),\partial D')}{d(z,\partial D)}.$$

This pointwise result, combined with integration along curves, easily gives

$$k_{D'}(f(z_1), f(z_2)) \asymp k_D(z_1, z_2).$$

□

Problem 1. Is Theorem 1 true in dimensions $n \geq 3$?

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