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BIFURCATION ANALYSIS OF MINIMIZING HARMONIC MAPS DESCRIBING THE EQUILIBRIUM OF NEMATIC PHASES BETWEEN CYLINDERS

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Bifurcation Analysis of Minimizing Harmonic Maps Describing the Equilibrium of Nematic Phases between Cylinders

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Introduction

Consider a nematic liquid crystal confined between two coaxial circular cylinders centered on the z-axis. Suppose that each cylinder imposes a strong anchoring boundary condition requiring the director field to be normal to it. In experiments on a nematic material[#] in such a configuration, Williams, Cladis, and Kléman [1973, 2]

The material was MBBA, *i.e.*, methoxy-benzilidene-butylaniline, a frequently studied low molecular weight nematic.

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observed that, for the range of cylinder diameters studied, at equilibrium the director lies along radial lines in the (x, y)-plane with no component in the z-direction. We shall here show that such is the case in the one-constant theory, *i.e.*, for minimizing harmonic maps into S^2, provided the ratio ρ of the radius of the inner cylinder to that of the outer cylinder is equal to or greater than $e^{-\pi}$; however, if ρ is less than $e^{-\pi}$, then the free-energy has a minimizer that preserves radial symmetry in the sense that the projection of u onto the (x, y)-plane lies along radial lines, but this minimizer has a component along the z-axis at every point off the bounding surfaces.

It is a consequence of an elementary remark, which is here called "Theorem 0" and whose proof is given in Appendix I, that the present minimization problems for functions on a three-dimensional region can be reduced to problems for functions on two-dimensional regions.

Let Ω be a smooth bounded domain in $\mathbb{R}^2$, put $D = \Omega \times [0, 1]$, and let φ be a given smooth map from $\partial\Omega$ to S^2. Define the sets $\mathcal{F}$ and $\mathcal{E}$ by

$$\mathcal{F} = \{\ u \in H^1(D; S^2);\ u(x, y, z) = \varphi(x, y) \text{ for } (x, y, z) \in \partial\Omega \times [0, 1]\ \},$$

$$\mathcal{E} = \{\ u \in H^1(\Omega; S^2);\ u(x, y) = \varphi(x, y) \text{ for } (x, y) \in \partial\Omega\ \}.$$

Theorem 0. *If u is a minimizer for*

$$\min_{u \in \mathcal{F}} \int_D |\nabla u|^2\, dx\, dy\, dz\ ,$$

then u is independent of z and is a minimizer for

$$\min_{u \in \mathcal{E}} \int_\Omega |\nabla u|^2\, dx\, dy\ .$$

In Section 1, where we identify the minimizers for the coaxial case, the domain Ω will be $B_1 \setminus \overline{B_\rho}$, where ρ is in (0, 1) and

$$B_t = \{ (x, y) \in \mathbb{R}^2;\ x^2+y^2<t^2 \},$$

and the strong-anchoring boundary condition mentioned above will be expressed by the following choice for the function φ:

$$\varphi(x, y) = (x/r, y/r, 0), \qquad (r^2 = x^2 + y^2).$$

Our main result, Theorem 1, asserts: *(i)* If $\rho \geq e^{-\pi}$, the energy

$$E(u) = \int_\Omega |\nabla u|^2 \, dx\, dy$$

is minimized on $\mathcal{E}$ by the function u_0 defined on Ω by

$$u_0(x, y) = (x/r, y/r, 0).$$

(ii) If $\rho < e^{-\pi}$, u_0 does not minimize E; there is then a minimizer with radial symmetry whose range does not intersect the equator of S^2, *i.e.*, the set

$$S^1 = \{ (x, y, z) \in \mathbb{R}^3;\ x^2+y^2+z^2 = 1,\ z = 0 \};$$

to within a reflexion through the (x, y)-plane, the minimizer of E is unique in the class of radially symmetric maps.

As our discussion will show, for $\rho < e^{-\pi}$, the magnitude of the z-component, u_3, of the radially symmetric minimizer u of E has its maximum value at $r = \rho^{1/2}$. Because $|u_3(\rho^{1/2})|$ approaches zero as ρ approaches $e^{-\pi}$ from below, experiments to test our new form of "escape to the third-dimension" will require care: when the maximum of $|u_3|$ is large, it occurs near a bounding surface.#

This and related matters will be discussed in a paper on physical applications of the present analysis.

In Section 2, for the general case in which the hole B_ρ is not centered at the origin, we study the behavior of the minimizer and its energy as ρ approaches zero. We there describe the sense in which the minimizing map from $B_1 \backslash \overline{B_\rho}$ to S^2 approaches the radially symmetric minimizing map from B_1 to S^2 whose importance to liquid crystal theory was brought out Cladis & Kléman [1972, 1] and Meyer [1973, 1].

Several results of independent interest are given in the appendices. Among these is the following: any harmonic map from a smooth bounded domain in $\mathbb{R}^2$ to S^2 whose range lies in the northern hemisphere,

$$S^2_+ = \{ (x, y, z) \in \mathbb{R}^3;\ x^2+y^2+z^2 = 1,\ z \geq 0 \},$$

but is not a subset of the equator S^1, is minimizing.

1. Identification of the Minimal Energy Maps

Here $\Omega = B_1 \setminus \overline{B_\rho}$, and

$$\mathcal{E} = \{\ u \in H^1(\Omega, S^2);\ u_0(x, y) = (x/r, y/r, 0) \text{ on } \partial\Omega\ \}.$$

Recall that the function $u_0(x, y) = (x/r, y/r, 0)$ satisfies the equation

$$-\Delta u_0 = u_0 |\nabla u_0|^2$$

on Ω and hence is a harmonic map from Ω to S^2. Of course u_0 is in $\mathcal{E}$, but, as we show below, this critical point of the energy E on $\mathcal{E}$ is not necessarily a minimizer.

Let $\mathcal{E}_0$ be set of functions in $\mathcal{E}$ which have *radial symmetry, i.e.*, have the form

$$u(x, y) = c(r) u_0(x, y) + s(r)(0, 0, 1)$$

with $c^2 + s^2 = 1$.

Theorem 1.

(1) *If* $\rho \geq e^{-\pi}$, *then* u_0 *is a minimizer of* E *on* $\mathcal{E}$; *if* $\rho > e^{-\pi}$, u_0 *is the unique minimizer in* $\mathcal{E}$.

(2) *If* $\rho < e^{-\pi}$, *then* u_0 *is not a minimizer of* E *on* $\mathcal{E}$, *but there is a minimizer of* E *on* $\mathcal{E}$ *that belongs to* $\mathcal{E}_0$ *and has* $s(r) > 0$ *for every* r *in* $(\rho, 1)$; *this minimizer is unique in* $\mathcal{E}_0$ *to within replacement of* $s(r)$ *by* $-s(r)$.

Remark 1. For $\rho < e^{-\pi}$ Meyer [1973, 1, pp. 409-411] found a harmonic map in $\mathcal{E}_0$ for which $|u_3|$ has a non-zero maximum at $r = \sqrt{\rho}$; as the proof of Theorem 1 shows, for each $\rho < e^{-\pi}$ that radially symmetric harmonic map gives a minimizer of E on $\mathcal{E}$ and, to within a reflection through the (x, y)-plane, is the unique such minimizer in $\mathcal{E}_0$. After the present study was completed, Sandier [1991, 2] proved that for each ρ in (0, 1) the minimizer of E on $\mathcal{E}$ is unique (to within reflection) and is radially symmetric.

To prove Theorem 1 we employ two lemmas.

Lemma 1. *For each* ρ *in* (0, 1), $\min_{u \in \mathcal{E}} E(u)$ *is attained at a map in* $\mathcal{E}_0$.

Proof. For an $\varepsilon > 0$, put

$$\Psi_\varepsilon(x, y) = \left(\frac{x}{r}\cos\varepsilon, \frac{y}{r}\cos\varepsilon, \sin\varepsilon\right)$$

for (x, y) in $\partial\Omega$, and define

$$\mathcal{E}_\varepsilon = \{\, u \in H^1(\Omega, S^2);\ u = \Psi_\varepsilon \text{ on } \partial\Omega \,\}.$$

Let u^ε be a minimizer for E on $\mathcal{E}_\varepsilon$. By a standard argument, we may assume that $u_3^\varepsilon \geq \sin\varepsilon$ in Ω. [First we may replace (u_1, u_2, u_3) by $(u_1, u_2, |u_3|)$ which is again a minimizer, and as we then have

$$-\Delta|u_3| = |u_3||\nabla u|^2 \geq 0,$$

[1979, 1], there is a unique harmonic map from Ω into $S^2 \cap \{u_3 \geq \sin \varepsilon\}$ with a given boundary value. This map u^ε is radially symmetric, since any minimizer of E in the class $\mathcal{E}_0$ is harmonic. We claim that for some sequence $\varepsilon_n \to 0$, u^{ε_n} converges strongly in H^1 to a minimizer of E on $\mathcal{E}$, and this minimizer is radially symmetric. To prove this assertion it suffices to check that

$$\limsup_{\varepsilon \to 0} \min_{\mathcal{E}_\varepsilon} E \leq \min_{\mathcal{E}} E .$$

Given any v in $\mathcal{E}$, consider $\mathrm{v}^\varepsilon = R_\varepsilon(\mathrm{v})$, where $R_\varepsilon : S^2 \to S^2$ is defined by

$$R_\varepsilon(\xi_1, \xi_2, \xi_3) = (a_\varepsilon \xi_1,\ a_\varepsilon \xi_2,\ \max\{|\xi_3|,\ \sin \varepsilon\})$$

with

$$a_\varepsilon = \left[\frac{1 - \max\ (|\xi_3|^2, \sin^2 \varepsilon)}{1 - |\xi_3|^2}\right]^{1/2} .$$

Clearly v^ε is in $\mathcal{E}_\varepsilon$, $E(\mathrm{v}^\varepsilon) \geq E(u^\varepsilon)$, and $E(\mathrm{v}^\varepsilon) \to E(\mathrm{v})$.

Lemma 2. *For each* ρ *in* $(0, 1)$ *there is a unique minimizer of E in the class* $\mathcal{E}_0$ *(modulo a reflection through the* (x, y)*-plane). This minimizer is* u_0 *if and only if* $\rho \geq e^{-\pi}$.

Proof of Lemma 2. Let $u = (u_1, u_2, u_3)$ be a minimizer of E in $\mathcal{E}_0$. We claim that either $u_3 \geq 0$ on Ω or $u_3 \leq 0$ on Ω. Suppose, to the contrary, that u_3 takes both positive and negative values in Ω. Then $\tilde{u} = (u_1, u_2, |u_3|)$ is also a minimizer of E on $\mathcal{E}_0$ and coincides with u on some open set. As both u and $\tilde{u}$ are minimizers, they are analytic functions from Ω to $\mathbb{R}^3$, and hence $u = \tilde{u}$ on Ω, *i.e.*, $u_3 \geq 0$ on Ω, which is a contradiction.

Without loss of generality we assume $u_3 \geq 0$ and we write

$$u(r\cos\theta,\ r\sin\theta) = (\cos\theta\cos\varphi(r),\ \sin\theta\cos\varphi(r),\ \sin\varphi(r))$$

for some smooth function $\varphi : [\rho, 1] \to [0, \pi/2]$ with $\varphi(\rho) = \varphi(1) = 0$.

The energy $E(u)$ is given by

$$E(u) = 2\pi \int_{\rho}^{1} \left[\left(\frac{d\varphi}{dr}\right)^2 + \frac{\cos^2\varphi}{r^2} \right] r\, dr .$$

It is convenient to change variables and let $t = \log r$, with t in $[\log\rho, 0]$, so that

$$E(u) = 2\pi \int_{\log\rho}^{0} \left[\left(\frac{d\varphi}{dt}\right)^2 + \cos^2\varphi \right] dt .$$

As u_0 corresponds to $\varphi \equiv 0$,

$$E(u) - E(u_0) = 2\pi \int_{\log \rho}^{0} \left[\left(\frac{d\varphi}{dt}\right)^2 - \sin^2\varphi\right] dt .$$

We now distinguish two cases.

(1) In the case $\rho \geq e^{-\pi}$, *i.e.*, $|\log \rho| \leq \pi$, we have

$$E(u) - E(u_0) = 2\pi \int_{\log \rho}^{0} \left[\left(\frac{d\varphi}{dt}\right)^2 - \varphi^2\right] dt + 2\pi \int_{\log \rho}^{0} [\varphi^2 - \sin^2\varphi]\, dt \geq 0 ,$$

where the inequality is strict unless $\varphi \equiv 0$, because, for φ in $H_0^1(\log \rho, 0)$,

$$\int_{\log \rho}^{0} \left(\frac{d\varphi}{dt}\right)^2 dt \geq \lambda_1 \int_{\log \rho}^{0} \rho^2\, dt ,$$

where $\lambda_1 = \pi^2(\log \rho)^{-2}$, which is not less than 1, is the first eigenvalue of $-\varphi''$. It follows that u_0 is the unique minimizer of E on $\mathcal{E}_0$ when $\rho \geq e^{-\pi}$.

(2) In the case of $\rho < e^{-\pi}$, *i.e.*, $|\log \rho| > \pi$, u_0 is not a minimizer, for the choice

$$\varphi(t) = \beta \sin \frac{\pi t}{|\log \rho|}$$

with β sufficiently small yields $E(u) < E(u_0)$. For each minimizer of E on $\mathcal{E}_0$, φ obeys the variational equation

$$-\varphi'' = \sin\varphi \ \cos\varphi,$$

and hence $\psi = 2\varphi$ obeys the following boundary-value problem for the pendulum equation:

$$\left.\begin{aligned} -\psi'' &= \sin\psi \qquad \text{in } (\log\rho,\, 0), \\ \psi(\log\rho) &= \psi(0) = 0. \end{aligned}\right\} \tag{1}$$

As $0 \le \varphi \le \pi/2$, we have $0 \le \psi \le \pi$, and thus $\sin\psi \ge 0$. By the strong maximum principle, either $\psi \equiv 0$ or $\psi > 0$ on $(\log\rho, 0)$. As u_0 is not a minimizer, $\psi \equiv 0$ is excluded. The uniqueness of the positive solution of (1) is a classical result and follows from the concavity of the sine function on $[0, \pi]$ (see, e.g., Krasnoselskii [1964, 1] or Brezis & Oswald [1987, 1] and the references therein).

We remark in passing that the unique positive solution of (1) attains its maximum at $(\log\rho)/2$ and is symmetric about that point. Thus,

$$\int_{B_{\sqrt{\rho}} \setminus B_{\rho}} |\nabla u|^2 \, dx\, dy = \int_{B_1 \setminus B_{\sqrt{\rho}}} |\nabla u|^2 \, dx\, dy$$

and *the maximum value of* $|u_3| = |\sin \varphi(r)|$ *is attained at* $r = \rho^{1/2}$.

Proof of Theorem 1.

(1) By Lemma 1, there is a minimizer u^* of E on $\mathcal{E}$ that belongs to $\mathcal{E}_0$, and since u^* is in $\mathcal{E}_0$, Lemma 2 yields $E(u^*) \geq E(u_0)$; thus u_0 is a minimizer of E not just on $\mathcal{E}_0$, but on all of $\mathcal{E}$. We now show that u_0 is the unique minimizer of E on $\mathcal{E}$ if $\rho > e^{-\pi}$.

When $\rho > e^{-\pi}$, we may choose $R > 1$ with $\rho/R > e^{-\pi}$. For any minimizer v of E on $\mathcal{E}$ we may consider the map w: $B_R \setminus B_\rho \to S^2$ defined by

$$w(x, y) = \begin{cases} v(x, y) \text{ on } B_1 \setminus B_\rho, \\ u_0(x, y) \text{ on } B_R \setminus B_1. \end{cases}$$

Clearly w is in $H^1(B_R \setminus B_\rho; S^2)$ and

$$\int_{B_R \setminus B_\rho} |\nabla w|^2 \, dx \, dy = \int_{B_R \setminus B_\rho} |\nabla u_0|^2 \, dx \, dy .$$

Recall that u_0 is a minimizer for the energy

$$\tilde{E}(u) = \int_{B_R \setminus B_\rho} |\nabla u|^2 \, dx \, dy ,$$

as $\rho/R > e^{-\pi}$. Thus w is also a minimizer of $\widetilde{E}$, and hence is analytic on $B_R \setminus B_\rho$. Because w coincides with u_0 on $B_R \setminus B_1$, it coincides with u_0 everywhere, *i.e.*, v $= u_0$ on $B_1 \setminus B_\rho$.

(2) The assertion in Theorem 1 for the case $\rho \le e^{-\pi}$ follows directly from the two Lemmas.

Remark 2. There is an alternative proof of assertion (2) in Theorem 1. One can use Lemma 1 together with a general fact, proved in Appendix II: Any harmonic map to S^2 from a smooth bounded domain in $\mathbb{R}^2$ whose range is a subset of S^2_+ but not a subset of S^1 is minimizing.

Remark 3. Theorem 1 has an extension to dimension n for $1 \le n \le 6$. More precisely, put

$$B_t = \{ x \in \mathbb{R}^n ;\ r = |x| < t \},$$

$\Omega = B_1 \setminus B_\rho$, and

$$\mathcal{E} = \{u \in H^1(\Omega, S^n);\ u(x) = (x/r, 0) \text{ on } \partial\Omega\},$$

and let $\mathcal{E}_0$ be the set of functions in $\mathcal{E}$ with radial symmetry, which here means that

$$u(x) = c(r)u_0(x) + s(r)(0, 0, \ldots, 1)$$

where $c^2 + s^2 = 1$ and $u_0(x) = (x/r, 0)$. We recall (see Jäger and Kaul [1983, 2]) that, for $n \geq 7$ and $\Omega = B_1$, u_0 is a minimizer of E and that it is not a minimizer of E for $3 \leq n \leq 6$. Using the same arguments as in the proof of Theorem 1 we have

Theorem 1′. *Assume* $2 \leq n \leq 6$, *so that* $P(n) = -\frac{1}{4}n^2 + 2n - 2$ *is nonnegative.*

(1) *If* $\rho \geq e^{-\pi/P(n)}$, *then* u_0 *is a minimizer of E on* $\mathcal{E}_0$. *Moreover* u_0 *is the unique minimizer of E in* $\mathcal{E}$ *if* $\rho > e^{-\pi/P(n)}$.

(2) *If* $\rho < e^{-\pi/P(n)}$, *then* u_0 *is not a minimizer of E on* $\mathcal{E}_0$. *In this case there is a minimizer of E on* $\mathcal{E}$ *which belongs to* $\mathcal{E}_0$ *and has* $s(r) > 0$ *for all r in* $(\rho, 1)$. *Moreover, to within replacement of* $s(r)$ *by* $-s(r)$, *this minimizer is unique in* $\mathcal{E}_0$.

2. Asymptotic Analysis for Small ρ

We consider here the more general case in which the hole B_ρ need not be centered at the origin and thus

$$\Omega = \Omega_\rho(a) = B_1 \setminus \overline{B_\rho}(a),$$

where $a = (a_1, a_2)$ is a given point in B_1 and

$$B_\rho(a) = \{(x, y) \in \mathbb{R}^2;\ (x - a_1)^2 + (y - a_2)^2 = r_a^2 < \rho^2\} .$$

We let $\rho \to 0$, and are concerned with the behavior of minimizers of the energy in Ω_ρ subject to the boundary condition that u be normal to the boundary of Ω_ρ, *i.e.*,

$$\left.\begin{aligned} u(x,y) &= (x,\ y,\ 0) \quad \text{on } \partial B_1, \\ u(x,y) &= ((x-a_1)/\rho,\ (y-a_2)/\rho,\ 0) \quad \text{on } \partial B_\rho(a). \end{aligned}\right\} \tag{2}$$

As it is natural to expect that these minimizers converge to a minimizer of the energy in the full disc B_1, it is useful to recall some known facts about the problem in B_1. On the set

$$\mathcal{E}_1 = \{\ u \in H^1(B_1; S^2);\ u = u_0 \text{ on } \partial B_1\ \},$$

with u_0 as in Section 1, the minimum,

$$\min_{u \in \mathcal{E}_1} \int_{B_1} |\nabla u|^2 \, dx\, dy ,$$

is attained uniquely, modulo a reflection through the (x, y)-plane, and the minimizer is given by#

$$\bar{u}\,(x, y) \;=\; \frac{2}{1+r^2}\,(x, y, 1) \;+\; (0, 0, -1)\;.$$

For the convenience of the reader we sketch the proof following the method of Brezis & Coron [1983, 1]. We have

$$E(\bar{u}) = 4\pi,$$

which may be verified by direct computation or by use of the conformality of $\bar{u}$. Let $\tilde{u}$ be a minimizer, and assume, without loss of generality, $\tilde{u}_3 \geq 0$. Put

$$\mathrm{v}(x,y) \;=\; \begin{cases} \tilde{u}\,(x,y) & \text{for } r \leq 1, \\ \underline{u}(x/r^2,\, y/r^2) & \text{for } r > 1, \end{cases}$$

where

This map $\bar{u}$ is the same as that studied by Cladis & Kléman [1972, 1] and Meyer [1973, 1]; *i.e.*, in terms of polar coordinates r, θ on B_1,

$\bar{u}(r\cos\theta,\; r\sin\theta) \;=\; (\cos\theta\cos\varphi(r),\;\; \sin\theta\cos\varphi(r),\;\; \sin\varphi(r))$ where $\cot\!\left(\frac{\pi}{4}+\frac{\varphi}{2}\right) \;=\; r$. Alternatively, $\bar{u}$ is the stereographic projection of B_1 onto S^2_+ with vertex at the south pole $(0, 0, -1)$.

$$\underline{u}(x,y) = \frac{2}{1+r^2}(x, y, -1) + (0, 0, 1).$$

The degree of v (as a map from $\mathbb{R}^2 \cup \{\infty\} \simeq S^2$) is 1, because each point in the southern hemisphere has only one counterimage. This implies that

$$1 = \frac{1}{4\pi}\int_{\mathbb{R}^2} \mathrm{v} \cdot \mathrm{v}_x \wedge \mathrm{v}_y \, dx\, dy \le \frac{1}{8\pi}\int_{\mathbb{R}^2} |\nabla \mathrm{v}|^2 \, dx dy$$

$$= \frac{1}{8\pi}[E(\tilde{u}) + E(\underline{u})] = \frac{1}{8\pi}[E(\tilde{u}) + 4\pi];$$

hence $E(\tilde{u}) \ge 4\pi$, and therefore $E(\tilde{u}) = 4\pi$; *i.e.*, $\underline{u}$ and $\bar{u}$ are minimizers. Consequently, v is a minimizer of the energy on $\mathbb{R}^2 \cup \{\infty\}$ among maps of degree 1, and thus v is analytic. This implies that $\tilde{u} = \bar{u}$ and completes the proof.

With $\Omega_\rho(a)$ as above, let $\mathcal{E}_\rho(a)$ be the set of functions in $H^1(\Omega_\rho(a); S^2)$ that obey the boundary condition (2), and let u_ρ be a minimizer of

$$E_{\rho,a}(u) = \int_{\Omega_\rho(a)} |\nabla u|^2$$

on $\mathcal{E}_\rho(a)$. Without loss generality we may assume that u_ρ takes its values in the northern hemisphere S^2_+.

Theorem 2. *For each fixed a in* B_1, *as* $\rho \to 0$, u_ρ *converges to* $\bar{u}$ *uniformly on every compact subset of* $B_1 \setminus \{a\}$; *moreover,*

$$\lim_{\rho \to 0} E_{\rho, a}(u_\rho) = 8\pi = \int_{B_1} |\nabla \bar{u}|^2 + 4\pi .$$

Remark 4. It is an obvious consequence of this theorem that, for any location $a = (a_1, a_2)$ of the center of the hole $B_\rho(a)$, if ρ is small enough, u_ρ "escapes to the third dimension," *i.e.*, has values off the equator S^1.

The proof of Theorem 1 is constructed in stages requiring proof of the following four assertions about limits as $\rho \to 0$:

(1) $\limsup E_{\rho, a}(u_\rho) \le 8\pi$

(2) $u_\rho \to \bar{u}$ weakly in $H^1_{loc}(B_1 \setminus \{a\})$;

(3) $\liminf E_{\rho, a}(u_\rho) \ge 8\pi$;

(4) $u_\rho \to \bar{u}$ uniformly on every compact subset of $B_1 \setminus \{a\}$.

Proof. **(1)** To show that

$$\limsup_{\rho \to 0} \int_{\Omega_\rho(a)} |\nabla u_\rho|^2 \leq 8\pi$$

we seek, for each (small) $\rho > 0$, a map v from Ω_ρ to S^2 such that

$$\int_{\Omega_\rho(a)} |\nabla \mathrm{v}|^2 \leq 8\pi + o(1). \tag{3}$$

To this end, let

$$\mathrm{v} = \begin{cases} \bar{u}\big((x-a_1)\rho/r_a^2,\ (y-a_2)\rho/r_a^2 \big) & \text{in } B_{\sqrt{\rho}}(a)\setminus \overline{B}_\rho(a), \\ w(x,y)/|w(x,y)| & \text{in } B_\delta(a)\setminus \overline{B}_{\sqrt{\rho}}(a), \\ \bar{u}(x,y) & \text{in } B_1\setminus B_\delta(a), \end{cases}$$

where $\delta > \sqrt{\rho}$ will be determined later and w is the solution of

$$\left.\begin{aligned} \Delta w &= 0 \quad \text{in } B_\delta(a)\setminus \bar{B}_{\sqrt{\rho}}(a), \\ w &= \bar{u}(x-a_1, y-a_2) \quad \text{on } \partial B_{\sqrt{\rho}}(a), \\ w &= \bar{u}(x,y) \quad \text{on } \partial B_\delta(a). \end{aligned}\right\}$$

Now,

$$E_{\rho,a}(\mathrm{v}) = I + II + III$$

with

$$I = \int_{B_{\sqrt{\rho}}(a)\setminus B_\rho(a)} |\nabla \mathrm{v}|^2 , \qquad II = \int_{B_\delta(a)\setminus B_{\sqrt{\rho}}(a)} |\nabla \mathrm{v}|^2 , \qquad III = \int_{B_1(a)\setminus B_\delta(a)} |\nabla \mathrm{v}|^2 .$$

Clearly,

$$I \le \int_{B_1} |\nabla \bar{u}|^2 = 4\pi \tag{4}$$

and

$$III \le 4\pi . \tag{5}$$

We claim that when δ is chosen appropriately,

$$II \to 0 \quad \text{as } \rho \to 0 . \tag{6}$$

By the maximum principle, w has its values outside of a fixed neighborhood of 0 that depends only on a, and, therefore, there is a number C , independent of ρ, for which

$$\int_{B_\delta(a)\setminus B_{\sqrt{\rho}}(a)} |\nabla v|^2 \leq C \int_{B_\delta(a)\setminus B_{\sqrt{\rho}}(a)} |\nabla w|^2 .$$

To prove (6) it suffices to show that as $\rho \to 0$

$$\int_{B_\rho(a)\setminus \bar{B}_{\sqrt{\rho}}(a)} |\nabla w|^2 \to 0 . \tag{7}$$

To this end we observe that

$$w = w_1 + w_2 + w_3 ,$$

where w_1 is defined by

$$\left.\begin{aligned} &\Delta w_1 = 0 \quad \text{in } B_\delta(a)\setminus\bar{B}_{\sqrt{\rho}}(a), \\ &w_1 = (0, 0, 1) \quad \text{on } \partial B_{\sqrt{\rho}}(a), \\ &w_1 = \bar{u}(a) \quad \text{on } \partial B_\delta(a), \end{aligned}\right\}$$

w_2 by

$$\left.\begin{aligned} &\Delta w_2 = 0 \quad \text{in } B_\delta(a)\setminus\bar{B}_{\sqrt{\rho}}(a), \\ &w_2 = 0 \quad \text{on } \partial B_{\sqrt{\rho}}(a), \\ &w_3 = \bar{u}(x, y) - \bar{u}(a) \quad \text{on } \partial B_\delta(a), \end{aligned}\right\}$$

and w_3 by

$$\left.\begin{aligned} &\Delta w_3 = 0 \quad \text{in } B_\delta(a)\setminus\bar{B}_{\sqrt{\delta}}(a), \\ &w_3 = \bar{u}(x-a_1, y-a_2) - (0, 0, 1) \quad \text{on } \partial B_{\sqrt{\rho}}(a), \\ &w_3 = 0 \quad \text{on } \partial B_\delta(a). \end{aligned}\right\}$$

Let us now choose $\delta = \delta(\rho)$ so that $\delta/\sqrt{\rho} \to \infty$ as $\rho \to 0$, (for example, we may put $\delta = \rho^{1/3}$). By an explicit calculation,

$$\int_{B_\delta(a)\setminus B_{\sqrt{\rho}}(a)} |\nabla w_1|^2 = \frac{2\pi|(0,0,1) - \bar{u}(a)|^2}{|\log(\delta/\sqrt{\rho})|} \to 0 .$$

Moreover, there is c, independent of δ, for which

$$\int_{B_\delta(a)\setminus B_{\sqrt{\rho}}(a)} |\nabla w_1|^2 \leq c\, ||w_2||^2{}_{H^{1/2}(\partial B_\delta(a))}$$

$$\leq c\, ||w_2||^2{}_{L^2(\partial B_\delta(a))}\, ||w_2||_{H^1(\partial B_\delta(a))}$$

$$\leq c\delta^2 ,$$

and a similar result holds for w_3. Thus (7) follows, and hence v obeys (6). In view of (4), (5), and (6), the assertion **(1)** is proven.

(2) Consider a sequence of minimizers u_ρ of $E_{\rho,a}$ with $\rho \to 0$. As $E_{\rho,a}(u_\rho)$ is bounded there is a subsequence u_{ρ_n} converging weakly in $H^1_{loc}(B_1\setminus\{a\})$ to a limit $\hat{u}$ in $H^1(B_1)$ with $\hat{u} = u_0$ on ∂B_1. Fix $r > 0$ arbitrarily small. In view of a result of Schoen [1984, 2], u_{ρ_n} converges to $\hat{u}$ in the C^2-norm on every compact subset of

$B_1\backslash(B_r(a)\cap S)$ where S is a finite set, and it then follows from a result of Sacks & Uhlenbeck [1988, 1] that $\hat{u}$ is a smooth harmonic map on B_1. In view of Theorem A.1, proven in Appendix II, to show that $\hat{u}$ is a minimizing harmonic map on B_1, it suffices to show that its range is not in S^1. If, to the contrary, $\hat{u}(B_1)\subset S^1$, then $\hat{u}$ is a continuous map from B_1 to S^1 that equals the identity map on ∂B_1, which is impossible by the Brouwer fixed point theorem. As $\hat{u}$ is the unique minimizing S^2_+-valued harmonic map in $\mathcal{E}_1$, $\hat{u} = \bar{u}$, and thus we have subsequence u_{ρ_n} converging weakly H^1_{loc} to $\bar{u}$. By the uniqueness of $\bar{u}$, the full sequence converges weakly in H^1_{loc} to $\bar{u}$.

(3) To show that the number $L = \liminf E_{\rho,a}(u_\rho)$ is not less than 8π, we note that there is a sequence of minimizers u_{ρ_n} for which $L = \lim E_{\rho_n,a}(u_{\rho_n})$ and, by (2), $u_{\rho_n} \to \bar{u}$ weakly in $H^1_{\text{loc}}(B_1\backslash\{a\})$. Employing again the result of Schoen mentioned above, we can assert (after a second selection of a subsequence if necessary) that $u_{\rho_n} \to \bar{u}$ in the C^2-norm except at isolated points of $B_1\backslash\{a\}$. We now *fix* a circle Σ of small radius α centered at a with

$$\|u_{\rho_n} - \bar{u}\|_{L^\infty(\Sigma)} \to 0.$$

This can be done for arbitrary α, with the possible exception of a countable set of values of α. In the following discussion c denotes various constants independent of α.

As $|\bar{u}(x,y) - \bar{u}(a)| \leq c\alpha$ for all (x, y) in Σ, there holds

$$\|u_{\rho_n} - \bar{u}(a)\|_{L^\infty(\Sigma)} \leq c\alpha + o(1)\,. \tag{8}$$

Because $u_{\rho_n} \to \bar{u}$ weakly in $H^1(B_1 \setminus \bar{B}_\alpha(a))$,

$$\liminf E_{\alpha,a}(u_{\rho_n}) \geq E_{\alpha,a}(\bar{u}) \geq 4\pi - c\alpha^2\,. \tag{9}$$

In view of (8),

$$u_{\rho_n}(\Sigma) \subset B_{c\alpha + o(1)}(\bar{u}(a))\,.$$

From the observation that

$$u_{\rho_n}(x,y) = \left((x-a_1)/\rho_n,\ (y-a_2)/\rho_n,\ 0 \right) \quad \text{on } \partial B_{\rho_n}(a)$$

and a degree argument, it follows that if we put

$$\Gamma_n = B_\alpha(a) \setminus B_{\rho_n}(a)$$

then

$$u_{\rho_n}(\Gamma_n) \supset S^2_+ \setminus B_{c\alpha + o(1)}(\bar{u}(a))\,,$$

and this yields

$$\frac{1}{2}\int_{\Gamma_n} |\nabla u_{\rho_n}|^2 \geq \int_{\Gamma_n} \left| \frac{\partial u_{\rho_n}}{\partial x} \wedge \frac{\partial u_{\rho_n}}{\partial y} \right| = \text{area } u_{\rho_n}(\Gamma_n)$$

$$\geq 2\pi - c\alpha^2\,. \tag{10}$$

As $\Omega_{\rho_n}(a) = \Gamma_n \cup (B_1 \setminus B_\alpha(a))$, (9) and (10) yield

$$L = \lim_{n\to\infty} \int_{\Omega_{\rho_n}(a)} |\nabla u_{\rho_n}|^2 \geq 8\pi - c\alpha^2,$$

and as this holds for a set of α with 0 as a limit point, we conclude that $L \geq 8\pi$.

(4) To prove that $u_\rho \to \bar{u}$ uniformly on every compact subset of $B_1 \setminus \{a\}$, we show that $u_\rho \to \bar{u}$ strongly in $H^1_{\text{loc}}(B_1 \setminus \{a\})$ and employ a result of Schoen [1984, 2]. In view of (10)

$$\int_{\Gamma_n} |\nabla u_{\rho_n}|^2 \geq 4\pi - 2c\alpha^2$$

and by **(1)**,

$$\lim \int_{\Omega_{\rho_n}(a)} |\nabla u_{\rho_n}|^2 \leq 8\pi .$$

Thus,

$$\begin{aligned} \limsup_{\rho_n\to 0} \int_{B_1 \setminus B_\alpha(a)} |\nabla u_{\rho_n}|^2 &\leq 4\pi + 2c\alpha^2, \\ &= \int_{B_1} |\nabla \bar{u}|^2 + 2c\alpha^2 . \end{aligned}$$

Fix a compact subset K of $B_1 \backslash \{a\}$. For every $\alpha <$ dist (a, K) we have

$$\int_K |\nabla u_{\rho_n} - \nabla \bar{u}|^2 \le \int_{B_1 \backslash B_\alpha(a)} |\nabla u_{\rho_n} - \nabla \bar{u}|^2 .$$

Therefore,

$$\limsup_{\rho_n \to 0} \int_K |\nabla u_{\rho_n} - \nabla \bar{u}|^2 \le \int_{B_1} |\nabla \bar{u}|^2 + 2c\alpha^2 - \int_{B_1 \backslash B_\alpha(a)} |\nabla \bar{u}|^2$$

$$= \int_{B_\alpha(a)} |\nabla \bar{u}|^2 + 2c\alpha^2 .$$

Finally, we let α tend to zero and conclude that $u_{\rho_n} \to \bar{u}$ strongly in $H^1(K)$.

Remark 5. Rivière and Shafir [1991, 1] recently investigated the behavior of minimizers of $E_{\rho, a}$ as ρ approaches 0 and a does not remain constant but approaches the boundary of B_1. They show that min $E_{\rho, a}$ remains bounded and identify the limit in terms of lim $\rho/(\text{dist}\,(a, \partial B_1))$.

Appendix I. Proof of Theorem 0

Proof. Let

$$\alpha = \min_{v \in \mathcal{E}} \int_{\Omega} |\nabla v|^2 \, dx \, dy$$

be attained at v_0 in $\mathcal{E}$. For each u in $\mathcal{F}$, we have

$$\int_D |\nabla_{x,y} u|^2 \, dx \, dy \, dz \;\geq\; \alpha \;, \tag{I.1}$$

and since we can choose u in $\mathcal{F}$ so that $u(x, y, z) = v_0(x, y)$, we have

$$\alpha = \min_{u \in \mathcal{F}} \int_D |\nabla u|^2 \, dx \, dy \, dz \;.$$

Moreover, if a given u in $\mathcal{F}$ is a minimizer, *i.e.*, obeys

$$\int_D |\nabla u|^2 \, dx \, dy \, dz = \alpha \;,$$

then, by (I.1),

$$\int_D |\nabla_{x,y} u|^2 \, dx \, dy \, dz \;\geq \alpha \;,$$

which yields $\partial u/\partial z = 0$ and proves the theorem.

Remark 6. Sandier [1991, 3] has recently shown that if one replaces the energy,

$$E(u) = \int_D |\nabla u|^2 \, dx\, dy\, dz \ ,$$

by the Oseen-Frank energy of liquid-crystal theory, *i.e.*, by

$$\hat{E}(u) = \int_D \left(k_1(\text{div } u)^2 + k_2(u \cdot \text{curl } u)^2 + k_3|u \wedge \text{curl}|^2\right) dx\, dy\, dz$$

with the k_i positive constants, then the conclusion of Theorem 0 does not hold when $k_2 = k_3$. In particular, if $k_1 = k_2 = k_3 = 1$, then minimizers of $\hat{E}$ depend on z. This does not contradict Theorem 0, for in the equiconstant case $\hat{E}$ reduces to

$$\hat{E}(u) = E(u) - \int_D \text{div}\left((\nabla u)u - (\text{div } u)u\right) dx\, dy\, dz \ ,$$

where the added integral depends only on the values of u on the boundary of D. For the minimization problem considered here and by Sandier, values of u are not prescribed on the surfaces $z = 0$ and $z = 1$, and thus the minimizers of E and $\hat{E}$ differ.

Appendix II. Harmonic Maps with Values in S^2_+

Let Ω be a smooth bounded domain in $\mathbb{R}^2$.

Theorem A.1. *Let u be a harmonic map from Ω into S^2 with $u(\Omega) \subset S^2_+$. If $u(\Omega)$ is not contained in S^1, then u is minimizing.*

Remark 7. When Ω is not simply connected, the assumption that $u(\Omega)$ is not contained in S^1 is essential. There are harmonic maps $u : \Omega \to S^2$ which do not minimize the energy in $H^1(\Omega; S^1)$ and therefore in $H^1(\Omega; S^2)$. To have an example, let $\Omega = B_1 \setminus \bar{B}_{1/2}$ and

$$u(x, y) = (\cos f(x, y),\ \sin f(x, y),\ 0).$$

This map $u : \Omega \to S^1$ is harmonic whenever $\Delta f = 0$. If one chooses for f the boundary conditions

$$\left.\begin{aligned} f &= 0 \quad \text{on } \partial B_1, \\ f &= 2\pi k \quad \text{on } \partial B_{1/2}, \end{aligned}\right\}$$

with k a nonzero integer, then $|\nabla u|$ is not zero on Ω, *i.e.*, the energy corresponding to u is positive. But the minimum energy subject to the condition that u equal the constant $(1, 0, 0)$ on ∂B_1 and $\partial B_{1/2}$ is zero, and hence u is not minimizing in $H^1(\Omega; S^1)$.

Corollary to Theorem A.1 *If u is a harmonic map from B^1 to S^2 that reduces to the identity map on ∂B_1 and has its range, $u(B_1)$, in S^2_+, then u is the stereographic projection with vertex* $(0, 0, -1)$, *i.e.*, $u = \bar{u}$ *where*

$$\bar{u}(x, y) = \frac{2}{1 + r^2} (x, y, 1) + (0, 0, -1) .$$

Proof. As $\bar{u}$ is the unique S^2_+-valued minimizer of the energy in the class $\mathcal{E}_1$, *i.e.*, in the class of H^1-functions that reduce to the identity on ∂B_1, it here suffices to show that u is a minimizer. By Theorem A.1, it suffices to check that $u(B_1)$ is not contained in S^1. Suppose, to the contrary, that $u(B_1) \subset S^1$; then, as u can be deformed continuously into a constant map, the degree of $u|_{\partial B_1}$ is zero, which is impossible, since u is the identity map on ∂B_1.

Remark 8.[#] It is an open problem whether an arbitrary harmonic map u from

[#] See [1983, 1].

B^1 to S^2 that reduces to the identity on ∂B_1 must equal either $\bar{u}$ or $\underline{u} = (\bar{u}_1, \bar{u}_2, -\bar{u}_3)$.

Proof of Theorem A.1. In view of a result of Hélein [1990, 1], we know that u is smooth in Ω. We have

$$\Delta u_3 = u_3 |\nabla u|^2 \geq 0 \quad \text{on } \Omega,$$

and $u_3 \geq 0$ on Ω. As $u_3 \not\equiv 0$, it follows from the maximum principle that $u_3 > 0$ in Ω.

Let

$$\Omega_\delta = \{ (x, y) \in \Omega;\ \text{dist}\,((x, y), \partial\Omega) > \delta \} .$$

Let u° be a minimizer for

$$\min_{\substack{\mathrm{v} \in H^1(\Omega; S^2) \\ \mathrm{v} = u \text{ on } \partial\Omega}} \int |\nabla \mathrm{v}|^2 .$$

We shall prove that $\int |\nabla u|^2 = \int |\nabla u^\circ|^2$.

Clearly,

$$\|u - u^\circ\|_{H^{1/2}(\partial\Omega_\delta)} \to 0 \quad \text{as } \delta \to 0 . \tag{II.1}$$

Let φ_δ be the solution of

$$\left.\begin{aligned} \Delta \varphi_\delta &= 0 \quad \text{in } \Omega_\delta , \\ \varphi_\delta &= u - u^\circ \quad \text{on } \partial\Omega_\delta . \end{aligned}\right\}$$

By (II.1),

$$\int_{\Omega_\delta} |\nabla \varphi_\delta|^2 \;\to\; 0\,. \tag{II.2}$$

Set

$$u_\delta \;=\; \frac{u^\circ + t\varphi_\delta}{|u^\circ + t\varphi_\delta|} \quad \text{in } \Omega_\delta .$$

As $u_\delta = u$ on $\partial\Omega_\delta$, and $u(\Omega_\delta)$ is in a compact subset of S^2_+, it follows from a result of Jäger & Kaul [1979, 1] that

$$\int_{\Omega_\delta} |\nabla u_\delta|^2 \;\geq\; \int_{\Omega_\delta} |\nabla u\,|^2.$$

By (II.2),

$$\int_{\Omega_\delta} |\nabla u_\delta|^2 \;\to\; \int_{\Omega} |\nabla u^\circ|^2 \quad \text{as } \delta \to 0$$

and hence

$$\int_{\Omega} |\nabla u|^2 \leq \int_{\Omega} |\nabla u^\circ|^2,$$

which, as u° is a minimizer, proves our assertion.

Appendix III. Uniqueness and Nonuniqueness of Minimizers

Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain and let $\varphi : \partial\Omega \to S^2_+$ be given.

Consider the problem of finding

$$\min_{u \in H^1_\varphi} \int |\nabla u|^2 , \qquad \text{(III.1)}$$

where

$$H^1_\varphi = \{ u \in H^1(\Omega; S^2);\ u = \varphi \text{ on } \partial\Omega \} .$$

Note that, by the maximum principle, each minimizer takes all its values in S^2_+ (or in S^2_- if $u(\partial\Omega) \subset S^1$). Let u be a minimizer for (III.1) such that $u(\Omega) \subset S^2_+$.

Theorem A.2 *If $u(\Omega)$ is not a subset of S^1, we have the following alternative.*

Either

(*i*) *u is a strict local minimizer in which case it is the unique minimizer,*

or

(*ii*) *or there is a continuum of minimizers containing u.*

Remark 9. Both cases occur. Here are two examples:

1. $\Omega = B_1$ and $\varphi(x, y) = (x, y, 0)$. Then in accord with the discussion given early in Section 2, the unique minimizer is $u = \bar{u}$.

2. $\Omega = B_1 \setminus \overline{B}_{1/2}$ and φ is constant on each component of the boundary, namely

$$\left.\begin{array}{lll} \varphi = (1,0,0) & \text{on} & \partial B_1, \\ \varphi = (-1,0,0) & \text{on} & \partial B_{1/2}. \end{array}\right\}$$

In this case, we have a family of minimizers. If f is a harmonic function on Ω with

$$\left.\begin{array}{lll} f = 0 & \text{on} & \partial B_1, \\ f = \pi & \text{on} & \partial B_{1/2}, \end{array}\right\}$$

then

$$u_\theta = (\cos f, \ \sin f \cos \theta, \ \sin f \sin \theta)$$

is a minimizer for every θ in $(0, \pi)$.

Proof of Theorem A.2. We use the notation of the proof of Theorem A.1. It suffices to show that if there is another minimizer u_1 (besides u), u_1 and u can be connected by a branch of minimizers. Let

$$u_\delta = \frac{u_1 + \varphi_\delta}{|u_1 + \varphi_\delta|} \quad \text{on } \Omega_\delta .$$

Note that $u_\delta = u$ on $\partial\Omega_\delta$. The flow,

$$\left.\begin{aligned} \frac{\partial \mathrm{v}}{\partial t} - \Delta \mathrm{v} &= \mathrm{v}|\nabla \mathrm{v}|^2 \quad \text{in } \Omega_\delta \times [0,\infty), \\ \mathrm{v}(x,0) &= u_\delta(x) \quad \text{on } \Omega_\delta , \\ \mathrm{v}(x,t) &= u(x) \quad \text{on } \partial\Omega_\delta \times [0,\infty), \end{aligned}\right\}$$

has a smooth global solution, as $u_\delta(\Omega_\delta)$ is contained in a compact subset of S^2_+ (see, e.g., [1982, 1] [1985, 1] [1989, 1].) Moreover, as $t \to \infty$, $\mathrm{v}(t) \to u|_{\Omega_\delta}$ in $H^1(\Omega_\delta)$; indeed u restricted to Ω_δ is the only harmonic map that has values in S^2_+ and the same boundary data. A standard estimate gives

$$E(\mathrm{v}(T_2)) + 2\int_{T_1}^{T_2} \left| \frac{\partial \mathrm{v}}{\partial t} \right|^2 \leq E(\mathrm{v}(T_1)), \qquad T_1 \leq T_2 ,$$

where all integrals are evaluated on Ω_δ. Thus, for all $t > 0$,

$$E(u) \leq E(\mathrm{v}(t)) \leq E(u_\delta) . \tag{III.2}$$

Clearly we have, as $\delta \to 0$,

$$\|u - u_\delta\|_{H^1(\Omega_\delta)} = \alpha(\delta) \to \|u - u_1\|_{H^1(\Omega)} = \alpha > 0 .$$

Fix ε such that $0 < \varepsilon < \alpha$; for δ sufficiently small, there is, by continuity, a $T = T(\varepsilon, \delta)$ such that

$$\|u - \mathrm{v}(T)\|_{H^1(\Omega_\delta)} = \varepsilon . \qquad \text{(III.3)}$$

Note that as $\delta \to 0$ (at fixed ε), we have by (III.2)

$$E(\mathrm{v}(T)) \to E(u) ,$$

and $\mathrm{v}(T) \to \mathrm{v}_\varepsilon$ weakly in H^1 for a subsequence $\delta_n \to 0$ and some $\mathrm{v}_\varepsilon \in H^1$. The assertion that

$$\mathrm{v}(T) \to \mathrm{v}_\varepsilon \text{ strongly in } H^1$$

follows from the observation that

$$E(\mathrm{v}(T)) \to E(u) \leq E(\mathrm{v}_\varepsilon) .$$

By passing to the limit of small δ in (III.2), we obtain

$$E(\mathrm{v}_\varepsilon) = E(u) ,$$

and hence v_ε is a minimizer; by passing to the same limit in (III.3), we find that

$$\|u - \mathrm{v}_\varepsilon\|_{H^1} = \varepsilon .$$

We now present an application of Theorem A.2: There is an $\alpha < e^{-\pi}$ such that the minimizer in Theorem 1 is unique for every $\rho \in [\alpha, e^{-\pi})$.

It suffices to check that the second variation of the energy is positive definite at the (unique) radial minimizer. Let u be the radial minimizer in $\Omega = B_1 \setminus \overline{B_\rho}$, and write

$$u(r\cos\theta,\ r\sin\theta) = (\cos\theta\cos\varphi(r),\ \sin\theta\cos\varphi(r),\ \sin\varphi(r)).$$

Let u_t be a perturbation of u of the form

$$u_t(r\cos\theta, r\sin\theta) = (\cos(\theta + t\tau)\cos(\varphi + t\psi),\ \sin(\theta + t\tau)\cos(\varphi + t\psi),\ \sin(\varphi + t\psi)),$$

where $\tau(r, \theta)$ and $\psi(r, \theta)$ are smooth admissible perturbations, *i.e.*,

$$\tau(1, \theta) = \tau(\rho, \theta) = \psi(1, \theta) = \psi(\rho, \theta) = 0.$$

We have

$$E(u_t) = E(u) + t^2 \int_\rho^1 \int_0^{2\pi} \left[|\nabla\psi|^2 - \frac{\psi^2}{r^2}\cos 2\varphi + |\nabla\tau|^2\cos^2\varphi - \frac{2}{r^2}\psi\frac{\partial\tau}{\partial\theta}\sin 2\varphi \right] r\,dr\,d\theta + o(t^2);$$

that is,

$$\delta^2 E = \int_\rho^1 \int_0^{2\pi} \left[|\nabla \psi|^2 - \frac{\psi^2}{r^2}\cos 2\varphi + |\nabla \tau|^2 \cos^2 \varphi - \frac{2}{r^2}\psi \frac{\partial \tau}{\partial \theta}\sin 2\varphi \right] r\,dr\,d\theta .$$

On taking $t = \log r$ as a new variable we obtain

$$\delta^2 E = \int_{\log \rho}^{0} \int_0^{2\pi} \left[|\nabla \psi|^2 - \psi^2 \cos 2\varphi + |\nabla \tau|^2 \cos^2\varphi - 2\psi \frac{\partial \tau}{\partial \theta} \sin 2\varphi \right] dt\,d\theta .$$

To show that for an α in $(0, e^{-\pi})$ (to be determined later), $\delta^2 E$ is positive definite for each ρ in $(\alpha, e^{-\pi})$, we split $\delta^2 E$ into three integrals:

$$\delta^2 E = I + II + III ,$$

where

$$I = -\int_0^{2\pi} \int_{\log \rho}^{0} 2\psi \frac{\partial \tau}{\partial \theta} \sin 2\varphi\, dt\, d\theta ,$$

$$II = \int_0^{2\pi} \int_{\log \rho}^{0} \left[\left(\frac{\partial \psi}{\partial \theta}\right)^2 + |\nabla \tau|^2 \cos^2\varphi \right] dt\, d\theta ,$$

$$III = \int_0^{2\pi} \int_{\log \rho}^{0} \left[\left(\frac{\partial \psi}{\partial t}\right)^2 - \psi^2 \cos 2\varphi \right] dt\, d\theta .$$

Put

$$\bar{\psi}(t) = \frac{1}{2\pi} \int_0^{2\pi} \psi(t, \theta)\, d\theta .$$

We have

$$I = - \int_0^{2\pi} \int_{\log \rho}^{0} 2(\psi - \bar{\psi}) \frac{\partial \tau}{\partial \theta} \sin 2\varphi \, dt\, d\theta$$

$$= -4 \int_0^{2\pi} \int_{\log \rho}^{0} \left[(\psi - \bar{\psi}) \sin \varphi \right] \left[\frac{\partial \tau}{\partial \theta} \cos \varphi \right] dt\, d\theta$$

$$\geq - \int_0^{2\pi} \int_{\log \rho}^{0} \left[4(\psi - \bar{\psi})^2 \sin^2 \varphi + \left(\frac{\partial \tau}{\partial \theta} \right)^2 \cos^2 \varphi \right] dt\, d\theta .$$

We now choose α such that if $\alpha < \rho < e^{-\pi}$ then for $\log \rho \leq t \leq 0$,

$$\sin \varphi(t) \leq \frac{1}{2} .$$

We obtain

$$I \geq - \int_0^{2\pi} \int_{\log \rho}^{0} \left[(\psi - \overline{\psi})^2 + \left(\frac{\partial \tau}{\partial \theta}\right)^2 \cos^2 \varphi \right] dt\, d\theta$$

$$\geq - \int_0^{2\pi} \int_{\log \rho}^{0} \left[\left(\frac{\partial \psi}{\partial \theta}\right)^2 + \left(\frac{\partial \tau}{\partial \theta}\right)^2 \cos^2 \varphi \right] dt\, d\theta$$

and

$$I + II \geq \int_0^{2\pi} \int_{\log \rho}^{0} \left(\frac{\partial \tau}{\partial \theta}\right)^2 \cos^2 \varphi \, dt\, d\theta \, .$$

From the choice of α it follows that $\cos^2 \varphi \geq 3/4$. Hence,

$$\delta^2 E \geq \frac{3}{4} \int_0^{2\pi} \int_{\log \rho}^{0} \left(\frac{\partial \tau}{\partial t}\right)^2 dt\, d\theta + \int_0^{2\pi} \int_{\log \rho}^{0} \left[\left(\frac{\partial \psi}{\partial t}\right)^2 - \psi^2 \cos 2\varphi \right] dt\, d\theta \, . \qquad \text{(III.4)}$$

As the second integral in the right hand side of (III.4) is precisely the second variation of the energy E when u is restricted to the set of radially symmetric maps in $\mathcal{E}_0$, our conclusion follows from (III.4) and from the fact that the second variation of

$$E_0(\varphi) = 2\pi \int_{\log \rho}^{0} \left[\left(\frac{d\varphi}{dt}\right)^2 - \sin^2 \varphi \right] dt$$

is positive definite at a minimizer, which is a consequence of the concavity of the function $\psi \mapsto \sin\psi$ for $\psi \in [0, \pi]$ (e.g., [1992,2]).

Remark 10. The above argument fails when $\rho = e^{-\pi}$, because $\delta^2 E$ is not positive definite. To see this, note that for $(\tau, \psi) = (0, \sin t)$, $\delta^2 E = 0$.

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