Bicubic Interpolation Over Triangles

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1. Introduction. In finite element solutions of partial differential equations, triangular elements are introduced naturally when curved boundaries are approximated by polygons, [5, 6]. The purpose of this paper is to present cubic and bicubic interpolation formulae for triangles. Three interpolation schemes are presented, in §2, for partitionings involving both rectangular and triangular elements. An interpolation formula is also presented, in §3, for a complete triangulation of the polygon. The order of approximation of each of these interpolating polynomials is established in §4.

Consider a function f of two variables having continuous derivatives through order four on a domain $\mathcal{R}$ of the x–y plane, *i.e.*, $f \in C^4[\mathcal{R}]$. Assume that the boundary of $\mathcal{R}$ is a polygon whose vertices are grid-points of a rectangular network of lines π. Thus $\mathcal{R}$ is partitioned into a union of rectangles and right triangles by the mesh π (Figure 1). We will denote by $(\mathcal{R}, \pi)$ the region $\mathcal{R}$ or polygon $\mathcal{R}$ with associated mesh π.

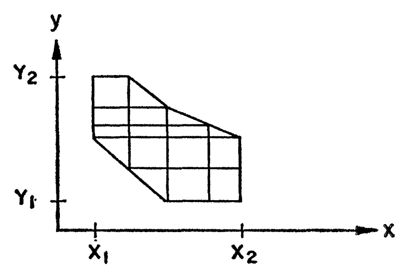

FIGURE 1

If the mesh π is determined by $X_1 = x_0 < x_1 < \cdots < x_n = X_2$ and $Y_1 = y_0 < y_1 < \cdots < y_m = Y_2$, then define $\bar{h} = \min_i (x_i - x_{i-1})$, $\bar{h}' = \min_i (y_i - y_{i-1})$ and $h = \max_{ij} (x_i - x_{i-1}, y_i - y_{i-1})$.

Let $(\mathcal{R}, \pi)$ be a partitioned polygon and $\mathcal{P}^2(\mathcal{R}, \pi)$ be the space of piecewise polynomials w, such that in each finite element of $(\mathcal{R}, \pi)$, $w(x, y)$ is a bicubic

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polynomial. For *rectangular* polygons the *smooth Hermite Space* $H^2(\mathcal{R}, \pi) \equiv \mathcal{P}^2(\mathcal{R}, \pi) \cap C^1[\mathcal{R}]$ has been investigated by Birkhoff, Schultz and Varga [1]. If $\mathcal{R}$ is a *non-rectangular* polygon, then the space $\mathcal{P}^2(\mathcal{R}, \pi) \cap C^1[\mathcal{R}]$ is still well defined. Interpolation Schemes A and B in §2 resulted from an attempt to determine good approximations to f from $H^2(\mathcal{R}, \pi)$. Interpolation Scheme C in §2 relaxes the continuity requirement to $w \in C[\mathcal{R}]$, but involves a more simple *cubic* polynomial. Finally interpolation Scheme D in §3 resulted from an attempt to determine good approximations to f from $H^2(\mathcal{R}, \pi)$ where π consists only of triangular elements, (Figure 3).

2. Interpolation formulae. For a rectangle R, with vertices $P_{ij}:(x_i, y_j)$; $1 \leqq i, j \leqq 2$ (Figure 2), define the bicubic polynomial

$$(1) \qquad u(x, y) = \sum_{i=1}^{2} \sum_{j=1}^{2} \{H_i(x)G_j(y)f_{ij} + H_{i+2}(x)G_j(y)f_{ij}^{(1,0)} + H_i(x)G_{j+2}(y)f_{ij}^{(0,1)} + H_{i+2}(x)G_{j+2}(y)f_{ij}^{(1,1)}\}$$

where

$$f_{ij}^{(k,l)} \equiv \partial^{(k+l)}f/\partial x^k \, \partial y^l (x_i, y_j)$$

and

$$\begin{aligned} H_1(x) &= (1/a^3)(2x^3 - 3ax^2 + a^3), & G_1(y) &= (1/b^3)(2y^3 - 3by^2 + b^3), \\ H_2(x) &= (-1/a^3)(2x^3 - 3ax^2), & G_2(y) &= (-1/b^3)(2y^3 - 3by^2), \\ H_3(x) &= (1/a^2)(x^3 - 2ax^2 + a^2x), & G_3(y) &= (1/b^2)(y^3 - 2by^2 + b^2y), \\ H_4(x) &= (1/a^2)(x^3 - ax^2), & G_4(y) &= (1/b^2)(y^3 - by^2). \end{aligned}$$

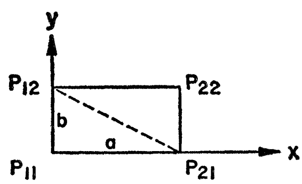

FIGURE 2. $\mathbf{R}: 0 \leqq x \leqq a,\ 0 \leqq y \leqq b$. $\mathbf{T}: 0 \leqq x \leqq a,\ 0 \leqq y \leqq -\frac{b}{a}x + b$.

It is known [4] that u is the unique bicubic polynomial such that

$$(2) \qquad u_{ij}^{(k,l)} = f_{ij}^{(k,l)}, \qquad 0 \leqq k, l \leqq 1; \quad 1 \leqq i, j \leqq 2.$$

Further, if $w \in \mathcal{P}^2(\mathcal{R}, \pi)$, for a *rectangular* polygon $\mathcal{R}$, is defined in each rectangular element as in (1), then $w \in H^2(\mathcal{R}, \pi)$, [1, 4]. In fact $w^{(k,l)} \in C[\mathcal{R}]$ for $0 \leqq k, l \leqq 1$. When such is the case we say that the functions $w^{(k,l)}$, $0 \leqq k, l \leqq 1$ are *compatible* between finite elements.

The fact that discontinuities (or non-compatibilities) in functional values and/or derivatives between elements can lead to "erratic performance" of finite element methods using triangular elements has been demonstrated for structural mechanics problems by R. Clough, [5, p. 112].

We now seek interpolation formulae for the triangular elements of a general polygon $(\mathcal{R}, \pi)$, (Figure 1). There are two motivating factors: (i) we desire such polynomials (and certain derivatives) to be compatible with (1) in adjoining rectangular elements, *i.e.*, the resulting $w \in H^2(\mathcal{R}, \pi)$, and (ii) we desire such formulae, when restricted to a boundary segment of $\mathcal{R}$, to be independent of values of f and its derivatives at interior mesh points. This latter factor will allow us to match exactly certain boundary conditions in the applications to finite element methods. In particular, if f is cubic in a linear parameter along a boundary segment, then $w \equiv f$ on that segment.

Let $\{f_{ij}^{(k,l)} : 0 \leqq k, l \leqq 1, 2 \leqq i + j \leqq 3\}$ be given and define

$$(3) \qquad v(x, y) = \sum_{\substack{i=1 \\ (2 \leqq i+j \leqq 3)}}^{2} \sum_{j=1}^{2} \{\varphi_{ij}(x, y) f_{ij} + \xi_{ij}(x, y) f_{ij}^{(1,0)} + \eta_{ij}(x, y) f_{ij}^{(0,1)} + \psi_{ij}(x, y) f_{ij}^{(1,1)}\}$$

$0 \leqq x \leqq a, 0 \leqq y \leqq -bx/a + b$. We now present three interpolation schemes, denoted Schemes A, B, and C, where the appropriate polynomial coefficients for (3) are given in Table 1.

2.1 Interpolation Scheme A. Using the A-coefficients in (3) we have:

Theorem 1. *Let $w \in \mathcal{P}^2(\mathcal{R}, \pi)$ be defined on each rectangular element as in* (1) *discarding the interpolation of $f_{ij}^{(1,1)}$, and on each triangular element by Scheme* A. *Then $w \in H^2(\mathcal{R}, \pi)$ and v, the restriction of w to the triangle* T, *satisfies*

$$(4) \qquad v^{(k,l)}(x_i, y_j) = f_{ij}^{(k,l)}, \qquad 0 \leqq k + l \leqq 1; \quad 2 \leqq i + j \leqq 3.$$

Finally, if f is a cubic polynomial in a linear parameter along $\overline{P_{12}P_{21}}$, then $v \equiv f$ on $\overline{P_{12}P_{21}}$.

Proof. Conditions (4) can be verified directly. Since

$$G_1\left(-\frac{b}{a}x + b\right) = H_2(x), \qquad G_2\left(-\frac{b}{a}x + b\right) = H_1(x)$$

$$G_3\left(-\frac{b}{a}x + b\right) = -\frac{b}{a}H_4(x), \qquad G_4\left(-\frac{b}{a}x + b\right) = -\frac{b}{a}H_3(x)$$

$0 \leqq x \leqq a$, it follows directly that $v(x, -bx/a + b)$ depends only on $f_{ij}^{(k,l)}$, $0 \leqq k + l \leqq 1$ for $i + j = 3$. Further, $v(x, -bx/a + b)$ is cubic in x, and thus cubic in a linear parameter t along $\overline{P_{12}P_{21}}$. Since v and $\partial v/\partial t$ agree with f and $\partial f/\partial t$ respectively at P_{12} and P_{21}, if f is also cubic in t, we must have $v \equiv f$ along $\overline{P_{12}P_{21}}$.

TABLE 1

	A	B	C		A	B	C
$\varphi_{11}(x, y) = 1 - G_2(y) - H_2(x)$	*	*	*	$\varphi_{12}(x, y) = G_2(y)$	*	*	*
				$\varphi_{21}(x, y) = H_2(x)$	*	*	*
$\xi_{11}(x, y) = H_3(x)G_1(y) + (a/b)H_2(x)G_4(y)$	*	*		$\xi_{21}(x, y) = H_4(x)$	*	*	*
$= H_3(x) - (1/b^2)xy^2$			*				
$\xi_{12}(x, y) = H_3(x)G_2(y) - (a/b)H_2(x)G_4(y)$	*	*					
$= (1/b^2)xy^2$			*				
$\eta_{11}(x, y) = H_1(x)G_3(y) + (b/a)H_4(x)G_2(y)$	*	*		$\eta_{12}(x, y) = G_4(y)$	*	*	*
$= G_3(y) - (1/a^2)x^2y$			*				
$\eta_{21}(x, y) = H_2(x)G_3(y) - (b/a)H_4(x)G_2(y)$	*	*					
$= (1/a^2)x^2y$			*				

	A	B	C
$\psi_{11}(x, y) = H_3(x)G_3(y) - H_4(x)G_4(y)$		*	
$= (1/ab)xy(ab - bx - ay)$			*
$= 0$	*		
$\psi_{12}(x, y) = xG_4(y) + (a/2)H_2(x)[G_4(y) + bG_2(y)] + (b/2)H_4(x)G_2(y)$		*	
$= 0$	*		*
$\psi_{21}(x, y) = yH_4(x) + (b/2)G_2(y)[H_4(x) + aH_2(x)] + (a/2)H_2(x)G_4(y)$		*	
$= 0$	*		*

Next, note that

$$v^{(k,l)}(0, y) = u^{(k,l)}(0, y), \quad 0 \leqq k + l \leqq 1; \quad 0 \leqq y \leqq b \tag{5}$$
$$v^{(k,l)}(x, 0) = u^{(k,l)}(x, 0), \quad 0 \leqq k + l \leqq 1, \quad 0 \leqq x \leqq a,$$

where $u^{(k,l)}$ is computed using (1), discarding the $f_{ij}^{(1,1)}$ terms. Thus the piecewise bicubic polynomial $w \in C^1[\mathcal{R}]$ and the proof is complete.

Note that $v^{(1,0)}(x, -bx/a + b)$ and $v^{(0,1)}(x, -bx/a + b)$ depend on f_{11}, $f_{11}^{(1,0)}$ and $f_{11}^{(0,1)}$. The following corollary establishes that Scheme A cannot be modified so as to have the restriction of $v^{(1,0)}$ to the hypotenuse independent of f_{11}.

Corollary 1. *The bicubic $v(x, y)$ in Scheme* A *cannot be modified so that $v^{(1,0)}(x, -bx/a + b)$ depends only on $f_{ij}^{(k,l)}$, $0 \leqq k + l \leqq 1$; $i + j = 3$.*

Proof. Assume the contrary and let the coefficient of f_{11} be the bicubic polynomial

$$p(x, y) = \sum_{i=0}^{3} \sum_{j=0}^{3} \beta_{ij} x^i y^j.$$

From (4) and $w \in C^1[\mathcal{R}]$ we must have

$$p(x, y) = 1 - H_2(x) - G_2(y) + \beta_{22}x^2y^2 + \beta_{23}x^2y^3 + \beta_{32}x^3y^2 + \beta_{33}x^3y^3.$$

Further, $p(x, -bx/a + b) \equiv 0$ for $0 \leqq x \leqq a$ implies (by equating the coefficient of x^2 to zero)

$$(6) \qquad \beta_{22} + b\beta_{23} = 0.$$

But $p^{(1,0)}(x, -bx/a + b) \equiv 0$ for $0 \leqq x \leqq a$ implies (by equating the coefficient of x to zero)

$$(7) \qquad \beta_{22} + b\beta_{23} = (3/a^2b^2).$$

Equations (6) and (7) are inconsistent. Thus the desired coefficient $p(x, y)$ does not exist.

We can modify interpolation Scheme A however so as to interpolate $f^{(1,1)}$ at the three vertices.

2.2 Interpolation Scheme B. Using the B-coefficients in (3) one can prove:

Theorem 2. *Let $w \in \mathcal{P}^2(\mathcal{R}, \pi)$ be defined on each rectangular element as in* (1) *and on each triangular element by Scheme* B. *Then $w \in H^2(\mathcal{R}, \pi)$, and v, the restriction of w to the triangle* **T**, *satisfies*

$$(8) \qquad v^{(k,l)}(x_i, y_i) = f_{ij}^{(k,l)}, \qquad 0 \leqq k, l \leqq 1; \quad 2 \leqq i + j \leqq 3.$$

Finally, if f is a cubic polynomial in a linear parameter along $\overline{P_{12}P_{21}}$, then $v \equiv$ *f on $\overline{P_{12}P_{21}}$*.

We can simplify our interpolation formula if we relax the condition that $w \in C^1[\mathcal{R}]$. The result is the following *cubic* interpolation scheme which was related to the author by Garrett Birkhoff.

2.3 Interpolation Scheme C. Using the C-coefficients in (3) one can verify directly as in the previous schemes:

Theorem 3. *Let $w \in \mathcal{P}^2(\mathcal{R}, \pi)$ be defined on each rectangular element of* $(\mathcal{R}, \pi)$ *as in* (1) *and on each triangular element by Scheme* C. *Then $w \in C[\mathcal{R}]$ and v, the restriction of w to the triangle* **T**, *is* cubic *and satisfies*

$$(9) \qquad \begin{aligned} &v^{(k,l)}(x_i, y_i) = f_{ij}^{(k,l)}, \qquad 0 \leqq k + l \leqq 1; \qquad 2 \leqq i + j \leqq 3, \qquad \text{and} \\ &v^{(1,1)}(x_1, y_1) = f_{11}^{(1,1)}. \end{aligned}$$

Further, if f is a cubic polynomial in a linear parameter along $\overline{P_{12}P_{21}}$, then $v \equiv f$ on $\overline{P_{12}P_{21}}$.

Corollary 1. *There exists one and only one cubic polynomial $v(x, y)$ satisfying* (9).

Proof. The existence was established in Theorem 3. As for the uniqueness, let $v_1(x, y)$ be another cubic polynomial satisfying (9). Then the cubic polynomial $p(x, y) = (v - v_1)(x, y)$ is such that $p^{(i,j)}$, $0 \leqq i + j \leqq 1$, vanish at all vertices of **T**. The restriction of p to each side of **T** is cubic in any linear parameter t on that side. Hence p must vanish identically on the sides of **T**, since it is given by cubic Hermite interpolation in the parameter t. Thus from the unique factorization theorem for polynomials, $p(x, y) = cxy(bx + ay - ab)$, where c is a constant. But $p^{(1,1)}(0, 0) = 0$ implies $c = 0$. Thus $v_1 \equiv v$ and the uniqueness is established.

Though $w^{(1,0)}$ and $w^{(0,1)}$ may be discontinuous across an interface common to a rectangular and triangular element, the discontinuity is slight (as will be shown in Section 4).

3. Complete Triangulations. Suppose that each rectangle of (R, π) has been refined by triangulation such that the diagonals of adjoining rectangles coalesce at the same mesh point, (Figure 3).

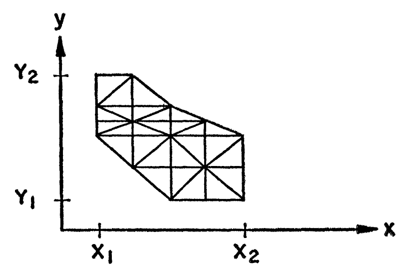

FIGURE 3

We now inquire whether or not we can construct a piecewise bicubic $w \in \mathcal{P}^2(\mathcal{R}, \pi)$ such that $w_{ij}^{(k,l)} = f_{ij}^{(k,l)}$ for $0 \leqq k, l \leqq 1$ at each mesh point (x_i, y_j) and such that the functions $w^{(k,l)} \in C[\mathcal{R}]$, $0 \leqq k, l \leqq 1$. First consider:

3.1 Interpolation Scheme D. Consider the polynomial in (3) with coefficients determined as in Scheme C *except*

$$\varphi_{11}(x, y) = 1 - G_2(y) - H_2(x) + (6/a^2b^2)xy(bx + ay - ab),$$

$$\xi_{11}(x, y) = H_3(x) + (1/ab^2)xy(2bx + ay - 2ab),$$

and

$$\eta_{11}(x, y) = G_3(y) + (1/a^2b)xy(2ay + bx - 2ab).$$

We then have:

Theorem 4. *Let $w \in \mathcal{P}^2(\mathcal{R}, \pi)$ be defined on each triangular element by Scheme* D. *Then $w \in C[\mathcal{R}]$ and v, the restriction of w to the triangle* **T**, *is cubic and satisfies* (4). *Furthermore, $w \in H^2(\mathcal{R}, \pi)$ if and only if all triangles of π are congruent to* **T**. *In this case the polynomial coefficients of $f_{ij}^{(k,l)}$ in* (3) *are unique.*

Proof. The continuity of w and properties (4) follow directly from (3) and the D-coefficients. Since $\varphi_{11}^{(1,0)}(0, y)$ depends on "a" and $\varphi_{11}^{(0,1)}(x, 0)$ depends on "b", a necessary condition that $w \in C^1[\mathfrak{R}]$ is that all triangles in π be congruent to **T**. It follows directly from (3) that this condition is also sufficient. (Note that even if $v(x, y)$ is taken to be bicubic, this condition is still necessary.)

As in the proof of Corollary 1 to Theorem 3, if v_1 is another cubic polynomial satisfying (4), then $p(x, y) \equiv (v - v_1)(x, y) = cxy(bx + ay - ab)$, where c is a constant. Then note that $p^{(1,0)}(0, y) \neq 0$, $p^{(0,1)}(x, 0) \neq 0$ and $p^{(1,0)}(x, -bx/a + b) \neq 0$. Thus if $w \in C^1[\mathfrak{R}]$, c is necessarily independent of $f_{ij}^{(k,l)}$, $0 \leqq k + l \leqq 1$; $2 \leqq i + j \leqq 3$. The constant c could be chosen for instance to interpolate f at the centroid of T, in which case v would be uniquely determined. But then $w^{(1,0)}$ would be discontinuous on $\overline{P_{11}P_{12}}$. Thus, in general, c must be a constant, which can be chosen as 0.

Note that even if all triangles are congruent to **T**, Schemes A, B and C will *not* yield piecewise polynomials of class $C^1[\mathfrak{R}]$ if π consists only of triangular elements.

We next show that Scheme D cannot be modified to include the interpolation of $f^{(1,1)}$ at all three vertices of T, without destroying the compatibility of the functions $w^{(k,l)}$, $0 \leqq k + l \leqq 1$.

Corollary 1. *There does* not *exist a* $w \in H^2(\mathfrak{R}, \pi)$ *such that* $w^{(k,l)}(x_i, y_j) = f_{ij}^{(k,l)}$, $0 \leqq k, l \leqq 1$ *at each mesh point* $(x_i, y_j) \in \pi$.

Proof. Assume the contrary, and suppose the coefficient of f_{11} in (3) is the bicubic

$$c(x, y) = \sum_{i=0}^{3} \sum_{j=0}^{3} \alpha_{ij} x^i y^j.$$

Then the conditions $c(0, 0) = 1$, $c^{(1,0)}(0, 0) = c^{(0,1)}(0, 0) = c^{(1,1)}(0, 0) = 0$ imply respectively that $\alpha_{00} = 1$, $\alpha_{10} = \alpha_{01} = \alpha_{11} = 0$. Also, $c^{(1,0)}(0, b) = 0$, $c^{(1,1)}(0, b) = 0$ imply that $\alpha_{12} = \alpha_{13} = 0$.

Since $c^{(0,1)}(x, -bx/a + b) = 0$ for all $0 \leqq x \leqq a$, we can equate constant and linear coefficients to zero to obtain respectively $2\alpha_{02} + 3b\alpha_{03} = 0$ and $\alpha_{02} + 3b\alpha_{03} = 0$ from which it follows that $\alpha_{02} = \alpha_{03} = 0$. But then $c(0, b) = 1$ which implies $v(0, b) \neq f_{12}$.

Thus for complete triangulations, one must sacrifice the interpolation of $f^{(1,1)}$ at the mesh points for compatibility of the functions $w^{(k,l)}$, $0 \leqq k + l \leqq 1$, across diagonal interfaces.

4. Orders of approximation. The orders of approximation to f of the piecewise polynomial functions developed in §2 and §3 are established in the following theorems. For g defined and suitably differentiable on $\mathfrak{R}$, let $\|g^{(k,l)}\| = \max \{|g^{(k,l)}(x, y)|: (x, y) \in \mathfrak{R}\}$ and $\|g^{(r)}\| = \max \{\|g^{(k,l)}\|: k + l = r\}$. We then have the following:

Theorem 5. *For* $f \in C^4[\mathfrak{R}]$, *let* $e(x, y) = f(x, y) - w(x, y)$ *be the error in the* *approximation of f by the piecewise polynomial w. Then,*

$$(10)\quad \|e^{(k,l)}\| \leqq \begin{cases} \alpha_{kl}\, \|f^{(2)}\|\, h^2/\bar{h}^k(\bar{h}')^l, & 0 \leqq k + l \leqq 1, \quad \textit{Scheme A} \\ \beta_{kl}\, \|f^{(4)}\|\, h^4/\bar{h}^k(\bar{h}')^l, & 0 \leqq k + l \leqq 3, \quad \textit{Scheme B} \end{cases}$$

where the α_{kl} and β_{kl} are constants.

Proof. The Taylor Formula associated with a function $g \in C^m[\mathcal{R}]$ is $g(c + h, d + k) = T_m(g; c, d; c + h; d + k) + R_m(g; c, d; c + h; d + k)$ where

$$T_m(g; c, d; c + h, d + k) = g(c, d) + \sum_{i=1}^{m-1} (1/i!)\left[\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^i g(x, y)\right]_{x=c,\,y=d}$$

$$R_m(g; c, d; c + h, d + k) = (1/m!)\left[\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^m g(x, y)\right]_{x=c+\theta h,\,y=d+\theta k}$$

$0 \leqq \theta \leqq 1$ and (c, d) is some fixed point in $\mathcal{R}$. The formula is valid for all h, k such that $(c + h, d + k) \in \mathcal{R}$.

We now prove (10) for Scheme B. Replace $f_{ij}^{(k,l)}$ in (3) by their appropriate Taylor Formula representations about some arbitrary, but fixed point (c, d) in the triangle **T**, as follows:

$$\begin{aligned} v(x, y) = \sum_{\substack{i=1\\(2\leqq i+j\leqq 3)}}^{2} \sum_{j=1}^{2} \{&\varphi_{ij}(x, y)[T_4(f; c, d; x_i, y_j) + R_4(f; c, d; x_i, y_j)] \\ &+ \xi_{ij}(x, y)[T_3(f_x; c, d; x_i, y_j) + R_3(f_x; c, d; x_i, y_j)] \\ &+ \eta_{ij}(x, y)[T_3(f_y; c, d; x_i, y_j) + R_3(f_y; c, d; x_i, y_j)] \\ &+ \psi_{ij}(x, y)[T_2(f_{xy}; c, d; x_i, y_j) + R_2(f_{xy}; c, d; x_i, y_j)]\}. \end{aligned}$$

By regrouping the terms on the right, one can verify that $v(x, y) = T_4(f; c, d; x, y) + R(x, y)$, where

(11)

$$\begin{aligned} R(x, y) = \sum_{\substack{i=1\\(2\leqq i+j\leqq 3)}}^{2} \sum_{j=1}^{2} \{&\varphi_{ij}(x, y)R_4(f; c, d; x_i, y_j) + \xi_{ij}(x, y)R_3(f_x; c, d; x_i, y_j) \\ &+ \eta_{ij}(x, y)R_3(f_y; c, d; x_i, y_j) + \psi_{ij}(x, y)R_2(f_{xy}; c, d; x_i, y_j)\}. \end{aligned}$$

For $0 \leqq x \leqq a$, $0 \leqq y \leqq b$, as $a \to 0$ and $b \to 0$

$$H_i^{(r)}(x) = \begin{cases} O(1/a^r) & i = 1, 2 \\ O(1/a^{r-1}) & i = 3, 4 \end{cases}, \qquad G_i^{(r)}(y) = \begin{cases} O(1/b^r) & i = 1, 2 \\ O(1/b^{r-1}) & i = 3, 4 \end{cases}$$

$(r = 0, 1, 2, 3)$. Since $R_4(f; c, d; x_i, y_j)$, $R_3(f_x; c, d; x_i, y_j)$, $R_3(f_y, c, d; x_i, y_j)$ and $R_2(f_{xy}; c, d; x_i, y_j)$ do not depend on (x, y) it follows from (11) that

$$(12)\qquad \|R^{(k,l)}\| \leqq \beta_{kl}\, \|f^{(4)}\|\, h^4/a^k b^l, \qquad 0 \leqq k + l \leqq 3$$

where the β_{kl} are constants, and (10) is established for a point (c, d) in **T**.

One can prove (10) for a point P in a rectangular element in an analogous manner using (1). For such points however the result is also given in [1, Theorem 5].

The proof of (10) for Scheme A is similar and is thus omitted. One can also establish in the above manner that the bounds in (10) for Scheme B are also valid for Scheme C, however we will obtain a stronger result in Theorem 7. Before dismissing Scheme A though, note that the expression for Scheme A analogous to (11) is actually:

$$(13) \qquad v(x, y) = T_3(f; c, d; x, y) - \{xy - (x - x^2/a)bG_2(y) - (y - y^2/b)aH_2(x)\}f^{(1,1)}(c, d) + O(h^3)$$

and the corresponding bicubic in a rectangular element **R** is

$$(14) \qquad u(x, y) = T_3(f; c, d; x, y) - \{(bG_2(y) - y)(x - aH_2(x))\}f^{(1,1)}(c, d) + O(h^3).$$

The coefficients of $f^{(1,1)}(c, d)$ in (13) and (14) are zero for points on the boundary of the triangular element **T**, and the rectangular element **R**, respectively. Thus (10) can be strengthened as follows:

Corollary 1. [*Scheme* A] *For* (x, y) *on a mesh line of* π

$$|e(x, y)| \leqq \alpha_{00} \, ||f^{(3)}|| \, h^3.$$

The bicubic v of Scheme D interpolates the function $f(x, y) \equiv 1$ by the polynomial $v(x, y) = 1 + (6/a^2b^2)xy(bx + ay - ab)$. Thus interior to **T** we can conclude only that $||v - f||$ is bounded. However we do have

Theorem 6. *For* $f \in C^3[\mathcal{R}]$, *let* $e(x, y) = f(x, y) - w(x, y)$ *be the error in the approximation of f by the piecewise cubic polynomial w* (*Scheme* D). *For* (x, y) *on the mesh lines determining* π,

$$|e(x, y)| \leqq \delta_{00} \, ||f^{(3)}|| \, h^3,$$

where δ_{00} *is a constant independent of* (x, y).

Proof. The expansion analogous to (11) is

$$w(x, y) = T_3(f; c, d; x, y) + xy(bx + ay - ab)O(1) + O(h^3),$$

from which Theorem 6 follows.

Unfortunately, the error bounds in (10) for $k + l > 0$ depend on the ratio $(h/\bar{h})$ and $(h/\bar{h}')$, thus placing restrictions on the way in which a mesh can be successively refined. *I.e.*, to be able to conclude from (10) Scheme B, that $||e^{(1,0)}|| = O(h^3)$ as $h \to 0$, we must guarantee that $(h/\bar{h})$ remain bounded as $h \to 0$. Garrett Birkhoff has related to the author the following result for Scheme C:

Theorem 7. *For* $f \in C^4[\mathcal{R}]$, *let* $e(x, y) = f(x, y) - w(x, y)$ *be the error in the* *approximation of f by the piecewise polynomial $w(x, y)$, (Scheme* C). *Then*

$$\|e^{(k,l)}\| \leqq \gamma_{kl} \, \|f^{(4)}\| \, h^{4-(k+l)}, \qquad 0 \leqq k, l \leqq 1 \tag{15}$$

where the γ_{kl} are constants independent *of $(h/\bar{h})$ and $(h/\bar{h}')$.*

Remark. For rectangular elements, the assertions are given in [1, Theorem 10].

Proof. For triangular elements, *e.g.*, **T** in Figure 2, we first note that $|e^{(4,0)}(x, 0)| = |f^{(4,0)}(x, 0)| \leqq \|f^{(4,0)}\|$. Hence applying to $e^{(j,0)}(x, 0)$ a standard theorem of Lagrange interpolation error [7, p. 187–8] we have for $0 \leqq x \leqq a$,

$$|e^{(j,0)}(x, 0)| \leqq \|f^{(4,0)}\| \, a^{4-j}, \qquad j = 0, 1, 2, 3. \tag{16}$$

Similarly, for $0 \leqq y \leqq b$,

$$|e^{(0,j)}(0, y)| \leqq \|f^{(0,4)}\| \, b^{4-j}, \qquad j = 0, 1, 2, 3. \tag{17}$$

From $|e^{(3,1)}(x, 0)| = |f^{(3,1)}(x, 0)| \leqq \|f^{(3,1)}\|$ it follows that for $0 \leqq x \leqq a$

$$|e^{(j,1)}(x, 0)| \leqq \|f^{(3,1)}\| \, a^{3-j}, \qquad j = 0, 1, 2. \tag{18}$$

Similarly, for $0 \leqq y \leqq b$,

$$|e^{(1,j)}(0, y)| \leqq \|f^{(1,3)}\| \, b^{3-j}, \qquad j = 0, 1, 2. \tag{19}$$

To extend these error bounds from the horizontal and vertical sides to the interior of **T**, we now recall the four-point partial difference formula

$$\begin{aligned} \triangle_{xy}(e^{(1,1)}) &= e^{(1,1)}(c, d) - e^{(1,1)}(c, 0) - e^{(1,1)}(0, d) + e^{(1,1)}(0, 0) \\ &= \int_0^c \int_0^d e^{(2,2)}(x, y) \, dx \, dy = \int_0^c \int_0^d f^{(2,2)}(x, y) \, dx \, dy, \end{aligned}$$

whence $|\triangle_{xy}(e^{(1,1)})| \leqq \|f^{(2,2)}\| \, cd$. Transposing, and using (18)–(19), we get for $(c, d) \in \mathbf{T}$

$$\begin{aligned} |e^{(1,1)}(c, d)| &\leqq |e^{(1,1)}(c, 0)| + |e^{(1,1)}(0, d)| + \|f^{(2,2)}\| \, cd \\ &\leqq \|f^{(3,1)}\| \, a^2 + \|f^{(1,3)}\| \, b^2 + \|f^{(2,2)}\| \, ab, \end{aligned} \tag{20}$$

Thus establishing (15) for $k = l = 1$. Consequently,

$$\begin{aligned} |e^{(1,0)}(c, d)| &\leqq |e^{(1,0)}(c, 0)| + \int_0^c |e^{(1,1)}(c, y)| \, dy \\ &\leqq \|f^{(4,0)}\| \, a^3 + \|f^{(3,1)}\| \, a^2 b + \|f^{(2,2)}\| \, ab^2/2 + \|f^{(1,3)}\| \, b^3/3, \end{aligned} \tag{21}$$

establishing (15) for $k = 1, l = 0$. Interchanging the roles of x and y one can establish (15) for $k = 0, l = 1$.

Next note that

$$\begin{aligned} |e(c, d)| &\leqq |e(0, d) + \int_0^c |e^{(1,0)}(x, d)| \, dx \\ &\leqq \|f^{(0,4)}\| \, h^4 + Kh^4 \end{aligned}$$

where $K = \|f^{(4,0)}\|/4 + \|f^{(3,1)}\|/3 + \|f^{(2,2)}\|/4 + \|f^{(1,3)}\|/3.$ *Q.E.D.*

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