Bi-Unitary Perfect Polynomials Over $GF(g)$ (*).

JACOB T. B. BEARD, JR.

Summary. – *This paper continues the author's excursions into the arithmetic of polynomials over finite fields. For monic polynomials A, $B \in GF[q, x]$ where p is a prime, $q = p^d$ and $d \geq 1$: The divisor B of A is a* bi-unitary divisor *of A provided 1 is the greatest common unitary divisor of the polynomials B and A/B, and we say that A is* bi-unitary perfect (b.u.p.) *over $GF(q)$ provided A equals the sum $\sigma^{**}(A)$ of the distinct bi-unitary divisors of A in $GF[q, x]$. A diversity of b.u.p. polynomials over $GF(q)$ is found, some of which are neither perfect nor unitary perfect. For $p > 2$ we can only conjecture a characterization of the b.u.p. polynomials which split in $GF[p, x]$, so several open questions remain. Examples of non-splitting b.u.p. polynomials over $GF(p)$ are given for $p = 2, 3, 5$ which, in turn, allow the construction of such examples over $GF(p^d)$ for these p.*

1. – Introduction and notation.

This paper continues the author's excursions into the arithmetic of polynomials over finite fields [1]-[6]. As before, we are concerned with monic polynomials A, B, $C \in GF[q, x]$ where p is a prime, $q = p^d$ and $d \geq 1$. Whereas a divisor C of A is a unitary divisor of A provided $1 = (C, A/C)$ [2], a divisor B of A is called a *bi-unitary divisor* of A provided 1 is the greatest common unitary divisor of the polynomials B and A/B. Accordingly, we say that A is *bi-unitary perfect* (*b.u.p.*) over $GF(q)$ provided A equals the sum $\sigma^{**}(A)$ of the distinct bi-unitary divisors of A in $GF[q, x]$. The rational integer concept is due to CHARLES R. WALL, who found the first three unitary perfect numbers 6, 60, 90 to be the only bi-unitary perfect numbers [9]. In contrast we find a variety of b.u.p. polynomials even over $GF(p)$, some of which are neither perfect [1], [3], [6] nor unitary perfect [2]-[5]. At present, for $p > 2$ we can only conjecture a characterization (§ 3) of the non-perfect b.u.p. polynomials which split in $GF[p, x]$, so several questions remain open. Examples of non-splitting b.u.p. polynomials over $GF(p)$ are given for $p = 2, 3, 5$ (§ 4) which, in turn, allow the construction of such examples over $GF(p^d)$ for these primes.

(*) Entrata in Redazione il 2 novembre 1985.
(**) Written while a Visiting Professor at The University of Tennessee-Knoxville.
Indirizzo dell'A.: Tennessee Technological University, Cookeville, Tennessee 38505, USA.

2. – Preliminary results.

Evidently, the function σ^{**} is multiplicative and degree-preserving, and for a prime (monic irreducible) polynomial P we have

$$\sigma^{**}(P^{2n}) = 1 + P + \dots + P^{n-1} + P^{n+1} + \dots + P^{2n} = \\ = (1 + P^{n+1})(1 + P + \dots + P^{n-1})\,.$$

Thus the fundamental identities for σ^{**} on powers of prime polynomials are given by

$$\sigma^{**}(P^{2n+1}) = \sigma(P^{2n+1}) = \frac{P^{2(n+1)}-1}{P-1}\,, \tag{1}$$

$$\sigma^{**}(P^{2n}) = (P^{n+1}+1)\sigma(P^{n-1}) = (P^{n+1}+1)\frac{P^{n}-1}{P-1}\,, \tag{2}$$

where σ is the sum of the divisors function (studied initially over $GF[2, x]$ by CANADAY [8]). Next, observe that a polynomial A is b.u.p. if and only if $\sigma^{**}(A) - A = 0$.

THEOREM 1. – If the polynomial A is b.u.p. over $GF(p^d)$, then at least p distinct prime polynomials in $GF[p^d, x]$ divide A.

PROOF. – Consider the canonical decomposition of A as the product of positive powers of distinct primes, $A = \prod_{i=1}^{n} P_i^{\alpha(i)}$. The admissible summands of $\sigma^{**}(A) - A$ having maximum degree are monic and number at most n, hence $n \geq p$. ∎

COROLLARY. – If the polynomial A is b.u.p. and splits over $GF(p)$, then $(x^p - x)|A$.

One can prove our next result (as we have its analogs for perfect and unitary perfect polynomials [3; Theorem 7]) using another basic fact: If $A(x) \in GF[q, x]$ has canonical decomposition $A(x) = \prod_{i=1}^{n} (P_i(x))^{\alpha(i)}$, then for any $b \in GF(q)$ the polynomial $A_b(x) = A(x-b)$ has canonical decomposition $A_b(x) = \prod_{i=1}^{n} (Q_i(x))^{\alpha(i)}$ where $Q_i(x) = P_i(x-b)$. For those $d \geq 1$ such that the primes $P_i(x)$ over $GF(p)$ remain irreducible over $GF(p^d)$, one can use Theorem 2 to construct examples of b.u.p. polynomials over $GF(p^d)$ from those already known over $GF(p)$. Moreover, with no restriction except $d > 1$, Theorem 2 can be applied to obtain relatively prime b.u.p. polynomials over $GF(p^d)$ which, in turn, yield their product as yet another b.u.p. polynomial over $GF(p^d)$. (Recall that σ^{**} is multiplicative.)

THEOREM 2. – If the polynomial $A(x)$ is b.u.p. over $GF(q)$, then $A_b(x) = A(x-b)$ is b.u.p. over $GF(q)$ for each $b \in GF(q)$.

3. – Splitting bi-unitary perfect polynomials.

Hereafter, we write $A \to B$ to denote $\sigma^{**}(A) = B$. Thus from (2) we have

$$\prod_{i=0}^{p-1}(x-i)^{2n} \to \left(\prod_{i=0}^{p-1}[(x-i)^{n+1}+1]\right)\left(\prod_{i=0}^{p-1}\frac{(x-i)^n-1}{x-i-1}\right). \tag{3}$$

In view of the Corollary to Theorem 1, we begin our study of b.u.p. splitting polynomials over $GF(p)$ by focusing on (3) via $\sigma^{**}(x^{2n})$ and the factorizations of $x^k \pm 1$. In this context we see that n must have the form $n = Np^d$, since $(x-i)^n - 1$ does not split in $GF[p, x]$ otherwise. (E.g., see the proofs of Theorems 3, 4 in [1].) Accordingly, we make this assumption on n hereafter, and have

$$\begin{aligned}\prod_{i=0}^{p-1}(x-i)^{2Np^d} &\to \left(\prod_{i=0}^{p-1}[(x-i)^{Np^d+1}+1]\right)\sigma\left(\prod_{i=0}^{p-1}(x-i)^{Np^d-1}\right) = \\ &= \left(\prod_{i=0}^{p-1}[(x-i)^{Np^d+1}+1]\right)\left(\prod_{i=0}^{p-1}(x-i)^{Np^d-1}\right),\end{aligned} \tag{4}$$

the equality on the right holding since the argument of σ is perfect [1; Theorem 5].

THEOREM 3. – The polynomial $(x^2-x)^{2n} = x^{2n}(x-1)^{2n}$ is b.u.p. over $GF(2)$ if and only if $n = 1$.

PROOF. – The condition $n = 1$ is clearly sufficient. For the converse, invoke (4) with $N = 1$ and note that $(x-i)^{2^d+1}+1$ does not split over $GF(2)$ unless 2^d+1 is a power of 2. ∎

For odd primes p, the cases $p \equiv 1, 3 \pmod 4$ are distinguished but overlap.

THEOREM 4. – The polynomial $A = \prod_{i=0}^{p-1}(x-i)^{2n} = (x^p - x)^{2n}$ is b.u.p. over $GF(p)$ if and only if either

i) $n = p-1$;

ii) $n \equiv 0 \pmod 2$ and $n(n+1)|(p-1)$; or

iii) $p \equiv 1 \pmod 4$, $n \equiv 1 \pmod 2$ and $2n(n+1)|(p-1)$.

PROOF. – The result holds for $p = 2$ by Theorem 3, thus let $p > 2$. For the necessity, assume A is b.u.p. over $GF(p)$, so that (4) holds with $n = Np^d$ and $N|(p-1)$. Consider, from (2),

$$x^{2Np^d} \to (x^{Np^d+1}+1)\frac{x^{Np^d}-1}{x-1} = (x^{Np^d+1}+1)\frac{(x^N-1)^{p^d}}{x-1},$$

and let $g(x) = x^{Np^d+1} + 1$. Then $g(x)$ splits in $GF[p, x]$ since A is b.u.p. *Case 1.* Let $p \equiv 3 \pmod 4$. The Legendre symbol $((-1)/p) = -1$ and $g(x)$ splits in $GF[p, x]$, hence $N \equiv 0 \pmod 2$. If $g(x)$ has repeated roots in $GF(p)$, then $g(x) \in GF[p, x^p]$, from which $N = p - 1$ and $d = 0$. If $g(x)$ has distinct roots in $GF(p)$, then $d = 0$ and $(N + 1)|(p - 1)$, so that $n(n + 1) = N(N + 1)|(p - 1)$ since $(N, N + 1) = 1$. *Case 2.* Let $p \equiv 1 \pmod 4$. Except for « $((-1)/p) = -1$ implies $N \equiv 0 \pmod 2$ », the necessity arguments in the preceding case hold. Thus assume the alternative: $n = Np^d$, $N|(p - 1)$ and $n \equiv 1 \pmod 2$. Then $g(x) \notin GF[p, x^p]$, so $g(x)$ has distinct roots in $GF(p)$. Since $g(x)$ splits in $GF[p, x]$, then $d = 0$ and $n(n + 1)|(p - 1)$ as before. Thus $x^{2(n+1)} - 1 = (x^{n+1} - 1)(x^{n+1} + 1)$ has $2(n + 1)$ distinct roots in $GF(p)^*$. These roots constitute a subgroup of $GF(p)^*$, so $2(n + 1)|(p - 1)$. Since $(n, 2(n + 1)) = 1$ and $n(n + 1)|(p - 1)$, then $2n(n + 1)|(p - 1)$.

For the converse, the cases $p \equiv 1, 3 \pmod 4$ are argued simultaneously. First, for $n = p - 1$ we have from (4),

$$\prod_{i=0}^{p-1} (x - i)^{2(p-1)} \to \left(\prod_{i=0}^{p-1} [(x - i)^p + 1]\right)\left(\prod_{i=0}^{p-1} (x - i)^{p-2}\right) = \\ = \left(\prod_{i=0}^{p-1} (x - i + 1)^p\right)\left(\prod_{i=0}^{p-1} (x - i)^{p-2}\right) = \prod_{i=0}^{p-1} (x - i)^{2(p-1)} .$$

When $n \equiv 1 \pmod 2$ and $2n(n + 1)|(p - 1)$, the facts that $2(n + 1)|(p - 1)$ and $GF(p)^*$ is cyclic imply that $x^{2(n+1)} - 1$ splits over $GF(p)^*$ with distinct roots a_j such that $x^{n+1} + 1 = \prod_{j=1}^{n+1} (x - a_j)$. Thus

$$\prod_{i=0}^{p-1} [(x - i)^{n+1} + 1] = \prod_{i=0}^{p-1} \prod_{j=1}^{n+1} (x - a_j - i) = \prod_{j=1}^{n+1} \prod_{i=0}^{p-1} (x - a_j - i) = \prod_{i=0}^{p-1} (x - i)^{n+1} ,$$

$$\prod_{i=0}^{p-1} (x - i)^{2n} \to \left(\prod_{i=0}^{p-1} [(x - i)^{n+1} + 1]\right)\left(\prod_{i=0}^{p-1} (x - i)^{n+1}\right) = \prod_{i=0}^{p-1} (x - i)^{2n} .$$

For $n \equiv 0 \pmod 2$ and $n(n + 1)|(p - 1)$, we re-examine the factor $g(x) = x^{n+1} + 1$ of $\sigma^{**}(x^{2n})$. Since n is even, then $g(-x) = 1 - x^{n+1}$; and since $(n + 1)|(p - 1)$, $g(-x)$ splits over $GF(p)^*$. Moreover, for each $a \in GF(p)^*$, $a \neq -a$. Hence $g(x)$ splits and has $n + 1$ distinct roots in $GF(p)^*$, say $x^{n+1} + 1 = \prod_{i=0}^{n+1} (x - a_i)$, $a_j \in GF(p)^*$. Thus

$$\prod_{i=0}^{p-1} [(x - i)^{n+1} + 1] = \prod_{i=0}^{p-1} \prod_{j=1}^{n+1} (x - a_j - i) = \prod_{j=1}^{n+1} \prod_{i=0}^{p-1} [x - (a_j + i)] = \\ = \prod_{j=1}^{n+1} \prod_{i=0}^{p-1} (x - i) = \prod_{i=0}^{p-1} (x - i)^{n+1} ,$$

so that

$$\prod_{i=0}^{p-1} (x - i)^{2n} \to \left(\prod_{i=0}^{p-1} [(x - i)^{n+1} + 1]\right)\left(\prod_{i=0}^{p-1} (x - i)^{n-1}\right) = \prod_{i=0}^{p-1} (x - i)^{2n} . \quad \blacksquare$$

THEOREM 5. – The polynomial $A = x^\alpha(x-1)^\beta$ is b.u.p. over $GF(2)$ if and only if either (i) $\alpha = \beta = 2^n - 1$ and $n \geqslant 0$ (in which case A is perfect); or (ii) $\alpha = \beta = 2$.

PROOF. – It was established in [8] that $\sigma^{**}(x^{2n+1}) = \sigma(x^{2n+1})$ does not split over $GF(2)$ unless $2n+1 = 2^d - 1$. Suppose A is b.u.p. over $GF(2)$ and α, β have mixed parities. By Theorem 2 we can assume $A = x^{2n+1}(x-1)^{2m}$, in which case $m = 1$ as noted in the proof of Theorem 3. Thus for every $d \geqslant 0$ we have $A = x^{2^d-1}(x-1)^2 \to \to x^2(x-1)^{2^d-1} \neq A$, so that A is not b.u.p. The equality of the exponents α, β having the same parity is covered by Theorem 3 and [8]. ∎

At present, for $p > 2$ we can only conjecture and partially establish a characterization of all b.u.p. polynomials which split over $GF(p)$. The results of this paper and [1], [6] establish all aspects of the Conjecture except the necessity for the equality of the exponents $\alpha(i)$ when $p > 2$. For completeness and comparison, we restate the known perfection [1], [2], [6], [8] of splitting polynomials over $GF(p)$:

FACT. – Let $A = \prod_{i=0}^{p-1} (x-i)^{\alpha(i)} \in GF[p, x]$.

1) The polynomial A is perfect over $GF(p)$ if and only if $\alpha(0) = \ldots = \alpha(p-1) = Np^n - 1$ for some $N|(p-1)$ and some $n \geqslant 0$.

2) The polynomial A is unitary perfect over $GF(p)$ if and only if $\alpha(0) = \ldots = \alpha(p-1) = Np^n$ for some $n \geqslant 0$ where either $p = 2$ and $N = 1$ or else $(p-1)/N \equiv 0 \pmod 2$.

CONJECTURE. – The polynomial $A = \prod_{i=0}^{p-1} (x-1)^{\alpha(i)}$ is b.u.p. over $GF(p)$ if and only if $\alpha(0) = \ldots = \alpha(p-1)$ and one of the following conditions holds:

i) $\alpha(0) = Np^n - 1 \equiv 1 \pmod 2$, $N|(p-1)$ and $n \geqslant 0$ (in which case A is perfect);

ii) $\alpha(0) = 2(p-1)$;

iii) $\alpha(0) = 2N$, $N \equiv 0 \pmod 2$ and $N(N+1)|(p-1)$; or

iv) $p \equiv 1 \pmod 4$, $\alpha(0) = 2N$, $N \equiv 1 \pmod 2$ and $2N(N+1)|(p-1)$.

Recalling WALL's result [9] that 6, 60, 90 are the only b.u.p. numbers, note the Conjecture implies that the number $SNPBUP(p)$ of splitting non-perfect b.u.p. polynomials over $GF(p)$ satisfies $0 < SNPBUP(p) < \infty$, in contrast to $SP(p) = SUP(p) = \infty$ which the Fact demonstrates. In this connection we are reminded of the equivalence relations which can be defined on the sets of splitting perfect polynomials [6] and unitary perfect polynomials [3].

Partial Proof of the Conjecture ($\alpha(0) = \ldots = \alpha(p-1)$ *whenever each* $\alpha(i) = 2n(i)$ *and either* $\min\{n(i)\} = 1$, *or* $\max\{n(i)\} = p-1$ *and each* $n(i) \equiv 0 \pmod 2$). Suppose $1 = \min\{n(i)\} < \max\{n(i)\}$. From (1) and (2), the admissible summands A_j of

$\sigma^{**}(A) - A$ having maximum degree are the monic polynomials given by

$$A_j = (x-j)^{2n(j)-1} \prod_{i \neq j} (x-i)^{2n(i)}, \qquad n(j) > 1 .$$

By our supposition these A_j number fewer than p, so $\sigma^{**}(A) - A \neq 0$ and A is not b.u.p.

Now assume $2 \leqslant \min\{n(i)\} < \max\{n(i)\} = p-1$ with each $n(i) \equiv 0 \pmod 2$. Then $n(i)[n(i)+1] | (p-1)$ whenever $n(i) < p-1$. By Theorem 2 we can assume w.l.o.g. that $n(0) = p-1$, so that $x^{2(p-1)}|A$. (I.e., $x^m|A$ but $x^{m+1}|A$.) We will argue on the value of $n(1)$, but digress temporarily.

From (2), whenever $n(i) = p-1$ we have

$$(x-i)^{2(p-1)} \to (x-i+1)^p \prod_{l=2}^{p-1} (x-l) . \tag{5}$$

To detail the factorization of $\sigma^{**}([x-i]^{2n(i)})$ when $n(i) \equiv 0 \pmod 2$ and $n(i)[n(i)+1] | (p-1)$, let H_m be the subgroup of $GF(p)^*$ of order m, $m | (p-1)$. From our arguments (proof of Theorem 4) on the factorizations of $g(x) = x^{n+1} + 1$ and $g(-x) = (1-x)(x^{n+1}-1)/(x-1)$ for n even, we have

$$\frac{x^{n+1}-1}{x-1} = \prod_{\substack{c \in H_{n+1} \\ c \neq 1}} (x-c) ,$$

and

$$x^{n+1} + 1 = (1+x) \prod_{\substack{c \in H_{n+1} \\ c \neq 1}} (x+c) = \prod_{c \in H_{n+1}} (x+c) . \tag{6}$$

When n is even then $-1 \in H_n$, so that

$$\frac{x^n-1}{x-1} = \prod_{\substack{b \in H_n \\ b \neq 1}} (x-b) = \prod_{\substack{b \in H_n \\ a \neq -1}} (x+b) . \tag{7}$$

Thus from (2), (6), (7) and the fact that $H_n \cap H_{n+1} = 1$, for $n(i)$ even and $n(i)[n(i)+1] | (p-1)$ we have

$$(x-i)^{2n(i)} \to [(x-i)^{n(i)+1} + 1] \frac{(x-i)^{n(i)} - 1}{x-i-1} = (x-i+1)^2 \prod_{\substack{c \in H_{n(i)} \cup H_{n(i)+1} \\ c \neq -1, 1}} (x-i+c) . \tag{8}$$

Continuing, let $k = |\{i : n(i) < p-1\}|$. First, assume $n(1) < p-1$. Then the factor x is not contributed to $\sigma^{**}(A)$ via (5), and x^2 is contributed via (8) when $i = 1$. Thus if A is b.u.p., precisely $x^{2(p-1)-2} = x^{2p-4}$ must be contributed to $\sigma^{**}(A)$ via (8) with $i \neq 1$. I.e., there must exist precisely $2p-4$ elements i in $GF(p)^*$ with $n(i) < p-1$ and satisfying

$$i \in H_{n(i)} \cup H_{n(i)+1} - \{-1, 1\}. \tag{9}$$

Since $n(1) < p-1$, then $k-1 \geqslant 2p-4$ so that $k \geqslant 2(p-1)-1$. But $0 < k \leqslant p-1$, hence $k < 2k \leqslant 2(p-1)$. Thus $k = 2(p-1)-1 \leqslant p-1$ so that $p \leqslant 2$, a contradiction. Now assume $n(1) = p-1$. Then the factor x^p is contributed to $\sigma^{**}(A)$ via (5), and x is not contributed to $\sigma^{**}(A)$ via (8) for $i = 1$. Thus if A is b.u.p., precisely

TABLE. – *Non-splitting bi-unitary perfect polynomials.*

p	*deg*	*Complete Factorization*
2	9	$x^3(1+x)^4(1+x+x^2)$
	12	$x^3(1+x)^5(1+x+x^2)^2$
	12	$x^4(1+x)^4(1+x+x^2)^2$
	15	$x^4(1+x)^5(1+x+x^2)^3$
	16	$x^6(1+x)^6(1+x+x^2)^2$
	19	$x^7(1+x)^6(1+x^3+x^4)$
	24	$x^7(1+x)^9(1+x^3+x^4)^2$
	25	$x^7(1+x)^{10}(1+x+x^2)^2(1+x^3+x^4)$
	29	$x^7(1+x)^{12}(1+x+x^2)^2(1+x+x^3)(1+x^2+x^3)$
	32	$x^7(1+x)^{13}(1+x+x^3)^2(1+x^2+x^3)^2$
	24	$x^8(1+x)^8(1+x^3+x^4)(1+x+x^2+x^3+x^4)$
	29	$x^8(1+x)^9(1+x^3+x^4)^2(1+x+x^2+x^3+x^4)$
	30	$x^8(1+x)^{10}(1+x+x^2)^2(1+x^3+x^4)(1+x+x^2+x^3+x^4)$
	34	$x^8(1+x)^{12}(1+x+x^2)^2(1+x+x^3)(1+x^2+x^3)(1+x+x^2+x^3+x^4)$
	37	$x^8(1+x)^{13}(1+x+x^3)^2(1+x^2+x^3)^2(1+x+x^2+x^3+x^4)$
	34	$x^9(1+x)^9(1+x^3+x^4)^2(1+x+x^2+x^3+x^4)^2$
	35	$x^9(1+x)^{10}(1+x+x^2)^2(1+x^3+x^4)(1+x+x^2+x^3+x^4)^2$
	39	$x^9(1+x)^{12}(1+x+x^2)^2(1+x+x^3)(1+x^2+x^3)(1+x+x^2+x^3+x^4)^2$
	42	$x^9(1+x)^{13}(1+x+x^3)^2(1+x^2+x^3)^2(1+x+x^2+x^3+x^4)^2$
	47	$x^{12}(1+x)^{13}(1+x+x^2)^2(1+x+x^3)^3(1+x^2+x^3)^3$
	62	$x^{13}(1+x)^{13}(1+x+x^2)^2(1+x+x^3)^4(1+x^2+x^3)^4(1+x^3+x^4)$ $(1+x+x^2+x^3+x^4)$
	40	$x^{14}(1+x)^{14}(1+x+x^3)^2(1+x^2+x^3)^2$
3	15	$x(1+x)^4(2+x)^6(1+x^2)(2+2x+x^2)$
	12	$x^2(1+x)^2(2+x)^2(1+x^2)(2+x+x^2)(2+2x+x^2)$
	14	$x^3(1+x)^4(2+x)^5(1+x^2)$
	22	$x^3(1+x)^4(2+x)^7(1+x^2)^2(2+x+x^2)(2+2x+x^2)$
	20	$x^3(1+x)^5(2+x)^6(1+x^2)^2(2+2x+x^2)$
	26	$x^4(1+x)^8(2+x)^5(2+2x+x^2)(1+2x+x^2+x^3)(1+2x+x^2+x^4)$
	34	$x^4(1+x)^9(2+x)^5(1+x^2)(1+x+2x^2+x^3)(1+2x+x^2+x^3)$ $(1+2x+x^2+x^4)(2+x+x^2+2x^3+x^4)$
5	40	$x^4(1+x)^4(2+x)^4(3+x)^4(4+x)^4(1+x+x^2)(1+4x+x^2)(2+x^2)$ $(2+x+x^2)(2+4x+x^2)(3+x^2)(3+2x+x^2)(3+3x+x^2)(4+2x+x^2)$ $(4+3x+x^2)$

$x^{2(p-1)-p} = x^{p-2}$ must be contributed to $\sigma^{**}(A)$ via (8), (9) with $i \neq 1$. Hence the absurdity that $GF(p)^*$ contains $p-2$ distinct elements i with $-1 \neq i \neq 1$. ■

4. – Non-splitting bi-unitary perfect polynomials over $GF(g)$.

The non-splitting b.u.p. polynomials given in the Table for $p = 2, 3, 5$ were hand-calculated, using an « algorithm » like that described in [3; p. 298] and computer-generated factorization tables [7] obtained over ten years ago. For brevity, the Table does not include any translates over $GF(p)$ except self-translates; from these examples, others can be obtained by using Theorem 2. The peculiarities of the « algorithm » [3] preclude any claims for completeness of the Table. However, we have pursued examples A « up to the form » $(x^p - x)^d | A$ where $d \leqslant 15, 9, 5$ for $p = 2, 3, 5$ respectively, subject to A having a prime factor of even multiplicity.

REFERENCES

[1] J. T. B. BEARD, JR. - J. R. O CONNELL - K. I. WEST, *Perfect polynomials over GF(q)*, Atti Acad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur., (8), **62** (1977), pp. 283-291.
[2] J. T. B. BEARD, JR., *Unitary perfect polynomials over GF(q)*, Atti Acad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur., (8), **62** (1977), pp. 417-422.
[3] J. T. B. BEARD, JR. - A. T. BULLOCK - M. S. HARBIN, *Infinitely many perfect and unitary perfect polynomials over GF(q)*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur., (8), **63** (1977), pp. 294-303.
[4] J. T. B. BEARD, JR. - M. S. HARBIN, *Non-splitting unitary perfect polynomials over GF(q)*, Atti Acad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur., (8), **66** (1979), pp. 179-185.
[5] J. T. B. BEARD, JR. - J. K. DOYLE - K. I. MANDELBERG, *Square-separable primes and unitary perfect polynomials*, Atti Acad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur., (8), **68** (1980), pp. 397-401.
[6] J. T. B. BEARD, JR., *Perfect polynomials revisited*, submitted.
[7] J. T. B. BEARD, JR. - K. I. WEST, *Factorization tables for GF[q, x]*, unpublished.
[8] E. F. CANADAY, *The sum of the divisors of a polynomial*, Duke Math. J., **7** (1941), pp. 721-737.
[9] C. R. WALL, *Bi-unitary perfect numbers*, Proc. Amer. Mat. Soc., **33** (1972), pp. 39-42.