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BI-PARACONTACT STRUCTURES AND LEGENDRE FOLIATIONS

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Abstract

We study almost bi-paracontact structures on contact manifolds. We prove that if an almost bi-paracontact structure is defined on a contact manifold (M, η), then under some natural assumptions of integrability M carries two transverse bi-Legendrian structures. Conversely, if two transverse bi-Legendrian structures are defined on a contact manifold, then M admits an almost bi-paracontact structure. We define a canonical connection on an almost bi-paracontact manifold and we study its curvature properties, which resemble those of the Obata connection of a para-hypercomplex (or complex-product) manifold. Further, we prove that any contact metric manifold whose Reeb vector field belongs to the (κ, μ)-nullity distribution canonically carries an almost bi-paracontact structure and we apply the previous results to the theory of contact metric (κ, μ)-spaces.

1. Introduction

The study of Legendre foliations on contact manifolds is very recent in literature, being initiated in the early 90's by the work of Libermann, Pang et alt. (cf. [16], [22]). Lately, the notion of "bi-Legendrian" structure has made its appearance, especially with regard to its applications to Cartan geometry ([17]) and Monge-Ampère equations ([20]) and to other geometric structures associated with a contact manifold, such as paracontact metrics. In particular, in [10] the author studied the interplays between bi-Legendrian manifolds and paracontact geometry, whereas in [11] the theory of bi-Legendrian structures was applied for the study of a remarkable class of contact Riemannian manifolds, namely contact metric (κ, μ)-spaces. We recall that a contact metric (κ, μ)-space is a contact Riemannian manifold (M, ϕ, ξ, η, g) such that the Reeb vector field ξ belongs to the (κ, μ)-nullity distribution, i.e. the following condition holds

$$R^g(X, Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY),$$

for some real numbers κ, μ and for any $X, Y \in \Gamma(TM)$, where R^g denotes the curvature tensor field of the Levi Civita connection and $2h$ is the Lie derivative of

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the structure tensor ϕ in the direction of the Reeb vector field. This definition, which has no analogue in even dimension, was introduced by Blair, Koufor-giorgos and Papantoniou in [4], as a generalization both of the well-known Sasakian condition $R^g(X,Y)\xi = \eta(Y)X - \eta(X)Y$ and of those contact metric manifolds verifying $R^g(X,Y)\xi = 0$ which were studied by Blair in [2]. A notable class of examples of contact metric (κ,μ)-spaces is given by the tangent sphere bundle of Riemannian manifold of constant curvature.

One of the main results in [4] was that any non-Sasakian contact metric (κ,μ)-space is foliated by two mutually orthogonal Legendre foliations $\mathscr{D}_h(\lambda)$ and $\mathscr{D}_h(-\lambda)$, given by the eigendistributions of the symmetric operator h corresponding to the eigenvalues λ and $-\lambda$, respectively, where $\lambda := \sqrt{1-\kappa}$. Thus any contact metric (κ,μ)-space is canonically a bi-Legendrian manifold.

In this paper we show that this is only a part of the story. In fact we prove that also the operator ϕh is diagonalizable and admits the same eigenvalues as h. Overall, the corresponding eigendistributions $\mathscr{D}_{\phi h}(\lambda)$ and $\mathscr{D}_{\phi h}(-\lambda)$ are integrable and define two mutually orthogonal Legendre foliations, as well. Thus any contact metric (κ,μ)-space carries two bi-Legendrian structures and, moreover, any foliation of each bi-Legendrian structure is transversal to the foliations of the other one. This geometrical structure resembles the concept, in even dimension, of 3-*web* ([21]) together with its closely linked tensorial notion, *para-hypercomplex or complex-product structure* ([1], [15], [18]). In fact, let ϕ_1, ϕ_2, ϕ_3 denote the $(1,1)$-tensor fields defined by

$$\phi_1 := \frac{1}{\sqrt{1-\kappa}}\phi h, \quad \phi_2 := \frac{1}{\sqrt{1-\kappa}}h, \quad \phi_3 := \phi. \tag{1.1}$$

Then one can check that ϕ_1 and ϕ_2 are anti-commuting almost paracontact structures on M such that $\phi_1\phi_2 = \phi_3$.

Thus we are motivated in the study of this new geometric structure, which we call *almost bi-paracontact structure*. An almost bi-paracontact structure on a contact manifold (M,η) is by definition any triplet (ϕ_1,ϕ_2,ϕ_3), where ϕ_1 and ϕ_2 are anti-commuting tensor fields satisfying $\phi_1^2 = \phi_2^3 = I - \eta\otimes\xi$ and $\phi_3 = \phi_1\phi_2$ is an almost contact structure on (M,η). Then one can prove that ϕ_1 and ϕ_2 are in fact almost paracontact structures and the eigendistributions corresponding to ± 1 define, under some natural assumptions, four mutually transversal Legendre foliations.

When the structure is *normal*, that is when the Nijenhuis tensors of ϕ_1, ϕ_2, ϕ_3 vanish, the leaves of such foliations admit an affine structure. This is due to the existence of a unique linear connection ∇^c which preserves ϕ_1, ϕ_2, ϕ_3. ∇^c is called the *canonical connection* of the almost bi-paracontact manifold (M,ϕ_1,ϕ_2,ϕ_3) and it can be considered, in some sense, as the odd-dimensional counterpart of the Chern connection of an almost para-hypercomplex manifold ([18]), as well as of the connection studied by Andrada for a complex-product manifold ([1]), and of the Obata connection of a manifold endowed with an almost quaternion structure of the second kind ([27]). In fact we prove that in any normal almost bi-paracontact manifold the 1-dimensional foliation $\mathscr{F}_\xi$ defined by the Reeb vector field is transversely para-hypercomplex, i.e. the almost bi-paracontact structure (ϕ_1, ϕ_2, ϕ_3) is projectable to a local para-hypercomplex structure on the leaf space.

We further investigate the curvature properties of this connection, proving that, under the assumption of normality, its curvature tensor field R^c is of type $(1,1)$ with respect to ϕ_1, ϕ_2, ϕ_3, i.e. $R^c(\phi_1 X, \phi_1 Y) = R^c(\phi_2 X, \phi_2 Y) = -R^c(\phi_3 X, \phi_3 Y) = -R^c(X, Y)$ for all $X, Y \in \Gamma(TM)$.

In the second part of the paper we apply our general results on almost bi-paracontact structures to the theory of contact metric (κ, μ)-spaces. First, we study the bi-Legendrian structure $(\mathscr{D}_{\phi h}(\lambda), \mathscr{D}_{\phi h}(-\lambda))$. We prove that the Legendre foliations $\mathscr{D}_{\phi h}(\lambda)$ and $\mathscr{D}_{\phi h}(-\lambda)$ are either non-degenerate or flat, according to the Pang's classification of Legendre foliations (cf. [22]). In particular, $\mathscr{D}_{\phi h}(\lambda)$ and $\mathscr{D}_{\phi h}(-\lambda)$ are positive definite if and only if $I_M > 0$, negative definite if and only if $I_M < 0$, flat if and only if $I_M = 0$, where

$$I_M := \frac{1 - \frac{\mu}{2}}{\sqrt{1-\kappa}}$$

is the invariant introduced by Boeckx for classifying contact metric (κ, μ)-structures. This provides a new geometrical interpretation of such invariant in terms of Legendre foliations (another one was given in [11]).

Then we consider the almost bi-paracontact structure (ϕ_1, ϕ_2, ϕ_3) defined by (1.1) and prove that the semi-Riemannian metrics g_1 and g_2, given by

$$g_1 := d\eta(\cdot, \phi_1 \cdot) + \eta \otimes \eta, \quad g_2 := d\eta(\cdot, \phi_2 \cdot) + \eta \otimes \eta,$$

define two associated paracontact metrics satisfying

$$R^{g_\alpha}(X, Y)\xi = \kappa_\alpha(\eta(Y)X - \eta(X)Y) + \mu_\alpha(\eta(Y)h_\alpha X - \eta(X)h_\alpha Y)$$

where

$$\kappa_1 = \left(1 - \frac{\mu}{2}\right)^2 - 1, \quad \mu_1 = 2(1 - \sqrt{1-\kappa}),$$

$$\kappa_2 = \kappa - 2 + \left(1 - \frac{\mu}{2}\right)^2, \quad \mu_2 = 2.$$

Mreover, $I_M = 0$ if and only if (ϕ_1, ξ, η, g_1) is para-Sasakian. Furthermore, we prove that any contact metric (κ, μ)-space such that $I_M \neq \pm 1$ admits a supplementary non-normal almost bi-paracontact structure, although one of the two paracontact structures is normal (cf. Theorem 5.14). In this way we obtain a class of examples of strictly non-normal, integrable almost bi-paracontact structures.

Finally, we deal with the following question, which generalizes the well-known problem of finding conditions ensuring the existence of Sasakian structures compatible with a given contact form: let (M, η) be a contact manifold; then does (M, η) admit a compatible contact metric (κ, μ)-structure? As a matter of fact, the answer to this question involves the standard almost bi-paracontact structure (1.1) of contact metric (κ, μ)-spaces. In particular, using the properties of the canonical connection ∇^c, we find necessary conditions for a contact manifold (M, η) endowed with an almost bi-paracontact structure to admit a compatible contact metric (κ, μ)-structure (cf. Theorem 5.13).

2. Preliminaries

2.1. Almost contact and paracontact structures. A *contact manifold* is a $(2n+1)$-dimensional smooth manifold M which carries a 1-form η, called *contact form*, satisfying the condition $\eta \wedge (d\eta)^n \neq 0$ everywhere on M. It is well known that given η there exists a unique vector field ξ, called *Reeb vector field*, such that

$$i_\xi \eta = 1, \quad i_\xi \, d\eta = 0. \tag{2.1}$$

From (2.1) it follows that $\mathscr{L}_\xi \, d\eta = 0$, i.e. the 1-dimensional foliation $\mathscr{F}_\xi$ defined by the Reeb vector field is transversely symplectic. In the sequel we will denote by $\mathscr{D}$ the $2n$-dimensional distribution defined by $\ker(\eta)$, called the *contact distribution*. It is easy to see that the Reeb vector field is an infinitesimal automorphism with respect to the contact distribution and the tangent bundle of M splits as the direct sum $TM = \mathscr{D} \oplus \mathbf{R}\xi$.

Given a contact manifold (M, η) one can consider two different geometric structures associated with the contact form η, namely a "contact metric structure" and a "paracontact metric structure".

An *almost contact structure* on a $(2n+1)$-dimensional smooth manifold M is nothing but a triplet (ϕ, ξ, η), where ϕ is a tensor field of type $(1,1)$, η a 1-form and ξ a vector field on M satisfying the following conditions

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \tag{2.2}$$

where I is the identity mapping. From (2.2) it follows that $\phi\xi = 0$, $\eta \circ \phi = 0$ and the $(1,1)$-tensor field ϕ has constant rank $2n$ ([3]). Given an almost contact manifold (M, ϕ, ξ, η) one can define an almost complex structure J on the product $M \times \mathbf{R}$ by setting $J\left(X, f\frac{d}{dt}\right) = \left(\phi X - f\xi, \eta(X)\frac{d}{dt}\right)$ for any $X \in \Gamma(TM)$ and $f \in C^\infty(M \times \mathbf{R})$. Then the almost contact manifold is said to be *normal* if the almost complex structure J is integrable. The computation of the Nijenhuis tensor of J gives rise to the four tensors defined by

$$N_\phi^{(1)}(X, Y) = [\phi, \phi](X, Y) + 2\, d\eta(X, Y)\xi, \tag{2.3}$$

$$N_\phi^{(2)}(X, Y) = (\mathscr{L}_{\phi X}\eta)(Y) - (\mathscr{L}_{\phi Y}\eta)(X), \tag{2.4}$$

$$N_\phi^{(3)}(X) = (\mathscr{L}_\xi \phi)X, \tag{2.5}$$

$$N^{(4)}(X) = (\mathscr{L}_\xi \eta)(X), \tag{2.6}$$

where $[\phi, \phi]$ is the Nijenhuis tensor of ϕ, defined by

$$[\phi, \phi](X, Y) := \phi^2[X, Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y],$$

and $\mathscr{L}_X$ denotes the Lie derivative with respect to the vector field X. One finds that the structure (ϕ, ξ, η) is normal if and only if $N_\phi^{(1)}$ vanishes identically; in particular, if $N_\phi^{(1)} = 0$ then also the other tensors $N_\phi^{(2)}$, $N_\phi^{(3)}$ and $N_\phi^{(4)}$ vanish (cf. [24]). By a long but straightforward computation one can prove the following lemma which will turn out very useful in the sequel.

Lemma 2.1. *In any almost contact manifold (M, ϕ, ξ, η) for any $X, Y \in \Gamma(TM)$,*

$$\phi N_\phi^{(1)}(X, Y) + N_\phi^{(1)}(\phi X, Y) = N_\phi^{(2)}(X, Y)\xi + \eta(X) N_\phi^{(3)}(Y). \tag{2.7}$$

Any almost contact manifold (M, ϕ, ξ, η) admits a *compatible metric*, i.e. a Riemannian metric g satisfying

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2.8}$$

for all $X, Y \in \Gamma(TM)$. The manifold M is said to be an *almost contact metric manifold* with structure (ϕ, ξ, η, g). From (2.8) it follows immediately that $\eta = g(\cdot, \xi)$ and $g(\cdot, \phi\cdot) = -g(\phi\cdot, \cdot)$. Then one defines the 2-form Φ on M by $\Phi(X, Y) = g(X, \phi Y)$, called the *fundamental* 2*-form* of the almost contact metric manifold. If $\Phi = d\eta$ then η becomes a contact form, with ξ its corresponding Reeb vector field, and (M, ϕ, ξ, η, g) is called *contact metric manifold.*

In a contact metric manifold one has

$$\nabla^g \xi = -\phi - \phi h \tag{2.9}$$

$$\begin{aligned} N_\phi^{(1)}(X, Y) = {} & (\nabla^g_{\phi X}\phi)Y - (\nabla^g_{\phi Y}\phi)X + (\nabla^g_X\phi)\phi Y \\ & - (\nabla^g_Y\phi)\phi X - \eta(Y)\nabla^g_X\xi + \eta(X)\nabla^g_Y\xi \end{aligned} \tag{2.10}$$

where ∇^g is the Levi Civita connection of (M, g) and $h := \frac{1}{2}N_\phi^{(3)}$. The tensor field h is symmetric with respect to g and vanishes identically if and only if the Reeb vector field is Killing, and in this case the contact metric manifold is said to be *K-contact*. A normal contact metric manifold is called *Sasakian manifold*. Any Sasakian manifold is also K-contact and the converse holds only in dimension 3. A contact metric manifold is said to be *integrable* if and only if the following condition is fulfilled

$$(\nabla^g_X\phi)Y = g(X + hX, Y)\xi - \eta(Y)(X + hX). \tag{2.11}$$

Any Sasakian manifold satisfies such condition. By replacing (2.11) and (2.9) in (2.10) one can prove the following

Proposition 2.2. *In an integrable contact metric manifold*

$$N_\phi^{(1)}(X, Y) = 2(\eta(Y)\phi h X - \eta(X)\phi h Y). \tag{2.12}$$

Corollary 2.3. *Any integrable K-contact manifold is Sasakian.*

On the other hand on a contact manifold (M,η) one can consider also compatible paracontact metric structures. We recall (cf. [14]) that an *almost paracontact structure* on a $(2n+1)$-dimensional smooth manifold M is given by a $(1,1)$-tensor field $\tilde{\phi}$, a vector field ξ and a 1-form η satisfying the following conditions

(i) $\eta(\xi)=1,\ \tilde{\phi}^2 = I - \eta\otimes\xi$,

(ii) the tensor field $\tilde{\phi}$ induces an almost paracomplex structure on each fibre on $\mathscr{D}=\ker(\eta)$.

Recall that an almost paracomplex structure on a $2n$-dimensional smooth manifold is a tensor field $\tilde{J}$ of type $(1,1)$ such that $\tilde{J}\neq I$, $\tilde{J}^2=I$ and the eigendistributions T^+, T^- corresponding to the eigenvalues 1, -1 of $\tilde{J}$, respectively, have dimension n.

As an immediate consequence of the definition one has that $\tilde{\phi}\xi=0$, $\eta\circ\tilde{\phi}=0$ and the field of endomorphisms $\tilde{\phi}$ has constant rank $2n$. As for the almost contact case, one can consider the almost paracomplex structure on $M\times\mathbf{R}$ defined by $\tilde{J}\left(X, f\frac{d}{dt}\right) = \left(\tilde{\phi}X + f\xi, \eta(X)\frac{d}{dt}\right)$, where X is a vector field on M and f a C^∞ function on $M\times\mathbf{R}$. By definition, if $\tilde{J}$ is integrable, the almost paracontact structure $(\tilde{\phi},\xi,\eta)$ is said to be *normal*. The computation of $\tilde{J}$ in terms of the tensors of the almost paracontact structure leads us to define four tensors

$$N^{(1)}_{\tilde{\phi}}(X,Y) = [\tilde{\phi},\tilde{\phi}](X,Y) - 2\,d\eta(X,Y)\xi, \tag{2.13}$$

$$N^{(2)}_{\tilde{\phi}}(X,Y) = (\mathscr{L}_{\tilde{\phi}X}\eta)(Y) - (\mathscr{L}_{\tilde{\phi}Y}\eta)(X), \tag{2.14}$$

$$N^{(3)}_{\tilde{\phi}}(X) = (\mathscr{L}_{\xi}\tilde{\phi})X, \tag{2.15}$$

$$N^{(4)}(X) = (\mathscr{L}_{\xi}\eta)(X), \tag{2.16}$$

The almost paracontact structure is then normal if and only if these four tensors vanish. However, as it is shown in [28], the vanishing of $N^{(1)}_{\tilde{\phi}}$ implies the vanishing of the remaining tensors.

Any almost paracontact manifold admits a semi-Riemannian metric $\tilde{g}$ such that

$$\tilde{g}(\tilde{\phi}X,\tilde{\phi}Y) = -\tilde{g}(X,Y) + \eta(X)\eta(Y) \tag{2.17}$$

for all $X, Y\in\Gamma(TM)$. Then $(M,\tilde{\phi},\xi,\eta,\tilde{g})$ is called an *almost paracontact metric manifold.* Notice that any such a semi-Riemannian metric is necessarily of signature $(n+1,n)$. Moreover, as in the almost contact case, from (2.17) it follows easily that $\eta = g(\cdot,\xi)$ and $\tilde{g}(\cdot,\tilde{\phi}\cdot) = -\tilde{g}(\tilde{\phi}\cdot,\cdot)$. Hence one defines the *fundamental 2-form* of the almost paracontact metric manifold by $\tilde{\Phi}(X,Y) = \tilde{g}(X,\tilde{\phi}Y)$. If $d\eta = \tilde{\Phi}$, η becomes a contact form and $(M,\tilde{\phi},\xi,\eta,\tilde{g})$ is said to be a *paracontact metric manifold.*

On a paracontact metric manifold $(M, \tilde{\phi}, \xi, \eta, \tilde{g})$ one has

$$\nabla^{\tilde{g}} \xi = -\tilde{\phi} + \tilde{\phi}\tilde{h} \tag{2.18}$$

$$N_{\tilde{\phi}}^{(1)}(X, Y) = (\nabla^{\tilde{g}}_{\tilde{\phi}X}\tilde{\phi})Y - (\nabla^{\tilde{g}}_{\tilde{\phi}Y}\tilde{\phi})X + (\nabla^{\tilde{g}}_X\tilde{\phi})\tilde{\phi}Y - (\nabla^{\tilde{g}}_Y\tilde{\phi})\tilde{\phi}X + \eta(Y)\nabla^{\tilde{g}}_X\xi - \eta(X)\nabla^{\tilde{g}}_Y\xi \tag{2.19}$$

where $\tilde{h} := \frac{1}{2}N_{\tilde{\phi}}^{(3)}$. One proves (see [28]) that $\tilde{h}$ is symmetric with respect to $\tilde{g}$ and $\tilde{h}$ vanishes identically if and only if ξ is a Killing vector field and in such case $(M, \tilde{\phi}, \xi, \eta, \tilde{g})$ is called a *K-paracontact manifold*. By using (2.18) one can prove (cf. [12]) the formula

$$R^{\tilde{g}}(X, Y)\xi = -(\nabla^{\tilde{g}}_X\tilde{\phi})Y + (\nabla^{\tilde{g}}_Y\tilde{\phi})X + (\nabla^{\tilde{g}}_X\tilde{\phi})\tilde{h}Y + \tilde{\phi}((\nabla^{\tilde{g}}_X\tilde{h})Y) - (\nabla^{\tilde{g}}_Y\tilde{\phi})\tilde{h}X - \tilde{\phi}((\nabla^{\tilde{g}}_Y\tilde{h})X). \tag{2.20}$$

A normal paracontact metric manifold is said to be a *para-Sasakian manifold*. Also in this context the para-Sasakian condition implies the K-paracontact condition and the converse holds in dimension 3. In terms of the covariant derivative of $\tilde{\phi}$ the para-Sasakian condition may be expressed by

$$(\nabla^{\tilde{g}}_X\tilde{\phi})Y = -\tilde{g}(X, Y)\xi + \eta(Y)X. \tag{2.21}$$

In any paracontact metric manifold Zamkovoy introduced a canonical connection which plays the same role in paracontact geometry of the generalized Tanaka-Webster connection ([25]) in a contact metric manifold. In fact the following result holds.

Theorem 2.4 ([28]). *On a paracontact metric manifold there exists a unique connection* ∇^{pc}, *called the* canonical paracontact connection, *satisfying the following properties*:

(i) $\nabla^{pc}\eta = 0$, $\nabla^{pc}\xi = 0$, $\nabla^{pc}\tilde{g} = 0$,
(ii) $(\nabla^{pc}_X\tilde{\phi})Y = (\nabla^{\tilde{g}}_X\tilde{\phi})Y - \eta(Y)(X - \tilde{h}X) + \tilde{g}(X - \tilde{h}X, Y)\xi$,
(iii) $T^{pc}(\xi, \tilde{\phi}Y) = -\tilde{\phi}\tilde{T}^{pc}(\xi, Y)$,
(iv) $T^{pc}(X, Y) = 2\,d\eta(X, Y)\xi$ *for all* $X, Y \in \Gamma(\mathscr{D})$.

The explicit expression of this connection is the following

$$\nabla^{pc}_X Y = \nabla^{\tilde{g}}_X Y + \eta(X)\tilde{\phi}Y + \eta(Y)(\tilde{\phi}X - \tilde{\phi}\tilde{h}X) + \tilde{g}(X - \tilde{h}X, \tilde{\phi}Y)\xi. \tag{2.22}$$

Moreover, the torsion tensor field is given by

$$T^{pc}(X, Y) = \eta(X)\tilde{\phi}\tilde{h}Y - \eta(Y)\tilde{\phi}\tilde{h}X + 2\tilde{g}(X, \tilde{\phi}Y)\xi. \tag{2.23}$$

If the paracontact metric connection preserves the structure tensor $\tilde{\phi}$, that is the Levi Civita connection satisfies

$$(\nabla^{\tilde{g}}_X\tilde{\phi})Y = \eta(Y)(X - \tilde{h}X) - \tilde{g}(X - \tilde{h}X, Y)\xi \tag{2.24}$$

for any $X, Y \in \Gamma(TM)$, then the paracontact metric structure $(\tilde{\phi}, \xi, \eta, \tilde{g})$ is said to be *integrable*. This is the case, in particular, when the eigendistributions $T^{\pm}$ of $\tilde{\phi}$ associated to the eigenvalues ± 1 are involutive. Moreover, from (2.24) and (2.21) it follows that any para-Sasakian manifold is integrable. By replacing (2.24) and (2.18) in (2.19) one can straightforwardly prove the following proposition.

Proposition 2.5. *In an integrable paracontact metric manifold*

$$N^{(1)}_{\tilde{\phi}}(X, Y) = 2(\eta(Y)\tilde{\phi}\tilde{h}X - \eta(X)\tilde{\phi}\tilde{h}Y). \tag{2.25}$$

Corollary 2.6. *Any integrable K-paracontact manifold is para-Sasakian.*

2.2. Bi-Legendrian manifolds. Let (M, η) be a $(2n+1)$-dimensional contact manifold. It is well known that the contact condition $\eta \wedge (d\eta)^n \neq 0$ geometrically means that the contact distribution $\mathscr{D}$ is as far as possible from being integrable. In fact one can prove that the maximal dimension of an involutive subbundle of $\mathscr{D}$ is n. Such n-dimensional integrable distributions are called *Legendre foliations* of (M, η). More generally a *Legendre distribution* on a contact manifold (M, η) is an n-dimensional subbundle L of the contact distribution not necessarily integrable but verifying the weaker condition that $d\eta(X, X') = 0$ for all $X, X' \in \Gamma(L)$. The theory of Legendre foliations has been extensively investigated in recent years from various points of views. In particular Pang ([22]) provided a classification of Legendre foliations by using a bilinear symmetric form $\Pi_{\mathscr{F}}$ on the tangent bundle of the foliation $\mathscr{F}$, defined by

$$\Pi_{\mathscr{F}}(X, X') = -(\mathscr{L}_X \mathscr{L}_{X'}\eta)(\xi) = 2\, d\eta([\xi, X], X'). \tag{2.26}$$

He called a Legendre foliation *positive* (*negative*) *definite*, *non-degenerate*, *degenerate* or *flat* according to the circumstance that the bilinear form $\Pi_{\mathscr{F}}$ is positive (negative) definite, non-degenerate, degenerate or vanishes identically, respectively. By (2.26) it follows that $\mathscr{F}$ is flat if and only if ξ is "foliate", i.e. $[\xi, X] \in \Gamma(T\mathscr{F})$ for any $X \in \Gamma(T\mathscr{F})$.

If (M, η) is endowed with two transversal Legendre distributions L_1 and L_2, we say that (M, η, L_1, L_2) is an *almost bi-Legendrian manifold*. Thus, in particular, the tangent bundle of M splits up as the direct sum $TM = L_1 \oplus L_2 \oplus \mathbf{R}\xi$. When both L_1 and L_2 are integrable we refer to a *bi-Legendrian manifold*. An (almost) bi-Legendrian manifold is said to be flat, degenerate or non-degenerate if and only if both the Legendre distributions are flat, degenerate or non-degenerate, respectively. Any contact manifold (M, η) endowed with a Legendre distribution L admits a canonical almost bi-Legendrian structure. Indeed let (ϕ, ξ, η, g) be a compatible contact metric structure. Then the relation $d\eta(\phi X, \phi Y) = \Phi(\phi X, \phi Y) = d\eta(X, Y)$ easily implies that $Q := \phi L$ is a Legendre distribution on M which is g-orthogonal to L. Q is usually referred as the *conjugate Legendre distribution* of L and in general is not involutive, even if L is. In [7] the existence of a canonical connection on an almost bi-Legendrian manifold has been proven:

THEOREM 2.7 ([7]). *Let* (M, η, L_1, L_2) *be an almost bi-Legendrian manifold. There exists a unique linear connection* ∇^{bl} *called the* bi-Legendrian connection, *satisfying the following properties*:

(i) $\nabla^{bl} L_1 \subset L_1$, $\nabla^{bl} L_2 \subset L_2$,

(ii) $\nabla^{bl} \xi = 0$, $\nabla^{bl}\, d\eta = 0$,

(iii) $T^{bl}(X, Y) = 2\, d\eta(X, Y)\xi$ *for all* $X \in \Gamma(L_1)$, $Y \in \Gamma(L_2)$, $T^{bl}(X, \xi) = [\xi, X_{L_1}]_{L_2} + [\xi, X_{L_2}]_{L_1}$ *for all* $X \in \Gamma(TM)$,

where X_{L_1} *and* X_{L_2} *denote the projections of* X *onto the subbundles* L_1 *and* L_2 *of* TM, *respectively. Furthermore, the torsion tensor field* T^{bl} *of* ∇^{bl} *is explicitly given by*

$$(2.27)\quad T^{bl}(X, Y) = -[X_{L_1}, Y_{L_1}]_{L_2 \oplus \mathbf{R}\xi} - [X_{L_2}, Y_{L_2}]_{L_1 \oplus \mathbf{R}\xi} + 2\, d\eta(X, Y)\xi + \eta(Y)([\xi, X_{L_1}]_{L_2} + [\xi, X_{L_2}]_{L_1}) - \eta(X)([\xi, Y_{L_1}]_{L_2} + [\xi, Y_{L_2}]_{L_1}).$$

In [10] the interplays between paracontact geometry and the theory of bi-Legendrian structures have been studied. More precisely it has been proven the existence of a biunivocal correspondence $\Psi : \mathscr{B} \to \mathscr{P}$ between the set $\mathscr{B}$ of almost bi-Legendrian structures and the set $\mathscr{P}$ of paracontact metric structures on the same contact manifold (M, η). This bijection maps bi-Legendrian structures onto integrable paracontact structures, flat almost bi-Legendrian structures onto K-paracontact structures and flat bi-Legendrian structures onto para-Sasakian structures. For the convenience of the reader we recall more explicitly how the above biunivocal correspondence is defined. If (L_1, L_2) is an almost bi-Legendrian structure on (M, η), the corresponding paracontact metric structure $(\tilde{\phi}, \xi, \eta, \tilde{g}) = \Psi(L_1, L_2)$ is given by

$$(2.28)\quad \tilde{\phi}|_{L_1} = I, \quad \tilde{\phi}|_{L_2} = -I, \quad \tilde{\phi}\xi = 0, \quad \tilde{g} := d\eta(\cdot, \tilde{\phi}\cdot) + \eta \otimes \eta.$$

Moreover, the relationship between the bi-Legendrian and the canonical paracontact connections has been investigated, proving that in the integrable case they in fact coincide:

THEOREM 2.8 ([10]). *Let* (M, η, L_1, L_2) *be an almost bi-Legendrian manifold and let* $(\tilde{\phi}, \xi, \eta, \tilde{g}) = \Psi(L_1, L_2)$ *be the paracontact metric structure induced on* M *by* (2.28). *Let* ∇^{bl} *and* ∇^{pc} *be the corresponding bi-Legendrian and canonical paracontact connections. Then*

(a) $\nabla^{bl} \tilde{\phi} = 0$, $\nabla^{bl} \tilde{g} = 0$,

(b) *the bi-Legendrian and the canonical paracontact connections coincide if and only if the induced paracontact metric structure is integrable.*

3. Almost bi-paracontact structures on contact manifolds

DEFINITION 3.1. Let (M, η) be a contact manifold. An *almost bi-paracontact structure* on (M, η) is a triplet (ϕ_1, ϕ_2, ϕ_3) where ϕ_3 is an almost contact structure compatible with η, and ϕ_1, ϕ_2 are two anti-commuting tensors on M such that $\phi_1^2 = \phi_2^2 = I - \eta \otimes \xi$ and $\phi_1 \phi_2 = \phi_3$.

The manifold M endowed with such a geometrical structure is called an *almost bi-paracontact manifold*. From the definition it easily follows that $\phi_1\phi_3 = -\phi_3\phi_1 = \phi_2$ and $\phi_3\phi_2 = -\phi_2\phi_3 = \phi_1$.

For each $\alpha \in \{1,2,3\}$ let $\mathscr{D}_\alpha^+$ and $\mathscr{D}_\alpha^-$ denote the eigendistributions of ϕ_α corresponding, respectively, to the eigenvalues 1 and -1. Notice that, as $\phi_\alpha\xi = 0$, $\mathscr{D}_\alpha^+$ and $\mathscr{D}_\alpha^-$ are in fact subbundles of the contact distribution. In the following proposition we collect some properties of those distributions.

Proposition 3.2. *Let $(M, \eta, \phi_1, \phi_2, \phi_3)$ be an almost bi-paracontact manifold. Then*

1. $\phi_1(\mathscr{D}_2^+) = \mathscr{D}_2^-$, $\phi_1(\mathscr{D}_2^-) = \mathscr{D}_2^+$,
2. $\phi_2(\mathscr{D}_1^+) = \mathscr{D}_1^-$, $\phi_2(\mathscr{D}_1^-) = \mathscr{D}_1^+$,
3. $\phi_3(\mathscr{D}_\alpha^+) = \mathscr{D}_\alpha^-$, $\phi_3(\mathscr{D}_\alpha^-) = \mathscr{D}_\alpha^+$ *for each* $\alpha \in \{1,2\}$,
4. $\phi_1 : \mathscr{D}_2^+ \to \mathscr{D}_2^-$ *and* $\phi_2 : \mathscr{D}_1^+ \to \mathscr{D}_1^-$ *are isomorphisms*,
5. *the tangent bundle of M splits up as the direct sum* $TM = \mathscr{D}_\alpha^+ \oplus \mathscr{D}_\alpha^- \oplus \mathbf{R}\xi = \mathscr{D}_\alpha^\pm \oplus \mathscr{D}_\beta^\pm \oplus \mathbf{R}\xi$ *for all* $\alpha, \beta \in \{1,2\}$, $\alpha \neq \beta$,
6. $\dim(\mathscr{D}_1^+) = \dim(\mathscr{D}_1^-) = \dim(\mathscr{D}_2^+) = \dim(\mathscr{D}_2^-) = n$. *In particular, ϕ_1 and ϕ_2 are almost paracontact structures.*

Proof. For any $X \in \Gamma(\mathscr{D}_2^+)$ one has $\phi_2\phi_1 X = -\phi_1\phi_2 X = -\phi_1 X$, so that $\phi_1(\mathscr{D}_2^+) \subset \mathscr{D}_2^-$. On the other hand, let Y be a vector field tangent to $\mathscr{D}_2^-$ and set $X := \phi_1 Y$. Then $\phi_1 X = \phi_1^2 Y = Y$, so that it remains only to prove that $X \in \Gamma(\mathscr{D}_2^+)$. Indeed, $\phi_2 X = \phi_2\phi_1 Y = -\phi_1\phi_2 Y = \phi_1 Y = X$. Thus $\phi_1(\mathscr{D}_2^+) = \mathscr{D}_2^-$ and analogously one can prove that $\phi_1(\mathscr{D}_2^-) = \mathscr{D}_2^+$. In a similar way one proves the other identities, as well as the fourth property. In order to prove the fifth property it is enough to show that $\mathscr{D} = \mathscr{D}_\alpha^+ \oplus \mathscr{D}_\alpha^- = \mathscr{D}_\alpha^\pm \oplus \mathscr{D}_\beta^\pm$ for all $\alpha, \beta \in \{1,2\}$. Let us consider the case $\alpha = 1$. Then we can decompose every $X \in \Gamma(\mathscr{D})$ as $X = \frac{1}{2}(X - \phi_1 X) + \frac{1}{2}(X + \phi_1 X)$. An immediate computation shows that $\frac{1}{2}(X - \phi_1 X) \in \mathscr{D}_1^-$ and $\frac{1}{2}(X + \phi_1 X) \in \mathscr{D}_1^+$. Next, if $X \in \mathscr{D}_1^+ \cap \mathscr{D}_1^-$ then $\phi_1 X = X = -\phi_1 X$, from which it follows that $\phi_1^2 X = -\phi_1^2 X$, hence $X = 0$. Thus $\mathscr{D} = \mathscr{D}_1^+ \oplus \mathscr{D}_1^-$. In a similar way one can prove that $\mathscr{D} = \mathscr{D}_2^+ \oplus \mathscr{D}_2^-$. Now we prove the identity $\mathscr{D} = \mathscr{D}_1^+ \oplus \mathscr{D}_2^+$. If $X \in \mathscr{D}_1^+ \cap \mathscr{D}_2^+$ then $\phi_1 X = X = \phi_2 X$, hence $X = \phi_1\phi_2 X = \phi_3 X$ and this implies that $X = 0$. On the other hand, note that from 4, since $\mathscr{D} = \mathscr{D}_\alpha^+ \oplus \mathscr{D}_\alpha^-$, $\alpha \in \{1,2\}$, it follows that, for each $\alpha \in \{1,2\}$, $\dim(\mathscr{D}_\alpha^+) = \dim(\mathscr{D}_\alpha^-) = n$. Hence $\dim(\mathscr{D}_1^+ + \mathscr{D}_2^+) = 2n$ and we conclude that $\mathscr{D} = \mathscr{D}_1^+ \oplus \mathscr{D}_2^+$. The other identities can be proven similarly. □

Proposition 3.3. *In any almost bi-paracontact manifold one has $\mathscr{D}_1^\pm = \{X + \phi_3 X \mid X \in \mathscr{D}_2^\pm\}$ and $\mathscr{D}_2^\pm = \{X + \phi_3 X \mid X \in \mathscr{D}_1^\mp\}$.*

Proof. We show that $\mathscr{D}_1^+ = \{X + \phi_3 X \mid X \in \mathscr{D}_2^+\}$ by proving the two inclusions. Let $Y \in \mathscr{D}_1^+$. We have to prove the existence of $X \in \mathscr{D}_2^+$ such that $Y = X + \phi_3 X$. We put $X := \frac{1}{2}(Y - \phi_3 Y)$. Firstly we verify that in fact $X \in \mathscr{D}_2^+$. We have $\phi_2 X = \frac{1}{2}(\phi_2 Y - \phi_2\phi_3 Y) = \frac{1}{2}(\phi_2 Y + \phi_1 Y) = \frac{1}{2}(\phi_2 Y + Y) = \frac{1}{2}(Y + \phi_2\phi_1 Y)$

$= \frac{1}{2}(Y - \phi_1\phi_2 Y) = X$, hence $X \in \mathscr{D}_2^+$. Next, one can easily check that $Y = X + \phi_3 X$. Conversely, let X be a vector field belonging to $\mathscr{D}_2^+$. Then $\phi_1(X + \phi_3 X) = \phi_1 X + \phi_1\phi_3 X = \phi_1 X + \phi_2 X = \phi_3\phi_2 X + X = \phi_3 X + X$, so that $X + \phi_3 X \in \mathscr{D}_1^+$. In a similar manner one can prove the other equality. □

Example 3.4. Consider $\mathbf{R}^{2n+1}$ with global coordinates $\{x_1, \dots, x_n, y_1, \dots, y_n, z\}$ and the standard contact form $\eta = dz - \sum_{i=1}^n y_i\, dx_i$. Put, for each $i \in \{1, \dots, n\}$, $X_i := \dfrac{\partial}{\partial y_i}$ and $Y_i := \dfrac{\partial}{\partial x_i} + y_i \dfrac{\partial}{\partial z}$. Then the contact distribution $\mathscr{D}$ is spanned by the vector fields $X_1, \dots, X_n$, $Y_1, \dots, Y_n$. We define three tensor fields ϕ_1, ϕ_2, ϕ_3 by setting

$$\phi_1 X_i := X_i, \quad \phi_1 Y_i := -Y_i, \quad \phi_1\xi := 0,$$
$$\phi_2 X_i := -Y_i, \quad \phi_2 Y_i := -X_i, \quad \phi_2\xi := 0,$$
$$\phi_3 X_i := Y_i, \quad \phi_3 Y_i := -X_i, \quad \phi_3\xi := 0,$$

for all $i \in \{1, \dots, n\}$. Some straightforward computations show that (ϕ_1, ϕ_2, ϕ_3) defines a bi-paracontact structure on the contact manifold $(\mathbf{R}^{2n+1}, \eta)$. In this case the canonical distributions $\mathscr{D}_1^+$, $\mathscr{D}_1^-$, $\mathscr{D}_2^+$, $\mathscr{D}_2^-$ are given by

$$\mathscr{D}_1^+ = \operatorname{span}\{X_1, \dots, X_n\}, \quad \mathscr{D}_1^- = \operatorname{span}\{Y_1, \dots, Y_n\},$$
$$\mathscr{D}_2^+ = \operatorname{span}\{X_1 - Y_1, \dots, X_n - Y_n\}, \quad \mathscr{D}_2^- = \operatorname{span}\{X_1 + Y_1, \dots, X_n + Y_n\}.$$

In order to find some more examples we prove the following proposition.

Proposition 3.5. *Let (M, ϕ, ξ, η, g) be a contact metric manifold endowed with a Legendre distribution L. Then M admits a canonical almost bi-paracontact structure.*

Proof. Let Q be the conjugate Legendre distribution of L, i.e. the Legendre distribution on M defined by $Q := \phi(L)$ (see §2.2). We define the $(1,1)$-tensor field ψ on M by setting $\psi|_L = I$, $\psi|_Q = -I$, $\psi\xi = 0$. Then if we put $\phi_1 := \phi\psi$, $\phi_2 := \psi$, $\phi_3 := \phi$, it is not difficult to check that (ϕ_1, ϕ_2, ϕ_3) is in fact an almost bi-paracontact structure on (M, η). □

As a consequence of Proposition 3.5 we obtain a canonical almost bi-paracontact structure on the cotangent sphere bundle of a Riemannian manifold (M, g) and on any contact metric (κ, μ)-space ([4]). We will examine carefully this last example in the last section of the paper.

Definition 3.6. An almost bi-paracontact structure such that $\mathscr{D}_1^\pm$ and $\mathscr{D}_2^\pm$ are Legendre distributions is called a *Legendrian bi-paracontact structure*. If $\mathscr{D}_1^\pm$ and $\mathscr{D}_2^\pm$ define Legendre foliations of (M, η) then the almost bi-paracontact structure is called *integrable*.

We present some characterizations of the integrability of an almost bi-paracontact manifold.

PROPOSITION 3.7. *An almost bi-paracontact structure* (ϕ_1, ϕ_2, ϕ_3) *is Legendrian if and only if for each* $\alpha \in \{1, 2\}$ *the tensor field* $N^{(2)}_{\phi_\alpha}$ *vanishes identically. Furtheremore, in any Legendrian almost bi-paracontact structure also the tensor field* $N^{(2)}_{\phi_3}$ *vanishes identically. In particular, one has, for any* $X, Y \in \Gamma(TM)$,

$$d\eta(\phi_1 X, \phi_1 Y) = d\eta(\phi_2 X, \phi_2 Y) = -d\eta(\phi_3 X, \phi_3 Y) = -d\eta(X, Y) \tag{3.1}$$

Proof. First of all we have, for all $X \in \Gamma(\mathscr{D})$, $N^{(2)}_{\phi_\alpha}(\xi, X) = -\eta([\xi, \phi_\alpha X]) = 2\, d\eta(\xi, \phi_\alpha X) = 0$ by (2.1). Next, in order to prove that $N^{(2)}_{\phi_\alpha}$ vanishes on $\mathscr{D}$, we distinguish the cases $X, Y \in \Gamma(\mathscr{D}^+_\alpha)$, $X, Y \in \Gamma(\mathscr{D}^-_\alpha)$ and $X \in \Gamma(\mathscr{D}^\pm_\alpha)$, $Y \in \Gamma(\mathscr{D}^\mp_\alpha)$. In the first case we have $N^{(2)}_{\phi_\alpha}(X, Y) = \phi_\alpha X(\eta(Y)) - \eta([\phi_\alpha X, Y]) - \phi_\alpha Y(\eta(X)) + \eta([\phi_\alpha Y, X]) = 2\, d\eta(\phi_\alpha X, Y) + 2\, d\eta(X, \phi_\alpha Y) = 4\, d\eta(X, Y) = 0$, where the last equality is due to the fact that $\mathscr{D}^+_\alpha$ is a Legendre distribution. The case $X, Y \in \Gamma(\mathscr{D}^-_\alpha)$ is similar. Next, for any $X \in \Gamma(\mathscr{D}^\pm_\alpha)$, $Y \in \Gamma(\mathscr{D}^\mp_\alpha)$, we have $N^{(2)}_{\phi_\alpha}(X, Y) = -\eta([\phi_\alpha X, Y]) + \eta([\phi_\alpha Y, X]) = \mp\eta([X, Y]) \pm \eta([X, Y]) = 0$. Conversely, if $N^{(2)}_\alpha \equiv 0$ then, for any $X, Y \in \Gamma(\mathscr{D}^+_\alpha)$, $0 = N^{(2)}_\alpha(X, Y) = 2\, d\eta(\phi_\alpha X, Y) + 2\eta(X, \phi_\alpha Y) = 4\, d\eta(X, Y)$, so that $d\eta(X, Y) = 0$. Consequently, as, by Proposition 3.2, $\mathscr{D}^+_\alpha$ is n-dimensional, it is a Legendre distribution. In a similar way one can prove that also $\mathscr{D}^-_\alpha$ is a Legendre distribution. In order to prove the second part of the proposition, notice that since $N^{(2)}_{\phi_1}$ and $N^{(2)}_{\phi_2}$ vanish, for each $\alpha \in \{1, 2\}$, $d\eta(\phi_\alpha \cdot, \cdot) = -d\eta(\cdot, \phi_\alpha \cdot)$. Now, for any $X, Y \in \Gamma(TM)$, $d\eta(\phi_3 X, Y) = d\eta(\phi_1 \phi_2 X, Y) = -d\eta(\phi_2 X, \phi_1 Y) = d\eta(X, \phi_2 \phi_1 Y) = -d\eta(X, \phi_3 Y)$. Hence,

$$\begin{aligned} N^{(2)}_{\phi_3}(X, Y) &= \phi_3 X(\eta(Y)) - \eta([\phi_3 X, Y]) - \phi_3 Y(\eta(X)) + \eta([\phi_3 Y, X]) \\ &= 2\, d\eta(\phi_3 X, Y) - 2\, d\eta(\phi_3 Y, X) = 0. \end{aligned}$$

□

PROPOSITION 3.8. *An almost bi-paracontact structure* (ϕ_1, ϕ_2, ϕ_3) *is Legendrian* (*respectively, integrable*) *if and only if, for each* $\alpha \in \{1, 2\}$, $N^{(1)}_{\phi_\alpha}(X, X') \in \Gamma(\mathscr{D}^\mp_\alpha)$ (*respectively,* $N^{(1)}_{\phi_\alpha}(X, X') = 0$) *for any* $X, X' \in \Gamma(\mathscr{D}^\pm_\alpha)$.

Proof. By (2.13) we have, for any $X, X' \in \Gamma(\mathscr{D}^+_\alpha)$,

$$\begin{aligned} N^{(1)}_{\phi_\alpha}(X, X') &= [X, X] + [X, X] - \phi_\alpha[X, X'] - \phi_\alpha[X, X'] \\ &= 2[X, X'] - 2\phi_\alpha[X, X']. \end{aligned} \tag{3.2}$$

Hence, applying ϕ_α one obtains

$$\begin{aligned} \phi_\alpha N^{(1)}_{\phi_\alpha}(X, X') &= 2\phi_\alpha[X, X'] - 2[X, X'] + 2\eta([X, X'])\xi \\ &= -N^{(1)}_{\phi_\alpha}(X, X') - 4\, d\eta(X, X')\xi \end{aligned} \tag{3.3}$$

Then (3.3) implies that $d\eta(X, X') = 0$ if and only if $N^{(1)}_{\phi_\alpha}(X, X') \in \Gamma(\mathscr{D}^-_\alpha)$ and (3.2) that $\mathscr{D}^+_\alpha$ is involutive if and only if $N^{(1)}_{\phi_\alpha}(X, X') = 0$. Analogous arguments work for $\mathscr{D}^-_\alpha$. □

Corollary 3.9. *An almost bi-paracontact structure* (ϕ_1, ϕ_2, ϕ_3) *is integrable if and only if the tensor fields* $N^{(1)}_{\phi_1}$, $N^{(1)}_{\phi_2}$ *vanish on the contact distribution* $\mathscr{D}$. *Furthermore, in an integrable almost bi-paracontact manifold also the tensor field* $N^{(1)}_{\phi_3}$ *vanishes on* $\mathscr{D}$.

Proof. The proof is trivial in one direction. Conversely, notice that, for any $X \in \Gamma(\mathscr{D}^+_\alpha)$, $Y \in \Gamma(\mathscr{D}^-_\alpha)$, $N^{(1)}_{\phi_\alpha}(X, Y) = [X, Y] + [X, -Y] - \phi_\alpha[X, Y] - \phi_\alpha[X, -Y] = 0$. Then by Proposition 3.8 we have that $N^{(1)}_{\phi_1}$ and $N^{(1)}_{\phi_2}$ vanish on $\mathscr{D}$. Now for ending the proof it remains to demonstrate that if $N^{(1)}_{\phi_1}$ and $N^{(1)}_{\phi_2}$ vanish on $\mathscr{D}$ then also $N^{(1)}_{\phi_3}$ vanishes on $\mathscr{D}$. Let X, X' be sections of $\mathscr{D}^+_1$. By Proposition 3.2, $\phi_2 X$ and $\phi_2 X'$ are sections of $\mathscr{D}^-_1$. Then the integrability of $\mathscr{D}^+_1$ and $\mathscr{D}^-_1$ yields

$$
\begin{aligned}
(3.4) \qquad 0 &= \phi_1 N^{(1)}_{\phi_2}(X, X') \\
&= \phi_1[X, X'] + \phi_1[\phi_2 X, \phi_2 X'] - \phi_3[\phi_2 X, X'] - \phi_3[X, \phi_2 X'] \\
&= [X, X'] - [\phi_2 X, \phi_2 X'] - \phi_3[\phi_2 X, X'] - \phi_3[X, \phi_2 X'].
\end{aligned}
$$

Using (3.4) we have that

$$
\begin{aligned}
N^{(1)}_{\phi_3}(X, X') &= -[X, X'] + [\phi_3\phi_1 X, \phi_3\phi_1 X'] - \phi_3[\phi_3\phi_1 X, X'] - \phi_3[X, \phi_3\phi_1 X'] \\
&= -[X, X'] + [\phi_2 X, \phi_2 X'] + \phi_3[\phi_2 X, X'] + \phi_3[X, \phi_2 X'] = 0.
\end{aligned}
$$

Arguing in the same way one can prove that $N^{(1)}_{\phi_3}(Y, Y') = 0$ for all $Y, Y' \in \Gamma(\mathscr{D}^-_1)$. Next, for any $X \in \Gamma(\mathscr{D}^+_1)$ and $X \in \Gamma(\mathscr{D}^-_1)$, by (2.7) we get

$$\phi_3 N^{(1)}_{\phi_3}(X, Y) = -N^{(1)}_{\phi_3}(\phi_3 X, Y) + 2(d\eta(\phi_3 X, Y) + d\eta(X, \phi_3 Y))\xi = 0,$$

because $\phi_3 \mathscr{D}^\pm_1 = \mathscr{D}^\mp$ and by (3.1). On the other hand, since the almost bi-paracontact structure (ϕ_1, ϕ_2, ϕ_3) is integrable, in particular Legendrian, $\eta(N^{(1)}_{\phi_3}(X, Y)) = -\eta([X, Y]) + \eta([\phi_3 X, \phi_3 Y]) = N^{(2)}_{\phi_3}(X, \phi_3 Y) = 0$ by Proposition 3.7. Therefore, as $\mathscr{D} = \mathscr{D}^+_1 \oplus \mathscr{D}^-_1$, we conclude that $N^{(1)}_{\phi_3}(Z, Z') = 0$ for any $Z, Z' \in \Gamma(\mathscr{D})$. □

A notion stronger than integrability is that of "normal almost bi-paracontact structure".

Definition 3.10. Let $(M, \eta, \phi_1, \phi_2, \phi_3)$ be an almost bi-paracontact manifold. If the almost paracontact structures (ϕ_1, ξ, η), (ϕ_2, ξ, η) and the almost contact structure (ϕ_3, ξ, η) are normal, i.e. $N^{(1)}_{\phi_\alpha} = 0$ for each $\alpha \in \{1, 2, 3\}$, (ϕ_1, ϕ_2, ϕ_3) is called a *normal almost bi-paracontact structure*.

By arguing as in Corollary 3.9 one can prove that if $N^{(1)}_{\phi_1}$ and $N^{(1)}_{\phi_2}$ vanish identically, then also $N^{(1)}_{\phi_3} = 0$ and the almost bi-paracontact structure is normal. Moreover, since, for each $\alpha \in \{1, 2\}$ and any $X \in \Gamma(\mathscr{D})$, $N^{(1)}_{\phi_\alpha}(\xi, X) = N^{(3)}_{\phi_\alpha}(\phi_\alpha X)$, using Corollary 3.9 one can prove the following proposition.

Proposition 3.11. *An almost bi-paracontact structure is normal if and only if it is integrable and* $N^{(3)}_{\phi_1}$ *and* $N^{(3)}_{\phi_2}$ *vanish identically.*

As a consequence we are able to give a geometrical interpretation to normality in terms of Legendre foliations.

Corollary 3.12. *An almost bi-paracontact structure is normal if and only, for each* $\alpha \in \{1, 2\}$, *both* $\mathscr{D}^+_\alpha$ *and* $\mathscr{D}^-_\alpha$ *are flat Legendre foliations.*

Proof. Taking the definition of $N^{(3)}_{\phi_\alpha}$ into account, one can easily prove that ξ is foliate with respect both to $\mathscr{D}^+_\alpha$ and $\mathscr{D}^-_\alpha$ if and only if $N^{(3)}_{\phi_\alpha} = 0$. Then the assertion follows from this remark and Proposition 3.11. □

Thus we have seen that, under some natural assumptions, an almost bi-paracontact structure on a contact manifold gives rise to a pair of transverse (almost) bi-Legendrian structures $(\mathscr{D}^+_1, \mathscr{D}^-_1)$ and $(\mathscr{D}^+_2, \mathscr{D}^-_2)$. Conversely we have the following result.

Proposition 3.13. *Let* (L, Q) *and* (L', Q') *be two transverse almost bi-Legendrian structures on the contact manifold* (M, η). *Then there exists a Legendrian almost bi-paracontact structure* (ϕ_1, ϕ_2, ϕ_3) *such that* L, Q *and* L', Q' *are, respectively, the eigendistributions of* ϕ_1 *and* ϕ_2.

Proof. We define $\phi_1|_L = I$, $\phi_1|_Q = -I$, $\phi_1\xi = 0$ and $\phi_2|_{L'} = I$, $\phi_2|_{Q'} = -I$, $\phi_2\xi = 0$. Then we set $\phi_3 := \phi_1\phi_2$. One can easily check that (ϕ_1, ϕ_2, ϕ_3) is in fact an almost bi-paracontact structure on (M, η) such that, by construction, $\mathscr{D}^+_1 = L$, $\mathscr{D}^-_1 = Q$ and $\mathscr{D}^+_2 = L'$, $\mathscr{D}^-_2 = Q'$. In particular, (ϕ_1, ϕ_2, ϕ_3) is Legendrian and it is integrable if and only if L, Q, L', Q' are involutive. □

4. Canonical connections on bi-paracontact manifolds

In this section we attach to any almost bi-paracontact manifold some canonical connections and then we study their nice properties. To this end, we prove the following preliminary lemma.

Lemma 4.1. *Let* (ϕ_1, ϕ_2, ϕ_3) *be an almost bi-paracontact structure on the contact manifold* (M, η). *For each* $\alpha \in \{1, 2, 3\}$ *let* h_α *be the operator defined by* $h_\alpha := \frac{1}{2}\mathscr{L}_\xi\phi_\alpha = \frac{1}{2}N^{(3)}_{\phi_\alpha}$. *Then*

(a) $h_\alpha\phi_\alpha = -\phi_\alpha h_\alpha$ *for each* $\alpha \in \{1,2,3\}$,
(b) $\phi_1 h_2 + h_1\phi_2 = h_3 = -h_2\phi_1 - \phi_2 h_1$, $\phi_1 h_3 + h_1\phi_3 = h_2 = -h_3\phi_1 - \phi_3 h_1$, $\phi_2 h_3 + h_2\phi_3 = -h_1 = -h_3\phi_2 - \phi_3 h_2$.

Proof. (a) Let us assume that $\alpha \in \{1,2\}$. Then $(\mathscr{L}_\xi\phi_\alpha)\circ\phi_\alpha + \phi_\alpha\circ(\mathscr{L}_\xi\phi_\alpha) = \mathscr{L}_\xi(\phi_\alpha^2) = \mathscr{L}_\xi(I - \eta\otimes\xi) = -(\mathscr{L}_\xi\eta)\otimes\xi - \eta\otimes(\mathscr{L}_\xi\xi) = 0$, since $\mathscr{L}_\xi\eta = i_\xi\, d\eta + di_\xi\eta = 0$ by (2.1). Thus $h_\alpha\circ\phi_\alpha = -\phi_\alpha\circ h_\alpha$. The case $\alpha = 3$ is similar.

(b) $2h_3 = \mathscr{L}_\xi\phi_3 = \mathscr{L}_\xi(\phi_1\phi_2) = (\mathscr{L}_\xi\phi_1)\phi_2 + \phi_1(\mathscr{L}_\xi\phi_2) = 2h_1\phi_2 + 2\phi_1 h_2$. The other equalities can be proved in a similar way. □

THEOREM 4.2. *Let* (ϕ_1,ϕ_2,ϕ_3) *be an almost bi-paracontact structure on the contact manifold* (M,η). *For each* $\alpha\in\{1,2,3\}$ *there exists a unique linear connection* ∇^α *on* M *satisfying the following properties*:
(i) $\nabla^\alpha\xi = 0$,
(ii) $\nabla^1\phi_1 = 0$, $\nabla^1\phi_2 = \eta\otimes(2h_2 - h_1\phi_3 + \phi_3 h_1)$, $\nabla^1\phi_3 = \eta\otimes(2h_3 - h_1\phi_2 + \phi_2 h_1)$, $\nabla^2\phi_1 = \eta\otimes(2h_1 + h_2\phi_3 - \phi_3 h_2)$, $\nabla^2\phi_2 = 0$, $\nabla^2\phi_3 = \eta\otimes(2h_3 + h_2\phi_1 - \phi_1 h_2)$, $\nabla^3\phi_1 = \eta\otimes(2h_1 - h_3\phi_2 + \phi_2 h_3)$, $\nabla^3\phi_2 = \eta\otimes(2h_2 + h_3\phi_1 - \phi_1 h_3)$, $\nabla^3\phi_3 = 0$,
(iii) $T^\alpha(\phi_\alpha X, Y) - T^\alpha(X,\phi_\alpha Y) = 2(d\eta(\phi_\alpha X, Y) - d\eta(X,\phi_\alpha Y))\xi + \eta(Y)h_\alpha X + \eta(X)h_\alpha Y$ *for any* $X, Y \in \Gamma(TM)$,

where T^α *denotes the torsion tensor field of* ∇^α. ∇^1, ∇^2, ∇^3 *are explicitly given by*

$$\begin{aligned}\nabla^1_X Y = \frac{1}{4}(&[X,Y] - [\phi_1 X,\phi_1 Y] + \phi_1[X,\phi_1 Y] - \phi_1[\phi_1 X, Y] + \phi_2[X,\phi_2 Y]\\ &- \phi_3[X,\phi_3 Y] + \phi_3[\phi_1 X,\phi_2 Y] - \phi_2[\phi_1 X,\phi_3 Y]\\ &+ 2\eta(X)(-h_1\phi_1 Y + h_2\phi_2 Y - h_3\phi_3 Y) + 2\eta(Y)h_1\phi_1 X\\ &- \eta([X,Y])\xi + \eta([\phi_1 X,\phi_1 Y])\xi) + X(\eta(Y))\xi,\end{aligned} \tag{4.1}$$

$$\begin{aligned}\nabla^2_X Y = \frac{1}{4}(&[X,Y] - [\phi_2 X,\phi_2 Y] + \phi_2[X,\phi_2 Y] - \phi_2[\phi_2 X, Y] + \phi_1[X,\phi_1 Y]\\ &- \phi_3[X,\phi_3 Y] - \phi_3[\phi_2 X,\phi_1 Y] + \phi_1[\phi_2 X,\phi_3 Y]\\ &+ 2\eta(X)(h_1\phi_1 Y - h_2\phi_2 Y - h_3\phi_3 Y) + 2\eta(Y)h_2\phi_2 X\\ &- \eta([X,Y])\xi + \eta([\phi_2 X,\phi_2 Y])\xi) + X(\eta(Y))\xi,\end{aligned} \tag{4.2}$$

$$\begin{aligned}\nabla^3_X Y = \frac{1}{4}(&[X,Y] + [\phi_3 X,\phi_3 Y] + \phi_1[X,\phi_1 Y] + \phi_2[X,\phi_2 Y] - \phi_3[X,\phi_3 Y]\\ &+ \phi_3[\phi_3 X, Y] + \phi_2[\phi_3 X,\phi_1 Y] - \phi_1[\phi_3 X,\phi_2 Y]\\ &+ 2\eta(X)(h_1\phi_1 Y + h_2\phi_2 Y + h_3\phi_3 Y) - 2\eta(Y)h_3\phi_3 X\\ &- \eta([X,Y])\xi - \eta([\phi_3 X,\phi_3 Y])\xi) + X(\eta(Y))\xi.\end{aligned} \tag{4.3}$$

Proof. First of all we prove the uniqueness. Fix an $\alpha\in\{1,2,3\}$ and suppose that ∇ and ∇' are two linear connections satisfying (i), (ii) and (iii). Let us define the tensor $A := \nabla - \nabla'$. For any $X, Y \in \Gamma(\mathscr{D})$, since both ∇ and ∇' preserve the almost bi-paracontact structure, one has

$$A(X, \phi_\beta Y) = \phi_\beta A(X, Y) \tag{4.4}$$

for each $\beta \in \{1, 2, 3\}$. Because of (i), we have $A(X, \xi) = 0$ for all $X \in \Gamma(TM)$. Next, for all $Y \in \Gamma(\mathscr{D})$,

$$\begin{aligned} A(\xi, Y) &= \nabla_\xi Y - \nabla'_\xi Y \\ &= \nabla_Y \xi + T(\xi, Y) + [\xi, Y] - \nabla'_Y \xi - T'(\xi, Y) - [\xi, Y] \\ &= \varepsilon(T(\xi, \phi_\alpha^2 Y) - T'(\xi, \phi_\alpha^2 Y)) \\ &= \varepsilon(T(\phi_\alpha \xi, \phi_\alpha Y) - 2(d\eta(\phi_\alpha \xi, \phi_\alpha Y) - d\eta(\xi, \phi_\alpha^2 Y))\xi - \eta(\phi_\alpha^2 Y) h_\alpha \xi \\ &\quad - \eta(\xi) h_\alpha \phi_\alpha^2 Y - T'(\phi_\alpha \xi, \phi_\alpha Y) + 2(d\eta(\phi_\alpha \xi, \phi_\alpha Y) \\ &\quad - d\eta(\xi, \phi_\alpha^2 Y))\xi + \eta(\phi_\alpha^2 Y) h_\alpha \xi + \eta(\xi) h_\alpha \phi_\alpha^2 Y) = 0, \end{aligned}$$

where we have applied (ii) and (iii), and we have put $\varepsilon = 1$ if $\alpha \in \{1, 2\}$, $\varepsilon = -1$ if $\alpha = 3$. Further, from (iii) it follows that $T(\phi_\alpha X, Y) - T(X, \phi_\alpha Y) = T'(\phi_\alpha X, Y) - T'(X, \phi_\alpha Y)$, that is $\nabla_{\phi_\alpha X} Y - \nabla_Y \phi_\alpha X - \nabla_X \phi_\alpha Y + \nabla_{\phi_\alpha Y} X = \nabla'_{\phi_\alpha X} Y - \nabla'_Y \phi_\alpha X - \nabla'_X \phi_\alpha Y + \nabla'_{\phi_\alpha Y} X$. Consequently,

$$A(\phi_\alpha X, Y) - A(Y, \phi_\alpha X) - A(X, \phi_\alpha Y) + A(\phi_\alpha Y, X) = 0. \tag{4.5}$$

If in (4.5) we take $X \in \Gamma(\mathscr{D}_\alpha^+)$ and $Y \in \Gamma(\mathscr{D}_\alpha^-)$ we obtain

$$A(X, Y) = A(Y, X). \tag{4.6}$$

By virtue of (ii), for each $Z \in \Gamma(\mathscr{D})$, ∇_Z and ∇'_Z preserve the distributions $\mathscr{D}_\alpha^\pm$. Thus $A(X, Y) \in \Gamma(\mathscr{D}_\alpha^-)$ and $A(Y, X) \in \Gamma(\mathscr{D}_\alpha^+)$. This together with (4.6) and 5. of Proposition 3.2 imply that

$$A(X, Y) = A(Y, X) = 0. \tag{4.7}$$

Now let us consider $X, X' \in \Gamma(\mathscr{D}_\alpha^+)$ and let $\beta \in \{1, 2\}$, $\beta \neq \alpha$. Note that, by 1.–2. of Proposition 3.2, $\phi_\beta X' \in \Gamma(\mathscr{D}_\alpha^-)$. Then, by (4.4) and (4.7), $A(X, X') = A(X, \phi_\beta^2 X') = \phi_\beta A(X, \phi_\beta X') = 0$. In a similar way one can prove that $A(X', X) = 0$. Thus the tensor A vanishes identically and so ∇ and ∇' coincide.

In order to prove the existence, for each $\alpha \in \{1, 2, 3\}$, of a connection ∇^α satisfying (i), (ii), (iii), we distinguish the cases $\alpha \in \{1, 2\}$ and $\alpha = 3$. Let us consider $\alpha \in \{1, 2\}$. First of all, we put, by definition, $\nabla^\alpha \xi := 0$. Next, notice that by (iii) we have that $T^\alpha(\phi_\alpha X, \xi) = h_\alpha X$, for all $X \in \Gamma(TM)$. In particular, for any $X \in \Gamma(\mathscr{D})$, $T^\alpha(X, \xi) = T(\phi_\alpha^2 X, \xi) = h_\alpha \phi_\alpha X$. It follows that necessarily

$$\nabla_\xi^\alpha X = -h_\alpha \phi_\alpha X + [\xi, X] \tag{4.8}$$

for all $X \in \Gamma(\mathscr{D})$. In particular,

$$\nabla_\xi^\alpha X = \begin{cases} [\xi, X]_{\mathscr{D}_\alpha^+}, & \text{if } X \in \Gamma(\mathscr{D}_\alpha^+); \\ [\xi, X]_{\mathscr{D}_\alpha^-}, & \text{if } X \in \Gamma(\mathscr{D}_\alpha^-). \end{cases} \tag{4.9}$$

Further, for any $X \in \Gamma(\mathscr{D}_\alpha^+)$ and $Y \in \Gamma(\mathscr{D}_\alpha^-)$,

$$\begin{aligned} T^\alpha(X, Y) &= T^\alpha(\phi_\alpha X, Y) \\ &= T^\alpha(X, \phi_\alpha Y) + 2(d\eta(\phi_\alpha X, Y) - d\eta(X, \phi_\alpha Y))\xi \\ &= -T^\alpha(X, Y) + 4\, d\eta(X, Y)\xi, \end{aligned}$$

from which it follows that $T^\alpha(X, Y) = 2\, d\eta(X, Y)\xi$. Hence, $2\, d\eta(X, Y)\xi = \nabla^\alpha_X Y - \nabla^\alpha_Y X - [X, Y]_{\mathscr{D}_\alpha^+} - [X, Y]_{\mathscr{D}_\alpha^-} - \eta([X, Y])\xi$, that is

$$\nabla^\alpha_X Y - [X, Y]_{\mathscr{D}_\alpha^-} = \nabla^\alpha_Y X - [X, Y]_{\mathscr{D}_\alpha^+}. \tag{4.10}$$

Since, due to (ii), $\nabla^\alpha_X Y \in \Gamma(\mathscr{D}_\alpha^-)$ and $\nabla^\alpha_Y X \in \Gamma(\mathscr{D}_\alpha^+)$, both the sides of (4.10) must vanish and we conclude that

$$\nabla^\alpha_X Y = [X, Y]_{\mathscr{D}_\alpha^-}, \quad \nabla^\alpha_Y X = [Y, X]_{\mathscr{D}_\alpha^+}. \tag{4.11}$$

Moreover, taking 1.–2. of Proposition 3.2 into account, for any $X, X' \in \Gamma(\mathscr{D}_\alpha^+)$ we have

$$\nabla^\alpha_X X' = \nabla^\alpha_X \phi_\beta^2 X' = \phi_\beta \nabla^\alpha_X \phi_\beta X' = \phi_\beta [X, \phi_\beta X']_{\mathscr{D}_\alpha^-} \tag{4.12}$$

and, for any $Y, Y' \in \Gamma(\mathscr{D}_\alpha^-)$,

$$\nabla^\alpha_Y Y' = \nabla^\alpha_Y \phi_\beta^2 Y' = \phi_\beta \nabla^\alpha_Y \phi_\beta Y' = \phi_\beta [Y, \phi_\beta Y']_{\mathscr{D}_\alpha^+}, \tag{4.13}$$

where $\beta \in \{1,2\}$, $\beta \neq \alpha$. Now we decompose any $X, Y \in \Gamma(TM)$ as $X = X_+ + X_- + \eta(X)\xi$ and $Y = Y_+ + Y_- + \eta(Y)\xi$, where X_+, Y_+ and X_-, Y_- denote the projections onto the subbundles $\mathscr{D}_\alpha^+$ and $\mathscr{D}_\alpha^-$ of TM, respectively. Then by (4.10), (4.11), (4.12) and (4.13) we get

$$\begin{aligned} \nabla^\alpha_X Y = {} & \phi_\beta [X_+, \phi_\alpha Y_+]_{\mathscr{D}_\alpha^-} + [X_+, Y_-]_{\mathscr{D}_\alpha^-} + [X_-, Y_+]_{\mathscr{D}_\alpha^+} + \phi_\beta [X_-, \phi_\alpha Y_-]_{\mathscr{D}_\alpha^+} \\ & + X(\eta(Y))\xi + \eta(X)[\xi, Y_+]_{\mathscr{D}_\alpha^+} + \eta(X)[\xi, Y_-]_{\mathscr{D}_\alpha^-}. \end{aligned} \tag{4.14}$$

Notice that, as one can easily check,

$$X_+ = \frac{1}{2}(X + \phi_\alpha X - \eta(X)\xi), \quad X_- = \frac{1}{2}(X - \phi_\alpha X - \eta(X)\xi). \tag{4.15}$$

Then, applying (4.15) to (4.14), after some very long but straightforward computations, we get

$$\begin{aligned} \nabla^\alpha_X Y = X(\eta(Y))\xi + \frac{1}{4}(& [X, Y] - [\phi_\alpha X, \phi_\alpha Y] - \phi_\alpha [\phi_\alpha X, Y] + \phi_\alpha [X, \phi_\alpha Y] \\ & + \phi_\beta [X, \phi_\beta Y] - \phi_\beta \phi_\alpha [X, \phi_\beta \phi_\alpha Y] - \phi_\beta \phi_\alpha [\phi_\alpha X, \phi_\beta Y] \\ & + \phi_\beta [\phi_\alpha X, \phi_\beta \phi_\alpha Y] + \eta(X)\phi_\alpha [\xi, \phi_\alpha Y] - \eta(Y)\phi_\alpha [\xi, \phi_\alpha X] \\ & - \eta(X)\phi_\beta [\xi, \phi_\beta Y] + \eta(X)\phi_\beta \phi_\alpha [\xi, \phi_\beta \phi_\alpha Y] + \eta(Y)[\xi, X] \\ & + \eta(X)[\xi, Y] - \eta([X, Y])\xi + \eta([\phi_\alpha X, \phi_\alpha Y])\xi \\ & - \eta(Y)\eta([\xi, X])\xi - \eta(X)\eta([\xi, Y])\xi). \end{aligned} \tag{4.16}$$

Then we can take (4.16) as a definition and one can easily check that, for each $\alpha \in \{1, 2\}$, ∇^α satisfies (i), (ii) and (iii). Moreover, taking the definition of the operators h_1, h_2, h_3 into account, it is not difficult to verify that (4.16) implies (4.1)–(4.2). It remains to prove the theorem for $\alpha = 3$. In that case the same construction as for $\alpha \in \{1, 2\}$ can be repeated, but now arguing on the eigendistributions $\mathscr{D}_3^+$ and $\mathscr{D}_3^-$ of ϕ_3 corresponding to i and $-i$, respectively, and replacing (4.15) with

$$p_{\mathscr{D}_3^+} = \frac{1}{2}(I - i\phi_3 - \eta \otimes \xi), \quad p_{\mathscr{D}_3^-} = \frac{1}{2}(I + i\phi_3 - \eta \otimes \xi).$$

Then after very long computations one obtains

$$\begin{aligned}\nabla_X^3 Y = X(\eta(Y))\xi + \frac{1}{4}([X, Y] &+ [\phi_3 X, \phi_3 Y] + \phi_1[X, \phi_1 Y] + \phi_2[X, \phi_2 Y] \\ &- \phi_3[X, \phi_3 Y] + \phi_3[\phi_3 X, Y] + \phi_2[\phi_3 X, \phi_1 Y] - \phi_1[\phi_3 X, \phi_2 Y] \\ &- \eta(X)\phi_1[\xi, \phi_1 Y] + \eta(Y)\phi_3[\xi, \phi_3 X] - \eta(X)\phi_2[\xi, \phi_2 Y] - \eta(X)\phi_3[\xi, \phi_3 Y] \\ &- \eta(Y)[\xi, \phi_3^2 X] + \eta(X)[\xi, \phi_1^2 Y] - \eta([X, Y])\xi - \eta([\phi_3 X, \phi_3 Y])\xi),\end{aligned}$$

from which (4.3) follows. □

Proposition 4.3. *The torsion tensor fields of the linear connections ∇^1, ∇^2, ∇^3 stated in Theorem 4.2 are given by*

$$\begin{aligned}T^1(X, Y) = \frac{1}{4}((N_{\phi_3}^{(1)} - N_{\phi_2}^{(1)})(X, Y) &+ (N_{\phi_3}^{(1)} - N_{\phi_2}^{(1)})(\phi_1 X, \phi_1 Y)) \\ &+ (d\eta(X, Y) - d\eta(\phi_1 X, \phi_1 Y))\xi \\ &+ \frac{1}{2}(\eta(X)(-2h_1\phi_1 Y + h_2\phi_2 Y - h_3\phi_3 Y) \\ &- \eta(Y)(-2h_1\phi_1 X + h_2\phi_2 X - h_3\phi_3 X)),\end{aligned} \tag{4.17}$$

$$\begin{aligned}T^2(X, Y) = \frac{1}{4}((N_{\phi_3}^{(1)} - N_{\phi_1}^{(1)})(X, Y) &+ (N_{\phi_3}^{(1)} - N_{\phi_1}^{(1)})(\phi_2 X, \phi_2 Y)) \\ &+ (d\eta(X, Y) - d\eta(\phi_2 X, \phi_2 Y))\xi \\ &+ \frac{1}{2}(\eta(X)(h_1\phi_1 Y - 2h_2\phi_2 Y - h_3\phi_3 Y) \\ &- \eta(Y)(h_1\phi_1 X - 2h_2\phi_2 X - h_3\phi_3 X)),\end{aligned} \tag{4.18}$$

$$\begin{aligned}T^3(X, Y) = -\frac{1}{4}((N_{\phi_1}^{(1)} + N_{\phi_1}^{(1)})(X, Y) &- (N_{\phi_1}^{(1)} + N_{\phi_2}^{(1)})(\phi_3 X, \phi_3 Y)) \\ &+ (d\eta(X, Y) + d\eta(\phi_3 X, \phi_3 Y))\xi \\ &+ \frac{1}{2}(\eta(X)(h_1\phi_1 Y + h_2\phi_2 Y + 2h_3\phi_3 Y) \\ &- \eta(Y)(h_1\phi_1 X + h_2\phi_2 X + 2h_3\phi_3 X)).\end{aligned} \tag{4.19}$$

Proof. The proof follows from (4.1)–(4.3) by a straightforward computation. □

The connections stated in Theorem 4.2 give rise to a canonical connection on an almost bi-paracontact manifold that can be considered as an odd-dimensional counterpart of the Obata connection of an para-hypercomplex (or complex-product) manifold (cf. [1], [15], [18], [23], [27]).

THEOREM 4.4. *Let $(M, \eta, \phi_1, \phi_2, \phi_3)$ be an almost bi-paracontact manifold. There exists a unique linear connection ∇^c on M such that*

(i) $\nabla^c \xi = 0$,

(ii) $\nabla^c \phi_\alpha = \frac{2}{3}\eta \otimes h_\alpha$ *for each* $\alpha \in \{1, 2, 3\}$,

(iii) $T^c = d\eta + \frac{1}{3}(-d\eta(\phi_1\cdot, \phi_1\cdot) - d\eta(\phi_2\cdot, \phi_2\cdot) + d\eta(\phi_3\cdot, \phi_3\cdot)) + \frac{1}{6}(-N_{\phi_1}^{(1)} - N_{\phi_2}^{(1)} + N_{\phi_3}^{(1)})$.

Proof. We first prove the uniqueness of a linear connection satisfying the conditions (i), (ii) and (iii). Let ∇ and ∇' be two linear connections satisfying (i), (ii), (iii). Let us define the tensor $A := \nabla - \nabla'$. Because the expressions of the torsion tensor fields of ∇ and ∇' coincide, one has immediately that $A(X, Y) = A(Y, X)$ for all $X, Y \in \Gamma(TM)$. Hence A is symmetric. Then, due to (ii), one has $A(X, \phi_1 Y) = \phi_1 A(X, Y) = \phi_1 A(Y, X) = A(Y, \phi_1 X) = A(\phi_1 X, Y)$ and, analogously, $A(X, \phi_2 Y) = \phi_2 A(X, Y) = A(\phi_2 X, Y)$. Therefore

$$A(\phi_1 X, \phi_2 Y) = \phi_1 A(X, \phi_2 Y) = \phi_1\phi_2 A(X, Y) = \phi_3 A(X, Y). \tag{4.20}$$

On the other hand

$$A(\phi_1 X, \phi_2 Y) = \phi_2 A(\phi_1 X, Y) = \phi_2\phi_1 A(X, Y) = -\phi_3 A(X, Y). \tag{4.21}$$

Thus comparing (4.20) and (4.21) we get $\phi_3 A(X, Y) = -\phi_3 A(X, Y)$. Applying ϕ_3 to both the sides of the previous identity we obtain

$$-A(X, Y) + \eta(A(X, Y))\xi = A(X, Y) - \eta(A(X, Y))\xi. \tag{4.22}$$

Notice that as, for each $Z \in \Gamma(\mathscr{D})$, ∇_Z and ∇'_Z preserve ϕ_1, they also preserve the eigendistributions $\mathscr{D}_1^{\pm}$ and hence the contact distribution $\mathscr{D} = \mathscr{D}_1^+ \oplus \mathscr{D}_1^-$. This implies that $\eta(A(X, Y)) = 0$ whenever $X, Y \in \Gamma(\mathscr{D})$. Moreover, $A(X, \xi) = 0$ and $A(\xi, Y) = A(\xi, \phi_1^2 Y) = A(\phi_1\xi, \phi_1 Y) = 0$. Consequently (4.22) yields that A is anti-symmetric. Since it is also symmetric, it necessarily vanishes identically. This proves that $\nabla = \nabla'$.

In order to define a (necessarily unique) linear connection satisfying the conditions (i), (ii), (iii), we consider the barycenter of the canonical connections ∇^1, ∇^2, ∇^3 stated in Theorem 4.2. Thus we define, for all $X, Y \in \Gamma(TM)$,

$$\nabla_X^c Y := \frac{1}{3}(\nabla_X^1 Y + \nabla_X^2 Y + \nabla_X^3 Y).$$

We have immediately that $\nabla^c \xi = 0$. By the expressions in (ii) of Theorem 4.2 and by (b) of Lemma 4.1 we have

$$\begin{aligned}\nabla^c \phi_1 &= \frac{1}{3}(\nabla^2 \phi_1 + \nabla^3 \phi_1) = \frac{1}{3}\eta \otimes (2h_1 + h_2\phi_3 - \phi_3 h_2 + 2h_1 - h_3\phi_2 + \phi_2 h_3)\\ &= \frac{2}{3}\eta \otimes h_1\end{aligned}$$

and, analogously, $\nabla^c \phi_2 = \frac{2}{3}\eta \otimes h_2$, $\nabla^c \phi_3 = \frac{2}{3}\eta \otimes h_3$. Using (4.17)–(4.19) we can easily find the expression of the torsion of ∇^c:

$$\begin{aligned}T^c(X,Y) &= T^1(X,Y) + T^2(X,Y) + T^3(X,Y)\\ &= d\eta(X,Y)\xi\\ &\quad + \frac{1}{3}(-d\eta(\phi_1 X, \phi_1 Y) - d\eta(\phi_2 X, \phi_2 Y) + d\eta(\phi_3 X, \phi_3 Y))\xi\\ &\quad + \frac{1}{6}(-N^{(1)}_{\phi_1}(X,Y) - N^{(1)}_{\phi_2}(X,Y) + N^{(1)}_{\phi_3}(X,Y)).\end{aligned} \tag{4.23}$$

□

The unique connection ∇^c stated in Theorem 4.4 will be called the *canonical connection* of the almost bi-paracontact manifold $(M, \eta, \phi_1, \phi_2, \phi_3)$. Using (4.1)–(4.3), after a long computation, one finds that the explicit expression of ∇^c is the following:

$$\begin{aligned}\nabla^c_X Y = \frac{1}{12}(&3[X,Y] - [\phi_1 X, \phi_1 Y] - [\phi_2 X, \phi_2 Y] + [\phi_3 X, \phi_3 Y]\\ &+ 3\phi_1[X, \phi_1 Y] + 3\phi_2[X, \phi_2 Y] - 3\phi_3[X, \phi_3 Y] - \phi_1[\phi_1 X, Y]\\ &- \phi_2[\phi_2 X, Y] + \phi_3[\phi_3 X, Y] + \phi_1[\phi_2 X, \phi_3 Y] - \phi_1[\phi_3 X, \phi_2 Y]\\ &- \phi_2[\phi_1 X, \phi_3 Y] + \phi_2[\phi_3 X, \phi_1 Y] + \phi_3[\phi_1 X, \phi_2 Y] - \phi_3[\phi_2 X, \phi_1 Y]\\ &+ 2\eta(X)(h_1\phi_1 Y + h_2\phi_2 Y - h_3\phi_3 Y) + 2\eta(Y)(h_1\phi_1 X + h_2\phi_2 X - h_3\phi_3 X)\\ &+ (\eta([\phi_1 X, \phi_1 Y]) + \eta([\phi_2 X, \phi_2 Y]) - \eta([\phi_3 X, \phi_3 Y])\\ &- 3\eta([X,Y]))\xi) + X(\eta(Y))\xi.\end{aligned}$$

Corollary 4.5. *Let $(M, \eta, \phi_1, \phi_2, \phi_3)$ be a normal almost bi-paracontact manifold.*

1. *There exists a unique linear connection ∇^c on M preserving the almost bi-paracontact structure and whose torsion is given by*

$$T^c = 2\, d\eta \otimes \xi. \tag{4.24}$$

2. *The curvature tensor field of ∇^c satisfies*

$$R^c(\phi_1 \cdot, \phi_1 \cdot) = R^c(\phi_2 \cdot, \phi_2 \cdot) = -R^c(\phi_3 \cdot, \phi_3 \cdot) = -R^c. \tag{4.25}$$

In particular, for all $X \in \Gamma(TM)$

$$R^c(X, \xi) = 0 \tag{4.26}$$

3. *The Ricci tensor of* ∇^c, *defined as* $\mathrm{Ric}^c(X, Y) := \mathrm{trace}(Z \mapsto R^c(Z, X) Y)$, *is given by*

$$\mathrm{Ric}^c(X, Y) = -\frac{1}{2}\,\mathrm{trace}(R^c(X, Y)). \tag{4.27}$$

In particular, the Ricci tensor is skew-symmetric and $\mathrm{Ric}^c(\phi_1\cdot, \phi_1\cdot) = \mathrm{Ric}^c(\phi_2\cdot, \phi_2\cdot) = -\mathrm{Ric}^c(\phi_3\cdot, \phi_3\cdot) = \mathrm{Ric}^c$.

4. *The connection* ∇^c *and the connections* ∇^1, ∇^2, ∇^3 *coincide.*

Proof. 1. As in any normal almost bi-paracontact manifold the tensor fields h_1, h_2, h_3 vanish identically, by (ii) of Theorem 4.4, ∇^c preserves the tensor fields ϕ_1, ϕ_2, ϕ_3. Moreover, by (3.1) the expression (4.23) of the torsion simplifies in (4.24).

2. First of all notice that, since $\nabla^c \phi_\alpha = 0$, for each $\alpha \in \{1, 2, 3\}$ we have

$$R^c(X, Y) \circ \phi_\alpha = \phi_\alpha \circ R^c(X, Y). \tag{4.28}$$

for all $X, Y \in \Gamma(TM)$. Now the Bianchi identity yields

$$\begin{aligned} R^c(X, Y)Z + R^c(Y, Z)X + R^c(Z, X)Y &= T^c(T^c(X, Y), Z) + (\nabla^c_X T^c)(Y, Z) + T^c(T^c(Y, Z), X) \\ &\quad + (\nabla^c_Y T^c)(Z, X) + T^c(T^c(Z, X), Y) + (\nabla^c_Z T^c)(X, Y). \end{aligned} \tag{4.29}$$

We examine the terms in the right-hand-side of (4.29). Notice that, by (4.24), $T^c(T^c(X, Y), Z) = 4\, d\eta(X, Y)\, d\eta(\xi, Z)\xi = 0$ and

$$\begin{aligned} (\nabla^c_X T^c)(Y, Z) &= \nabla^c_X(2\, d\eta(Y, Z)\xi) - 2\, d\eta(\nabla^c_X Y, Z)\xi - 2\, d\eta(Y, \nabla^c_X Z)\xi \\ &= 2X(d\eta(Y, Z))\xi + 2\, d\eta(Y, Z)\nabla^c_X \xi \\ &\quad - 2\, d\eta(\nabla^c_X Y, Z)\xi - 2\, d\eta(Y, \nabla^c_X Z)\xi \\ &= 2(\nabla^c_X\, d\eta)(Y, Z)\xi. \end{aligned}$$

Hence (4.29) simplifies in

$$\begin{aligned} R^c(X, Y)Z + R^c(Y, Z)X + R^c(Z, X)Y &= 2((\nabla^c_X\, d\eta)(Y, Z) + (\nabla^c_Y\, d\eta)(Z, X) + (\nabla^c_Z\, d\eta)(X, Y))\xi. \end{aligned} \tag{4.30}$$

Now in (4.30) consider $X, Y \in \Gamma(\mathcal{D}^+_\alpha)$ and $Z \in \Gamma(\mathcal{D}^-_\alpha)$, $\alpha \in \{1, 2\}$. Then, as ∇^c preserves the contact distribution, the left-hand-side of (4.30) is tangent to $\mathcal{D}$ whereas the right-hand-side is transversal to $\mathcal{D}$. Hence they both vanish. Thus, in particular

$$R^c(X, Y)Z = -R^c(Y, Z)X - R^c(Z, X)Y. \tag{4.31}$$

But the left-hand-side of (4.31) is a section of $\mathcal{D}_\alpha^-$, whereas the right-hand-side is a section of $\mathcal{D}_\alpha^+$. Consequently, $R^c(X,Y)Z = 0$ for all $X, Y \in \Gamma(\mathcal{D}_\alpha^+)$ and $Z \in \Gamma(\mathcal{D}_\alpha^-)$. Since by Proposition 3.2, for any $\beta \neq \alpha$, ϕ_β maps $\mathcal{D}_\alpha^-$ onto $\mathcal{D}_\alpha^+$, applying (4.28) we get that $R^c(X,Y)Z = 0$ also for $Z \in \Gamma(\mathcal{D}_\alpha^+)$. Moreover, obviously, $R^c(X,Y)\xi = 0$, so that we can conclude that

$$R^c(X,Y) = 0 \tag{4.32}$$

for any $X, Y \in \Gamma(\mathcal{D}_\alpha^+)$. In a similar way one can prove that (4.32) holds for $X, Y \in \Gamma(\mathcal{D}_\alpha^-)$. Thus in both cases the relation $R^c(\phi_\alpha X, \phi_\alpha Y) = -R^c(X,Y)$, $\alpha \in \{1,2\}$, is trivially satisfied. Moreover, if $X \in \Gamma(\mathcal{D}_\alpha^+)$ and $Y \in \Gamma(\mathcal{D}_\alpha^-)$, $R^c(\phi_\alpha X, \phi_\alpha Y) = R^c(X, -Y) = -R^c(X,Y)$. In order to complete the proof in the case $\alpha \in \{1,2\}$ it remains to prove that $R^c(X,\xi) = 0$ for any $X \in \Gamma(\mathcal{D})$. Notice that, as $\nabla^c\xi = 0$ and $T^c(X,\xi) = 2\,d\eta(X,\xi) = 0$, $\nabla^c_\xi X = [\xi, X]$. By applying again the Bianchi identity (4.29) we obtain, for all $Z \in \Gamma(\mathcal{D})$,

$$\begin{aligned} R^c(X,\xi)Z + R^c(\xi,Z)X &= (\nabla^c_\xi T^c)(Z,X) \\ &= \nabla^c_\xi(T^c(Z,X)) - T^c([\xi,Z],X) - T^c(Z,[\xi,X]) \\ &= 2(\mathcal{L}_\xi\, d\eta)(Z,X)\xi = 0. \end{aligned}$$

Consequently $R^c(X,\xi)Z = -R^c(\xi,Z)X$. If in the last equality we take $X \in \Gamma(\mathcal{D}_\alpha^+)$ and $Z \in \Gamma(\mathcal{D}_\alpha^-)$, the left-hand-side is a section of $\mathcal{D}_\alpha^-$ while the right-hand-side is a section of $\mathcal{D}_\alpha^+$. Thus they both vanish and taking (4.28) into account we conclude that $R^c(X,\xi) = 0$ for all $X \in \Gamma(\mathcal{D})$. Finally, for any $X, Y \in \Gamma(TM)$, $R^c(\phi_3 X, \phi_3 Y) = R^c(\phi_1\phi_2 X, \phi_1\phi_2 Y) = -R^c(\phi_2 X, \phi_2 Y) = R^c(X,Y)$.

3. For simplifying the notation, let r_{XY} denote the endomorphism $Z \mapsto R^c(Z,X)Y$, so that $\mathrm{Ric}^c(X,Y) = \mathrm{trace}(r_{XY})$. From (4.25) it follows immediately that $r_{XY}(\xi) = 0$. Let $\{E_1, \ldots, E_n, E_{n+1}, \ldots, E_{2n}, \xi\}$ be a local basis such that, for each $i \in \{1, \ldots, n\}$, $E_i \in \Gamma(\mathcal{D}_1^+)$ and $E_{n+i} = \phi_2 E_i \in \Gamma(\mathcal{D}_1^-)$. In order to prove (4.27) we distinguish the cases (i) $X, Y \in \Gamma(\mathcal{D}_1^+)$, (ii) $X, Y \in \Gamma(\mathcal{D}_1^-)$, (iii) $X \in \Gamma(\mathcal{D}_1^+)$, $Y \in \Gamma(\mathcal{D}_1^-)$, (iv) $X \in \Gamma(TM)$, $Y = \xi$. In the first case, due to (4.32), $r_{XY}(E_i) = R^c(E_i, X)Y = 0$. Moreover, $r_{XY}(E_{n+i}) = R^c(E_{n+i}, X)Y \in \Gamma(\mathcal{D}_1^+)$ so that it has no components along the direction of $E_{n+1}, \ldots, E_{2n}, \xi$. Hence $\mathrm{Ric}^c(X,Y) = \mathrm{trace}(r_{XY}) = 0$. On the other hand, since $R^c(X,Y) = 0$, also the right-hand-side of (4.27) vanishes. The case (ii) being analogous, we pass to the case (iii). First of all, by (4.32), $r_{XY}(E_i) = R^c(E_i, X)Y = 0$. Next, by the Bianchi identity used before,

$$\begin{aligned} r_{XY}(E_{n+i}) = R^c(E_{n+i}, X)Y &= -R^c(X,Y)E_{n+i} - R^c(Y, E_{n+i})X \\ &= -R^c(X,Y)E_{n+i}, \end{aligned}$$

as $R^c(Y, E_{n+i}) = 0$.

Since $R^c(X,Y)E_{n+i} = R^c(X,Y)\phi_1 E_i = \phi_1(R^c(X,Y)E_i)$, we conclude that $\mathrm{trace}(r_{XY}) = -\frac{1}{2}\,\mathrm{trace}\, R^c(X,Y)$. The last case is obvious since, due to (4.25), $\mathrm{Ric}^c(X,\xi) = 0 = -\frac{1}{2}\,\mathrm{trace}(R^c(X,\xi))$.

4. Proposition 4.3, (4.24) and the normality of the almost bi-paracontact structure imply that $T^1(X,Y) = T^2(X,Y) = T^3(X,Y) = 2\,d\eta(X,Y)\xi = T^c(X,Y)$

for all $X, Y \in \Gamma(TM)$. Moreover, according to (ii) of Theorem 4.2, because of the vanishing of the tensor fields h_1, h_2, h_3, each connection ∇^1, ∇^2, ∇^3 preserves the tensor fields ϕ_1, ϕ_2, ϕ_3. Consequently, for each $\alpha \in \{1,2,3\}$, ∇^α fulfils all the conditions of Theorem 4.4 and hence coincides with ∇^c. □

Corollary 4.6. *Every normal almost bi-paracontact manifold carries four mutually transverse Legendre foliations whose leaves are totally geodesic and admit an affine structure.*

Proof. Since the almost bi-paracontact structure is normal, it is in particular integrable, so that the eigendistributions $\mathscr{D}_1^+$, $\mathscr{D}_1^-$, $\mathscr{D}_2^+$, $\mathscr{D}_2^-$ define four mutually transverse Legendre foliations on the manifold. The leaves of these foliations are auto-parallel with respect to the canonical connection ∇^c, so that they are totally geodesic. Finally, for each $\alpha \in \{1,2\}$, for any $X, X' \in \Gamma(\mathscr{D}_\alpha^\pm)$ we have, by (4.24), $T^c(X, X') = 0$ and, by (4.25), $R^c(X, X') = 0$. Thus ∇^c induces a flat, torsion-free connection on the leaves of the foliations $\mathscr{D}_1^+$, $\mathscr{D}_1^-$, $\mathscr{D}_2^+$, $\mathscr{D}_2^-$. □

We conclude the section by studying the transverse geometry of a normal almost bi-paracontact manifold with respect to the Reeb foliation. We show in fact that the space of leaves of a normal almost bi-paracontact manifold is para-hypercomplex (see [13] or, with different names, [1], [15], [18] [23], [27]). We recall that a *para-hypercomplex structure* on an even dimensional manifold is given by two anti-commuting product structures I, J and a complex structure K such that $IJ = K$. Then one can prove that the manifold admits a canonical connection, usually called the *Obata connection*, defined as the unique torsion-free connection preserving the para-hypercomplex structure.

Theorem 4.7. *Let $(M, \phi_1, \phi_2, \phi_3)$ be a normal almost bi-paracontact manifold. Then the* 1*-dimensional foliation defined by the Reeb vector field ξ is transversely para-hypercomplex. Furthermore, the canonical connection ∇^c is* (*locally*) *projectable to the Obata connection defined on the leaf space.*

Proof. First of all we have to prove that the tensor fields ϕ_1, ϕ_2, ϕ_3 are "foliated" objects, i.e. they are constant along the leaves of the Reeb foliation $\mathscr{F}_\xi$. Thus we have to show that $\mathscr{L}_\xi \phi_\alpha = 0$ for each $\alpha \in \{1,2,3\}$. In fact this condition is satisfied because, by assumption, $N_{\phi_\alpha}^{(1)} = 0$, so that also $N_{\phi_\alpha}^{(3)} = \mathscr{L}_\xi \phi_\alpha = 0$. Thus the tensor fields ϕ_1, ϕ_2, ϕ_3 are projectable. We prove that they (locally) project onto a para-hypercomplex structure. Let π be a local submersion defining the Reeb foliation. For each $\alpha \in \{1,2,3\}$ let J_α be the tensor field defined by $\pi_* \circ \phi_\alpha = J_\alpha \circ \pi_*$. Then it is clear that (J_1, J_2, J_3) is an almost para-hypercomplex structure. Moreover, for any two (local) vector fields X' and Y' in the leaf space, denoting by X and Y the unique basic vector fields on M such such that $\pi_* X = X'$ and $\pi_* Y = Y'$, we have

$$[J_\alpha, J_\alpha](X', Y') = \pi_*(N_{\phi_\alpha}^{(1)}(X, Y)) = 0,$$

so that the structure is integrable. For concluding the proof we prove that the canonical connection ∇^c projects onto the the Obata connection ∇^{Ob}. First we prove that ∇^c is projectable, i.e. it projects to connections of the local slice spaces of $\mathscr{F}_\xi$. The conditions for this are: a) for any basic vector fields $X \in \Gamma(\mathscr{D})$ and for any $V \in \Gamma(T\mathscr{F}_\xi)$ one has $\nabla^c_V X = 0$, b) if X and Y are basic vector fields then also $\nabla^c_X Y$ is a basic vector field ([19]). Here, by "basic vector field" we mean a vector field X transverse to the foliation $\mathscr{F}_\xi$ which is locally projectable to a vector field on the leaf space by means a local submersion defining $\mathscr{F}_\xi$; one can see that this is equivalent to require that $[X, V]$ is still tangent to the foliation for any $V \in \Gamma(T\mathscr{F}_\xi)$ (cf. [19], [26]). Now the condition (a) is easily verified since $\nabla^c_\xi X = [\xi, X] = 0$ because $[\xi, X]$ is tangent both to $\mathscr{D}$ and to $\mathscr{F}_\xi$ (X being basic). Also the second condition holds. Indeed first recall that, by construction, ∇^c preserves the contact distribution; next, by (4.26),

$$0 = R^c(X,\xi)Y = \nabla^c_X \nabla^c_\xi Y - \nabla^c_\xi \nabla^c_X Y - \nabla^c_{[X,\xi]} Y \tag{4.33}$$
$$= \nabla^c_X[\xi, Y] - \nabla^c_\xi \nabla^c_X Y = -\nabla^c_\xi \nabla^c_X Y$$

since $[X,\xi] = [Y,\xi] = 0$, X, Y being basic. Thus, by (4.33), $[\xi, \nabla^c_X Y] = \nabla^c_\xi \nabla^c_X Y = 0$ and hence $\nabla^c_X Y$ is basic. Therefore ∇^c locally projects along the leaves of $\mathscr{F}_\xi$ to a linear connection ∇' which parallelizes the induced complex and product structures, since $\nabla^c \phi_\alpha = 0$ for each $\alpha \in \{1,2,3\}$. It remains to prove that ∇' is symmetric. Let X', Y' be any local vector fields on the leaf space and let X, Y be the corresponding basic vector fields such that $\pi_* X = X'$ and $\pi_* Y = Y'$. Then $T'(X', Y') = \pi_* T^c(X, Y) = \pi_*(2\, d\eta(X, Y)\xi) = 0$. Thus ∇' coincides with the Obata connection. □

5. The standard bi-paracontact structure of a contact metric (κ, μ)-space

In this section we study one of the main examples of almost bi-paracontact manifolds, namely we show that any (non-Sasakian) contact metric (κ,μ)-space admits a canonical almost bi-paracontact structure which satisfies very interesting properties.

Recall that a contact metric (κ,μ)-space is a contact metric manifold (M,ϕ,ξ,η,g) such that the Reeb vector field ξ belongs to the "(κ,μ)-nullity distribution" i.e.

$$R^g(X, Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY), \tag{5.1}$$

This notion was introduced by Blair, Koufogiorgos and Papantoniou in [4], who proved the following fundamental results.

Theorem 5.1 ([4]). *Let (M,ϕ,ξ,η,g) be a contact metric (κ,μ)-space. Then necessarily $\kappa \le 1$. If $\kappa = 1$ then $h = 0$ and (M,ϕ,ξ,η,g) is Sasakian; if $\kappa < 1$, the contact metric structure is not Sasakian and M admits three mutually orthogonal totally geodesic distributions $\mathscr{D}(0) = \mathbf{R}\xi$, $\mathscr{D}_h(\lambda)$ and $\mathscr{D}_h(-\lambda) = \phi(\mathscr{D}_h(\lambda))$ corresponding to the eigenspaces of h, where $\lambda = \sqrt{1-\kappa}$.*

Furthermore, in [4] it is proved that any contact metric (κ,μ)-space satisfies (2.11), hence it is integrable, and for any $X \in \Gamma(\mathscr{D}_h(\lambda))$, $Y \in \Gamma(\mathscr{D}_h(-\lambda))$, $\nabla^g_X Y \in \Gamma(\mathscr{D}_h(-\lambda) \oplus \mathbf{R}\xi)$ and $\nabla^g_Y X \in \Gamma(\mathscr{D}_h(\lambda) \oplus \mathbf{R}\xi)$.

Given a non-Sasakian contact metric (κ,μ)-manifold (M,ϕ,ξ,η,g), Boeckx [5] proved that the number $I_M := \dfrac{1-\frac{\mu}{2}}{\sqrt{1-\kappa}}$, is an invariant of the contact metric (κ,μ)-structure, and he proved that two non-Sasakian contact metric (κ,μ)-manifolds $(M_1,\phi_1,\xi_1,\eta_1,g_1)$ and $(M_2,\phi_2,\xi_2,\eta_2,g_2)$ are locally isometric as contact metric manifolds if and only if $I_{M_1} = I_{M_2}$. Then the invariant I_M has been used by Boeckx for providing a local classification of contact metric (κ,μ)-spaces. An interpretation of the Boeckx invariant in terms of Legendre foliations is given in [11].

The standard example of contact metric (κ,μ)-manifolds is given by the tangent sphere bundle T_1N of a Riemannian manifold N of constant curvature c endowed with its standard contact metric structure. In this case $\kappa = c(2-c)$, $\mu = -2c$ and $I_{T_1N} = \dfrac{1+c}{|1-c|}$.

The link between contact metric (κ,μ)-spaces with the theory of Legendre foliations was pointed out in [9] and [11]. In fact any contact metric (κ,μ)-space (M,ϕ,ξ,η,g) is canonically a bi-Legendrian manifold with bi-Legendrian structure given by $(\mathscr{D}_h(\lambda),\mathscr{D}_h(-\lambda))$, and the corresponding bi-Legendrian connection preserves the tensors ϕ, h, g ([8], [9]). We prove now that a contact metric (κ,μ)-space admits a further bi-Legendrian structure which is transverse to $(\mathscr{D}_h(\lambda),\mathscr{D}_h(-\lambda))$.

Theorem 5.2. *In any non-Sasakian contact metric (κ,μ)-manifold the operator ϕh admits three eigenvalues, 0, of multiplicity 1, and λ, $-\lambda$, each of multiplicity n, where $\lambda := \sqrt{1-\kappa}$. The corresponding eigendistributions are given by $\mathscr{D}_{\phi h}(0) = \mathbf{R}\xi$ and*

$$\mathscr{D}_{\phi h}(\lambda) = \{X + \phi X \mid X \in \Gamma(\mathscr{D}_h(\lambda)\}, \tag{5.2}$$

$$\mathscr{D}_{\phi h}(-\lambda) = \{Y + \phi Y \mid Y \in \Gamma(\mathscr{D}_h(-\lambda)\}. \tag{5.3}$$

Furthermore, $\mathscr{D}_{\phi h}(\lambda)$ and $\mathscr{D}_{\phi h}(-\lambda)$ define two mutually orthogonal Legendre foliations which are transversal to the canonical bi-Legendrian structure $(\mathscr{D}_h(\lambda),\mathscr{D}_h(-\lambda))$.

Proof. That ϕh admits the eigenvalues 0 and $\pm\sqrt{1-\kappa}$ follows from the relation $h^2 = (\kappa-1)\phi^2$ ([4]). Since the operator h is symmetric and ϕ anti-commutes with h, also ϕh is symmetric and hence it is diagonalizable. Now, since the kernel of ϕh is generated by the Reeb vector field, we have that $\mathscr{D}_{\phi h}(0) = \mathbf{R}\xi$. Moreover, if $X \in \Gamma(\mathscr{D}_{\phi h}(\lambda))$, then $\phi h \phi X = -\phi\phi h X = -\lambda\phi X$, so that $\phi X \in \Gamma(\mathscr{D}_{\phi h}(-\lambda))$. This implies that $\mathscr{D}_{\phi h}(\lambda)$ and $\mathscr{D}_{\phi h}(-\lambda)$ have equal dimension n, if $2n+1$ is the dimension of M. $\mathscr{D}_{\phi h}(\lambda)$ and $\mathscr{D}_{\phi h}(-\lambda)$ are in fact mutually orthogonal. Indeed, for any $X \in \Gamma(\mathscr{D}_{\phi h}(\lambda))$ and $Y \in \Gamma(\mathscr{D}_{\phi h}(-\lambda))$, since the operator ϕh is symmetric, we have $\lambda g(X,Y) = g(\phi hX,Y) = g(X,\phi hY) = -\lambda g(X,Y)$, so that $g(X,Y) = 0$. In order to prove (5.2) first notice that, for any $X \in \Gamma(\mathscr{D}_h(\lambda))$, $\phi h(X+\phi X) = \lambda\phi X - \phi^2 hX = \lambda(X+\phi X)$ so that $X+\phi X \in \Gamma(\mathscr{D}_{\phi h}(\lambda))$. Thus it remains to show that, given $Y \in \Gamma(\mathscr{D}_{\phi h}(\lambda))$, there exists $X \in \Gamma(\mathscr{D}_h(\lambda))$ such that $Y = X + \phi X$. One can verify that $X := \frac{1}{2}(Y - \phi Y)$ has the required properties. In a similar way one proves (5.3). Now we are able to demonstrate the integrability of the distributions $\mathscr{D}_{\phi h}(\lambda)$ and $\mathscr{D}_{\phi h}(-\lambda)$. Any two sections of $\mathscr{D}_{\phi h}(\lambda)$ can be written as $X+\phi X$ and $X'+\phi X'$, for some $X, X' \in \Gamma(\mathscr{D}_h(\lambda))$. Then, by (2.11)

$$\begin{aligned}(5.4)\qquad \nabla^g_{X+\phi X}(X'+\phi X') &= \nabla^g_X X' + \nabla^g_{\phi X}X' + \phi\nabla^g_X X' + g(X+hX,X')\xi \\ &\quad + \phi\nabla^g_{\phi X}X' + g(\phi X + h\phi X, X')\xi \\ &= \nabla^g_X X' + \phi\nabla^g_X X' + \nabla^g_{\phi X}X' \\ &\quad + \phi\nabla^g_{\phi X}X' + (1+\lambda)g(X,X')\xi.\end{aligned}$$

Now, $\nabla^g_{\phi X}X' \in \Gamma(\mathscr{D}_h(\lambda) \oplus \mathbf{R}\xi)$, so that we can decompose $\nabla^g_{\phi X}X'$ along its component tangent to $\mathscr{D}_h(\lambda)$ and the one tangent to $\mathbf{R}\xi$, given by $\eta(\nabla^g_{\phi X}X')\xi = g(\nabla^g_{\phi X}X',\xi)\xi$. But, by (2.9), $g(\nabla^g_{\phi X}X',\xi) = -g(X',\nabla^g_{\phi X}\xi) = (\lambda-1)g(X,X')$, so that (5.4) becomes

$$\begin{aligned}(5.5)\qquad \nabla^g_{X+\phi X}(X'+\phi X') &= \nabla^g_X X' + \phi\nabla^g_X X' + (\nabla^g_{\phi X}X')_{\mathscr{D}_h(\lambda)} \\ &\quad + \phi(\nabla^g_{\phi X}X')_{\mathscr{D}_h(\lambda)} + 2\lambda g(X,X')\xi.\end{aligned}$$

Therefore

$$\begin{aligned}(5.6)\quad [X+\phi X, X'+\phi X'] &= [X,X'] + \phi[X,X'] + (\nabla^g_{\phi X}X')_{\mathscr{D}_h(\lambda)} - \phi(\nabla^g_{\phi X}X')_{\mathscr{D}_h(\lambda)} \\ &\quad - (\nabla^g_{\phi X'}X)_{\mathscr{D}_h(\lambda)} + \phi(\nabla^g_{\phi X'}X)_{\mathscr{D}_h(\lambda)}.\end{aligned}$$

Due to (5.2) each of the three terms $[X,X'] + \phi[X,X']$, $(\nabla^g_{\phi X}X')_{\mathscr{D}_h(\lambda)} + \phi(\nabla^g_{\phi X}X')_{\mathscr{D}_h(\lambda)}$ and $(\nabla^g_{\phi X'}X)_{\mathscr{D}_h(\lambda)} + \phi(\nabla^g_{\phi X'}X)_{\mathscr{D}_h(\lambda)}$ in the right-hand-side of (5.6) is a section of $\mathscr{D}_{\phi h}(\lambda)$. Thus we conclude that $\mathscr{D}_{\phi h}(\lambda)$ is involutive. In particular, being $\mathscr{D}_{\phi h}(\lambda)$ an integrable subbundle of $\mathscr{D}$, it defines a Legendre foliation of M. Analogous arguments work also for $\mathscr{D}_{\phi h}(-\lambda)$. It remains to prove that $\mathscr{D}_{\phi h}(\lambda)$ and $\mathscr{D}_{\phi h}(-\lambda)$ are transverse to each foliation of the bi-Legendrian structure $(\mathscr{D}_h(\lambda), \mathscr{D}_h(-\lambda))$. For instance we show that $TM = \mathscr{D}_{\phi h}(\lambda) \oplus \mathscr{D}_h(-\lambda) \oplus \mathbf{R}\xi$, the other cases being similar. If X is a vector field tangent both to $\mathscr{D}_{\phi h}(\lambda)$ and to $\mathscr{D}_h(-\lambda)$ then $\lambda X = \phi hX = -\lambda\phi X$ so that $X = -\phi X$. By applying ϕ we get $X = \phi X$, hence $X = 0$. Next, let Z be a vector field on M. Then there exist $X \in \Gamma(\mathscr{D}_h(\lambda))$ and $Y \in \Gamma(\mathscr{D}_h(-\lambda))$ such that $Z = X + Y + \eta(Z)\xi$. Adding and subtracting $\phi X \in \Gamma(\mathscr{D}_h(-\lambda))$ we obtain $Z = (X+\phi X) + (Y-\phi X) + \eta(Z)\xi$, where $X+\phi X \in \Gamma(\mathscr{D}_{\phi h}(\lambda))$ and $Y - \phi X \in \Gamma(\mathscr{D}_h(-\lambda))$. □

Theorem 5.2 implies that any (non-Sasakian) contact metric (κ,μ)-space is endowed with two transverse bi-Legendrian structures $(\mathscr{D}_h(\lambda), \mathscr{D}_h(-\lambda))$ and

$(\mathcal{D}_{\phi h}(\lambda), \mathcal{D}_{\phi h}(-\lambda))$ defined by the eigenspaces of the operators h and ϕh corresponding to the eigenvalues $\pm\lambda$. Thus by Proposition 3.13 we conclude that any (non-Sasakian) contact metric (κ, μ)-space M admits an integrable almost bi-paracontact structure which we call the *standard almost bi-paracontact structure* of the contact metric (κ, μ)-space M. One can easily prove the following result.

THEOREM 5.3. *Let (M, ϕ, ξ, η, g) be a non-Sasakian contact metric (κ, μ)-space. The standard almost bi-paracontact structure of M is given by (ϕ_1, ϕ_2, ϕ_3), where*

$$\phi_1 := \frac{1}{\sqrt{1-\kappa}}\phi h, \quad \phi_2 := \frac{1}{\sqrt{1-\kappa}}h, \quad \phi_3 := \phi.$$

According to the notation used in §3 we denote by $\mathcal{D}_1^{\pm}$ and $\mathcal{D}_2^{\pm}$ the eigendistributions of ϕ_1 and ϕ_2, respectively, corresponding to the eigenvalues ± 1. So $\mathcal{D}_1^{\pm} = \mathcal{D}_{\phi h}(\pm\lambda)$ and $\mathcal{D}_2^{\pm} = \mathcal{D}_h(\pm\lambda)$. Then, according to Theorem 5.3, (5.2)–(5.3) should be compared to Proposition 3.3.

Remark 5.4. For each $\alpha \in \{1, 2\}$ we can define a semi-Riemannian metric g_α by setting

$$g_\alpha(X, Y) := d\eta(X, \phi_\alpha Y) + \eta(X)\eta(Y) \tag{5.7}$$

for all $X, Y \in \Gamma(TM)$. Then it is easy to check that $(\phi_\alpha, \xi, \eta, g_\alpha)$ is a paracontact metric structure on M. In fact $(\phi_\alpha, \xi, \eta, g_\alpha) = \Psi(\mathcal{D}_\alpha^+, \mathcal{D}_\alpha^-)$ according to the notation used in §2.2. Let $\bar{\nabla}^{pc}$ and ∇^{pc} denote the canonical paracontact connections associated to the paracontact metric structures (ϕ_1, ξ, η, g_1) and (ϕ_2, ξ, η, g_2), respectively (cf. Theorem 2.4). Then, since $\mathcal{D}_1^{\pm}$ and $\mathcal{D}_2^{\pm}$ are integrable, Theorem 2.8 implies that $\nabla^{pc} = \nabla^{bl}$ and $\bar{\nabla}^{pc} = \bar{\nabla}^{bl}$, where ∇^{bl} denotes the bi-Legendrian connection corresponding to the bi-Legendrian structure $(\mathcal{D}_h(\lambda), \mathcal{D}_h(-\lambda))$ and $\bar{\nabla}^{bl}$ the bi-Legendrian connection associated to $(\mathcal{D}_{\phi h}(\lambda), \mathcal{D}_{\phi h}(-\lambda))$. In particular, by (2.23) we have that

$$\bar{T}^{bl}(\cdot, \xi) = -\phi_1 h_1, \quad T^{bl}(\cdot, \xi) = -\phi_2 h_2, \tag{5.8}$$

where $\bar{T}^{bl}$ and T^{bl} denote the torsion tensor fields of $\bar{\nabla}^{bl}$ and ∇^{bl}, respectively.

The bi-Legendrian structure $(\mathcal{D}_2^+, \mathcal{D}_2^-)$ was deeply studied in [9] and [11]. In the sequel we study the "new" bi-Legendrian structure, $(\mathcal{D}_1^+, \mathcal{D}_1^-)$.

THEOREM 5.5. *The Legendre foliations $\mathcal{D}_{\phi h}(\lambda)$ and $\mathcal{D}_{\phi h}(-\lambda)$ are either non-degenerate or flat. In particular, $\mathcal{D}_{\phi h}(\lambda)$ and $\mathcal{D}_{\phi h}(-\lambda)$ are positive definite if and only if $I_M > 0$, negative definite if and only if $I_M < 0$, flat if and only if $I_M = 0$.*

Proof. Let $X \in \Gamma(\mathcal{D}_{\phi h}(\lambda))$. Then the (κ, μ)-nullity condition becomes

$$R^g(X, \xi)\xi = \kappa X + \mu h X. \tag{5.9}$$

On the other hand,

$$(5.10)\qquad \begin{aligned} R^g(X,\xi)\xi &= -\nabla^g_\xi\nabla^g_X\xi - \nabla^g_{[X,\xi]}\xi \\ &= -\nabla^g_\xi\phi X + \nabla^g_\xi\phi hX + \phi[X,\xi] + \phi h[X,\xi] \\ &= \nabla^g_{\phi X}\xi + [\xi,\phi X] + \lambda\nabla^g_X\xi + \lambda[\xi,X] + \phi[X,\xi] \\ &\quad + \lambda[X,\xi]_{\mathscr{D}_{\phi h}(\lambda)} - \lambda[X,\xi]_{\mathscr{D}_{\phi h}(-\lambda)} \\ &= -\phi^2X - \phi h\phi X + [\xi,\phi X] + \lambda(-\phi X - \phi hX) \\ &\quad + \lambda[\xi,X] - \phi[\xi,X] - \lambda[\xi,X]_{\mathscr{D}_{\phi h}(\lambda)} + \lambda[\xi,X]_{\mathscr{D}_{\phi h}(-\lambda)} \\ &= X + \lambda\phi X + 2hX - \lambda\phi X - \lambda X + 2\lambda[\xi,X]_{\mathscr{D}_{\phi h}(-\lambda)}. \end{aligned}$$

Thus (5.9) and (5.10) imply

$$\kappa\phi X + \mu\phi hX = (1-\lambda)\phi X + 2\phi hX + 2\lambda\phi[\xi,X]_{\mathscr{D}_{\phi h}(-\lambda)},$$

from which it follows that

$$\phi[\xi,X]_{\mathscr{D}_{\phi h}(-\lambda)} = \frac{1-\sqrt{1-\kappa}}{2}\phi X - \frac{1-\frac{\mu}{2}}{\sqrt{1-\kappa}}X = \frac{1-\sqrt{1-\kappa}}{2}\phi X - I_M X.$$

Therefore, by (2.26), we have, for any $X, X' \in \Gamma(\mathscr{D}_{\phi h}(\lambda))$,

$$(5.11)\qquad \begin{aligned} \Pi_{\mathscr{D}_{\phi h}(\lambda)}(X,X') &= 2g([\xi,X]_{\mathscr{D}_{\phi h}(-\lambda)},\phi X') \\ &= -2g(\phi[\xi,X]_{\mathscr{D}_{\phi h}(-\lambda)},X') \\ &= -(1-\sqrt{1-\kappa})g(\phi X,X') + 2I_M g(X,X') \\ &= 2I_M g(X,X'). \end{aligned}$$

Similarly, one can prove that, for any $Y, Y' \in \Gamma(\mathscr{D}_{\phi h}(-\lambda))$,

$$(5.12)\qquad \Pi_{\mathscr{D}_{\phi h}(-\lambda)}(Y,Y') = 2I_M g(Y,Y').$$

The assertion of the theorem then easily follows from the expressions (5.11), (5.12) of the Pang invariant of the Legendre foliations $\mathscr{D}_{\phi h}(\lambda)$, $\mathscr{D}_{\phi h}(-\lambda)$. □

Since any (non-Sasakian) contact metric (κ,μ)-space (M,ϕ,ξ,η,g) is canonically endowed with an almost bi-paracontact manifold, it admits the linear connections ∇^1, ∇^2, ∇^3 stated in Theorem 4.2 and, moreover, the canonical connection ∇^c defined in Theorem 4.4. On the other hand, to M it is attached also the bi-Legendrian connection ∇^{bl} corresponding to the bi-Legendrian structure $(\mathscr{D}_h(\lambda),\mathscr{D}_h(-\lambda))$, as well as the bi-Legendrian connection $\overline{\nabla}^{bl}$ associated with $(\mathscr{D}_{\phi h}(\lambda),\mathscr{D}_{\phi h}(-\lambda))$. We now find the relations between these connections.

Lemma 5.6. *Let* (M, ϕ, ξ, η, g) *be a non-Sasakian contact metric* (κ, μ)*-space and* (ϕ_1, ϕ_2, ϕ_3) *its standard almost bi-paracontact structure. Then, for the operators* $h_\alpha := \frac{1}{2}\mathscr{L}_\xi \phi_\alpha$, $\alpha \in \{1, 2, 3\}$, *we have*

$$h_1 = -I_M h = -\left(1 - \frac{\mu}{2}\right)\phi_2, \tag{5.13}$$

$$h_2 = I_M \phi h + \sqrt{1-\kappa}\phi = \left(1 - \frac{\mu}{2}\right)\phi_1 + \sqrt{1-\kappa}\phi_3, \tag{5.14}$$

$$h_3 = h = \sqrt{1-\kappa}\phi_2. \tag{5.15}$$

Proof. The proof of (5.14) is given in [12, Lemma 4.5] whereas (5.15) is obvious. Then by using Lemma 4.1 one can prove (5.13). □

Substituting (5.13)–(5.15) in (ii) of Theorem 4.2 we get the following corollary.

Corollary 5.7. *Let* (M, ϕ, ξ, η, g) *be a non-Sasakian contact metric* (κ, μ)*-space and* (ϕ_1, ϕ_2, ϕ_3) *its standard almost bi-paracontact structure. The corresponding connections* ∇^1, ∇^2, ∇^3 *stated in Theorem* 4.2 *satisfy the following relations*:

$$\nabla^1\phi_1 = 0, \quad \nabla^1\phi_2 = 2\sqrt{1-\kappa}\eta \otimes \phi_3, \quad \nabla^1\phi_3 = 2\sqrt{1-\kappa}\eta \otimes \phi_2, \tag{5.16}$$

$$\nabla^2\phi_1 = 0, \quad \nabla^2\phi_2 = 0, \quad \nabla^2\phi_3 = 0, \tag{5.17}$$

$$\nabla^3\phi_1 = -(2-\mu)\eta \otimes \phi_2, \quad \nabla^3\phi_2 = (2-\mu)\eta \otimes \phi_1, \quad \nabla^3\phi_3 = 0. \tag{5.18}$$

Proposition 5.8. *With the notation above*, $\nabla^{bl} = \nabla^2$ *and* $\overline{\nabla}^{bl} = \nabla^1$.

Proof. First notice that ∇^{bl} satisfies the axioms (i), (ii), (iii) of Theorem 4.2 characterizing ∇^2. Indeed by definition $\nabla^{bl}\xi = 0$. Next, $\nabla^{bl}\phi = \nabla^{bl}h = 0$ ([8]) so that, tacking (5.17) into account, $\nabla^{bl}\phi_\alpha = 0 = \nabla^2\phi_\alpha$ for each $\alpha \in \{1, 2, 3\}$. Finally, by using the expression (2.27) of T^{bl}, a direct computation shows that also (iii) is satisfied. Then $\nabla^{bl} = \nabla^2$. As second step we prove that if S denotes the $(1,1)$-type tensor field given by $S(X, Y) := \nabla^{bl}_X Y - \overline{\nabla}^{bl}_X Y$, then we have

$$S(\cdot, \xi) = 0, \quad S(\xi, \cdot) = -\phi h, \quad S = 0 \text{ on } \mathscr{D}. \tag{5.19}$$

Obviously $S(\cdot, \xi) = 0$. In order to prove the remaining relations, let us define a linear connection ∇' on M by putting

$$\nabla'_E F := \begin{cases} \nabla^{bl}_E F, & \text{for } E \in \Gamma(\mathscr{D}),\ F \in \Gamma(TM); \\ \overline{\nabla}^{bl}_E F, & \text{for } E \in \Gamma(\mathbf{R}\xi),\ F \in \Gamma(TM). \end{cases}$$

We prove that $\nabla' = \overline{\nabla}^{bl}$ by checking that ∇' satisfies the axioms which characterize the bi-Legendrian connection associated with the bi-Legendrian structure $(\mathscr{D}_{\phi h}(\lambda), \mathscr{D}_{\phi h}(-\lambda))$. First, we prove that ∇' preserves the Legendre foliations $\mathscr{D}_{\phi h}(\lambda)$ and $\mathscr{D}_{\phi h}(-\lambda)$. Due to (5.2) any vector field tangent to $\mathscr{D}_{\phi h}(\lambda)$ has the form $X + \phi X$ for some $X \in \Gamma(\mathscr{D}_h(\lambda))$. Then, for any $Z \in \Gamma(\mathscr{D})$, we have

$$\nabla'_Z(X + \phi X) = \nabla'_Z X + \nabla'_Z \phi X = \nabla'_Z X + \nabla^{bl}_Z \phi X = \nabla'_Z X + \phi \nabla^{bl}_Z X = \nabla'_Z X + \phi \nabla'_Z X.$$

Since $\nabla'_Z X = \nabla^{bl}_Z X \in \Gamma(\mathscr{D}_h(\lambda))$, we conclude that $\nabla'_Z(X + \phi X) \in \Gamma(\mathscr{D}_{\phi h}(\lambda))$. Thus $\nabla'_Z \mathscr{D}_{\phi h}(\lambda) \subset \mathscr{D}_{\phi h}(\lambda)$. Moreover, $\nabla'_\xi \mathscr{D}_{\phi h}(\lambda) = \overline{\nabla}_\xi \mathscr{D}_{\phi h}(\lambda) \subset \mathscr{D}_{\phi h}(\lambda)$. Analogously one can prove that ∇' preserves $\mathscr{D}_{\phi h}(-\lambda)$. Next, $\nabla' \, d\eta = 0$ since $\nabla^{bl} \, d\eta = 0$ and $\overline{\nabla}^{bl} \, d\eta = 0$. Finally, one can easily prove that $T'(Z, \xi) = \overline{T}^{bl}(Z, \xi) = [\xi, Z_{\mathscr{D}_{\phi h}(\lambda)}]_{\mathscr{D}_{\phi h}(-\lambda)} + [\xi, Z_{\mathscr{D}_{\phi h}(-\lambda)}]_{\mathscr{D}_{\phi h}(\lambda)}$ and $T'(Z, Z') = T^{bl}(Z, Z') = 2 \, d\eta(Z, Z')\xi$ for any $Z, Z' \in \Gamma(\mathscr{D})$. Thus, by Theorem 2.7, $\nabla' = \overline{\nabla}^{bl}$ and hence $S = 0$ on $\mathscr{D}$. Finally, by (5.8)

$$\nabla^{bl}_\xi Z = \nabla^{bl}_Z \xi - T^{bl}(Z, \xi) - [Z, \xi] = \phi_2 h_2 Z + [\xi, Z]$$

and, analogously,

$$\overline{\nabla}^{bl}_\xi Z = \phi_1 h_1 Z + [\xi, Z].$$

Therefore, by using (5.13) and (5.14), one finds $S(\xi, Z) = \phi_2 h_2 Z - \phi_1 h_1 Z = -\phi h Z$. Thus (5.19) is completely proved. In particular, one obtains

$$\overline{\nabla}^{bl}_\xi \phi = \nabla^{bl}_\xi \phi + \phi h \phi - \phi^2 h = 2h \tag{5.20}$$

and

$$\overline{\nabla}^{bl}_\xi h = \nabla^{bl}_\xi h + \phi h^2 - h \phi h = 2\phi h^2 = 2(1 - \kappa)\phi. \tag{5.21}$$

Then $\overline{\nabla}^{bl}$ satisfies (5.16). Since it easily satisfies also the other two conditions which uniquely define the connection ∇^1, we conclude that $\overline{\nabla}^{bl} = \nabla^1$. □

The paracontact metric structure (ϕ_2, ξ, η, g_2) defined in Remark 5.4 was studied in [12]. Now we are able to study (ϕ_1, ξ, η, g_1). We show that both the paracontact metric structures satisfy a nullity condition.

Theorem 5.9. *Let (M, ϕ, ξ, η, g) be a non-Sasakian contact metric (κ, μ)-space and let (ϕ_1, ϕ_2, ϕ_3) be its standard almost bi-paracontact structure. Let g_1 and g_2 denote the semi-Riemannian metrics defined by* (5.7), *compatible with the almost paracontact structures ϕ_1 and ϕ_2, respectively. Then the paracontact metric structures $(\phi_\alpha, \xi, \eta, g_\alpha)$, $\alpha \in \{1, 2\}$, satisfy*

$$R^{g_\alpha}(X, Y)\xi = \kappa_\alpha(\eta(Y)X - \eta(X)Y) + \mu_\alpha(\eta(Y)h_\alpha X - \eta(X)h_\alpha Y)$$

where

$$\kappa_1 = \left(1 - \frac{\mu}{2}\right)^2 - 1, \quad \mu_1 = 2(1 - \sqrt{1-\kappa}), \tag{5.22}$$

$$\kappa_2 = \kappa - 2 + \left(1 - \frac{\mu}{2}\right)^2, \quad \mu_2 = 2. \tag{5.23}$$

Furthermore, $I_M = 0$ *if and only if* (ϕ_1, ξ, η, g_1) *is para-Sasakian.*

Proof. For the case $\alpha = 2$ the assertion was already proved in [12]. We prove the case $\alpha = 1$. First notice that, as $\mathscr{D}_1^+$ and $\mathscr{D}_1^-$ are involutive, the paracontact metric structure (ϕ_1, ξ, η, g_1) satisfies (2.24) (cf. [28]). Then by (2.22) we have that

$$\begin{aligned}(\nabla^{g_1}_X h_1) Y = {} & (\overline{\nabla}^{pc}_X h_1) Y - 2\eta(X)\phi_1 h_1 Y - \eta(Y)\phi_1 h_1 X + \eta(Y)\phi_1 h_1^2 X \\ & - g_1(X, \phi_1 h_1 Y)\xi + g_1(h_1 X, \phi_1 h_1 Y)\xi.\end{aligned} \tag{5.24}$$

Moreover, due to (5.13) and Proposition 5.8 we get

$$\begin{aligned}(\overline{\nabla}^{pc}_X h_1) Y = (\overline{\nabla}^{bl}_X h_1) Y = (\nabla^1_X h_1) Y &= -\left(1 - \frac{\mu}{2}\right)(\nabla^1_X \phi_2) Y \\ &= (\mu - 2)\sqrt{1-\kappa}\,\eta(X)\phi_3 Y.\end{aligned} \tag{5.25}$$

Thus, by replacing (2.24), (5.24) and (5.25) in (2.20) we find

$$\begin{aligned}R^{g_1}(X, Y)\xi = {} & -\eta(Y)(X - h_1 X) + g_1(X - h_1 X, Y)\xi + \eta(X)(Y - h_1 Y) \\ & - g_1(Y - h_1 Y, X)\xi - g_1(X - h_1 X, h_1 Y)\xi + \phi_1((\overline{\nabla}^{pc}_X h_1) Y) \\ & - 2\eta(X)\phi_1^2 h_1 Y + \eta(Y)\phi_1 h_1 \phi_1 X + \eta(Y)\phi_1^2 h_1 X \\ & + g_1(Y - h_1 Y, h_1 X)\xi - \phi_1((\overline{\nabla}^{pc}_Y h_1) X) + 2\eta(Y)\phi_1^2 h_1 X \\ & - \eta(X)\phi_1 h_1 \phi_1 Y - \eta(X)\phi_1^2 h_1 Y \\ = {} & -\eta(Y)X + \eta(X)Y + (\mu - 2)\sqrt{1-\kappa}\,\eta(X)\phi_1\phi_3 Y \\ & - 2\eta(X)\phi_1^2 h_1 Y + \eta(Y)\phi_1^2 h_1^2 X \\ & - (\mu - 2)\sqrt{1-\kappa}\,\eta(Y)\phi_1\phi_3 X + 2\eta(Y)\phi_1^2 h_1 X - \eta(X)\phi_1^2 h_1^2 Y \\ = {} & -\eta(Y)X + \eta(X)Y + (\mu - 2)\sqrt{1-\kappa}\,\eta(X)\phi_2 Y \\ & - 2\eta(X)h_1 Y - \left(1 - \frac{\mu}{2}\right)^2 \eta(Y)\phi_2^2 X \\ & - (\mu - 2)\sqrt{1-\kappa}\,\eta(Y)\phi_2 X + 2\eta(Y)h_1 X + \left(1 - \frac{\mu}{2}\right)^2 \eta(X)\phi_2^2 Y\end{aligned}$$

$$
\begin{aligned}
&= -\eta(Y)X + \eta(X)Y + 2\sqrt{1-\kappa}\,\eta(X)h_1 Y \\
&\quad - 2\eta(X)h_1 Y + \left(1-\frac{\mu}{2}\right)^2 \eta(Y)X \\
&\quad - 2\sqrt{1-\kappa}\,\eta(Y)h_1 X + 2\eta(Y)h_1 X - \left(1-\frac{\mu}{2}\right)^2 \eta(X)Y \\
&= \left(\left(1-\frac{\mu}{2}\right)^2 - 1\right)(\eta(Y)X - \eta(X)Y) \\
&\quad + 2(1-\sqrt{1-\kappa})(\eta(Y)h_1 X - \eta(X)h_1 Y).
\end{aligned}
$$

For the last assertion in the statement of the theorem, we have that $I_M = 0$ if and only if $\mu = 2$, i.e., by (5.13), if and only if $h_1 = 0$. As the paracontact metric structure (ϕ_1, ξ, η, g_1) is integrable, the assert follows from Corollary 2.6. □

We now study the special properties of the connection ∇^c (cf. Theorem 4.4) associated to the standard almost bi-paracontact structure (ϕ_1, ϕ_2, ϕ_3) of a (non-Sasakian) contact metric (κ, μ)-space (M, ϕ, ξ, η, g). We call ∇^c the *canonical connection of the contact metric* (κ, μ)*-space* M.

LEMMA 5.10. *The torsion tensor field of the canonical connection of a non-Sasakian contact metric* (κ, μ)*-space* (M, ϕ, ξ, η, g) *is given by*

$$
\begin{aligned}
T^c(X,Y) = \frac{2}{3}\Bigg(\eta(Y)\left(\left(1-\frac{\mu}{2}\right)\phi X + \phi h X\right) \\
- \eta(X)\left(\left(1-\frac{\mu}{2}\right)\phi Y + \phi h Y\right)\Bigg) + 2\,d\eta(X,Y)\xi.
\end{aligned} \tag{5.26}
$$

In particular,

$$
T^c(X,\xi) = \frac{2}{3}\left(\left(1-\frac{\mu}{2}\right)\phi X + \phi h X\right). \tag{5.27}
$$

Proof. First of all notice that, being the almost bi-paracontact structure (ϕ_1, ϕ_2, ϕ_3) integrable, (3.1) holds. Then by replacing (2.12), (2.25), (3.1) into (iii) of Theorem 4.4 we obtain

$$
\begin{aligned}
T^c(X,Y) = 2\,d\eta(X,Y)\xi + \frac{1}{6}(-2\eta(Y)\phi_1 h_1 X + 2\eta(X)\phi_1 h_1 Y - 2\eta(Y)\phi_2 h_2 X \\
+ 2\eta(X)\phi_2 h_2 Y + 2\eta(Y)\phi_3 h_3 X - 2\eta(X)\phi_3 h_3 Y).
\end{aligned} \tag{5.28}
$$

By substituting (5.13) and (5.14) in (5.28), a straightforward computation yields (5.26). □

PROPOSITION 5.11. *With the notation above, we have for any* $X, Y \in \Gamma(\mathscr{D})$,

$$\nabla^c_X Y = \nabla^1_X Y = \nabla^2_X Y = \nabla^3_X Y.$$

Proof. Let ∇' be the linear connection defined by

$$\nabla'_E F := \begin{cases} \nabla^{bl}_E F, & \text{if } E \in \Gamma(\mathscr{D}); \\ \nabla^c_E F, & \text{if } E \in \Gamma(\mathbf{R}\xi). \end{cases}$$

We check that ∇' satisfies (i), (ii), (iii) of Theorem 4.4. First of all, obviously $\nabla'\xi = 0$. Next, for all $X, Y \in \Gamma(\mathscr{D})$, by (5.26), $T'(X, Y) = T^{bl}(X, Y) = 2\,d\eta(X, Y)\xi = T^c(X, Y)$ and $T'(X, \xi) = T^c(X, \xi)$. Finally, for all $X, Y \in \Gamma(\mathscr{D})$, we have $(\nabla'_X \phi_\alpha)Y = (\nabla^{bl}_X \phi_\alpha)Y = 0 = (\nabla^c_X \phi_\alpha)Y$ for each $\alpha \in \{1, 2, 3\}$, since $\nabla^{bl}\phi = \nabla^{bl}h = 0$. Moreover, by definition, $(\nabla'_\xi \phi_\alpha)X = (\nabla^c_\xi \phi_\alpha)X$. Thus by the uniqueness of ∇^c we have that $\nabla' = \nabla^c$. Then, since by Proposition 5.8 $\nabla^{bl} = \nabla^2$, we have that ∇^2 and ∇^c coincide on the contact distribution. Moreover, Proposition 5.8 and (5.19) imply that also $\nabla^1 = \overline{\nabla}^{bl}$ and ∇^c coincide on $\mathscr{D}$. The same property is then necessarily satisfied by ∇^3 since ∇^c is the barycenter of ∇^1, ∇^2, ∇^3. □

COROLLARY 5.12. *The canonical connection* ∇^c *of a contact metric* (κ, μ)*-space* (M, ϕ, ξ, η, g) *is a contact connection, i.e.* $\nabla^c \eta = \nabla^c\,d\eta = 0$, *and satisfies*

$$\nabla^c \phi_1 = -\frac{2}{3}\left(1 - \frac{\mu}{2}\right)\eta \otimes \phi_2 \tag{5.29}$$

$$\nabla^c \phi_2 = \frac{2}{3}\left(1 - \frac{\mu}{2}\right)\eta \otimes \phi_1 + \frac{2}{3}\sqrt{1-\kappa}\eta \otimes \phi_3 \tag{5.30}$$

$$\nabla^c \phi_3 = \frac{2}{3}\sqrt{1-\kappa}\eta \otimes \phi_2 \tag{5.31}$$

Proof. By Proposition 5.8 and Proposition 5.11 we have, for all $X, Y, Z \in \Gamma(\mathscr{D})$, $(\nabla^c_X\,d\eta)(Y, Z) = (\nabla^2_X\,d\eta)(Y, Z) = (\nabla^{bl}_X\,d\eta)(Y, Z) = 0$ and, since $\nabla^c \xi = 0$, $(\nabla^c_X\,d\eta)(Y, \xi) = 0$. Moreover, from (5.27) it follows that

$$\nabla^c_\xi X = [\xi, X] - \frac{2}{3}\left(\left(1 - \frac{\mu}{2}\right)\phi X + \phi h X\right). \tag{5.32}$$

Then (5.32) yields

$$\begin{aligned}(\nabla^c_\xi\,d\eta)(X, Y) &= \xi(d\eta(X, Y)) - d\eta([\xi, X], Y) + \frac{2}{3}\left(1 - \frac{\mu}{2}\right) d\eta(\phi X, Y) \\ &\quad + \frac{2}{3}\,d\eta(\phi h X, Y) - d\eta(X, [\xi, Y]) \\ &\quad + \frac{2}{3}\left(1 - \frac{\mu}{2}\right) d\eta(X, \phi Y) + \frac{2}{3}\,d\eta(X, \phi h Y) \\ &= (\mathscr{L}_\xi\,d\eta)(X, Y) + \frac{2}{3}g(\phi h X, \phi Y) + \frac{2}{3}g(X, \phi^2 h Y) = 0,\end{aligned}$$

since $\mathscr{L}_\xi\, d\eta = 0$ and h is a symmetric operator. Finally, (5.29)–(5.31) follow from (ii) of Theorem 4.4 and from (5.13), (5.14). □

Conversely, we show that (5.29)–(5.31) in some sense characterize the existence of a contact metric (κ, μ)-structure on an almost bi-paracontact manifold.

THEOREM 5.13. *Let (ϕ_1, ϕ_2, ϕ_3) be an integrable almost bi-paracontact structure on the contact manifold (M, η) such that the associated canonical connection satisfies $\nabla^c\, d\eta = 0$ and*

$$\nabla^c \phi_1 = -a\eta \otimes \phi_2 \tag{5.33}$$

$$\nabla^c \phi_2 = a\eta \otimes \phi_1 + b\eta \otimes \phi_3 \tag{5.34}$$

$$\nabla^c \phi_3 = b\eta \otimes \phi_2 \tag{5.35}$$

for some $a > 0$ (respectively, $a < 0$) and $b > 0$. Let us define

$$g_1 := d\eta(\cdot, \phi_1 \cdot) + \eta \otimes \eta, \quad g_2 := d\eta(\cdot, \phi_2 \cdot) + \eta \otimes \eta, \tag{5.36}$$
$$g_3 := -d\eta(\cdot, \phi_3 \cdot) + \eta \otimes \eta$$

and assume that the symmetric bilinear form $\pi_1 := g_1(h_1 \cdot, \cdot)$ is positive definite (respectively, negative definite). Then, for each $\alpha \in \{1, 2\}$, $(\phi_\alpha, \xi, \eta, g_\alpha)$ is a paracontact metric $(\kappa_\alpha, \mu_\alpha)$-structure and (ϕ_3, ξ, η, g_3) is a contact metric (κ_3, μ_3)-structure, where

$$\kappa_1 := \frac{9}{4}a^2 - 1, \quad \mu_1 := 2 - 3b, \tag{5.37}$$

$$\kappa_2 := \frac{9}{4}(a^2 - b^2) - 1, \quad \mu_2 := 2, \tag{5.38}$$

$$\kappa_3 := 1 - \frac{9}{4}b^2, \quad \mu_3 := 2 + 3a. \tag{5.39}$$

Moreover, (ϕ_1, ϕ_2, ϕ_3) is the standard almost bi-paracontact structure of the contact metric (κ_3, μ_3)-manifold $(M, \phi_3, \xi, \eta, g_3)$.

Proof. Since the almost bi-paracontact structure is assumed to be integrable, we have in particular, by Proposition 3.7, that the bilinear forms g_1, g_2, g_3, defined by (5.36), are symmetric, so that the definition is well posed. Notice that, by construction, for each $\alpha \in \{1, 2, 3\}$, g_α is compatible with the corresponding structure, i.e.

$$g_\alpha(\phi_\alpha X, \phi_\alpha Y) = -\varepsilon(g_\alpha(X, Y) - \eta(X)\eta(Y))$$

where we have posed $\varepsilon = 1$ if $\alpha \in \{1, 2\}$ and $\varepsilon = -1$ if $\alpha = 3$. Moreover, each g_α is, by definition, an associated metric, i.e. $d\eta(X, Y) = g_\alpha(X, \phi_\alpha Y)$ for all $X, Y \in$

$\Gamma(TM)$. Furthermore, by comparing (5.33)–(5.35) with (ii) of Theorem 4.4 we have that

$$h_1 = -\frac{3}{2}a\phi_2, \quad h_2 = \frac{3}{2}(a\phi_1 + b\phi_3), \quad h_3 = \frac{3}{2}b\phi_2. \tag{5.40}$$

Hence, by (5.40), we have, for all $X, Y \in \Gamma(TM)$,

$$\begin{aligned} g_3(X,Y) &= -d\eta(X,\phi_3 Y) + \eta(X)\eta(Y) \\ &= -g_1(X,\phi_1\phi_3 Y) + \eta(X)\eta(Y) \\ &= -g_1(X,\phi_2 Y) + \eta(X)\eta(Y) \\ &= \frac{2}{3a}g_1(X,h_1 Y) \\ &= \frac{2}{3a}\pi_1(X,Y). \end{aligned}$$

Then the assumptions of positive definiteness of π_1 and $a > 0$ imply that g_3 is a Riemannian metric. It follows that $(\phi_\alpha, \xi, \eta, g_\alpha)$ is a paracontact metric structure for $\alpha \in \{1,2\}$ and a contact metric structure for $\alpha = 3$. Now, since the almost bi-paracontact structure (ϕ_1, ϕ_2, ϕ_3) is integrable, by Corollary 3.9, the tensor fields $N^{(1)}_{\phi_1}$, $N^{(1)}_{\phi_2}$, $N^{(1)}_{\phi_3}$ vanish on $\mathscr{D}$. Moreover, Proposition 3.7 implies that $d(\phi_1 X, \phi_1 Y) = d(\phi_2 X, \phi_2 Y) = -d\eta(\phi_3 X, \phi_3 Y) = -d\eta(X,Y)$ for any $X, Y \in \Gamma(\mathscr{D})$. Hence, taking (iii) of Theorem 4.4 into account, the torsion of the canonical connection is given by

$$T^c(X,Y) = 2\,d\eta(X,Y)\xi \tag{5.41}$$

for all $X, Y \in \Gamma(\mathscr{D})$. We now are able to prove that on the contact distribution the canonical connection and the Levi Civita connection of g_3 are related by the formula

$$\nabla^c_X Y = \nabla^{g_3}_X Y - \eta(\nabla^{g_3}_X Y)\xi. \tag{5.42}$$

Indeed, let us define a linear connection ∇' on M by

$$\nabla'_X Y := \begin{cases} \nabla^c_X Y + \eta(\nabla^{g_3}_X Y)\xi, & \text{if } X, Y \in \Gamma(\mathscr{D}); \\ \nabla^{g_3}_X Y, & \text{elsewhere.} \end{cases}$$

We prove that in fact ∇' coincides with the Levi Civita connection of (M, g_3). For any $X, Y, Z \in \Gamma(\mathscr{D})$ we have

$$\begin{aligned} (\nabla'_X g_3)(Y,Z) &= (\nabla^c_X g_3)(Y,Z) - \eta(Z)\eta(\nabla^{g_3}_X Y) - \eta(Y)\eta(\nabla^{g_3}_X Z) \\ &= -X(d\eta(Y,\phi_3 Z)) + d\eta(\nabla^c_X Y, \phi_3 Z) + d\eta(Y, \phi_3 \nabla^c_X Z) \\ &= -X(d\eta(Y,\phi_3 Z)) + d\eta(\nabla^c_X Y, \phi_3 Z) + d\eta(Y, \nabla^c_X \phi_3 Z) \\ &= -(\nabla^c_X\, d\eta)(Y, \phi_3 Z) = 0, \\ (\nabla'_X g_3)(Y,\xi) &= (\nabla^{g_3}_X g_3)(Y,\xi) - \eta(\nabla^c_X Y) = 0 \end{aligned}$$

and

$$(\nabla'_{\xi} g_3)(Y,Z) = (\nabla^{g_3}_{\xi} g_3)(Y,Z) = 0.$$

Next, by (5.41)

$$T'(X,Y) = T^c(X,Y) + \eta(\nabla^{g_3}_X Y)\xi - \eta(\nabla^{g_3}_Y X)\xi = 2\,d\eta(X,Y)\xi + \eta([X,Y])\xi = 0,$$

and $T'(X,\xi) = T^{g_3}(X,\xi) = 0$. Thus $\nabla' = \nabla^{g_3}$ and (5.42) follows. Then (5.34), (5.40) and (5.42) yield, for any $X, Y, Z \in \Gamma(\mathscr{D})$,

$$\begin{aligned} g_3((\nabla^{g_3}_X h_3)Y,Z) &= g_3((\nabla^c_X h_3)Y,Z) + \eta(\nabla^{g_3}_X h_3 Y)\eta(Z) \\ &= \frac{3}{2} b g_3((\nabla^c_X \phi_2)Y,Z) \\ &= \frac{3}{2} ab\eta(X) g_3(\phi_1 X,Z) + \frac{3}{2} b^2 \eta(X) g_3(\phi_3 X,Z) = 0. \end{aligned}$$

Therefore the tensor field h_3 is "η-parallel" (cf. [6]) and so, by [6, Theorem 4], (ϕ_3,ξ,η,g_3) is a contact metric (κ,μ)-space. The values of κ and μ can be found by comparing (5.33)–(5.35) with (5.29)–(5.31). After a straightforward computation it turns out that they are given by (5.39). The remaining part of the theorem follows from Theorem 5.9. In particular, (5.37) and (5.38) are consequence of (5.22) and (5.23), respectively. The case $a < 0$ can be proved in a similar way. □

Formulae (5.13)–(5.15) together with (a) of Lemma 4.1 allow us to define a supplementary almost bi-paracontact structure on a non-Sasakian contact metric (κ,μ)-space. In fact, by (5.14) we have

$$\begin{aligned} h_2^2 &= \left(1-\frac{\mu}{2}\right)^2 \phi_1^2 + \left(1-\frac{\mu}{2}\right)\sqrt{1-\kappa}\,\phi_1\phi_3 \\ &\quad + \left(1-\frac{\mu}{2}\right)\sqrt{1-\kappa}\,\phi_3\phi_1 + (1-\kappa)\phi_3^2 \\ &= \left(\left(1-\frac{\mu}{2}\right)^2 - (1-\kappa)\right)(I - \eta\otimes\xi). \end{aligned} \tag{5.43}$$

Therefore, under the assumption that $\left(1-\frac{\mu}{2}\right)^2 \neq 1-\kappa$, we are led to consider the tensor field

$$\begin{aligned} \psi &:= \frac{1}{\sqrt{\left|\left(1-\frac{\mu}{2}\right)^2 - (1-\kappa)\right|}} h_2 \\ &= \frac{1}{\sqrt{\left|\left(1-\frac{\mu}{2}\right)^2 - (1-\kappa)\right|}} \left(\left(1-\frac{\mu}{2}\right)\phi_1 + \sqrt{1-\kappa}\,\phi_3\right) \end{aligned} \tag{5.44}$$

By (5.43) we see that if $\left(1-\frac{\mu}{2}\right)^2-(1-\kappa)>0$ then the tensor field ψ satisfies $\psi^2=I-\eta\otimes\xi$, whereas if $\left(1-\frac{\mu}{2}\right)^2-(1-\kappa)<0$ we have $\psi^2=-I+\eta\otimes\xi$. Notice that $\left(1-\frac{\mu}{2}\right)^2-(1-\kappa)>0$ if and only if $|I_M|>0$. Therefore we are able to prove the following theorem.

THEOREM 5.14. *Let (M,ϕ,ξ,η,g) be a non-Sasakian contact metric (κ,μ)-space such that $I_M\neq\pm1$.*

(i) *If $|I_M|>1$ then M admits an integrable almost bi-paracontact structure $(\phi_1',\phi_2',\phi_3')$, given by*

$$\phi_1':=\frac{1}{\sqrt{\left(1-\frac{\mu}{2}\right)^2-(1-\kappa)}}(I_M\phi h+\sqrt{1-\kappa}\phi)$$

$$\phi_2':=\frac{1}{\sqrt{1-\kappa}}h$$

$$\phi_3':=\frac{1}{\sqrt{\left(1-\frac{\mu}{2}\right)^2-(1-\kappa)}}(I_M h+\sqrt{1-\kappa}\phi h).$$

(ii) *If $|I_M|<1$ then M admits an integrable almost bi-paracontact structure $(\phi_1'',\phi_2'',\phi_3'')$, given by*

$$\phi_1'':=\frac{1}{\sqrt{1-\kappa}}h$$

$$\phi_2'':=\frac{1}{\sqrt{1-\kappa-\left(1-\frac{\mu}{2}\right)^2}}(I_M h+\sqrt{1-\kappa}\phi h)$$

$$\phi_3'':=\frac{1}{\sqrt{1-\kappa-\left(1-\frac{\mu}{2}\right)^2}}(I_M\phi h+\sqrt{1-\kappa}\phi).$$

Proof. Let us assume $|I_M|>1$. In order to relieve the notation, we put $\alpha:=1-\frac{\mu}{2}$ and $\beta:=\sqrt{1-\kappa}$. As remarked before, by a direct computation one proves that $\phi_1'^2=I-\eta\otimes\xi$. Moreover, by (a) of Lemma 4.1, $\phi_2h_2=-h_2\phi_2$, so that $\phi_1'=\frac{1}{\sqrt{\alpha^2-\beta^2}}h_2$ and $\phi_2'=\phi_2$ anti-commute. Thus $(\phi_1',\phi_2',\phi_3'=\phi_1'\phi_2')$ is an almost bi-paracontact structure on (M,η). We prove that it is integrable, by showing that the eigendistributions $\mathscr{D}_1'^{\pm}$ associated to ϕ_1' define Legendre foliations, since we already know that $\mathscr{D}_2'^{\pm} = \mathscr{D}_2^{\pm}$ do. First we show that $\mathscr{D}_1'^{+}$ is a Legendrian distribution. For any $X, X' \in \Gamma(\mathscr{D}_1'^{+})$ we have

$$\begin{aligned} d\eta(X,X') &= d\eta(\phi_1'X, \phi_1'X') \\ &= \frac{1}{\alpha^2-\beta^2}(\alpha^2\, d\eta(\phi_1X,\phi_1X') + \alpha\beta\, d\eta(\phi_1X,\phi_3X') \\ &\quad + \alpha\beta\, d\eta(\phi_3X,\phi_1X') + \beta^2\, d\eta(\phi_3X,\phi_3X')). \end{aligned} \tag{5.45}$$

Now, notice that $d\eta(\phi_1X,\phi_1X') = -d\eta(\phi_3X,\phi_3X') = -d\eta(X,X')$, and $d\eta(\phi_1X,\phi_3X') = d\eta(\phi_1X,\phi_1\phi_2X') = -d\eta(X,\phi_2X') = -d\eta(\phi_3X,\phi_1X')$, so that (5.45) becomes

$$d\eta(X,X') = -\frac{\alpha^2-\beta^2}{\sqrt{\alpha^2-\beta^2}}\, d\eta(X,X') = -\sqrt{\alpha^2-\beta^2}\, d\eta(X,X').$$

Hence $d\eta(X,X') = 0$. It remains to prove that $\mathscr{D}_1'^{+}$ is involutive. Take $X, X' \in \Gamma(\mathscr{D}_1'^{+})$. By (5.26), the torsion of the canonical connection ∇^c of the contact metric (κ,μ)-space (M,ϕ,ξ,η,g) satisfies $T^c(X,X') = 2\, d\eta(X,X')\xi = 0$. Then, using (5.29)–(5.31), we have

$$\begin{aligned} \phi_1'[X,X'] &= \phi_1'(\nabla^c_X X' - \nabla^c_{X'} X) \\ &= \frac{1}{\sqrt{\alpha^2-\beta^2}}(\alpha\phi_1\nabla^c_X X' + \beta\phi_3\nabla^c_X X' - \alpha\phi_1\nabla^c_{X'}X - \beta\phi_3\nabla^c_{X'}X) \\ &= \frac{1}{\sqrt{\alpha^2-\beta^2}}(\alpha\nabla^c_X\phi_1X' + \beta\nabla^c_X\phi_3X' - \alpha\nabla^c_{X'}\phi_1X - \beta\nabla^c_{X'}\phi_3X) \\ &= \nabla^c_X\phi_1'X' - \nabla^c_{X'}\phi_1'X \\ &= [X,X']. \end{aligned}$$

In the same way one can prove that also $\mathscr{D}_1'^{-}$ is involutive. Thus we conclude that the almost bi-paracontact structure $(\phi_1',\phi_2',\phi_3')$ is integrable. The case $|I_M| < 1$ can be proved in a similar way. □

Remark 5.15. By a straightforward computation one obtains

$$h_1' = -\sqrt{I_M^2-1}h, \quad h_2' = I_M\phi h + \sqrt{1-\kappa}\phi, \quad h_3' = 0,$$
$$h_1'' = I_M\phi h + \sqrt{1-\kappa}\phi, \quad h_2'' = 0, \quad h_3'' = \sqrt{1-I_M^2}h.$$

Moreover, the integrability of the almost bi-paracontact structure yields, by Corollary 3.9, $N^{(1)}_{\phi_3'} = 0$ on $\mathscr{D}$. On the other hand, for any $X \in \Gamma(\mathscr{D})$, $N^{(1)}_{\phi_3'}(X,\xi) = -[X,\xi] - \phi_3'[\phi_3'X,\xi] = 2\phi_3'h_3' = 0$. Hence the almost contact structure (ϕ_3',ξ,η) is normal. Nevertheless the almost bi-paracontact itself is not normal because h_1' and h_2' do not vanish. Similar arguments hold for $(\phi_1'', \phi_2'', \phi_3'')$. Thus we have obtained a class of examples of integrable, non-normal almost bi-paracontact structures such that one structure is normal.

References

[1] A. Andrada, Complex product structures and affine foliations, Ann. Glob. Anal. Geom. **22** (2005), 377–405.

[2] D. E. Blair, Two remarks on contact metric structures, Tohoku Math. J. **28** (1976), 373–379.

[3] D. E. Blair, Riemannian geometry of contact and symplectic manifolds, Progress in mathematics **203**, Birkhäuser, Boston, 2002.

[4] D. E. Blair, T. Koufogiorgos and B. J. Papantoniou, Contact metric manifolds satisfyng a nullity condition, Israel J. Math. **91** (1995), 189–214.

[5] E. Boeckx, A full classification of contact metric (κ, μ)-spaces, Illinois J. Math. **44** (2000), 212–219.

[6] E. Boeckx and J. T. Cho, η-parallel contact metric spaces, Different. Geom. Appl. **22** (2005), 275–285.

[7] B. Cappelletti Montano, Bi-Legendrian connections, Ann. Polon. Math. **86** (2005), 79–95.

[8] B. Cappelletti Montano, Some remarks on the generalized Tanaka-Webster connection of a contact metric manifold, Rocky Mountain J. Math. **40** (2010), 1009–1037.

[9] B. Cappelletti Montano and L. Di Terlizzi, Contact metric (κ, μ)-spaces as bi-Legendrian manifolds, Bull. Austral. Math. Soc. **77** (2008), 373–386.

[10] B. Cappelletti Montano, Bi-Legendrian structures and paracontact geometry, Int. J. Geom. Meth. Mod. Phys. **6** (2009), 487–504.

[11] B. Cappelletti Montano, The foliated structure of contact metric (κ, μ)-spaces, Illinois J. Math., to appear.

[12] B. Cappelletti Montano and L. Di Terlizzi, Geometric structures associated to a contact metric (κ, μ)-space, Pacific J. Math. **246** (2010), 257–292.

[13] S. Ivanov and S. Zamkovoy, Parahermitian and paraquaternionic manifolds, Different. Geom. Appl. **23** (2005), 205–234.

[14] S. Kaneyuki and F. L. Williams, Almost paracontact and parahodge structures on manifolds, Nagoya Math. J. **99** (1985), 173–187.

[15] P. Libermann, Sur le problème d'équivalence de certaines structures infinitésimales, Ann. Mat. Pura Appl. **36** (1954), 27–120.

[16] P. Libermann, Legendre foliations on contact manifolds, Different. Geom. Appl. **1** (1991), 57–76.

[17] Y. Machida and T. Morimoto, On decomposable Monge-Ampère equations, Lobachevskii J. Math. **3** (1999), 185–196.

[18] S. Marchiafava and P. T. Nagy, (Anti-)hypercomplex structures and 3-webs on a manifold, Technical Report n. 38, University "La Sapienza" of Rome, 2003.

[19] P. Molino, Riemannian foliations, Progress in mathematics **73**, Birkhäuser, Boston, 1988.

[20] T. Morimoto, Differential equations associated to a representation of a Lie algebra from the viewpoint of nilpotent analysis, Developments of Cartan Geometry and Related Mathematical Problems, RIMS Kokyuroku **1502** (2006), 238–250.

[21] P. T. Nagy, Invariant tensorfields and the canonical connection of a 3-web, Aequationes Math. **35** (1988), 31–44.

[22] M. Y. Pang, The structure of Legendre foliations, Trans. Amer. Math. Soc. **320** (1990), 417–453.

[23] R. Santamaría Sánchez, Examples of manifolds with three supplementary distributions, Atti Sem. Mat. Fis. Univ. Modena **157** (1999), 419–428.

[24] S. Sasaki and Y. Hatakeyama, On differentiable manifolds with certain structures which are closely related to almost contact structure II, Tohoku Math. J. **13** (1961), 281–294.

[25] S. Tanno, Variational problems on contact Riemannian manifolds, Trans. Amer. Math. Soc. **314** (1989), 349–379.

[26] Ph. Tondeur, Geometry of foliations, Monographs in mathematics **90**, Birkhäuser, Basel, 1997.

[27] K. Yano and M. Ako, Almost quaternion structures of the second kind and almost tangent structures, Kodai Math. Sem. Rep. **25** (1973), 63–94.

[28] S. Zamkovoy, Canonical connections on paracontact manifolds, Ann. Glob. Anal. Geom. **36** (2009), 37–60.

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