Bidirectional Reflection Distribution Function expressed in terms of surface scattering modes

Jan J. Koenderink, Andrea J.van Doorn,
and Marigo Stavridi

Helmholtz Instituut, Universiteit Utrecht,
Princetonplein 5, PO Box 80 000, 3508 TA Utrecht, The Netherlands

Abstract. In many applications one needs a concise description of the Bidirectional Reflection Distribution Function (BRDF) of real materials. Because the BRDF depends on two independent directions (thus has four degrees of freedom) one typically has only a relatively sparse set of observations. In order to be able to interpolate these sparse data in a convenient and principled manner a series development in terms of an orthonormal basis is required. The elements of the basis should be ordered with respect to angular resolution. Moreover, the basis should automatically respect the inherent symmetries of the physics, *i.e.*, Helmholtz's reciprocity and (most often) surface isotropy. We indicate how to construct a set of orthonormal polynomials on the Cartesian product of the hemisphere with itself with the required symmetry and invariance properties. These "surface scattering modes" form a convenient basis for the description of BRDF's.

1 Introduction

Surface reflection (or rather: scattering) by natural materials[8] is conveniently described[13, 6, 7, 11] by the Bidirectional Reflection Distribution Function (or BRDF for short). The BRDF is the ratio of the radiance in the direction of the exit beam to the irradiance caused by the entrance beam. The BRDF depends on two directions, that is on four independent angles, two of them in the interval $(0°, 90°)$ and the other pair in the periodic range $(0°, 360°)$. The former two describe the deviation from the surface normal direction, the latter two the azimuth with respect to a fiducial direction on the surface. For an angular resolution of $\delta << 1$ one thus has to specify $4\pi^2/\delta^4$ independent samples; for an angular resolution of 10° that already amounts to more than fortythousand samples.

In view of these numbers it is perhaps not surprising that in practice few materials have been fully characterized in this way. For many important applications much lower resolutions suffice and then the BRDF description is a practical one. An example is graphics rendering, although hardly any attempt seems to exist where one uses empirical data. In practice one usually substitutes model approximations. In case empirical data are used the problem of interpolation or representation of the data comes up. One would prefer methods of representation that guarantee physically consistent results, *i.e.*, the BRDF should

satisfy certain symmetries that reflect elementary physical constraints such as invariance under permutation of entrance and exit beams (so called Helmholtz's reciprocity[5]). Thus the problem arises of how to represent empirical data in a numerically advantageous and physically acceptable, principled manner.

Some type of series development in terms of an orhonormal set of basis functions that are ordered with respect to angular resolution appears the obvious choice. Then the desired angular resolution can be simply set by *truncating* the series, whereas the structure guarantees that the approximation is always optimal in the least squares sense. Thus one has to construct the desired orthonormal basis. We proceed to show how to do this.

2 A set of orthonormal polynomials on the hemisphere

The hemisphere (here denoted as $\mathbf{H}^2$) has the topology of the unit disk $\mathbf{D}^2$. Thus it makes sense to try to adapt known systems on the unit disk to our present problem. The unique set of polynomials that are complete and orthogonal on the unit disk and have the desired invariance properties with respect to rotations about the symmetry center of the unit disk are the well known Zernike polynomials[2]. This basis was introduced by Zernike in order to construct a principled method of describing wavefront aberrations for circular pupils (so called Zernike–Nijboer theory of aberrations).

We consider the upper hemisphere $\mathbf{H}^2$ of the unit sphere $\mathbf{S}^2$ with the usual coordinate systems. In Cartesian coordinates $\{x, y, z\}$ of $\mathbf{R}^3$ the upper hemisphere is given by $x^2 + y^2 + z^2 = 1$, $z \geq 0$. In polar coordinates $\{\vartheta, \varphi\}$ of $\mathbf{R}^2$ (ϑ the polar distance, φ the azimuth), it is $\vartheta \leq \frac{\pi}{2}$. The polar coordinates $\{\varrho, \varphi\}$ of the unit disk $\mathbf{D}^2$ will also be used.

The invariance we require is "invariant form" with respect to rotations about the origin of $\mathbf{R}^2$, or the z–axis of $\mathbf{R}^3$, that is, changes of azimuth. Thus if

$$\begin{aligned} x' &= x\cos\varphi + y\sin\varphi \\ y' &= -x\sin\varphi + y\cos\varphi, \end{aligned}$$

then the polynomial $V(x, y)$ should be taken to $V(x', y')$ such that

$$V(x, y) = G(\varphi)V(x', y'),$$

where $G(\varphi)$ is a continuous function with period 2π such that $G(0) = 1$.

We require that the transformation reflects the properties of the rotation group faithfully, thus $G(\varphi_1 + \varphi_2) = G(\varphi_1)G(\varphi_2)$. This determines the function G fully, we have

$$G(\varphi) = e^{il\varphi},$$

where l is any integer. Thus we have $V(\varrho\cos\varphi, \varrho\sin\varphi) = R(\varrho)e^{il\varphi}$. By hypothesis $V(x, y)$ is a polynomial in x, y of degree n (say). It follows that $R(\varrho)$ is a polynomial of degree n which contains no power of ϱ of degree lower than $|l|$. $R(\varrho)$ is even or odd as $|l|$ is even or odd. The Zernike polynomials are the unique

choice that contains a member for each (n, l) and thus constitutes a complete basis.

The Zernike polynomials are denoted

$$V_n^l(\varrho\cos\varphi, \varrho\sin\varphi) = R_n^l(\varrho)e^{il\varphi}.$$

It has been shown that this is a complete basis for functions on the interior of the unit disk. This set contains $\frac{1}{2}(n+1)(n+2)$ linearly independent polynomials of degree $\leq n$. In the conventional normalization we have

$$\int_{\mathbf{D}^2} V_n^{l*} V_{n'}^{l'}\, dA = \frac{\pi}{n+1}\delta_{ll'}\delta_{nn'},$$

where δ_{pq} is the Kronecker symbol (*i.e.*, $\delta_{pp} = 1$ and $\delta_{pq} = 0$, $p \neq q$) and $dA = dx\, dy$.

The radial functions $R_n^l(\varrho)$ are closely related to Jacobi's polynomials, which are terminating hypergeometric series. A closed form formula is[2]:

$$R_n^{\pm m}(\varrho) = \sum_{s=0}^{\frac{n-m}{2}} (-1)^s \frac{(n-s)!}{s!(\frac{n+m}{2}-s)!(\frac{n-m}{2}-s)!}\varrho^{n-2s}.$$

The radial functions take the value unity on the boundary of $\mathbf{D}^2$ (due to the conventional normalization).

We show how to map $\mathbf{D}^2$ on $\mathbf{H}^2$ in such a way that Zernike's system can be mapped into a complete, orthonormal basis of functions on $\mathbf{H}^2$. The required system clearly has the general form

$$W(\vartheta, \varphi) = \Theta(\vartheta)G(\varphi),$$

for the very same reasons as discussed above. From elementary differential calculus we have $2\sin\frac{\vartheta}{2}\, d(2\sin\frac{\vartheta}{2}) = \sin\vartheta\, d\vartheta$. Thus we also have

$$\int_{\mathbf{D}^2} V(x,y)dA = \int_{\mathbf{H}^2} W(\vartheta,\varphi)d\Omega = \int_{\mathbf{H}^2} \frac{1}{2}R(\sqrt{2}\sin\frac{\vartheta}{2})G(\varphi)d\Omega,$$

where $d\Omega = \sin\vartheta d\vartheta d\varphi$. We use essentially the area true mapping of $\mathbf{S}^2$ on $\mathbf{R}^2$ due to Lambert.

When we define

$$K_n^l(\vartheta,\varphi) = \sqrt{\frac{2n+1}{2}}R_n^l(\sqrt{2}\sin\frac{\vartheta}{2})e^{il\varphi},$$

then we have

$$\int_{\mathbf{H}^2} K_n^{l\,*} K_{n'}^{l'} d\Omega = \delta_{nn'}\delta_{ll'},$$

i.e., the $K_n^l(\vartheta,\varphi)$ are a complete, orthonormal system on $\mathbf{H}^2$ with the desired invariance properties. Please notice that this system is different from that of the

spherical harmonics $Y_l^m(\vartheta, \varphi)$ which are an orthonormal basis for functions on *the whole* of $\mathbf{S}^2$.

This system should have many uses in radiometry and photometry (and transport theory in general) since it allows you to expand arbitrary functions of direction at one side of a planar interface. It might seem unlikely that such a system has not been proposed earlier, yet we have not been able to find an instance in the literature on radiometry and/or photometry.

3 An orthonormal basis for the description of BRDF's

The bidirectional reflection properties of a surface are clearly specified by the Bidirectional Reflection Distribution Function (BRDF), originally due to Edwards[4] and effectively introduced by Nicodemus *et al*[13]. One defines

$$f(\vartheta_i, \varphi_i, \vartheta_r, \varphi_r) = \frac{dN_r(\vartheta_r, \varphi_r)}{dH_i(\vartheta_i, \varphi_i)},$$

(the subscript i denotes the incident beam, the subscript r the reflected beam), thus the BRDF is the ratio of the scattered radiance to the incident irradiance.

The BRDF may become singular, especially for the case of grazing incidence. This happens, for instance, for perfect specular reflection. In such cases it is advantageous to deal with the function

$$g(\vartheta_i, \varphi_i, \vartheta_r, \varphi_r) = f(\vartheta_i, \varphi_i, \vartheta_r, \varphi_r) \cos \vartheta_i,$$

which is the scattered radiance for irradiance by a collimated source of constant intensity, instead of the BRDF. We then develop g rather than f in terms of an orthonormal basis and thus avoid singular behavior. For the case of natural materials this will be seldom necessary though.

The most general form of the BRDF in terms of the aforementioned orthonormal basis functions is:

$$f(\vartheta_i, \varphi_i, \vartheta_r, \varphi_r) = \sum_{klk'l'} a_{klk'l'} K_n^l(\vartheta_i, \varphi_i) K_{n'}^{l'}(\vartheta_r, \varphi_r).$$

However, various symmetries severely constrain this general form. We consider these constraints below. Because of the orthonomality we find the coefficients $a_{klk'l'}$ by integration:

$$a_{klk'l'} = \int_{\mathbf{H}^2 \times \mathbf{H}^2} f(\vartheta_i, \varphi_i, \vartheta_r, \varphi_r) K_n^l(\vartheta_i, \varphi_i) K_{n'}^{l'}(\vartheta_r, \varphi_r) \, d\Omega \, d\Omega'.$$

3.1 Helmholtz's reciprocity

"Helmholtz's reciprocity" simply expresses the fact that in the approximation of geometrical optics

$$f(\vartheta_i, \varphi_i, \vartheta_r, \varphi_r) = f(\vartheta_r, \varphi_r, \vartheta_i, \varphi_i),$$

the idea being simply that one counts rays irrespective of their direction[10]. The extreme generality of the idea ensures that it applies under the most various conditions.

Helmholtz's reciprocity enables us to write

$$f(\vartheta_i, \varphi_i, \vartheta_r, \varphi_r) = \sum_{klk'l'} a_{klk'l'}(K_n^l(\vartheta_i, \varphi_i)K_{n'}^{l'}(\vartheta_r, \varphi_r) + K_{n'}^{l'}(\vartheta_i, \varphi_i)K_n^l(\vartheta_r, \varphi_r)).$$

The symmetrical functions

$$H_{nn'}^{ll'}(\vartheta_i, \varphi_i, \vartheta_r, \varphi_r) = K_n^l(\vartheta_i, \varphi_i)K_{n'}^{l'}(\vartheta_r, \varphi_r) + K_{n'}^{l'}(\vartheta_i, \varphi_i)K_n^l(\vartheta_r, \varphi_r),$$

may be called "Helmholtz surface scattering modes". Their azimuthal dependence is

$$e^{i(l+l')\varphi},$$

which for photometric purposes may be written in terms of (real rather than complex) trigonometric functions. In the case of isotropic surfaces (see below) we need only keep the cosine (or even) components, the sine (or odd) components describe the surface anisotropy.

3.2 Surface isotropy

A more special type of symmetry is surface isotropy. Although not completely general, this condition applies often to a good approximation. It yields a very strong constraint on the general form of the BRDF. Indeed, the BRDF may depend only on $|\varphi_i - \varphi_j|$. This implies that the azimuthal dependence is in terms of $\cos l(\varphi_i - \varphi_j)$, $l = 0, 1, \ldots$. We have

$$f(\vartheta_i, \varphi_i, \vartheta_r, \varphi_r) = \sum_{nml} a_{nml}(\Theta_n^l(\vartheta_i)\Theta_m^l(\vartheta_r) + \Theta_m^l(\vartheta_i)\Theta_n^l(\vartheta_r)) \cos l(\varphi_i - \varphi_r),$$

where

$$\begin{aligned} n &\geq 0 \\ 0 \leq m &\leq n \\ 0 \leq l &\leq m \\ (n-l),\ (m-l) &\text{ even.} \end{aligned}$$

This brings down the number of basis functions that have to be taken into account enormously.

We calculate the coefficients a_{nml} simply as

$$a_{nml} = \int_{\mathbf{H}^2 \times \mathbf{H}^2} f(\vartheta_i, \varphi_i, \vartheta_r, \varphi_r)(\Theta_n^l(\vartheta_i)\Theta_m^l(\vartheta_r) + \Theta_m^l(\vartheta_i)\Theta_n^l(\vartheta_r)) \cos l(\varphi_i - \varphi_r)\, d\Omega\, d\Omega'.$$

We will denote this system as

$$S_{nm}^l(\vartheta_i, \vartheta_r, \Delta\varphi_{ir}) = (\Theta_n^l(\vartheta_i)\Theta_m^l(\vartheta_r) + \Theta_m^l(\vartheta_i)\Theta_n^l(\vartheta_r)) \cos l\Delta\varphi_{ir}$$

where we have set $\Delta\varphi_{ir} = |\varphi_i - \varphi_r|$.

3.3 Resulting basis polynomials

Due to the symmetries the number of components in the orthonormal basis of functions on $\mathbf{H}^2 \times \mathbf{H}^2$ is much reduced. There are only five for order two, fourteen for order four, fiftyfive for order eight. Yet the number increases fast when one raises the maximum order: One has already to take 285 basis functions into account for order sixteen, even in the case of the isotropic surface. Order eight might be a limit for most practical work. The angular resolution will then be about $360°/2 * 8 = 22\frac{1}{4}°$ which is amply sufficient for purposes of graphics rendering of diffusely reflecting materials. For work in which sharply articulated functions (such as true specular components) have to be represented accurately one needs to draw much higher orders into account of course.

All surface scattering modes up to order 4 have been depicted in figure 1.

Explicit expressions for the basis functions up to order two are

$$\begin{aligned}
S^0_{00}(\vartheta_1, \vartheta_2, \Delta\varphi_{12}) &= \tfrac{1}{\pi} \\
S^1_{11}(\vartheta_1, \vartheta_2, \Delta\varphi_{12}) &= \tfrac{4}{\pi} \sin\tfrac{\vartheta_1}{2} \sin\tfrac{\vartheta_2}{2} \cos\Delta\varphi_{12} \\
S^0_{20}(\vartheta_1, \vartheta_2, \Delta\varphi_{12}) &= \tfrac{\sqrt{3}}{\pi}(\cos\vartheta_1 + \cos\vartheta_1 - 1) \\
S^0_{22}(\vartheta_1, \vartheta_2, \Delta\varphi_{12}) &= \tfrac{3}{\pi}(1 + \cos\vartheta_1)(1 + \cos\vartheta_2) \\
S^2_{22}(\vartheta_1, \vartheta_2, \Delta\varphi_{12}) &= \tfrac{3}{\pi}(1 - \cos\vartheta_1)(1 - \cos\vartheta_2)\cos 2\Delta\varphi_{12}.
\end{aligned}$$

Although the basis functions become complicated for the higher orders it is easy enough to construct them automatically and the system is convenient enough for routine use.

4 Lambertian and specular components

The BRDF of a perfect Lambertian[9] surface has a BRDF that is constant, namely $f(\vartheta_1, \vartheta_2, \Delta\varphi_{12}) = 1/\pi$. The Lambertian BRDF is just the initial term of the series development, that is to say $S^0_{00}(\vartheta_1, \vartheta_2, \Delta\varphi_{12})$.

For a perfect mirror we have

$$g(\vartheta_i, \varphi_i, \vartheta_r, \varphi_r) = \delta(\vartheta_r - \vartheta_i)\delta(\varphi_i - \varphi_r + \pi)/\sin\vartheta_r.$$

The scattered radiance for a constant collimated source is easily expanded in terms of the basis, the coefficients are simply proportional with $(-1)^l\delta_{nm}$ for we have:

$$a_{nml} = \int_{\mathbf{H}^2 \times \mathbf{H}^2} g(\vartheta_i, \varphi_i, \vartheta_r, \varphi_r) S^l_{nm}(\vartheta_i, \vartheta_r, \Delta\varphi_{ir}) d\Omega_i d\Omega_r,$$

where we can immediately carry out one integration (because of the Dirac delta functions) and are left with the integral

$$\begin{aligned}
a_{nml} &= \textstyle\int_{\mathbf{H}^2} S^l_{nm}(\vartheta, \vartheta, \pi) d\Omega = \\
a_{nml} &= (-1)^l 4\pi\delta_{nm} \textstyle\int_{\mathbf{H}^2} \Theta^l_n(\vartheta)^2 \sin\vartheta \, d\vartheta,
\end{aligned}$$

which again is immediate because of the properties of the $\Theta^l_n(\vartheta)$. Thus we obtain a constant angular spectrum, much like the Fourier development of a Dirac impulse.

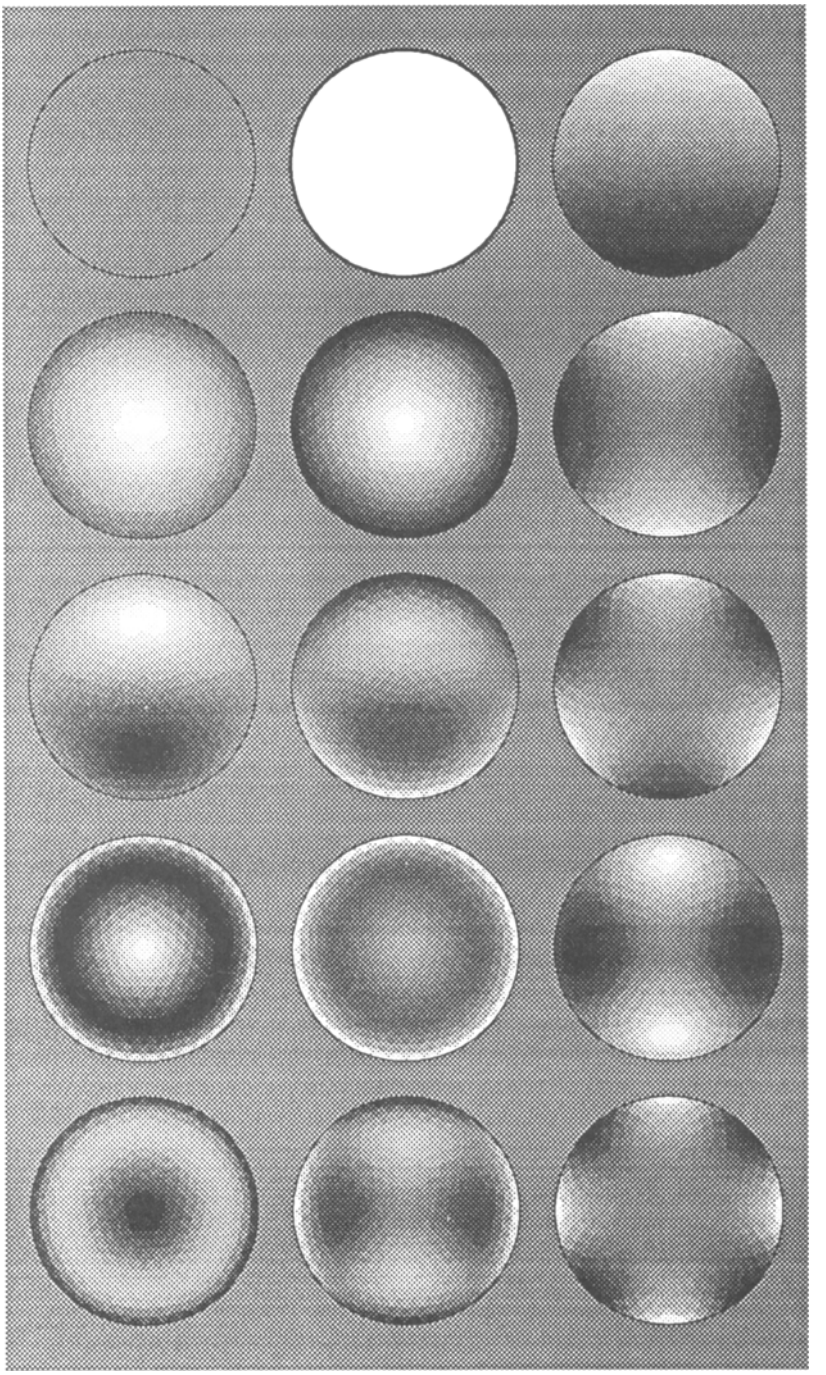

Fig. 1. *Plots of all surface scattering modes* $S^{l}_{nm}(\pi/4, \vartheta_r, \Delta\varphi_{ir})$ *up to order 4.*

By combination of the Lambertian and the specular expressions we can construct the BRDF for an ideal glossy paint layer. This is the classical description of glossy surfaces in terms of a purely diffuse and a purely specular component[3]. We assume that the pigment particles yield a Lambertian component, whereas the specular component is due to Fresnel reflection at the interface with air.

For the illustrations (figure 2) we took the specular reflection coefficient equal to 2% (thus we didn't take Fresnel's formulas into account). We show the scattering indicatrix for a collimated beam incident at $\vartheta_i = 45°$ in various approximations, orders 2 to 8. (Notice that order two preserves only the Lambertian component.) Even the low order approximations preserve the qualitative nature of the specularity quite well, the main degeneration being a loss of angular resolution. Adding higher order terms concentrates the forward scattering lobe more and more inside a small solid angle centered on the direction of the mirror reflection. For order 8 we have 55 degrees of freedom, thus the solid angle of a "pixel" is roughly $2\pi/55$, and we can estimate the diameter of the forward scattering lobe as $(180/\pi)\sqrt{(4/\pi)(2\pi/55)} \approx 22°$, that is the resolution of the order 8 approximation, *etc.*

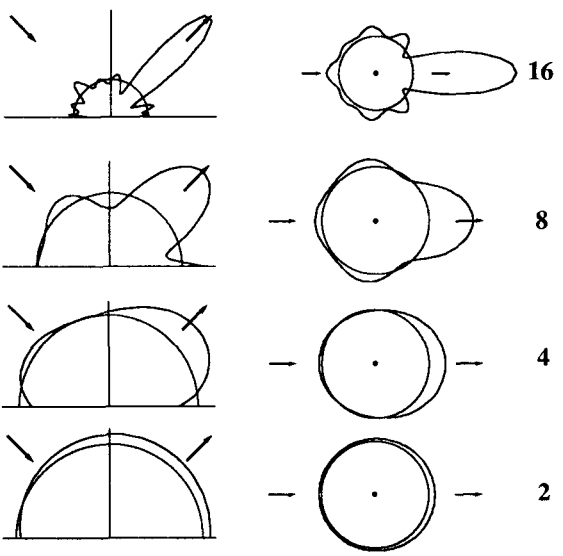

Fig. 2. *Various approximations (truncated series of Helmholtz surface scattering modes) of a model containing pure Lambertian and a pure mirror reflection term. The entrance beam enters at* $\vartheta_i = 45°$. *In the lefthand column we have meridional cross sections and in the righthand column azimuthal cross sections.*

5 Development of conventional expressions in terms of surface scattering modes

Many expressions have been proposed as models of the BRDF of generic materials[1, 12, 15, 16, 17]. Some of these fail to comply with Helmholtz's reciprocity, practically all are descriptions of isotropic surfaces. Such expressions can be roughly divided into two categories: First we have expressions derived on (phenomenological) physical principles for certain model surfaces such as randomly distributed micromirrors, *etc.* Secondly we have the category of formulas based on ease of numerical evaluation for datastructures readily available in graphics rendering pipelines. Although such expressions are designed not to be totally irrealistic, modelling any reasonable model surface on physical principles is a secondary objective.

One of the earliest and certainly simplest models is certainly the Lambertian diffusely scattering surface. We have seen that it can be described perfectly with only the initial term of a development into surface scattering modes. One may well ask how such a development for some of the other expressions turns out.

Many of the models proposed in the literature in fact are represented *exactly* (that is: the series expansion in terms of the Helmholtz surface reflection modes *terminates* and contains only contributions from a finite number of modes) by the development proposed by us. Examples are the reflection model proposed by Blinn[1] (popular in computer graphics) and the model proposed by Minnaert[10] (itself a generalization of a model proposed by Öpik[14]).

As an example we consider Minnaert's proposal, which was especially constructed to respect Helmholtz's reciprocity:

$$f(\vartheta_i, \varphi_i, \vartheta_r, \varphi_r) = \tfrac{k+1}{2\pi}(\cos\vartheta_i \cos\vartheta_r)^{(k-1)} \ (0 \leq k \leq 1).$$

Clearly this can be written as a polynomial in $\sin(\vartheta_i/2)$ and $\sin(\vartheta_r/2)$ of maximum degree $2(k-1)$ in either variable. Thus we can indeed write Minnaert's expression as a linear combination of a finite number of surface scattering modes.

6 Empirical BRDF's

In expressing the BRDF of materials that have to be measured in the laboratory we meet with at least two problems. First there is the practical problem of measuring the BRDF. Apart from the standard photometric problems we run here in the problem that the number of degrees of freedom is very large indeed. In principle we ought to sample a four–parameter space. Although Helmholtz's reciprocity (which may be expected to hold and can even be used as a check on the data) and surface isotropy (which need not pertain and has to be checked empirically) can be used to ameliorate this problem one still needs a great number of independent measurements. Then we have the problem of how to compute the development in terms of the basis.

The development in terms of the basis is primarily a problem of numerical integration. The coefficients of the development are defined as integrals over

$H^2 \times H^2$. We approximate the integrals by Riemann sums and therefore have to define a tesselation of the hemisphere, preferably one in which the tesseræ subtend identical solid angles. Since the hemisphere has to be evenly covered, a good method is to start with the tesselation defined by the faces of the regular icosahedron. We then can produce refinements by barycentric subdivision. Of course we only keep faces on the northern hemisphere. If we perform a single barycentric subdivision we obtain a triangulation of the hemisphere into 40 faces of identical area and of roughly equilateral shape.

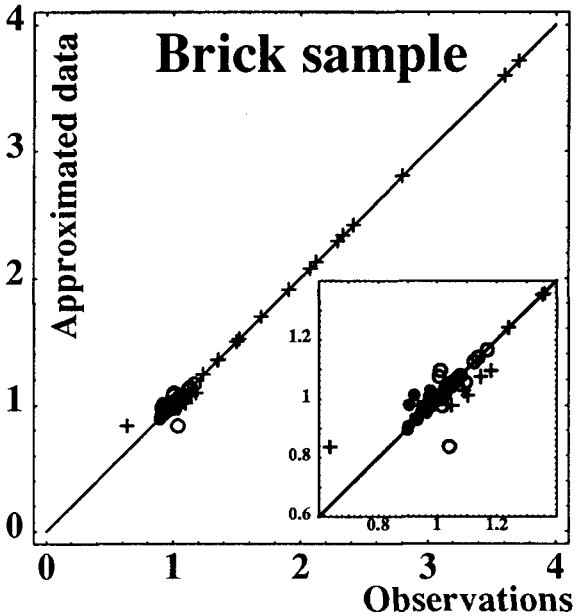

Fig. 3. *Scatterplot of the results from the 8^{th} order approximation of the brick data against the observations.*

The full specification of a BRDF measurement on this tesselation would be a 40×40 data matrix, involving 1600 independent measurements. Thus the effort involved in actually measuring a BRDF is quite appreciable. This is perhaps the reason why full BRDF measurements are rarely reported in the literature. Most measurements are confined to the plane of incidence. Helmholtz's reciprocity brings down the number of degrees of freedom to 820, surface isotropy to 66, which is a number that might be considered practical. With 66 independent measurements we are able to construct an approximation of the 8^{th} order (there are 55 independent functions in the 8^{th} order basis).

Since the data slightly overspecify the 8^{th} order approximation we use a pseudoinverse method to find the best fitting coefficients of the truncated series.

This representation of the BRDF by a 40×40 matrix (or larger by progressive

barycentric subdivision) is of interest by itself and may prove useful in many computer vision contexts. Very high angular resolutions will in fact seldom be required in applications and the matrix can be used as a lookup table.

In collecting the data one should take care to use entrance and exit beams that are roughly centered on the barycentra of the faces of the triangulation and have an aperture of the order of the solid angle of the faces (about 20° in diameter). The latter condition is necessary to prevent aliassing problems. Likewise, the data should be fitted by truncated series where the highest frequency terms can still be sampled by the tesselation.

Here we present data on the surface of a brick. The brick surface scatters roughly Lambertian for normal directions of incidence, whereas both forward and backscatter lobes develop for near grazing incidence. We find that the measurements are represented within the experimental tolerances by a series expansion truncated at order eight (in fact order four would do about equally well).

That this analytical expression really represents the measurements very well is borne out by the scatterdiagram presented in figure 3. The correlation is very high, the residuals can be ascribed to experimental error. In this case a lower order approximation might do as well and would lead to a somewhat simpler expression.

7 Conclusion

We have constructed a complete, orthonormal basis with desirable invariance properties for the representation of BRDF's taking Helmholtz's reciprocity and (if applicable) surface isotropy into account. The orthonormal basis of functions on the hemisphere ("surface scattering modes") is of wider interest though, since it applies to any situation where one encounters functions of direction at one side of a planar interface (transport phenomena, radiometry, photometry). The orthonormal basis for the BRDF's is defined on the direct product of the hemisphere with itself.

We have shown how the basis can be applied to express empirical BRDF data. This has a number of advantages. First of all, the use of a truncated expansion in orthonormal functions ensures that we find the best approximation in the least squares sense among all linear combinations of elements in this basis. When the high order elements are practically of less importance than the low order ones (as is the case here), we obtain automatically desirable approximations. Secondly, the approximations are guaranteed to satisfy Helmholtz's reciprocity and thus are physically realistic. This is an advantage over *ad hoc* approximations which often turn out to violate Helmholtz's reciprocity and are thus not even physically possible. Thirdly, the approximations are guaranteed to express surface isotropy when so desired.

The method of subdivision of the hemisphere in equal solid area facets used in the data sampling, which leads to a matrix representation of surface scattering, might be of considerable utility by itself in computer graphics as it can conveniently be implemented as a lookup table.

We have outlined how the method can be applied to empirical data and have presented results for the BRDF of a real material sample (a piece of brick).

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