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The Bianchi Models and New Inflation

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Abstract

The promise of the inflationary Universe scenario is to free the present state of the Universe from extreme dependence on initial data. Paradoxically, inflation is usually analyzed in the context of the homogeneous and isotropic Robertson-Walker (RW) cosmological models. We show that all but a small subset of the homogeneous models (the Bianchi models) undergo inflation. Any initial anisotropy is so strongly damped that if sufficient inflation occurs to solve the flatness/horizon problems the Universe today would still be very isotropic. Some of the Bianchi models will eventually (in the exponentially distant future) become very anisotropic again.

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Introduction

The most attractive feature of the inflationary Universe scenario[1] is that it offers the possibility of freeing the present state of the Universe, in regions as large as the present Hubble volume ($H^{-1} \simeq 10^{28} cm$), from extreme sensitivity to the initial state of the Universe. Since it is very unlikely that we will ever be privy to the initial data for the Universe this is indeed a very attractive attribute of inflation (for an alternative point of view, see ref. 2). This extreme reliance upon initial data was emphasized by Collins and Hawking[3] who demonstrated that without inflation the set of initial data which evolved into a model Universe which resembles ours at the current epoch is of measure zero. While inflation holds the potential to free us from the initial data, for simplicity, it is almost always analyzed within the context of an isotropic and homogeneous RW cosmological model. A key issue confronting inflation then, is which subset of initial data for Einstein's equations undergo sufficient inflation to evolve to a model universe with large regions which resemble our present Hubble volume. Clearly not all the initial data do. A trivial counterexample is a very closed RW model which recollapses before it can inflate.

Here we consider all of the homogeneous models, the nine Bianchi classes. These models are homogeneous, but do not expand isotropically. Wald[4] has shown that with the exception of a subset of Bianchi IX models, those which have very large positive spatial curvature, all Bianchi models with a positive cosmological constant asymptotically evolve to de Sitter space. In inflationary Universe models, the Universe does not in the strictest sense have a true cosmological term. Rather there is a vacuum energy density which depends upon an order parameter (usually the expectation value of some scalar field). So long as the scalar field is displaced from the zero energy minimum of its potential and is slowly evolving, the vacuum energy is approximately constant and behaves like a cosmological term. The issue then is a dynamical one; does the Universe evolve into a deSitter state before the scalar field reaches the minimum of its potential?

Previously it was shown that in the context of old inflation large anisotropy could prevent inflation by causing the scalar field to evolve too rapidly, thereby prematurely eliminating the effective cosmological term[5]. [If this did not occur, however, the authors showed that the initial anisotropy was indeed strongly damped.] More recently, Steigman and Turner[6] have shown that unlike old inflation, new inflation could not be prevented by anisotropy (i.e., models which would inflate in the absence of anisotropy, would inflate in the presence of anisotropy, regardless of how much anisotropy was present). Recently, several authors have claimed that, while inflation does occur in anisotropic models, growing modes of anisotropy will again make the Universe anisotropic after inflation, thereby defeating the best efforts of inflation[7]. This is the issue we will address in this Letter.

We will show that while some Bianchi models will indeed again become very anisotropic, inflation postpones this event to an exponentially distant time into the future and that models which inflate sufficiently to solve the horizon/flatness problems will today still be very isotropic. In this regard it has been known for a long while now that inflation does permanently render the Universe smooth within our Hubble volume; several authors have shown that if there were curvature perturbations (i.e., scalar density perturbations) present before inflation took place, then these perturbations will enter the horizon with the same amplitude as they would have in the absence of inflation, but at a much later time[8,9]. A finite epoch of inflation does not smooth the Universe globally, rather it creates large smooth patches, sufficiently large to encompass our Hubble volume at this late date in the history of the Universe.

Bianchi Inflation

For detailed discussions of the Bianchi classification and models we refer the interested reader to refs. 10-12. We denote the scale factors of the principal axes of the Universe by X_i, $i = 1 - 3$, the expansion rates in these directions by $h_i \equiv \dot{X}_i/X_i$, the proper volume of a unit comoving volume element by $V \equiv X_1X_2X_3$, and the mean expansion rate by $H \equiv \dot{R}/R \equiv \frac{1}{3}\dot{V}/V = (h_1 + h_2 + h_3)/3$, where $R \equiv V^{1/3}$ is the mean scale factor of the Universe. We will assume that the stress-energy in the Universe is described by a perfect, isotropic and homogeneous fluid with energy density ρ and isotropic pressure $p = \gamma\rho$. Such a fluid with $\gamma = 0$ corresponds to non-relativistic matter; with $\gamma = 1/3$ to a relativistic gas in thermal equilibrium; and with $\gamma = -1$ to vacuum energy. It follows from the conservation of stress-energy that

$$\rho \quad \propto \quad V^{-(1+\gamma)} \tag{1}$$

In these spacetimes the equation of motion for a homogeneous scalar field ϕ with lagrangian density $\mathcal{L} = \frac{1}{2}\dot{\phi}^2 - V(\phi)$ is

$$\ddot{\phi} + 3H\dot{\phi} + V' = 0 \tag{2}$$

where overdot denotes a time derivative, prime a derivative with respect to ϕ, and we use units where $\hbar = c = 8\pi G = 1$.

For all the Bianchi models except IV and VII_h the equations for the evolution of the scale factors can be written as[10,11]

$$\dot{h}_i + 3Hh_i = F_i + (\rho - p)/2 \tag{3}$$

$$\ddot{V}/V = F_1 + F_2 + F_3 + 3(\rho - p)/2 \tag{4}$$

where the F_i depend upon the scale factors X_i. For our purposes the crucial feature of the F_i is the fact that they decrease at least as fast as $V^{-2/3}$. That is the only property of the F_i that we will use. Physically, that corresponds to an effective energy density associated with anisotropy which decreases at least as fast as $V^{-2/3}$ (or R^{-2}) , which means that the fastest growing mode of anisotropy grows at the same rate as a curvature perturbation in a RW model. For Bianchi I models the $F_i = 0$; for Bianchi II, VI, VII, VIII, and IX the $F_i \propto V^{-2/3}$.

For the Bianchi IV and VII$_h$ models the equations cannot be put into diagonal form; there is an additional term on the rhs of Eqn(3), $G_i(\dot{h}_i, X_i)$ (see refs. 11, 12). The G_i decrease at least as fast as R^{-n} $(n \geq 1)$. For simplicity we will not specifically consider these terms as they do not qualitatively alter our analysis.

Throughout we will use the F_i to quantify the degree of anisotropy. To this end we will denote the typical size of the F_i as F and will assume that $F \propto V^{-2/3}$, that is we will take into account the most slowly-decaying mode of anisotropy. We will denote the relative size of the anisotropy by

$$\epsilon \equiv F/\rho \tag{5}$$

We will also assume that initially ϕ is displaced from the minimum of its potential (e.g., due to initial conditions or thermal effects), and that, in the absence of anisotropy during the time it takes ϕ to evolve to the minimum of its potential the scale factor of the Universe grows by a factor of exp(N), i.e., the Universe inflates by N e-folds. We will not be concerned with the detailed evolution of ϕ here; it will suffice to know just the total number of e-folds. We should emphasize again that we are not addressing the question of the initial value of ϕ; we assume, as is done in the usual RW inflationary analysis, that ϕ

is not initially at the zero energy minimum of its potential.

For all but the subset of very-highly positively curved Bianchi IX models, the anisotropy increases the mean expansion, thereby increasing the $3H\dot{\phi}$ friction term in Eqn(2). For all but these Bianchi IX models the work of Wald[4] and Steigman and Turner[6] implies that the Universe will become dominated by vacuum energy (and become deSitter like) before ϕ evolves appreciably from its initial value. Once the Universe is deSitter like it will take ϕ the usual N Hubble times to evolve to the minimum of its potential, during which time the Universe grows in size by a factor of $exp(N)$.

Wald's result implies that asymptotically $\epsilon \rightarrow 0$; for our purposes we will denote the beginning of inflation to be the time when ϵ is sufficiently small, say $\epsilon_b \simeq 0.1$, so that F can be treated as a perturbation in Eqns(3,4). During inflation we have

$$\rho \equiv \chi^2/3 \simeq V(\phi) \simeq M^4 \quad (6a)$$

$$\gamma \simeq -1 \quad (6b)$$

$$\ddot{V}/V \simeq \chi^2 \quad (6c)$$

$$V \;\propto\; exp(\chi t) \quad (6d)$$

$$\dot{h}_i + \chi h_i = F + \chi^2/3 \quad (6e)$$

where as usual we have ignored the kinetic term $(\frac{1}{2}\dot{\phi}^2)$ as it is much smaller than $V(\phi)$ during inflation.

Taking F to vary as $V^{-2/3} \;\propto\; exp(-2\chi/3t)$ and solving for h_i and X_i to order ϵ we obtain

$$h_i = [1 + 3\epsilon_b exp(-2\chi t/3)]\chi/3 + const\ exp(-\chi t) \quad (7a)$$

$$X_i \;\propto\; exp[\chi t/3 - 1.5\epsilon_b\ exp(-2\chi t/3) - const\ exp(-\chi t)/\chi] \quad (7b)$$

where *const* is an irrelevant integration constant.

As the Universe inflates all the h_i approach the usual RW inflationary value of $\chi/3$ exponentially fast, and F decreases as $exp(-2\chi/3t)$, so that all the Bianchi models evolve toward Bianchi I. Steigman and Turner[6] have shown that inflation will last the usual number of e-folds, and so at the end of inflation

$$\epsilon_e \le min[\epsilon_b, \epsilon_i]exp(-2N) \tag{8}$$

since ρ remains constant during inflation and F decreases at least as fast as $V^{-2/3}$. Note that the valueof ϵ_e is independent of the initial value of $\epsilon \equiv \epsilon_i$, provided that the initial value was larger than about 0.1 or so. If $\epsilon_i \le 0.1$, then ϵ_e depends upon ϵ_i and is even smaller, $\epsilon_e \simeq \epsilon_i exp(-2N)$.

Given ϵ at the end of inflation how does the anisotropy of the Universe then evolve. To answer this we shall assume that after inflation $p = \gamma\rho$ with $\gamma \neq -1$ and that the Universe goes through three subsequent phases: a post-inflation phase where the energy density is dominated by coherent oscillations of the ϕ field during which $\gamma = 0$; a radiation-dominated phase which begins when the ϕ particles decay, thereby reheating the Universe (to a temperature T_{RH}); and finally the current matter-dominated phase which begins when the Universe has a temperature of about $10eV$ and is about $10^{10}sec$ old.

We will now solve Eqns(3,4) for $\gamma \neq -1$ and $\epsilon << 1$. Clearly after inflation $\epsilon << 1$ and the Univeres is very nearly RW, so that F can be treated as a small perturbation. The relevant equations are then

$$\ddot{V}/V \simeq 3(1-\gamma)\rho/2 \tag{9}$$

$$\Rightarrow V \quad \propto \quad t^{2/(1+\gamma)}$$

$$\dot{h}_i + 2h_i t^{-1}/(1+\gamma) = F + (1-\gamma)\rho/2 \tag{10}$$

Treating F as a perturbation which varies as $V^{-2/3}$ and working to order ϵ we obtain:

$$h_i = 2t^{-1}/(3+3\gamma)[1 + 6\epsilon_o/(5+3\gamma)(t/t_o)^{(2+6\gamma)/(3+3\gamma)}] \tag{11}$$

where 'subscript o' refers to the value of the quantity at the reference time $t = t_o$. From Eqn(11) it is clear that ϵ grows as t (or R^2) during a radiation-dominated epoch and $t^{2/3}$ (or R) during a matter-dominated epoch. This is just as one would expect since the fastest-growing mode of anisotropy varies as R^{-2}, while $\rho_{rad} \propto R^{-4}$ and $\rho_{matter} \propto R^{-3}$.

We are now ready to compute the present anisotropy in the expansion of the Universe. At the end of inflation ϵ is of order $exp(-2N)$ or $\epsilon_i exp(-2N)$ if the Universe was never dominated by anisotropy. While the energy density of the Universe is dominated by coherent oscillations, from $\rho \simeq M^4$ to $\rho \simeq {T_{RH}}^4$, ϵ grows as R, or by a factor or $M^{4/3}/{T_{RH}}^{4/3}$. During the subsequent radiation-dominated phase, from $T \simeq T_{RH} - 10eV$, ϵ grows as R^2, or by a factor of $10^{36}T_{10}$, where $T_{RH} = T_{10}10^{10}GeV$. Finally, in the current matter-dominated phase which begins when the Universe had a temperature of about $10eV$, ϵ has grown by a factor of about 30,000. Bringing all of these factors together we find that the present level of anistropy is at most

$$\epsilon_{today} \simeq min[1, \epsilon_i]exp(-2N)10^{46}{M_{14}}^{4/3}{T_{10}}^{2/3} \tag{12}$$

The isotropy of the microwave background($\delta T/T \leq 10^{-4}$) constrains ϵ_{today} to be less ~~about 10^{-4}~~; sufficient inflation to guarantee this level of isotropy implies that

$$N \geq 57.5 + ln({M_{14}}^2 T_{10})/3 \tag{13}$$

which is precisely the amount of inflation required to solve the horizon/flatness problems[13]. This is not surprising as the fastest-growing mode of anisotropy varies as R^{-2}, just as the curvature of the Universe does. [For Bianchi IV and VII_h the effect of the G_i terms modifies Eqn(13): $N \geq 243/n - 64 + (4/n - 4/3)ln(M_{14}) + ln(T_{10})/3$.] If there are growing modes of anisotropy, then the Universe will ultimately become very anisotropic again. Assuming that the Universe continues to be matter-dominated and recalling that $\epsilon \propto R \propto t^{2/3}$ when the Universe is matter-dominated, it follows that ϵ will be of order unity when $t \simeq t_{anis}$, where

$$t_{anis} \simeq min[1, \epsilon_i]^{-3/2} M_{14}{}^{-2} T_{10}{}^{-1} exp(3N - 159) 10^{10} yr \tag{14}$$

Because of the interrelation between the isotropy and flatness problems, at about the same time, one would expect the Universe to become curvature-dominated, i.e., Ω deviate significantly from unity.

Summary

We have shown that all the Bianchi models, except for the subset of Bianchi IX models which recollapse before they inflate, will undergo inflation and in the process become highly isotropic. As in the RW inflationary model of inflation, we have assumed that the scalar field responsible for inflation is initially displaced from the minimum of its potential. Regardless of the initial level of anisotropy, all models will be isotropic today provided that sufficient inflation occured to solve the flatness and horizon problems. In the exponentially distant future the Universe may again become anisotropic provided that initially there were growing modes of anisotropy. As with the horizon and flatness problems inflation merely postpones the inevitable.

While inflation is almost always analyzed in the context of RW cosmological models, our analysis indicates that all homogeneous models, less the aforementioned subset of very-closed Bianchi IX models, undergo inflation in the usual way and in the process become highly isotropic. While this does not prove that inflation makes the present state of the Universe on scales as large as our current Hubble volume insensitive to the initial state of the Universe, it does go one step further toward establishing this conjecture.

Now that the homogeneous models seem to be in hand, the more difficult case of inhomogeneous models must be addressed. In this regard, it has already been shown that small inhomogeneities are no obstacle to inflation[8,9], and very recently Jensen and Stein-Schabes[14] have proven the analogue of Wald's result[4] for inhomogeneous models.

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