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BIFURCATION FROM INFINITY AND MULTIPLE SOLUTIONS FOR AN ELLIPTIC SYSTEM

RAFFAELE CHIAPPINELLI
Dipartimento di Matematica, Università degli Studi della Calabria
87036 Arcavacata di Rende, Italy

DJAIRO G. DE FIGUEIREDO†
IMECC-UNICAMP, Caixa Postal 6065, 13081 Campinas, S.P., Brazil

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Abstract. In this paper, we study multiplicity of solutions for a system of semilinear elliptic equations of the form

$$-\Delta u = \lambda u + f(x,u) - v$$
$$-\Delta v = \delta u - \gamma v$$

in some bounded smooth domain in $\mathbb{R}^N$, subject to homogeneous Dirichlet boundary conditions. The parameters δ and γ are positive and satisfy certain relations involving also the first eigenvalue λ_1 of $(-\Delta_0, H^1(\Omega))$. The parameter λ varies in a neighborhood of $\hat{\lambda}_1 := \lambda_1 + \delta/(\gamma+\lambda_1)$. We establish a priori bounds for solutions of the system when λ is an appropriate side of $\hat{\lambda}_1$, depending on the behavior of $f(x,s)$ and $s \to \pm\infty$. These bounds, together with a bifurcation from infinity, gives the multiplicity results.

Introduction. Let Ω be a bounded open subset of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. Consider the semilinear elliptic system depending on the real parameter λ

$$(S_\lambda) \qquad \begin{cases} -\Delta u = \lambda u + f(x,u) - v \\ -\Delta v = \delta u - \gamma v \end{cases} \qquad \text{in } \Omega$$

subject to Dirichlet boundary conditions $u = v = 0$ on $\partial\Omega$; here $f = f(x,s)$ is a real-valued continuous function on $\overline{\Omega} \times \mathbb{R}$ and γ, δ are nonnegative constants. The solutions (u,v) of (S_λ) represent steady-state solutions of reaction-diffusion systems of interest in Biology, see e.g., Rothe [14] and Lazer-McKenna [9].

The non-parametric system S_0 ($\lambda = 0$) was studied among others by De Figueiredo-Mitidieri [5], who proved the existence of one or even two [pairs (u,v) of] solutions under various assumptions on f, using both monotone iteration techniques and variational methods. In this paper, we study existence and multiplicity of solutions to (S_λ) when λ is near $\hat{\lambda}_1$,

$$\hat{\lambda}_1 := \lambda_1 + \frac{\delta}{\gamma + \lambda_1}$$

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with λ_1 the first eigenvalue of $-\Delta$ in Ω subject to zero Dirichlet boundary conditions. As it will be clear from the sequel, $\hat{\lambda}_1$ plays the role of first eigenvalue of the linear system ($f \equiv 0$) associated with (S_λ).

In order to set our problem in some more detail, observe that if $\delta = 0$, then $v = 0$ and (S_λ) thus reduces to the scalar problem

$$(P_\lambda^0) \qquad -\Delta u = \lambda u + f(x,u) \quad \text{in } \Omega, \quad u = 0 \quad \partial\Omega.$$

The existence of solutions of (P_λ^0) for λ near $\hat{\lambda}_1$ ($= \lambda_1$ in this case) was first proved by Landesman-Lazer —for a bounded f— under the classical conditions

$$\int_\Omega f^-(x)\varphi(x)\,dx < 0 < \int_\Omega f_+(x)\varphi(x)\,dx \tag{1}$$

or

$$\int_\Omega f_-(x)\varphi(x)\,dx > 0 > \int_\Omega f^+(x)\varphi(x)\,dx, \tag{1'}$$

where $f^\pm(x) \equiv \limsup_{s\to\pm\infty} f(x,s)$, $f_\pm(x) \equiv \liminf_{s\to\pm\infty} f(x,s)$ and φ is the positive and normalized eigenfunction of $-\Delta$ in Ω associated with λ_1. Roughly speaking, the role played by the above conditions is to prevent the possible solutions u_λ of (P_λ^0) from leaving a common bounded set — in $H_0^1(\Omega)$, say — when $\lambda \to \lambda_1^+$ (resp. $\lambda \to \lambda_1^-$); note that such a-priori bounds evidently do not exist if e.g., $f \equiv 0$. Landesman-Lazer result has since then been generalized in various directions, allowing, in particular, unbounded f's to come into play: see e.g., Brézis-Nirenberg [2] and references therein.

Very recently, Chiappinelli-Mawhin-Nugari [3] considered the problem of *multiplicity* of solutions to (P_λ^0) for λ near λ_1. Employing previous quite general ideas of Mawhin and Schmitt ([11], [12], [10]) on bifurcation from infinity, they showed that if besides (1′) f satisfies

$$\lim_{s\to+\infty} \frac{f(x,s)}{s} = 0 \tag{2}$$

then (P_λ^0) has at least two distinct solutions for $\lambda \to \lambda_1^+$ (i.e., λ converging to λ_1 from above), and in fact three such solutions if $f(x,s)/s \to 0$ as $|s| \to \infty$. Note the latter is the familiar condition ensuring the occurrence of asymptotic bifurcation at the simple eigenvalue $\lambda = \lambda_1$ (Rabinowitz [13]). However, [3] were unable to prove a similar result for $\lambda \to \lambda_1^-$ under the symmetric condition (1) rather than (1′). In this paper, we fill this gap by solving in fact a more general problem, i.e., considering the full system (S_λ) for any $\delta \geq 0$.

To do this, observe as in [5] that the second equation in (S_λ) can be solved for v in terms of u. If, for each given u, we let Bu denote the solution of the problem $-\Delta v + \gamma v = \delta u$ in Ω, $v = 0$ on $\partial\Omega$, then (S_λ) is equivalent to the single equation

$$(P_\lambda) \qquad -\Delta u + Bu = \lambda u + f(x,u) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega.$$

Note that (P_λ) is now an integrodifferential equation, for it contains the integral operator B. The presence of this nonlocal term makes things more difficult. Nevertheless, it is proved in [5] that $-\Delta + B$ has pure point spectrum in $L^2(\Omega)$, its eigenvalues being

$$\hat{\lambda}_k = \lambda_k + \frac{\delta}{\gamma + \lambda_k} \quad (k = 1, 2, \dots)$$

with λ_k the eigenvalues of $-\Delta$. Thus, (P_λ) retains the qualitative properties of (P_λ^0) in that it contains a linear operator with discrete spectrum together with a nonlinearity having asymptotic properties described by (1–1′) and (2).

One more basic fact proved in [5] is that $-\Delta + B$ enjoys a maximum principle: if $g \geq 0$ in Ω and u solves

$$-\Delta u + Bu - \lambda u = g \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega$$

then $u \geq 0$ in Ω provided λ restricted to an appropriate interval to the left of $\hat{\lambda}_1$ (depending on the constants γ, δ).

It is precisely by virtue of the spectral and maximum properties just described that we perform the main step of our work, i.e., the achievement of a-priori bounds for (P_λ) under [(2) and] (1)–(1′) for $\lambda \to \hat{\lambda}_1^+$, $\lambda \to \hat{\lambda}_1^-$, respectively. These two situations are in fact quite different — corresponding to whether or not one is inside the spectrum of $-\Delta + B$, and are dealt with (in Sections 3 and 2 respectively) by different methods.

Precisely, if (1) holds then by a familiar argument "ad absurdum" one is led to consider the linear problem

$$-\Delta u + Bu - \hat{\lambda}_1 u = m(x)u \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega \tag{$*$}$$

with $m \in L^\infty(\Omega)$ related to the asymptotic properties of f, and to prove that a nontrivial solution of $(*)$ is necessarily an eigenfunction of $-\Delta + B$ corresponding to $\hat{\lambda}_1$. For this to work however, one needs to restrict the class of nonlinearities by requiring linear growth:

$$|f(x,s)| \leq a|s| + c \quad (x \in \overline{\Omega},\ s \in \mathbb{R}) \tag{3}$$

a condition which is customary in the "scalar case" $B = 0$, ([2], [1]).

On the other hand, this restriction — as it is again well-known in the case $B = 0$, see e.g., Kazdan-Warner [8], pp. 574–575 — is unnecessary for "decreasing" nonlinearities, i.e., for f satisfying (1′). Here the method of sub- and supersolutions can be applied making full use of the aforementioned maximum principle, and in fact of a more general version of it in which the real parameter λ is replaced by an appropriate function on Ω. Also in this case however, the monotone iteration scheme requires an extra assumption on f; essentially, it consists in a lower bound on the derivative $f_s'(x,s)$ of f:

$$f_s'(x,s) \geq \alpha \quad (x \in \Omega,\ s \in \Omega) \tag{3′}$$

with α related to the constants γ, δ of the original problem.

Once the a-priori bounds are obtained, the existence and multiplicity of solution to (P_λ) — and thus of pairs of solutions (u,v) to (S_λ) — follows as in [3] by topological degree arguments together with the results on bifurcation from infinity quoted before. Our final result can then be briefly stated as follows:

Theorem. *Assume f satisfies (1), (2) and (3). Then for λ near $\hat{\lambda}_1$ (S_λ) has at least one pair of solutions if $\lambda \geq \hat{\lambda}_1$ and at least two distinct pairs of solutions (u,v) if $\lambda < \hat{\lambda}_1$, one such pair (u,v) consisting of functions which are positive in Ω. If on* *the other hand* (1′), (2), (3′) *hold, then a similar statement holds on reversing the side of* λ *with respect to* $\hat{\lambda}_1$.

1. Preliminaries. Let γ, δ be nonnegative numbers. Given $u \in L^p(\Omega)$, $1 < p < \infty$, let Bu denote the unique solution of the linear problem

$$-\Delta v + \gamma v = \delta u \quad \text{in } \Omega, \quad v = 0 \quad \text{on } \partial\Omega; \tag{1.1}$$

in other words, $B = \delta(-\Delta + \gamma)^{-1}$ under zero Dirichlet boundary conditions on $\partial\Omega$. By the L^p theory of linear elliptic equations, we can thus see B is a bounded linear operator of $L^p(\Omega)$ into $W^{2,p}(\Omega) \cap H_0^1(\Omega)$; also, by the Schauder theory, B maps the Hölder space $C^{0,\alpha}(\overline{\Omega})$ into $C^{2,\alpha}(\overline{\Omega})$.

Let us consider, in particular, B as an operator in $L^2(\Omega)$. Then B is symmetric and compact; also, the operator M defined in $L^2(\Omega)$ by

$$M \equiv -\Delta + B, \quad D(M) = H^2(\Omega) \cap H_0^1(\Omega) \tag{1.2}$$

is symmetric on its domain $D(M)$. If (λ_k) and (φ_k) denote the eigenvalues and eigenfunctions of $-\Delta$ in Ω with zero Dirichlet boundary conditions, then it is easily seen that the φ_k are also eigenfunctions of M corresponding to the modified eigenvalues

$$\hat{\lambda}_k \equiv \lambda_k + \frac{\delta}{\gamma + \lambda_k}, \quad k = 1, 2, \ldots. \tag{1.3}$$

(In particular, we shall heavily use the fact that $\hat{\lambda}_1$ is a *simple* eigenvalue of M whose associated eigenfunction $\varphi \equiv \varphi_1$ can be taken to be *positive* in Ω). A more detailed analysis [5] shows that in fact the spectrum $\sigma(M)$ of M is discrete and consists precisely of the above eigenvalues $\hat{\lambda}_k$; this corresponds to the fact that M has compact resolvent, i.e., $T_\lambda \equiv (M - \lambda I)^{-1}$ is compact whenever $\lambda \notin \sigma(M)$. Though this holds in the general case, it will be enough to restrict our attention to the case when

$$\gamma + \lambda_1 > \sqrt{\delta}, \tag{1.4}$$

an assumption which we hold throughout the paper without further mention.

More generally, T_λ, when considered for functions $u \in L^p(\Omega)$, $p > 1$, maps the latter space in $W^{2,p}(\Omega)$ and thus — by the Sobolev embedding theorem — into $C^{1,\alpha}(\overline{\Omega})$ if $p > N$, ($\alpha = 1 - \frac{N}{p}$). In particular, T_λ may be considered as a bounded linear operator of $C(\overline{\Omega})$ into $C^{1,\alpha}(\overline{\Omega})$.

Also, it is clear from the above discussion that the Fredholm alternative applies to the "resonant" problems

$$Mu - \lambda u = g, \quad \lambda = \hat{\lambda}_k \quad \text{for some } k \tag{1.5}$$

in the sense that (1.5) will be solvable if and only if g is L^2-orthogonal to the eigenfunction(s) φ_k corresponding to $\hat{\lambda}_k$; the solution is then made unique by requiring that itself be orthogonal to φ_k.

It was proved in [5] that T_λ is a positive operator if $-\gamma + 2\sqrt{\delta} \le \lambda \le \hat{\lambda}_1$. This is a maximum principle for the equation

$$-\Delta u + Bu - \lambda u = g(x) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega.$$

In this paper, however, we need a stronger maximum principle, namely Proposition 1.2 below. For completeness, we present its proof, which we learned from E. Mitidieri.

Proposition 1.2. *Let $a(x)$ be an L^∞ function in Ω such that*

$$-\gamma + 2\sqrt{\delta} \le a(x) < \hat{\lambda}_1 \quad (\text{a.e. } x \in \Omega) \tag{1.6}$$

and let u be a solution of the problem

$$-\Delta u + Bu - a(x)u = g(x) \quad in\ \Omega, \quad u = 0 \quad on\ \partial\Omega.$$

It is assumed that u is Lipschitz continuous in $\overline{\Omega}$ and that Ω satisfies the interior sphere condition. If $g \in C(\overline{\Omega})$ and $g \ge 0$ in Ω, $g \not\equiv 0$, then $u > 0$ in Ω and the outward normal derivative $\frac{\partial u}{\partial \nu} < 0$ on $\partial\Omega$.

Proof. We consider the problem in an equivalent form given by the system

$$\begin{cases} -\Delta u = a(x)u - v + g(x) \\ -\Delta v = \delta u - \gamma v. \end{cases}$$

Then we introduce a new variable $w = u - (1\sqrt{\delta})v$, and observe that u and w satisfy the system

$$\begin{cases} -\Delta u = (a(x) - \sqrt{\delta})u + \sqrt{\delta}w + g(x) \\ -\Delta w = (a(x) - 2\sqrt{\delta} + \gamma)u + (\sqrt{\delta} - \gamma)w + g(x), \end{cases}$$

which is cooperative in view of (1.6). Next we apply to the latter system Theorem 2 of [7]. For that matter we observe that this system satisfies Property Ψ with

$$\Psi(x) = (\varphi(x), t\varphi(x)), \quad t = \frac{\gamma + \lambda_1 - \sqrt{\delta}}{\gamma + \lambda_1}.$$

To conclude this section, we prove a technical result (to be used in Section 3) which follows from the spectral properties of M sketched above. It should be mentioned here that, under assumption (1.6), the eigenvalues $\hat{\lambda}_k$ of M form an *increasing* sequence ([5], Remark 1.2).

Proposition 1.3. *Let $\alpha^\pm \in L^\infty(\Omega)$ be such that for some $\varepsilon > 0$ and some $k \in \mathbb{N}$*

$$\hat{\lambda}_k + \varepsilon \le \alpha^\pm(x) \le \hat{\lambda}_{k+1} - \varepsilon \quad (a.e.\ x \in \Omega). \tag{1.7}$$

Then, if v satisfies

$$-\Delta v + Bv = \alpha^+ v^+ - \alpha^- v^- \quad in\ \Omega, \quad v = 0 \quad on\ \partial\Omega$$

(with v^+, v^- denoting the positive and negative parts of π), it follows that $v \equiv 0$.

Proof. Let $\mu := (\hat{\lambda}_{k+1} + \hat{\lambda}_k)/2$. Then,

$$-\Delta v + Bv - \mu v = (\alpha^+ - \mu)v^+ - (\alpha^- - \mu)v^- \equiv Fv \quad \text{in } \Omega \tag{1.8}$$

and $v = 0$ on $\partial\Omega$. Evidently, $\mu \notin \sigma(M)$ and $d := \text{dist}\,(\mu, \sigma(M)) = (\hat{\lambda}_{k+1} + \hat{\lambda}_k)/2$. As M is symmetric and with compact resolvent, $T_\mu = (M - \mu I)^{-1}$ has operator norm in $L^2(\Omega)$ equal to d^{-1}. On the other hand, the assumptions (1.7) on $\alpha^\pm$ imply that

$$\|\alpha^\pm - \mu\|_{L^\infty(\Omega)} \leq d - \varepsilon$$

which in turn implies $\|F(v)\|_{L^2(\Omega)} \leq (d-\varepsilon)\|v\|_{L^2(\Omega)}$. Thus, from (1.8),

$$\|v\|_{L^2(\Omega)} = \|T_\mu F(v)\|_{L^2(\Omega)} \leq d^{-1}(d-\varepsilon)\|v\|_{L^2(\Omega)}$$

which gives $v \equiv 0$.

2. Decreasing nonlinearities. In this section, we consider equation (P_λ) under hypothesis (1′) upon f. Our precise assumptions are as follows:

(A′) there exist c, $C \in L^1(\Omega)$ such that

$$\liminf_{s\to-\infty} f(x,s) \geq c(x), \qquad \int_\Omega c(x)\varphi(x)\,dx > 0$$

$$\limsup_{s\to+\infty} f(x,s) \leq C(x), \qquad \int_\Omega C(x)\varphi(x)\,dx < 0$$

(the above limits are meant to hold uniformly with respect to $x \in \Omega$).

(B′) $f = f(x,s)$ is continuous in both variables and C^1 in s, and

$$\eta > \beta - \hat{\lambda}_1, \tag{2.1}$$

where

$$\eta := \inf\{f_s'(x,s) : x \in \overline{\Omega},\ s \in \mathbb{R}\}, \quad \beta := -\gamma + 2\sqrt{\delta}.$$

Remark 2.1. The discussion below shows that, rather than (B′), it is enough to assume

$$f(x,s) - f(x,t) \geq \eta(s-t) \quad (s \geq t,\ x \in \Omega),$$

(i.e., $s \to f(x,s) - \eta s$ should be increasing) with η satisfying (2.1).

By a solution of (P_λ) we mean a strong solution $u \in W^{2,p}(\Omega) \cap W_0^{1,2}(\Omega)$ with $p > N$. In particular, a solution $u \in C^{1,\alpha}(\overline{\Omega})$.

Theorem 2.1. *Assume* (A′) *and* (B′), *and let* $\hat{\beta} := \max\{\beta, \beta - \eta\}$. *Then there exist a subsolution* $z < 0$ *of* (P_λ) *and a supersolution* $w > 0$ (*independent of* $\lambda : \hat{\beta} \leq \lambda \leq \hat{\lambda}_1$) *such that* $z \leq u \leq w$ *for all possible solutions* u *of* (P_λ) *with* $\hat{\beta} \leq \lambda \leq \hat{\lambda}_1$.

Proof. We prove the assertion concerning the subsolution, the other part being entirely similar. Let us first consider (P_λ) with $\lambda = \hat{\lambda}_1$. The "first half" of assumption (A′) above implies ([3], Lemma 1) that there exist a $d \in C(\overline{\Omega})$ with $\int_\Omega d(x)\varphi(x)\,dx > 0$ and $R > 0$ so that

$$f(x,s) \geq d(x) \quad \text{whenever} \quad s \leq -R\varphi(x). \tag{2.2}$$

Step a) Consider the linear problem

$$-\Delta u + Bu - \hat{\lambda}_1 u = d - \Big(\int_\Omega d\varphi\Big)\varphi \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \tag{2.3}$$

where the right-hand side v satisfies the orthogonality condition $\int_\Omega v\varphi = 0$, and let z_0 denote the solution of (2.3) with $\int_\Omega z_0\varphi = 0$.

The existence and uniqueness of z_0 follow by the Fredholm alternative recalled in Section 1; by regularity, we also have $z_0 \in C^{1,\alpha}(\overline{\Omega})$, since $d \in C(\overline{\Omega})$. Now this implies that $\mu\varphi < z_0 < \nu\varphi$ in Ω for some real constants μ, ν; and since any $z = z_0 + c\varphi$, $c \in \mathbb{R}$, also solves (2.3), we can thus choose c negative sufficiently large so that $z(x) \leq -R\varphi(x)$, $x \in \Omega$. But then by (2.2), $f(x, z(x)) \geq d(x)$ for all $x \in \Omega$ and, since $\int_\Omega d\varphi > 0$, we conclude that

$$-\Delta z + Bz - \hat{\lambda}_1 z < d \leq f(x, z) \quad \text{in } \Omega, \quad z = 0 \quad \text{on } \partial\Omega \tag{2.4}$$

which shows that z is a (strict) subsolution of (P_λ) with $\lambda = \hat{\lambda}_1$.

Step b) Let u be any given solution of (P_λ) with $\lambda = \hat{\lambda}_1$. Set $w = u - z$ and write the equation for w (here and henceforth we omit the boundary condition and write $\int$ for $\int_\Omega$):

$$-\Delta w + Bw - \hat{\lambda}_1 w = f(x, u) - d + \Big(\int d\varphi\Big)\varphi. \tag{2.5}$$

First, choose $\lambda_0 : \hat{\beta} \leq \lambda_0 < \hat{\lambda}_1$ so that

$$(\hat{\lambda}_1 - \lambda_0)w + \Big(\int d\varphi\Big)\varphi > 0 \quad \text{in } \Omega; \tag{2.6}$$

this is possible because — as $w \in C^{1,\alpha}(\overline{\Omega})$ — there exist again constants μ', ν' so that $\mu'\varphi < w < \nu'\varphi$ in Ω; (2.6) will hold as soon as λ_0 near enough to $\hat{\lambda}_1$. Now to prove our claim that $w = u - z \geq 0$, we use the maximum principle (Proposition 1.2) on distinguishing two cases:

Case $\eta \geq 0$. Then write (2.5) as

$$-\Delta w + Bw - \lambda_0 w = (\hat{\lambda}_1 - \lambda_0)w + \Big(\int d\varphi\Big)\varphi + f(x, u) - d := g_1(x).$$

As $\beta \leq \lambda_0 < \hat{\lambda}_1$, by the maximum principle, it will be enough to show that $g_1 \geq 0$ in Ω. To this purpose, first note that, by virtue of (2.6),

$$g_1(x) \geq f(x, u(x)) - d(x), \quad x \in \Omega.$$

Now in the set $\{x \in \Omega : u(x) \leq z(x)\}$ we have $u(x) < -R\varphi(x)$ (recall $z < -R\varphi$ in Ω) and thus by (2.2), $f(x, u(x)) \geq d(x)$. On the other hand, if $u(x) > z(x)$, write

$$g_1(x) \geq f(x, u(x)) - f(x, z(x)) + f(x, z(x)) - d(x)$$

so that, using the above argument for z and the mean value theorem,

$$g_1(x) \geq f'_s(x, \xi)[u(x) - z(x)]$$

for some $\xi = \xi(x) \in \mathbb{R}$. Therefore, on the set $\{x \in \Omega : u(x) > z(x)\}$, $g_1(x) \geq \eta w(x) \geq 0$ and the claim is proved in this case.

Case $\eta < 0$. Define an L^∞ function $a(x)$ on Ω by

$$a(x) = \begin{cases} \lambda_0 & \text{if } u(x) \leq z(x) \\ \hat{\lambda}_1 + \eta & \text{if } u(x) > z(x). \end{cases} \tag{2.7}$$

By assumption (B′),

$$\beta \leq a(x) < \hat{\lambda}_1 \quad \text{in } \Omega. \tag{2.8}$$

Write (2.5) as

$$-\Delta w + Bw - a(x)w = (\hat{\lambda}_1 - a(x))w + \Big(\int d\varphi\Big)\varphi + f(x,u) - d := g_2(x).$$

Because of (2.8), we may again invoke the maximum principle to conclude that $w \geq 0$ if $g_2 \geq 0$. Indeed, on the set $[u(x) \leq z(x)]$ we have $a(x) = \lambda_0$ and thus $g_2(x) \geq 0$ as already seen for the case $\eta \geq 0$. While if $u(x) > z(x)$, then

$$g_2(x) \geq -\eta w(x) + f(x, u(x)) - f(x, z(x)) + f(x, z(x)) - d(x)$$

so that, using again the mean value theorem and (B′), we get $g_2(x) \geq 0$. This proves Theorem 2.1 when λ is "frozen" to the value $\hat{\lambda}_1$. To complete the proof, i.e., to show that z is indeed a subsolution of (P_λ) for all $\lambda : \hat{\beta} \leq \lambda \leq \hat{\lambda}_1$, we merely observe that from (2.4), as $z < 0$,

$$-\Delta z + Bz - \hat{\lambda}_1 z < d + (\lambda - \hat{\lambda}_1)z \leq f(x,z) + (\lambda + \hat{\lambda}_1)z.$$

Moreover, if u is any solution of (P_λ) with $\hat{\beta} \leq \lambda \leq \hat{\lambda}_1$, then $w = u - z$ satisfies

$$\begin{aligned} -\Delta w + Bw - \lambda w &= (\lambda - \hat{\lambda}_1)z + f(x,u) - d + \Big(\int d\varphi\Big)\varphi \\ &\geq f(x,u) - d + \Big(\int d\varphi\Big)\varphi. \end{aligned}$$

The proof that $w \geq 0$ is now similar to the above, but it is simpler, as there is no need to introduce a λ_0 — as in (2.6) — to "shift to the left" of $\hat{\lambda}_1$. Accordingly, in case $\eta < 0$, the function a in (2.7) will be replaced by a_λ, where

$$a_\lambda(x) = \begin{cases} \lambda & \text{if } w(x) \leq 0 \\ \lambda + \eta & \text{if } w(x) > 0 \end{cases}$$

so that again $\beta \leq a_\lambda(x) < \hat{\lambda}_1$ in Ω, and the maximum principle will once more yield the result.

Corollary 2.1. *Under the assumption of Theorem* 2.1, *there exists* $R > 0$ *such that* $\|u\|_{1,\alpha} := \|u\|_{C^{1,\alpha}(\overline{\Omega})} < R$ *for all solutions of* (P_λ) *with* $\hat{\beta} \leq \lambda \leq \hat{\lambda}_1$.

Proof. By the regularity theory sketched in Section 1, for all $\lambda \in [\beta, \hat{\lambda}_1[$, $T_\lambda = (M - \lambda I)^{-1}$ exists and is a bounded linear operator of $C(\overline{\Omega})$ into $C^{1,\alpha}(\overline{\Omega})$. Thus, setting $T := T_\beta$ we may rewrite (P_λ) as

$$u = T[(\lambda - \beta)u + F(u)], \tag{2.9}$$

where $F(u)(x) = f(x, u(x))$, $x \in \overline{\Omega}$. Since $f = f(x, s)$ is continuous on $\overline{\Omega} \times \mathbb{R}$, F maps continuously $C(\overline{\Omega})$ into $C(\overline{\Omega})$ and is bounded on bounded sets. Now, let

$$K := \sup[|(\lambda - \beta)s + f(x, s)| : x \in \overline{\Omega},\ z(x) \leq s \leq w(x)],$$

where z, w are the sub and supersolutions of Theorem 2.1; by that theorem, we have $\sup_{x \in \overline{\Omega}} |(\lambda - \beta)u(x) + f(x, u(x))| \leq K$ for all solutions of (P_λ) with $\hat{\beta} \leq \lambda \leq \hat{\lambda}_1$. In other words,

$$\|(\lambda - \beta)u + F(u)\|_{L^\infty(\Omega)} \leq M$$

for all such solutions; the conclusion now follows from (2.9).

Theorem 2.2. *Assume* (A′), (B′) *and*

$$\lim_{s \to +\infty} \frac{f(x, s)}{s} = 0 \tag{C}$$

uniformly with respect to $x \in \Omega$. *Then, there exists* $\delta > 0$ *so that* (P_λ) *has at least one solution for* $\hat{\beta} \leq \lambda \leq \hat{\lambda}_1$ *and at least two solutions for* $\hat{\lambda}_1 < \lambda < \hat{\lambda}_1 + \delta$.

Proof. The proof follows similar lines to [3], where the scalar case $B = 0$ was treated. The main steps are sketched next.

Step 1: (Nonzero degree) As in (2.9), write (P_λ) in the form

$$u = (\lambda - \beta)Tu + TF(u) := C_\lambda(u), \quad u \in C^{1,\alpha}(\overline{\Omega}), \tag{2.10}$$

where C_λ is a compact operator in $C^{1,\alpha}(\overline{\Omega})$. Now it follows by Theorem 2.1 and its corollary that there exists a bounded open set U in $C^{1,\alpha}(\overline{\Omega})$ such that

$$\deg(I - C_\lambda, U, 0) = 1, \quad \hat{\beta} \leq \lambda \leq \hat{\lambda}, \tag{2.11}$$

where "deg" stands for the Leray-Schauder degree. This is a consequence of a fairly general result (see e.g., [4], Lemma 2.11 or [3], Proposition 2) concerning sub and supersolutions for equations possessing a *strong* maximum principle. In fact, U may be chosen as $\theta \cap B_R$, where

$$\theta = \left\{ u \in C^{1,\alpha}(\overline{\Omega}) : z < u < w \text{ in } \Omega,\quad \frac{\partial z}{\partial u} > \frac{\partial u}{\partial y} > \frac{\partial w}{\partial u} \text{ on } \partial\Omega \right\}$$

$$B_R = \{u \in C^{1,\alpha}(\overline{\Omega}) : \|u\|_{1,\alpha} < R\}$$

with R as in Corollary 2.1. Equation (2.11) proves in particular the first assertion of Theorem 2.2 in virtue of the solution property of the degree.

Remark 2.2. The mere existence of a solution to (P_λ) for $\hat{\beta} \leq \lambda \leq \hat{\lambda}_1$ could have been established earlier in force of the existing *ordered* pair $z < w$ of sub and supersolutions. Indeed, the method of monotone iteration (e.g., [4], Theorem 2.1) works for equation (P_λ) if one can find a real number c so that

i) $[-\Delta + B - (\lambda + c)\,I]^{-1}$ be a positive operator, and

ii) $s \to f(x,s) - cs$ be nondecreasing (for each $x \in \Omega$).

Both these requirements are accomplished (for the specified values of λ) if one chooses $c = \beta - \lambda$; indeed, i) then follows by the maximum principle, while ii) holds true because

$$f'_s(x,s) - (\beta - \lambda) \geq \eta - (\beta - \lambda) \geq 0$$

for the above $\lambda's$ (recall $\hat{\beta} = \max[\beta, \beta - \eta]$).

Step 2: (Continuation) Considering (2.11) with $\lambda = \hat{\lambda}_1$ we infer from the continuity of the degree that, for some $\delta_1 > 0$,

$$\deg(I - C_\lambda,\, U, 0) = 1 \tag{2.12}$$

for all $\lambda : \hat{\lambda}_1 < \lambda < \hat{\lambda}_1 + \delta_1$ (and in fact for all λ in an *open* interval centered at $\hat{\lambda}_1$). Thus, there exists a solution to (P_λ), lying in U, for all $\lambda : \hat{\lambda}_1 < \lambda < \hat{\lambda}_1 + \delta_1$.

Step 3: (Bifurcation from infinity) Let now assumption (C) come in. Define $\hat{f}$ on $\overline{\Omega} \times \mathbb{R}$ as follows:

$$\hat{f}(x,s) = \begin{cases} f(x,s) & s \geq 0 \\ f(x,0) & s < 0 \end{cases}$$

so that $\hat{f}$ is continuous $\overline{\Omega} \times \mathbb{R}$ and satisfies

$$\lim_{|s| \to \infty} \frac{\hat{f}(x,s)}{s} = 0. \tag{2.13}$$

Clearly, any solution $u \geq 0$ of the modified problem

$$-\Delta u + Bu - \lambda u = \hat{f}(x,u) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega \tag{2.14}$$

is a solution to (P_λ). On the other hand, (2.14) can be written as

$$u = (\lambda - \beta)Tu + T\hat{F}(u), \quad u \in C^{1,\alpha}(\overline{\Omega}) \tag{2.15}$$

and (2.13) now implies that $\|T\hat{F}(u)\|_{1,\alpha}/\|u\|_{1,\alpha} \to 0$ as $\|u\|_{1,\alpha} \to \infty$. (Precisely, from (2.13) we first infer that given $\varepsilon > 0$, there exists c_ε so that $\|\hat{F}(u)\|_\infty \leq \varepsilon\|u\|_\infty + c_\varepsilon$, where $\|u\|_\infty := \|u\|_{L^\infty(\Omega)}$. Then, as T is a bounded linear operator of $C(\overline{\Omega})$ into $C^{1,\alpha}(\overline{\Omega})$,

$$\|T\hat{F}(u)\|_{1,\alpha} \leq c\|\hat{F}(u)\|_\infty < c\varepsilon\|u\|_\infty + d \leq c'\varepsilon\|u\|_{1,\alpha} + d'$$

which proves the claim). Therefore, since $\hat{\lambda}_1 - \beta$ is a *simple* characteristic value of $T = T_\beta = (-\Delta + B - \beta)^{-1}$, we can employ the results of Rabinowitz [13] on bifurcation from infinity to deduce the existence of a connected set C in $\mathbb{R} \times C^{1,\alpha}(\overline{\Omega})$ of solution pairs (λ, u) of (2.15) with λ near $\hat{\lambda}_1$ and $\|u\|_{1,\alpha}$ near ∞. This last assertion precisely means that $C \cap U_r \neq \phi$ for all $r > 0$, where

$$U_r := \{(\lambda, u) \in \mathbb{R} \times C^{1,\alpha}(\overline{\Omega}) : |\lambda - \hat{\lambda}_1| < r,\ \|u\|_{1,\alpha} > r^{-1}\} \tag{2.16}$$

is a typical neighborhood of $(\hat{\lambda}_1, \infty)$ in $\mathbb{R} \times C^{1,\alpha}(\overline{\Omega})$.

Furthermore, since the eigenspace corresponding to $\hat{\lambda}_1$ is spanned by a positive function, C consists of two connected subsets C^+, C^- which both "meet $(\hat{\lambda}_1, \infty)$" in the sense specified above and are such that, for $r > 0$ sufficiently small,

$$(\lambda, u) \in C^+(\text{ resp. } C^-) \cap U_r \Longrightarrow u > 0 \text{ (resp. } u < 0) \quad \text{in } \Omega. \tag{2.17}$$

Step 4: (Multiple solutions) We now first deduce that there exists $r_0 > 0$ so that

$$(\lambda, u) \in C^+ \cap U_{r_0} \Longrightarrow \lambda > \hat{\lambda}_1. \tag{2.18}$$

Indeed, let $r > 0$ be such that $u > 0$ for $(\lambda, u) \in C^+ \cap U_r$ as in (2.17); those (λ, u) are thus solutions to the original problem (P_λ). But the a-priori bounds, Corollary 2.1, show that $\|u\| < R$ for all solutions of (P_λ) with $\hat{\lambda}_1 - \delta_0 \leq \lambda \leq \hat{\lambda}_1$ ($\delta_0 = \hat{\lambda}_1 - \hat{\beta}$); thus, the definition of U_r implies (2.18) as soon as $r_0 < \min[r, R^{-1}, \delta_0]$.

Next, let

$$\delta_2 := \sup\{\lambda : (\lambda, u) \in C^+ \cap U_{r_0}\};$$

in other words, $(\hat{\lambda}_1, \hat{\lambda}_1 + \delta_1)$ is the projection over $\mathbb{R}$ of the "piece" $C^+ \cap U_{r_0}$ of the bifurcation branch C^+. Thus by construction, to each $\lambda : \hat{\lambda}_1 < \lambda < \hat{\lambda}_1 + \delta_2$ there corresponds a solution $u = u_\lambda$ of (P_λ) with $\|u\| > r_0^{-1}$ and $u > 0$ in Ω. As $r_0^{-1} > R$ by the choice of r_0, this solution is distinct from the one found by continuation in Step 2 and lying inside the ball B_R. This ends the proof of Theorem 2.2 on taking $\delta = \min[\delta_1, \delta_2]$.

3. Increasing nonlinearities. In this section, we consider (P_λ) under the following set of assumptions on f:

(A) there exist γ, $\Gamma \in L^1(\Omega)$ such that

$$\limsup_{s \to -\infty} f(x, s) \leq \Gamma(x), \qquad \int_\Omega \Gamma(x)\varphi(x)\,dx < 0$$

$$\liminf_{s \to +\infty} f(x, s) \geq \gamma(x), \qquad \int_\Omega \gamma(x)\varphi(x)\,dx > 0.$$

(B) there exist positive constants c_1, c_2 so that

$$|f(x, s)| \leq c_1|s| + c_2, \quad (x, s) \in \overline{\Omega} \times \mathbb{R}$$

(C) $\lim_{s \to +\infty} \frac{f(x,s)}{s} = 0$.

Theorem 3.1. *Assume* (A), (B) *and* (C). *Then there exist* $\delta > 0$ *and* $M > 0$ *so that* $\|u\|_{H^1(\Omega)} \le M$ *for all solutions* u *of* (P_λ) *with* $\hat\lambda_1 \le \lambda \le \hat\lambda_1 + \delta$.

Proof. We argue by contradiction. Assume thus that there exist sequences (λ_n), (u_n) with $\lambda_n \ge \hat\lambda_1$, $\lambda_n \to \hat\lambda_1$, $\|u_n\|_{H^1(\Omega)} \to \infty$ such that

$$-\Delta u_n + Bu_n - \lambda_n u_n = f(x, u_n) \tag{3.1}$$

for all $n \in \mathbb{N}$. Let $v_n := \frac{u_n}{\|u_n\|}$ (in this section, $\|\cdot\|$ stands for $\|\cdot\|_{H^1(\Omega)}$); passing to subsequences, we can suppose that $v_n \rightharpoonup v_0$ in $H_0^1(\Omega)$, $v_n \to v_0$ in $L^2(\Omega)$ and almost everywhere in (Ω) ($\rightharpoonup$ and $\to$ denote weak and strong convergence respectively). Now, divide (3.1) by $\|u_n\|$, multiply by $z \in H_0^1(\Omega)$ and integrate to obtain

$$\int \nabla v_n \nabla z + \int (Bv_n)z - \lambda_n \int v_n z = \int w_n z, \quad \forall z \in H_0^1(\Omega), \tag{3.2}$$

where $w_n(x) := f(x, u_n(x))/\|u_n\|$. By assumption (B),

$$|w_n(x)| := c_1 |v_n(x)| + c_2 \|u_n\|^{-1} \le c_1 |v_n(x)| + c_2'$$

which implies that (w_n) is a bounded sequence in $L^2(\Omega)$ and thus converges weakly (through a subsequence) to some $w_0 \in L^2(\Omega)$. It is proved in [1, Lemma 4] that, in general,

$$\begin{aligned} \ell_+ v_0 \le w_0 \le k_+ v_0 \quad & \text{a.e. on the set } \{v_0(x) > 0\} \\ k_- v_0 \le w_0 \le \ell_- v_0 \quad & \text{a.e. on the set } \{v_0(x) < 0\} \\ w_0 = 0 \quad & \text{a.e. on the set } \{v_0(x) = 0\}, \end{aligned}$$

where

$$\ell_\pm(x) = \liminf_{s \to +\infty} \frac{f(x,s)}{s}, \quad k_\pm(x) = \limsup_{s \to \pm\infty} \frac{f(x,s)}{s}. \tag{3.3}$$

Thus, letting $m = m(x)$, denote the function defined in Ω by $w_0(x)/v_0(x)$ if $v_0(x) \ne 0$ and $m(x) = 0$ otherwise, one has $w_0 = mv_0$ in Ω with $\ell_+ \le m \le k_+$ in $\{v_0(x) > 0\}$ and $\ell_- \le m \le k_-$ in $\{v_0(x) < 0\}$.

In the present situation, (A) implies $\ell_+, \ell_- \ge 0$ and thus $m \ge 0$ in Ω; while (C) further implies that $m = 0$ on $\{v_0(x) > 0\}$ and in fact on $\{v_0(x) \ge 0\}$ by the above definition of m. Thus, $mv_0^+ \equiv 0$.

Now, letting $n \to \infty$ in (3.2), we obtain

$$\int \nabla v_0 \nabla z + \int (Bv_0)z - \hat\lambda_1 \int v_0 z = \int m v_0 z = -\int m v_0^- z, \quad z \in H_0^1(\Omega). \tag{3.4}$$

We first claim that $v_0 \not\equiv 0$. Indeed, taking $z = v_n$ in (3.2) and, letting $n \to \infty$, we obtain

$$1 + \int (Bv_0)v_0 - \hat\lambda_1 \int v_0^2 = \int m v_0^2,$$

which proves the claim.

Our next step is to show that $v_0 = c\varphi$ for some $c \in \mathbb{R}$. Indeed, from (3.4) with $z = \varphi$ we get

$$\int m v_0^- \varphi = 0. \tag{3.5}$$

Since $m \geq 0$ in Ω while $\varphi > 0$ in Ω, this shows that either $v_0^- \equiv 0$ (i.e., $v_0 \geq 0$), or $m = 0$ on the set $\{v_0 < 0\}$, so that at any rate $m = 0$ (a.e.) on the whole Ω. Thus, (3.4) says that v_0 is an eigenfunction of $-\Delta + B$ corresponding to the eigenvalue $\hat{\lambda}_1$ (recall $v_0 \not\equiv 0$). Then $v_0 = c\varphi$ for some $c \neq 0$.

Now, multiply the equation (3.1) for u_n by φ and integrate to get

$$(\hat{\lambda}_1 - \lambda_n) \int u_n \varphi = \int f(x, u_n)\varphi. \tag{3.6}$$

Assume, for instance, $c > 0$ in $v_0 = c\varphi$. Then $v_0 > 0$ in Ω; since we can show that in fact $v_n \to v_0$ in $C^{1,\alpha}(\overline{\Omega})$, it follows that $v_n > 0$ in Ω for n large enough. Then also $u_n = \|u_n\| v_n > 0$ in Ω, and (3.6) gives $0 \geq \int f(x, u_n)\varphi$ for large n. By Fatou's Lemma, since $u_n(x) \to +\infty$ for all $x \in \Omega$,

$$0 \geq \liminf_{s\to\infty} \int f(x, u_n(x))\varphi(x)\, dx \geq \int \liminf_{s\to+\infty} f(x, u_n(x))\varphi(x)\, dx \geq \int \gamma(x)\varphi(x)\, dx.$$

which contradicts assumption (A). Note that the use of Fatou's Lemma is justified in view of (A) and the bound on $f(x, u_n(x))$ for $0 \leq u_n(x) \leq s_0$, any $s_0 > 0$.

Similarly, the possibility $c < 0$ contradicts the other half of (A); this accomplishes the proof of the Theorem 3.1.

Corollary 3.1. *Under the same assumptions as in Theorem 3.1, there exists $R > 0$ so that $\|u\|_{1,\alpha} < R$ for all solutions of (P_λ) with $\hat{\lambda}_1 \leq \lambda \leq \hat{\lambda}_1 + \delta$.*

Proof. By a familiar bootstrap argument, the $H^1(\Omega)$ bounds found above yield bounds in $L^\infty(\Omega)$. Then the same argument follows as in Corollary 2.1.

Theorem 3.2. *Under the same assumptions, there exists $\delta > 0$ so that (P_λ) has at least one solution $\hat{\lambda}_1 \leq \lambda \leq \hat{\lambda}_1 + \delta$ and at least two solutions for $\hat{\lambda}_1 - \delta < \lambda < \hat{\lambda}_1$.*

Proof. It follows the same lines as Theorem 2.2, once we prove that

$$\deg(I - C_{\hat{\lambda}_1}, B_R, 0) \neq 0 \tag{3.7}$$

for a suitable chosen $R > 0$. Indeed, this is enough to ensure that

$$\deg(I - C_\lambda, B_R, 0) \neq 0 \tag{3.8}$$

when λ runs in a small interval around $\hat{\lambda}_1$ (Theorem 2.2, Step 2). To this purpose, we construct via a-priori bounds an homotopy of $I - C_{\hat{\lambda}_1}$ with the linear map $I - (\bar{\lambda} - \beta)T$ (notations as in Theorem 2.2) with $\hat{\lambda}_1 < \bar{\lambda} < \hat{\lambda}_2$; our claim will then follow because the latter is a linear homeomorphism and has therefore nonzero degree. We proceed in two steps along the propositions below; it suffices as usual to achieve the bounds in $H^1(\Omega)$ as they imply the bounds in $C^{1,\alpha}(\overline{\Omega})$.

Proposition 3.1. *Let μ be such that $0 < \mu < \hat{\lambda}_2 - \hat{\lambda}_1$. Then there exists $R_0 > 0$ so that $\|u\|_{H_0^1(\Omega)} < R_0$ for all possible solutions of*

$$-\Delta u + Bu = \hat{\lambda}_1 u + f(x, u^+) + [(1-t)f(x, -u^-) - t\mu u^-], \quad 0 \le t \le 1. \tag{3.9}$$

Proof. The argument is similar to that used in Theorem 3.1. Indeed if not, there exist sequences $(u_n) \subset H_0^1(\Omega)$, $(t_n) \subset [0,1]$, so that

$$-\Delta u_n + Bu_n = \hat{\lambda}_1 u_n + f(x, u_n^+) + [(1-t_n)f(x, -u_n^-) - t_n \mu u_n^-] \tag{3.10}$$

with $\|u_n\| \to \infty$. We can assume that $t_n \to \bar{t} \in [0,1]$ and also, setting $v_n = \frac{u_n}{\|u_n\|}$, that $v_n \rightharpoonup v_0$ in $H_0^1(\Omega)$, $v_n \to v_0$ in $L^2(\Omega)$ and, pointwisely, almost everywhere in Ω. As in Theorem 3.1, we obtain that $v_0 \not\equiv 0$ and satisfies

$$-\Delta v_0 + Bv_0 = \hat{\lambda}_1 v_0 + (1-\bar{t})(-m v_0^-) - \bar{t}\mu v_0^- \tag{3.11}$$

with $m \in L^\infty(\Omega)$, $m \ge 0$ in Ω. Multiply this by φ and integrate to obtain

$$\int [(1-\bar{t})m + \bar{t}\mu]\, v_0^- \varphi = 0. \tag{3.12}$$

This shows that either $v_0 \ge 0$ or $\bar{t} = 0$ and $m = 0$ in Ω. In either case it follows from (3.11) that $-\Delta v_0 + Bv_0 = \hat{\lambda}_1 v_0$, proving that $v_0 = c\varphi$, $c \ne 0$.

i) Suppose first $c > 0$. Since $v_n \to v_0$ in $C^{1,\alpha}(\overline{\Omega})$, then $u_n > 0$ for large n. Thus, (3.10) for large n bècomes

$$-\Delta u_n + Bu_n = \hat{\lambda}_1 u_n + f(x, u_n) \tag{3.13}$$

which gives, on multiplying by φ and integrating,

$$\int f(x, u_n(x))\varphi(x)\, dx = 0 \quad (n \text{ large}). \tag{3.14}$$

Now use Fatou's Lemma and (A) to get, as in Theorem 3.1, the contradiction

$$0 \ge \int \gamma\varphi > 0.$$

ii) If $c < 0$, then $u_n < 0$ for large n and thus, (3.10) reads

$$-\Delta u_n + Bu_n = \hat{\lambda}_1 u_n + (1-t_n)f(x, u_n) + t_n \mu u_n. \tag{3.15}$$

Thus,

$$(1-t_n)\int f(x, u_n)\varphi + t_n\mu \int u_n\varphi = 0. \tag{3.16}$$

However,

$$\lim_{n\to\infty} \int u_n\varphi = \lim_{n\to\infty} \|u_n\| \int v_n\varphi = -\infty,$$

while by (A)

$$\limsup_{s\to\infty} \int f(x, u_n)\varphi \le \int \Gamma\varphi < 0.$$

It is now easy to check that these relations contradict (3.16) whatever $\bar{t}$, the limit of (t_n). This ends the proof of Proposition 3.1.

Proposition 3.2. *With μ as in Proposition* 3.1, *there exists $R_1 > 0$ so that $\|u\| < R_1$ for all possible solutions of*

$$-\Delta u + Bu = \hat{\lambda}_1 u + [(1-t)f(x,u^+) + t\mu u^+] - \mu u^-, \quad 0 \le t \le 1. \tag{3.17}$$

Proof. Let (u_n), (t_n) be such that

$$-\Delta u_n + Bu_n = \hat{\lambda}_1 u_n + [(1-t_n)f(x,u_n^+) + t_n\mu u_n^+] - \mu u_n^- \tag{3.18}$$

with $\|u_n\| \to \infty$. As before, the normalized sequence $v_n = \frac{u_n}{\|u_n\|}$ tends to a limit $v_0 \neq 0$ which now satisfies

$$-\Delta v_0 + Bv_0 = \hat{\lambda}_1 v_0 + \bar{t}\mu v_0^+ - \mu v_0^- = (\hat{\lambda}_1 + \bar{t}\mu)v_0^+ - (\hat{\lambda}_1 + \mu)v_0^-. \tag{3.19}$$

If $\bar{t} \neq 0$, the coefficients of v_0^+, v_0^- are strictly within $\hat{\lambda}_1$ and $\hat{\lambda}_2$; thus, by Proposition 1.3 in the Introduction, we have $v_0 = 0$, a contradiction. While if $\bar{t} = 0$, then

$$-\Delta v_0 + Bv_0 = \hat{\lambda}_1 v_0 - \mu v_0^-; \tag{3.20}$$

thus, multiplying by φ and integrating, $\mu \int v_0^- \varphi = 0$, which first implies $v_0 \ge 0$ and then by (3.20) $v_0 = c\varphi$, $c > 0$.

Since $v_n \to v_0$ in $C^{1,\alpha}(\overline{\Omega})$, then $u_n > 0$ for large n and thus, by (3.17) we obtain $(1-t_n)\int f(x,u_n)\varphi + t_n\mu \int u_n\varphi = 0$, which contradicts (A).

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