Bianchi type I magnetic string cosmological model

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Abstract

A Bianchi type I massive string cosmological model in the presence of a magnetic field is investigated. Some exact solutions are produced using a few tractable assumptions usually accepted in the literature. The analytical solutions are supplemented with numerical computations. In the frame of the present model the evolution of the Universe and other physical aspects are discussed.

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1 Introduction

Since the observation of the current expansion of the Universe which has apparently accelerated in the recent past, the anomalies found in the cosmic microwave background (CMB) and the large structures observations it becomes obvious that a pure Friedmann-Lemaitre-Robertson-Walker (FLRW) cosmology should be amended.

Bianchi type I (BI) cosmological models are the simplest anisotropic Universe models playing an important role in understanding essential features of the Universe. In this class of models it is possible to accommodate the presence of cosmic strings as an example of an anisotropy of space-times generated by one dimensional topological defects.

In the last time cosmic strings have drawn considerable interest among the researchers for various aspects such as the study of the early Universe. The presence of cosmic strings in the early Universe could be explained using grand unified theories. These strings arise during the phase transition after the Big Bang explosion as the temperature goes down below some critical critical point. It is believed that the existence of strings in the early Universe gives rise to the density fluctuations which leads to the formation of the galaxies. Also the cosmic strings have been used in attempts to investigate anisotropic dark energy component including a coupling between dark energy and a perfect fluid (dark matter) [1]. The cosmic string has stress energy and it is couple to the gravitational field.

In what follows we shall investigate the evolution of a BI cosmological models in presence of a cosmic string and magnetic fluid [2]. The inclusion of the magnetic field is motivated by the observational cosmology and astrophysics indicating that many subsystems of the Universe possess magnetic fields (see e. g. the reviews [3, 4] and references therein).

The paper has the following structure. We shall review the basic equations of an anisotropic BI model in the presence of a system of cosmic string and magnetic field. In Section III we introduce a few plausible assumptions usually accepted in the literature and some exact solutions are produced. We present also some numerical results regarding the evolution of the Universe in the presence or absence of a magnetic string. At the end we shall summarize the results and outline future prospects.

2 Fundamental Equations and general solutions

The line element of a BI Universe is

$$ds^2 = (dt)^2 - a_1(t)^2(dx^1)^2 - a_2(t)^2(dx^2)^2 - a_3(t)^2(dx^3)^2 \,. \tag{2.1}$$

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There are three scale factors a_i $(i = 1,2,3)$ which are functions of time t only and consequently three expansion rates. In principle all these scale factors could be different and it is useful to express the mean expansion rate in terms of the average Hubble rate:

$$H = \frac{1}{3}\left(\frac{\dot a_1}{a_1} + \frac{\dot a_2}{a_2} + \frac{\dot a_3}{a_3}\right), \tag{2.2}$$

where over-dot means differentiation with respect to t.

In the absence of a cosmological constant, the Einstein's gravitational field equation has the form

$$\frac{\ddot a_2}{a_2} + \frac{\ddot a_3}{a_3} + \frac{\dot a_2}{a_2}\frac{\dot a_3}{a_3} = \kappa T_1^1, \tag{2.3a}$$

$$\frac{\ddot a_3}{a_3} + \frac{\ddot a_1}{a_1} + \frac{\dot a_3}{a_3}\frac{\dot a_1}{a_1} = \kappa T_2^2, \tag{2.3b}$$

$$\frac{\ddot a_1}{a_1} + \frac{\ddot a_2}{a_2} + \frac{\dot a_1}{a_1}\frac{\dot a_2}{a_2} = \kappa T_3^3, \tag{2.3c}$$

$$\frac{\dot a_1}{a_1}\frac{\dot a_2}{a_2} + \frac{\dot a_2}{a_2}\frac{\dot a_3}{a_3} + \frac{\dot a_3}{a_3}\frac{\dot a_1}{a_1} = \kappa T_0^0. \tag{2.3d}$$

where κ is the gravitational constant. The energy momentum tensor for a system of cosmic string and magnetic field in a comoving coordinate is given by

$$T_\mu^\nu = \rho u_\mu u^\nu - \lambda x_\mu x^\nu + E_\mu^\nu, \tag{2.4}$$

where ρ is the rest energy density of strings with massive particles attached to them and can be expressed as $\rho = \rho_p + \lambda$, where ρ_p is the rest energy density of the particles attached to the strings and λ is the tension density of the system of strings [5, 6, 7] which may be positive or negative. Here u_i is the four velocity and x_i is the direction of the string, obeying the relation

$$u_i u^i = -x_i x^i = 1, \quad u_i x^i = 0. \tag{2.5}$$

In (2.4) $E_{\mu\nu}$ is the electromagnetic field given by Lichnerowich [8]

$$E_\mu^\nu = \bar\mu\Big[|h|^2\Big(u_\mu u^\nu - \frac{1}{2}\delta_\mu^\nu\Big) - h_\mu h^\nu\Big]. \tag{2.6}$$

Here $\bar\mu$ is a constant characteristic of the medium and called the magnetic permeability. Typically $\bar\mu$ differs from unity only by a few parts in 10^5 ($\bar\mu > 1$ for paramagnetic substances and $\bar\mu < 1$ for diamagnetic). In (2.6) h_μ is the magnetic flux vector defined by

$$h_\mu = \frac{1}{\bar\mu} * F_{\nu\mu} u^\nu, \tag{2.7}$$

where $*F_{\mu\nu}$ is the dual electromagnetic field tensor defined as

$$*F_{\mu\nu} = \frac{\sqrt{-g}}{2}\varepsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}. \tag{2.8}$$

Here $F^{\alpha\beta}$ is the electromagnetic field tensor and $\varepsilon_{\mu\nu\alpha\beta}$ is the totally anti-symmetric Levi-Civita tensor with $\varepsilon_{0123} = +1$. Here the co-moving coordinates are taken to be $u^0 = 1$, $u^1 = u^2 = u^3 = 0$. We choose the incident magnetic field to be in the direction of x-axis so that the magnetic flux vector has only one nontrivial component, namely $h_1 \neq 0$. In view of the aforementioned assumption from (2.7) one obtains $F_{12} = F_{13} = 0$. We also assume that the conductivity of the fluid is infinite. This leads to $F_{01} = F_{02} = F_{03} = 0$. Thus We have only one non-vanishing component of $F_{\mu\nu}$ which is F_{23}. Then from the first set of Maxwell equation

$$F_{\mu\nu;\beta} + F_{\nu\beta;\mu} + F_{\beta\mu;\nu} = 0,, \tag{2.9}$$

where the semicolon stands for covariant derivative, one finds

$$F_{23} = \mathfrak{I}, \quad \mathfrak{I} = \text{const}. \tag{2.10}$$

Introducing (2.10) in (2.7) and taking into account (2.8) we get

$$h_1 = \frac{a_1 \mathfrak{I}}{\bar{\mu} a_2 a_3}. \tag{2.11}$$

The electromagnetic field has only the following non-trivial components

$$E_0^0 = E_1^1 = -E_2^2 = -E_3^3 = \frac{\mathfrak{I}^2}{2\bar{\mu} a_2^2 a_3^2}. \tag{2.12}$$

Choosing the string along x^1 direction and using co-moving coordinates we have the following components of energy momentum tensor [9]:

$$\begin{aligned}
T_0^0 &= \rho + \frac{\mathfrak{I}^2}{2\bar{\mu}} \frac{a_1^2}{\tau^2}, &(2.13a)\\
T_1^1 &= \lambda + \frac{\mathfrak{I}^2}{2\bar{\mu}} \frac{a_1^2}{\tau^2}, &(2.13b)\\
T_2^2 &= -\frac{\mathfrak{I}^2}{2\bar{\mu}} \frac{a_1^2}{\tau^2}, &(2.13c)\\
T_3^3 &= -\frac{\mathfrak{I}^2}{2\bar{\mu}} \frac{a_1^2}{\tau^2}, &(2.13d)
\end{aligned}$$

where we introduce the volume scale of the BI space-time

$$\tau = a_1 a_2 a_3 \,, \tag{2.14}$$

namely, $\tau = \sqrt{-g}$ [10]. It is interesting to note that the evolution in time of τ is connected with the Hubble rate (2.2):

$$\frac{\dot{\tau}}{\tau} = 3H\,. \tag{2.15}$$

In view of $T_2^2 = T_3^3$ from (2.3b), (2.3c) one finds

$$a_2 = a_3 D \exp\Bigl(X \int \frac{dt}{\tau}\Bigr), \tag{2.16}$$

with D and X being integration constants. Due to anisotropy of the source filed, in order to solve the remaining Einstein equation we have to impose some additional conditions. Here we give two different conditions. It can be shown that the metric functions can be expressed in terms of τ. So let us first derive the equation for τ. Summation of Einstein Eqs. (2.3a), (2.3b), (2.3c) and 3 times (2.3d) gives

$$\frac{\ddot{\tau}}{\tau} = \frac{3}{2}\kappa\Bigl(\rho + \frac{\lambda}{3} + \frac{\mathfrak{I}^2}{3\bar{\mu}} \frac{a_1^2}{\tau^2}\Bigr). \tag{2.17}$$

Taking into account the conservation of the energy-momentum tensor, i.e., $T^{\nu}_{\mu;\nu} = 0$, we get in our case

$$\frac{1}{\tau}\frac{d}{dt}\bigl(\tau T_0^0\bigr) - \frac{\dot{a}_1}{a_1} T_1^1 - \frac{\dot{a}_2}{a_2} T_2^2 - \frac{\dot{a}_3}{a_3} T_3^3 = 0. \tag{2.18}$$

After a little manipulation from (2.18) one obtains

$$\dot{\rho} + \frac{\dot{\tau}}{\tau}\rho - \frac{\dot{a}_1}{a_1}\lambda = 0. \tag{2.19}$$

3 Some examples and explicit solutions

The above equations involve some unknowns and we need some additional relations between them to have a tractable problem. In what follows we shall use a relation between ρ and λ in accordance with the state equations for strings used in the literature. The simplest one being a proportionality relation:

$$\rho = \alpha\lambda \tag{3.20}$$

with the most usual choices of the constant α

$$\alpha = \begin{cases} 1 & \text{geometric string} \\ 1+\omega & \omega \geq 0, \quad p \text{ string or Takabayasi string} \\ -1 & \text{Reddy string}. \end{cases} \tag{3.21}$$

In order to solve the Einstein equations completely, we need also to impose some additional conditions. As an example, we shall follow the condition introduced by Bali [11] which was used in [2]. Following this proposal let us assume that the average Hubble rate H (2.2) in the model is proportional to the eigenvalue σ_1^1 of the shear tensor σ_μ^ν. For the BI space-time we have

$$\sigma_1^1 = -\frac{1}{3}\Big(4\frac{\dot a_1}{a_1} + \frac{\dot a_2}{a_2} + \frac{\dot a_3}{a_3}\Big). \tag{3.22}$$

Writing the aforementioned condition as

$$H = n\sigma_1^1, \tag{3.23}$$

one comes to the following relation

$$a_1 = Z\big(a_2 a_3\big)^N, \tag{3.24}$$

where $N = -(n+1)/(4n+1)$ being the proportionality constant and Z is the integration constant.

From (2.16) and (3.24) after some manipulation for the metric functions one finds [9]

$$a_1 = Z^{1/(N+1)}\tau^{N/(N+1)}, \tag{3.25a}$$

$$a_2 = \sqrt{D}\big(\frac{\tau}{Z}\big)^{1/2(N+1)} \exp\Big[\frac{X}{2}\int\frac{dt}{\tau}\Big], \tag{3.25b}$$

$$a_3 = \frac{1}{\sqrt{D}}\big(\frac{\tau}{Z}\big)^{1/2(N+1)} \exp\Big[-\frac{X}{2}\int\frac{dt}{\tau}\Big]. \tag{3.25c}$$

In this case Eq. (2.19) takes the form

$$\dot\rho + \big(\rho - \frac{N}{N+1}\lambda\big)\frac{\dot\tau}{\tau} = 0. \tag{3.26}$$

Eq. (2.17) now reads

$$\ddot\tau = \frac{3}{2}\kappa(\rho + \frac{\lambda}{3})\tau + \mathcal{X}\tau^{(N-1)/(N+1)}, \quad \text{where} \quad \mathcal{X} = \kappa\frac{\mathcal{J}^2}{2\bar\mu}Z^{2/(N+1)}. \tag{3.27}$$

We will see later, the right hand side of (3.27) is the function of τ, hence can be written as

$$\ddot\tau = \mathcal{F}(\tau). \tag{3.28}$$

Eq. (3.28) admits first integral which can be written as

$$\dot\tau = \sqrt{2[\mathcal{E} - \mathcal{U}(\tau)]} \tag{3.29}$$

with

$$\mathcal{U}(\tau) = -\int\mathcal{F}(\tau)d\tau. \tag{3.30}$$

The expression (3.30) can be viewed as potential, while $\mathcal{E}$ as energy level. A detailed analysis of this mechanism can be seen in, e.g., [12].

Let us now study Eqs. (3.26) and (3.27) for different equations of state. Assuming the relation (3.20) between the pressure of the perfect fluid ρ and the tension density λ from Eq. (3.26) one finds

$$\frac{\dot{\rho}}{\rho} = (\frac{N}{\alpha(N+1)} - 1)\frac{\dot{\tau}}{\tau} \tag{3.31}$$

with the solution

$$\rho = C_0 \tau^{\frac{N}{\alpha(N+1)}-1}, \tag{3.32}$$

while the equation for τ reads

$$\ddot{\tau} = \frac{3}{2}\kappa C_0(1+\frac{1}{3\alpha})\tau^{\frac{N}{\alpha(N+1)}} + \mathcal{X}\tau^{\frac{N-1}{N+1}}. \tag{3.33}$$

This equation can be set in the following form

$$\dot{\tau} = \sqrt{\frac{\kappa C_0(3\alpha+1)(N+1)}{N+\alpha(N+1)}\tau^{1+N/\alpha(N+1)} + \frac{\mathcal{X}(N+1)}{N}\tau^{2N/(N+1)} + \mathcal{E}_0} \tag{3.34}$$

s related to $\mathcal{E}$ as $\mathcal{E}_0 = 2\mathcal{E}$.

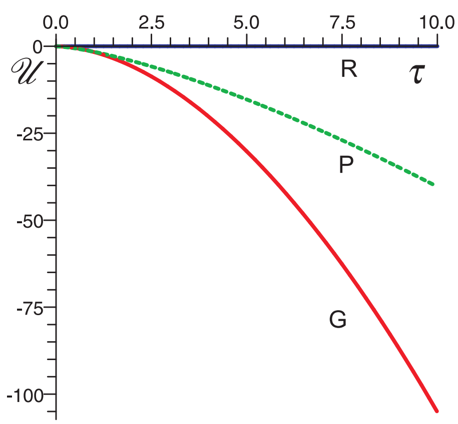

Figure 1: Potential corresponding to the different equations of state in absence of a magnetic field.

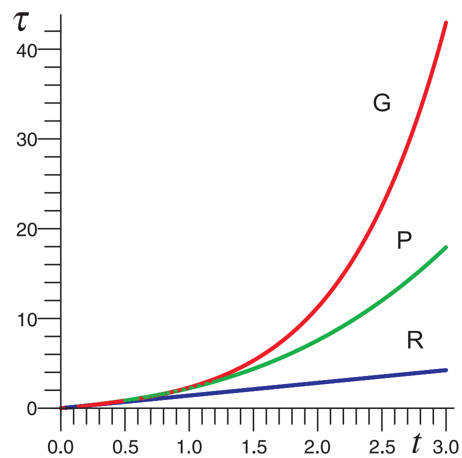

Figure 2: Evolution of the Universe for different equations of state in absence of a magnetic field.

In Figs. 1 and 3 we have illustrated the potential corresponding to the different equations of state without and with the magnetic field, respectively. The Figs. 2 and 4 show the evolution of τ for different cases. Here ”G”, ”P” and ”R” stand for geometric string, p string and Reddy string, respectively. The reason to illustrate figures with and without magnetic field is to show the role of magnetic field. As one sees, in all cases τ might be zero at the initial stage of evolution, thus giving rise to the initial singularity in one hand, τ is not bound from above which means in all three cases we have ever expanding Universe. But introduction of magnetic field into the system results in rapid growth of τ. In numerical analysis we used the following value for the problem parameters: $\kappa = 1$, $\omega = 1$, $N = 4$, $Z = 1$. In case of string only we set $\mathcal{I} = 0$, otherwise $\mathcal{I} = 1$. For magnetic permeability we choose $\bar{\mu} = 1.00001$ and $\bar{\mu} = 0.99999$, respectively. Since it doesn't make any significant change in the behavior of τ, we illustrate only case with $\bar{\mu} = 1.00001$. The initial value for τ is taken to be $\tau(0) = 0.0001$ and corresponding first derivative is calculated from (3.34) at $\mathcal{E}_0 = 1$.

4 Conclusions

In the present paper we investigated in the frame of BI models a string cosmological model in the presence of a magnetic field. We used some tractable assumptions concerning the parameters entering the model. In

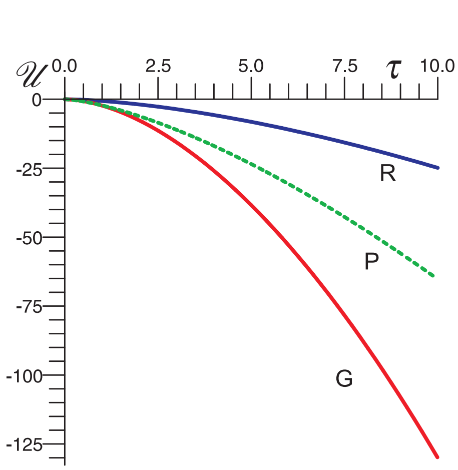

Figure 3: Potential corresponding to the different equations of state in presence of a magnetic field.

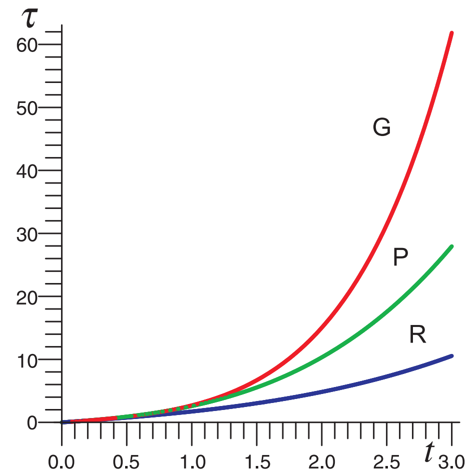

Figure 4: Evolution of the Universe for different equations of state in presence of a magnetic field.

doing so we consider the case when the trace of the expansion average Hubble rate H and the eigenvalue σ_1^1 of the shear tensor are proportional to each other. It should be noted that system in question can also be studied implying some other conditions. We plan to do that in some coming papers applying numerical methods and investigating the qualitative behaviour of the solutions [13]. Further we plan to examine the role of viscous fluid added to the system in question. We also plan to study the system within the scope of Bianchi type-VI cosmological model.

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