PHYSICAL REVIEW LETTERS

VOLUME 44 | 28 APRIL 1980 | NUMBER 17

Bifurcation and Symmetry Breaking in Nonlinear Dispersive Waves

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(Received 8 January 1980)

The equation governing four-wave interactions in a nonlinear dispersive system is studied. It is shown that a nonlinear steady-state plane wave can bifurcate into non-planar steady-state solutions. In the case of an isotropic medium, the bifurcation is degenerate and the bifurcated solutions may preserve or break the symmetry. An example is given of a symmetry-breaking solution for deep-water gravity waves and its stability is discussed.

PACS numbers: 03.40.Kf, 47.35.+i

The nonlinear equation

$$i\frac{\partial B(\vec{k},t)}{\partial t}=\iiint_{-\infty}^{\infty} T(\vec{k},\vec{k}_1,\vec{k}_2,\vec{k}_3)\delta(\vec{k}+\vec{k}_1-\vec{k}_2-\vec{k}_3)\exp\{i[\omega(\vec{k})+\omega(\vec{k}_1)-\omega(\vec{k}_2)-\omega(\vec{k}_3)]t\}\times B^*(\vec{k}_1,t)B(\vec{k}_2,t)B(\vec{k}_3,t)\,d^3k_1d^3k_2d^3k_3, \tag{1}$$

governs the time evolution of spectral components of a weakly nonlinear system in which four-wave interactions dominate in the time scale under consideration. The linear frequency $\omega(\vec{k})$ and the real interaction coefficient $T(\vec{k},\vec{k}_1,\vec{k}_2,\vec{k}_3)$ characterize the properties of the system of interest. Equation (1) describes a wide range of physical phenomena, such as plasmon interactions, spin waves in isotropic ferromagnets, and gravity waves on deep water, and is analogous to the Heisenberg equations of motion for a Bose gas, with B and B^* the classical analogs of quantum annihilation and creation operators.[1] For example,[2] the free surface $\eta(\vec{x},t)$ of weakly nonlinear deep-water gravity waves is given in terms of the spectral components $B(\vec{k},t)$ by

$$\eta(\vec{x},t)=\frac{1}{2\pi}\int\frac{|\vec{k}|^{1/4}}{2^{1/2}g^{1/4}}[B(\vec{k},t)e^{i(\vec{k}\cdot\vec{x}-\omega t)}+B^*(\vec{k},t)e^{-i(\vec{k}\cdot\vec{x}-\omega t)}]d^3k \tag{2}$$

where g is the acceleration due to gravity and ω is the frequency from linear theory,

$$\omega(\vec{k})=(g|\vec{k}|)^{1/2}. \tag{3}$$

The interaction coefficient T is real and given in Ref. 2.

Equation (1) admits exact solutions of the form

$$B(\vec{k},t)=a\delta(\vec{k}-\vec{k}_0)e^{-i\Omega(\vec{k})t}, \tag{4}$$

where a is real and

$$\Omega(\vec{k}) = T(\vec{k},\vec{k},\vec{k},\vec{k})a^2. \tag{5}$$

These correspond to nonlinear, steady-state, uniform, plane waves of wave vector $\vec{k}_0$ with a nonlinearly shifted frequency. We shall demonstrate that under appropriate conditions, this solution can bifurcate into nonplanar steady-state solutions at critical amplitudes a_0 dependent on an arbitrary modulational wave vector $\vec{p}$.

Consider the nonplanar steady solution propagating with the yet undetermined velocity $\vec{C} = C\vec{k}_0/|\vec{k}_0|$, described by

$$B(\vec{k},t) = \sum_{\vec{n}} a_{\vec{n}}\delta(\vec{k} - \vec{l}(\vec{n}))e^{i\omega(\vec{k})t}e^{-i\vec{k}\cdot\vec{C}t}, \tag{6}$$

where

$$\vec{l}(\vec{n}) = \vec{k}_0 + \vec{n}:\vec{p}; \tag{7}$$

$\vec{n}$ is a vector whose components run over all integers and $\vec{n}:\vec{p}$ is the dyadic product. This solves Eq. (1) provided that

$$\{\omega[\vec{l}(\vec{n})] - \vec{l}(\vec{n})\cdot\vec{C}\}a_{\vec{n}} + \sum_{\vec{n}+\vec{i}=\vec{j}+\vec{m}} T_{\vec{n},\vec{i},\vec{j},\vec{m}}\, a_{\vec{i}}^{\,*} a_{\vec{j}} a_{\vec{m}} = 0 \tag{8}$$

for all $\vec{n}$, where the summation is taken over all integer vectors $\vec{i}$, $\vec{j}$, and $\vec{m}$ satisfying $\vec{n}+\vec{i}=\vec{j}+\vec{m}$, with the notation $T_{\vec{n},\vec{i},\vec{j},\vec{m}} = T(\vec{l}(\vec{n}),\vec{l}(\vec{i}),\vec{l}(\vec{j}),\vec{l}(\vec{m}))$. The plane-wave solution is

$$\vec{C} = [\omega(\vec{k}_0) + \Omega(\vec{k}_0)]\vec{k}_0/|\vec{k}_0|^2,$$

and $a_{\vec{n}} = a$ for $\vec{n}=0$ and zero otherwise. Bifurcation can occur if the Fréchet derivation of Eq. (8) with respect to the vector $\{a_{\vec{n}}\}$ is singular.

A necessary condition for bifurcation to occur for the plane wave is that the following determinant should vanish for some particular $\vec{n}$:

$$\begin{vmatrix} \omega[\vec{l}(\vec{n})] - \vec{l}(\vec{n})\cdot\vec{C} + 2a^2T_{\vec{n},0,\vec{n},0} & a^2T_{\vec{n},-\vec{n},0,0} \\ a^2T_{-\vec{n},\vec{n},0,0} & \omega[\vec{l}(-\vec{n})] - \vec{l}(-\vec{n})\cdot\vec{C} + 2a^2T_{-\vec{n},0,-\vec{n},0} \end{vmatrix} = 0. \tag{9}$$

The condition is also sufficient if $\vec{C}$ is not stationary as a function of a.[3] The roots of this equation determine a_0 and the corresponding speed C_0 for a given $\vec{p}$ and $\vec{n}$. At the critical amplitude, neighboring solutions exist with the additional components

$$a_{\vec{n}} = a_{\vec{n}}^{\,*} = \epsilon\lambda_{\vec{n}},$$
$$a_{-\vec{n}} = a_{-\vec{n}}^{\,*} = \epsilon, \tag{10}$$

where

$$\lambda_{\vec{n}} = \frac{-T_{\vec{n},-\vec{n},0,0}a_0^2}{\omega[\vec{l}(\vec{n})] - \vec{l}(\vec{n})\cdot\vec{C}_0 + 2a_0^2T_{\vec{n},0,\vec{n},0}} = 1/\lambda_{-\vec{n}} \tag{11}$$

and $\epsilon \ll 1$.

If the medium is isotropic, the bifurcation is degenerate because rotation of $\vec{n}$ about the direction of propagation leaves Eqs. (9) invariant. In this case, the bifurcated solutions may either preserve or break the symmetry. To illustrate this phenomenon, we show results for two-dimensional gravity waves on deep water for which the dispersion relation Eq. (3) is isotropic in two-dimensional $\vec{k}$ space.

In Fig. 1, we show the critical wave steepness $|\vec{k}_0|a_0/\sqrt{2}\,\pi$ versus direction of the modulation wave vector $\vec{p}$ for various values of $\vec{p}\cdot\vec{k}_0/|k_0|^2$. It is interesting to note that for this case, bifurcation can occur at very small amplitudes if the modulational wave vector is sufficiently oblique. The bifurcation is degenerate since $\vec{n}$ can be (1, 1) or (1, −1) for the same value of critical wave steepness a. For a given modulational wave vector $\vec{p}$, three branches of two-dimensional solutions start from the critical amplitude, corresponding to the ratio $r = a_{(1,1)}/a_{(1,-1)}$ taking the values 1, 0, or ∞. The case $r = 1$ represents two-dimensional waves symmetrical about the direction of propagation. The cases $r = 0$ and $r = \infty$ represent skewed waves asymmetrical about the direction of propagation (as mirror images of each other), and describe states with broken symmetry.

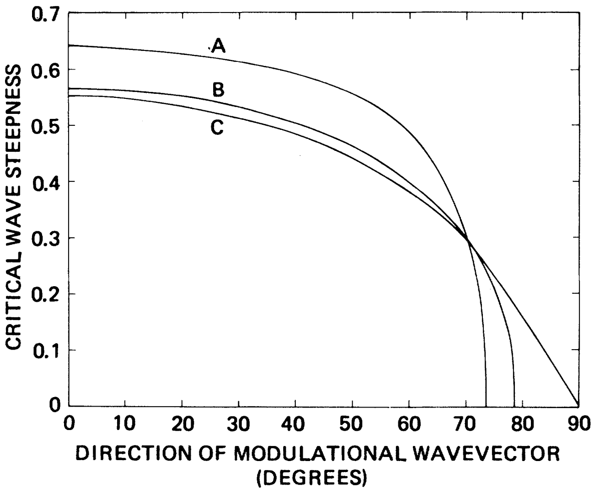

FIG. 1. Critical amplitude for bifurcation of deep-water gravity waves. Critical wave slope vs modulation wave-vector direction for three values of $\vec{p}\cdot\vec{k}_0/|\vec{k}_0|^2$: 0.5 (curve A), 0.1 (curve B), 10^{-5} (curve C).

For the skewed waves, only those terms with $\vec{n}=(n,n)$ or $\vec{n}=(n,-n)$ contribute. A truncated version of Eq. (6) was solved by Newton's method to give the coefficients $a_{\vec{n}}$ on the skewed-wave branch. An example of the free surface given by Eq. (2) associated with such a solution is shown in Fig. 2 for various values of $a_{(1,1)}$. Observations made by Su[4] strongly suggest that these waves do exist in nature. However, we do not know at this point what selection rules pick the particular values of $\vec{p}$ for those waves seen in nature.

The existence of bifurcation is associated with the existence of neutrally stable disturbances for the plane waves. Nonneutral distrubances, whether stable or unstable, will not change their characters as the solution moves to the new branches. We investigated the possibility that the neutral distrubances, which are stable for plane waves, could become unstable on the new branch. This requires solution of the eigenvalue problem for disturbances $v_{\vec{n}}e^{i\sigma t}$ to $a_{\vec{n}}$ and $w_{\vec{n}}e^{i\sigma t}$ to $a_{\vec{n}}{}^*$ given by

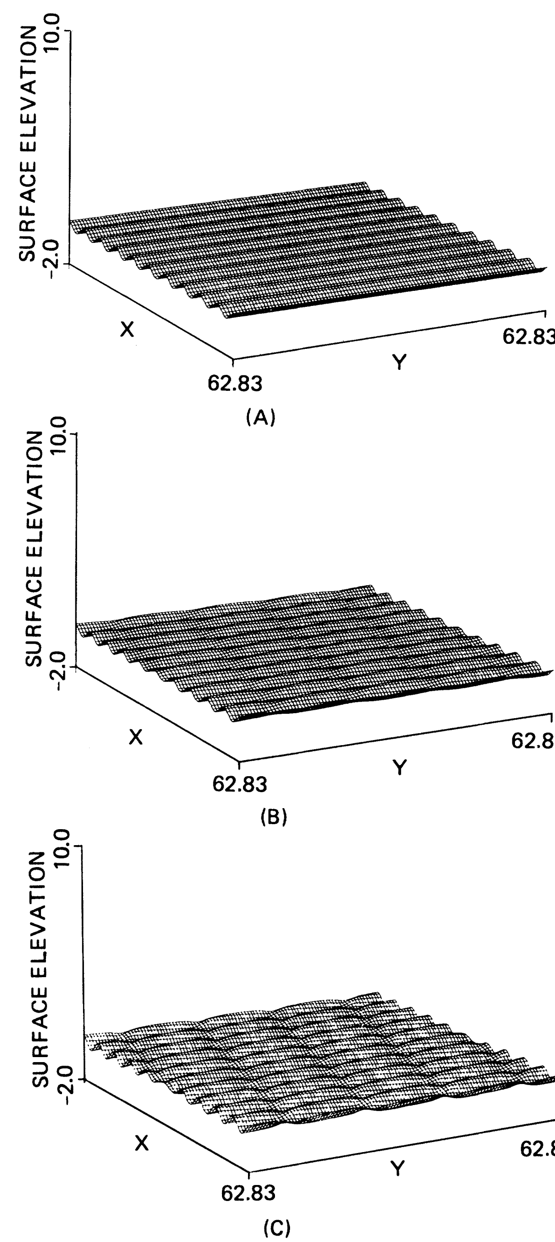

FIG. 2. Example of free surface associated with a skewed, bifurcated solution for deep-water gravity waves with $\vec{p}=(0.1,0.4)\vec{k}_0$ and $a_0=0.9606$: (a) $a_{(1,1)}/a_0$ [plane wave], (b) $a_{(1,1)}/a_0=0.2$, and (c) $a_{(1,1)}/a_0=0.4$. This choice of $\vec{p}$ gives modulations at an angle $\tan^{-1}4=76°$ with the direction of wave propagation.

$$\begin{aligned}&\{\sigma+\omega[\vec{l}(\vec{n})]-\vec{l}(\vec{n})\cdot\vec{C}\}v_{\vec{n}}+2\sum T_{\vec{n},\vec{l},\vec{j},\vec{m}}a_{\vec{l}}{}^*a_{\vec{j}}v_{\vec{m}}+\sum T_{\vec{n},\vec{l},\vec{j},\vec{m}}a_{\vec{j}}a_{\vec{k}}w_{\vec{l}}=0,\\&\{-\sigma+\omega[\vec{l}(\vec{n})]-\vec{l}(\vec{n})\cdot\vec{C}\}w_{\vec{n}}+2\sum T_{\vec{n},\vec{l},\vec{j},\vec{m}}a_{\vec{l}}{}^*a_{\vec{j}}w_{\vec{m}}+\sum T_{\vec{n},\vec{l},\vec{j},\vec{m}}a_{\vec{j}}a_{\vec{k}}v_{\vec{l}}=0.\end{aligned} \qquad (12)$$

It was found by analysis and numerical method that the skewed-wave branch for deep-water waves shown here does not exhibit exchange of stability and remains stable. Details will be published in a forthcoming paper.

[1]V. E. Zakharov, Zh. Eksp. Teor. Fiz. 51, 688 (1966) [Sov. Phys. JETP 24, 455 (1967)]; V. E. Zakharov, Zh. Eksp. Teor. Fiz. 51, 1107 (1966) [Sov. Phys. JETP 24, 740 (1967)]; V. E. Zakharov, V. S. L'vov, and S. S. Sarobrinets, Fiz. Tverd. Tela. 11, 2047 (1970) [Sov. Phys. Solid State 11, 2368 (1970)].

[2]D. R. Crawford, P. G. Saffman, and H. C. Yuen, Wave Motion 2, No. 2, 1 (1980).

[3]D. H. Sattinger, *Topics in Stability and Bifurcation Theory*, Lecture Notes in Mathematics No. 309 (Springer-Verlag, New York, 1973). The vanishing of the Fréchet derivative is necessary but not in general sufficient for bifurcation to occur as the vanishing is also a property of limit-point behavior. When $\vec{C}$ is not stationary the latter is not the case.

[4]M. Y. Su, private communication.

Spin $\frac{1}{2}$ from Gravity

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For a certain class of three-manifolds, the angular momentum of an asymptotically flat quantum gravitational field can have half-integral values.

PACS numbers: 04.60.+n, 04.20.Cv

There is a common expectation that at sufficiently small distances the topology of space-time is not Minkowskian.[1,2] Because such configurations cannot be reached by perturbing flat space, it is by no means obvious that the spin-2 character of linearized gravity will persist nor that the quantum gravitational field must in general have integral spin; and the possibility of spinorial manifolds was in fact suggested more than two decades ago by Finkelstein and Misner.[3] The present work may be regarded as a confirmation of their conjecture that gravity by itself can exhibit half-integral spin. We find in particular that the possible topologies of three-manifolds fall into two classes, those (including R^3, the topologically trivial space) which allow only integral spin, and those which give rise to a space of state vectors having both half-integral and integral spin sectors.

Our construction is somewhat analogous to the appearance of half-integral spin in the quantum mechanics of systems containing both magnetic and electric charges and in the constructions by Jackiw and Rebbi[4] and Hasenfratz and 't Hooft[5] of half-integral spin solitons from coupled Yang-Mills-Higgs and isospinor fields: In these cases and for us, the emergence of spin $\frac{1}{2}$ depends on a configuration space which, because of a gauge degree of freedom, is larger than the space of physical configurations, and on which a 2π rotation can therefore act nontrivially. A basic *difference* is that gravitational spin $\frac{1}{2}$ requires neither source fields nor charges [neither isospinors nor U(1) spinors].

We will work in the Schrödinger or "superspace" picture[6] (This is unrelated to supergravity: Superspace is a space of three-metrics in which one regards as equivalent metrics that differ only by a diffeomorphism, i.e., whose components are related by a coordinate transformation), taking as elements of configuration space asymptotically flat positive-definite metrics g_{ab} on a fixed three-manifold M. The Schrödinger state vector ψ is then a functional on this space of metrics. We begin by describing the kinematics of a theory of quantum gravity on an asymptotically flat space: We introduce the constraint equations and note that they are equivalent to demanding invariance of the wave function ψ under asymptotically trivial diffeomorphisms which are in the component of the identity. Next, we recall the classical Arnowitt-Deser-Misner (ADM) angular momentum[7] and obtain the corresponding quantum operators. We thereby acquire a representation of the rotation group SO(3) [or of its covering group SU(2)] on the space of wave functions, ψ, and so can formally ask whether half-integral spin occurs. We find that for Euclidean

