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BIFURCATION OF PERIODIC SOLUTIONS TO THE SINGULAR YAMABE PROBLEM ON SPHERES

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Abstract

We obtain uncountably many periodic solutions to the singular Yamabe problem on a round sphere that blow up along a great circle. These are (complete) constant scalar curvature metrics on the complement of $\mathbb{S}^1$ inside $\mathbb{S}^m$, $m \geq 5$, that are conformal to the round (incomplete) metric and *periodic* in the sense of being invariant under a discrete group of conformal transformations. These solutions come from bifurcating branches of constant scalar curvature metrics on compact quotients of $\mathbb{S}^m \setminus \mathbb{S}^1 \cong \mathbb{S}^{m-2} \times \mathbb{H}^2$.

1. Introduction

A major achievement in Geometric Analysis was the complete solution of the Yamabe problem, which asserts that any compact Riemannian manifold (M, g) of dimension $\dim M = m \geq 3$ admits a metric g' that has constant scalar curvature and is conformal to g. Attempts to generalize this statement in many directions have provided a fertile research area in the last decades. One such direction is to drop the compactness assumption on M. In this case, it is natural to consider only *complete* metrics, even though every conformal class obviously contains both complete and incomplete metrics. A well-studied version of this question, where the geometry at infinity is relatively tame, is the following:

Singular Yamabe Problem. *Let (M, g) be a compact Riemannian manifold and $\Lambda \subset M$ be a closed subset. Find a complete metric g' on $M \setminus \Lambda$ that has constant scalar curvature and is conformal to g.*

Although considerable progress was made in the general case, for many reasons the above problem is especially interesting in the particular case in which (M, g) is the unit round sphere $(\mathbb{S}^m, g_{\text{round}})$; see Schoen [**22**, §5]. This situation was initially considered in the 1970s by Loewner and Nirenberg [**14**], who obtained existence of solutions with scal < 0 in some cases where the Hausdorff dimension of Λ is $\geq (m-2)/2$. Most of the subsequent contributions to the problem, up

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to present, assume that Λ is smooth, with remarkable exceptions due to Finn [**10**, **9**]. Under this assumption, Aviles and McOwen [**3**] proved that for a general (M, g), a solution with scal < 0 exists if and only if $\dim \Lambda > (m-2)/2$, in which case one also has uniqueness of solutions and regularity results; see Mazzeo [**16**].

Analogously to the classical Yamabe problem, the case scal ≥ 0 is considerably more involved. The first major breakthroughs were obtained by Schoen [**21**] and Schoen and Yau [**23**]. The latter established that if $\mathbb{S}^m \setminus \Lambda$ admits a complete metric with scalar curvature bounded below by a positive constant, then the Hausdorff dimension of Λ is $\leq (m-2)/2$, and the former constructed several examples of domains $\mathbb{S}^m \setminus \Lambda$ that admit complete conformally flat metrics with constant positive scalar curvature, including the case where Λ is any finite set with at least two points. This existence result was greatly generalized by Mazzeo and Pacard [**17**, **18**], allowing Λ to be a disjoint union of submanifolds with dimensions between 1 and $(m-2)/2$ when (M, g) is a general compact manifold with constant nonnegative scalar curvature, and between 0 and $(m-2)/2$ in the case $(M, g) = (\mathbb{S}^m, g_{\mathrm{round}})$.

Some of the first solutions to the Singular Yamabe Problem on $\mathbb{S}^m \setminus \Lambda$ were constructed by lifting solutions to the classical Yamabe problem from compact quotients. In these constructions, Λ is the limit set of a Kleinian group, and hence either a round subsphere $\mathbb{S}^k \subset \mathbb{S}^m$ or totally unrectifiable. Since the corresponding metrics on $\mathbb{S}^m \setminus \Lambda$ are invariant under a discrete (cocompact) group of conformal transformations, we slightly abuse terminology and call these *periodic solutions*.

The purpose of the present paper is to apply bifurcation techniques to obtain many families of new periodic solutions when $\Lambda = \mathbb{S}^1$. Our central result is the following:

Main Theorem. *There exist uncountably many* branches *of periodic solutions to the singular Yamabe problem on $\mathbb{S}^m \setminus \mathbb{S}^1$, for all $m \geq 5$, having (constant) scalar curvature arbitrarily close to $(m-4)(m-1)$.*

The starting point for constructing such solutions is the existence of *trivial* periodic solutions in the case Λ is a round subsphere $\mathbb{S}^k \subset \mathbb{S}^m$ (cf. [**19**]). Through stereographic projection using a point in $\mathbb{S}^k$, there is a conformal equivalence $\mathbb{S}^m \setminus \mathbb{S}^k \cong \mathbb{R}^m \setminus \mathbb{R}^k$. Consider the flat metric in the latter, given in cylindrical coordinates by $\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + \mathrm{d}y^2$, where y is the coordinate in $\mathbb{R}^k$ and (r, θ) are polar coordinates in its orthogonal complement $\mathbb{R}^{m-k}$. Dividing by r^2, we obtain that this metric is conformal to

$$g_{\mathrm{prod}} = \mathrm{d}\theta^2 + \frac{\mathrm{d}r^2 + \mathrm{d}y^2}{r^2} = \frac{\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + \mathrm{d}y^2}{r^2},$$

which is clearly the ordinary product metric on $\mathbb{S}^{m-k-1} \times \mathbb{H}^{k+1}$, with the hyperbolic metric written in the upper half-plane model. Alternatively, recall that the subset of $\mathbb{S}^m$ at maximal distance from any round subsphere $\mathbb{S}^k$ is another round subsphere $\mathbb{S}^{m-k-1}$, and both submanifolds have trivial normal bundle. In particular, the exponential map is a diffeomorphism between $\mathbb{S}^m \setminus \mathbb{S}^k$ and the (trivial) normal disk bundle $D(\mathbb{S}^{m-k-1}) = \mathbb{S}^{m-k-1} \times D^{k+1}$. The pull-back of g_{round} is a doubly warped product metric on $[0, \frac{\pi}{2}) \times \mathbb{S}^{m-k-1} \times \mathbb{S}^k$. Dividing by the warping function of $\mathbb{S}^{m-k-1}$ and reparametrizing the $[0, \frac{\pi}{2})$ coordinate, one reobtains the product metric on $\mathbb{S}^{m-k-1} \times \mathbb{H}^{k+1}$, with the hyperbolic metric now written in the rotationally symmetric (Poincaré disk) model.

Thus, g_{prod} is a (smooth) solution to the Singular Yamabe Problem, since it is conformal to the (incomplete) round metric on $\mathbb{S}^m \setminus \mathbb{S}^k$ and has constant scalar curvature equal to

$$\mathrm{scal}_{m,k} := (m-k-1)(m-k-2) - (k+1)k = (m-2k-2)(m-1).$$

In particular, notice that $\mathrm{scal}_{m,k} > 0$ precisely when $k < (m-2)/2$. Mazzeo and Smale [**19**] used these trivial solutions to prove existence of infinitely many other solutions with scalar curvature $\mathrm{scal}_{m,k}$, perturbing the subsphere $\mathbb{S}^k \subset \mathbb{S}^m$ with a diffeomorphism of $\mathbb{S}^m$ close to the identity. These were the first nonperiodic solutions to be found on $\mathbb{S}^m \setminus \mathbb{S}^k$.

In order to find new periodic solutions, we concentrate on the exceptional case $k = 1$, which allows for paths of nonisometric compact quotients of $\mathbb{S}^m \setminus \mathbb{S}^1 \cong \mathbb{S}^{m-2} \times \mathbb{H}^2$ (if $k \geq 2$, then any compact quotients $\mathbb{S}^{m-k-1} \times \Sigma^{k+1}$ of $\mathbb{S}^m \setminus \mathbb{S}^k$ that have isomorphic fundamental groups must be isometric, by Mostow's Rigidity Theorem). These quotients arise from paths of (cocompact) lattices Γ_t on $\mathbb{H}^2$, which induce paths of hyperbolic surfaces $\Sigma_t = \mathbb{H}^2/\Gamma_t$, whose product with $\mathbb{S}^{m-2}$ gives paths of closed manifolds $\mathbb{S}^{m-2} \times \Sigma_t$. We prove that if the path Σ_t of hyperbolic surfaces degenerates in an appropriate way, then there exist nonproduct constant scalar curvature metrics on $\mathbb{S}^{m-2} \times \Sigma$ that *bifurcate* from $\mathbb{S}^{m-2} \times \Sigma_t$. These lift to metrics on $\mathbb{S}^m \setminus \mathbb{S}^1$ that are conformal to g_{prod} (and hence to g_{round}), providing new periodic solutions with *varying period* Γ_t. We obtain a very large quantity of solutions, with very general periods, since for any lattice Γ_0 on $\mathbb{H}^2$ we find a path Γ'_t of lattices (starting arbitrarily close to Γ_0) such that bifurcation occurs along $\mathbb{S}^{m-2} \times (\mathbb{H}^2/\Gamma'_t)$; see Theorem 3.4. Moreover, this can be done with lattices corresponding to hyperbolic surfaces of any desired genus ≥ 2.

Our bifurcating solutions have positive scalar curvature, and since $\mathrm{scal}_{m,1} > 0$ if and only if $m \geq 5$, the range for m considered in the above theorem is the largest possible. Regarding the case $m = 4$, notice that $\mathrm{scal}_{4,1} = 0$ and hence any two constant scalar curvature metrics on a compact quotient of $(\mathbb{S}^2 \times \mathbb{H}^2, g_{\text{prod}})$ must be homothetic [**2**, p. 175]. Thus, g_{prod} is the unique periodic solution on $\mathbb{S}^4 \setminus \mathbb{S}^1$.

Bifurcation of constant scalar curvature metrics from paths of product metrics on $\mathbb{S}^{m-2} \times \Sigma_t$ is obtained via a classical variational bifurcation criterion in terms of the Morse index of such metrics (as critical points of the Hilbert–Einstein functional). This Morse index can be computed explicitly in terms of eigenvalues of the Laplacian of Σ_t. Informed by well-known spectral results for hyperbolic surfaces, we choose an appropriate path Σ_t (which corresponds geometrically to pinching a noncontractible closed geodesic) to produce the desired effect on the Morse index that yields bifurcation. The only relevant information about the factor $\mathbb{S}^{m-2}$ is that the first eigenvalue of its Laplacian satisfies a certain bound in terms of scalar curvature; see (3.6). The same bifurcation result holds replacing $\mathbb{S}^{m-2}$ by any other closed manifold N that satisfies (3.6); see Theorem 3.4. In principle, this provides a method to construct periodic solutions to the Singular Yamabe problem on $M \setminus \Lambda$, for any M that admits a codimension 2 submanifold N satisfying (3.6), whose normal bundle is trivial and conformally equivalent to $M \setminus \Lambda$. For this reason, we prove all relevant bifurcation results on a general manifold $N \times \Sigma$, and only specialize to $N = \mathbb{S}^{m-2}$ in the last section of the paper. Determining other geometrically interesting occurrences of the above situation is an object of ongoing research by the authors.

The paper is organized as follows. In Section 2, we recall some spectral properties of hyperbolic surfaces. The appropriate variational framework for finding metrics of constant scalar curvature is discussed in Section 3 and the core bifurcation result (Theorem 3.4) is proved. Finally, Section 4 contains the final arguments necessary to prove the above Main Theorem.

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2. Spectrum of hyperbolic surfaces

Let Σ be a closed oriented smooth surface of genus $\mathrm{gen}(\Sigma) \geq 2$. Recall that, by the Uniformization Theorem, every conformal class of metrics on Σ contains a unique representative of constant sectional curvature -1, called a *hyperbolic metric*. We denote by $\mathcal{H}(\Sigma)$ the space of smooth hyperbolic metrics on Σ, endowed with the Whitney C^∞ topology. The Riemannian manifold (Σ, h), with $h \in \mathcal{H}(\Sigma)$, will be called a *hyperbolic* *surface*, and we denote the spectrum of the Laplacian operator on real-valued functions on (Σ, h) by

$$\operatorname{spec}(\Sigma, h) = \big\{0 < \lambda_1(\Sigma, h) \le \lambda_2(\Sigma, h) \le \cdots \le \lambda_k(\Sigma, h) \nearrow +\infty\big\},$$

where each eigenvalue is repeated according to its multiplicity. The goal of this section is to discuss some spectral properties of hyperbolic surfaces required for our applications; for general references, see [**6**, **7**].

REMARK 2.1. Observe that, in this particular situation, the choice of Whitney C^∞ topology still allows for the use of results usually available only for Banach manifolds. More precisely, denote by $\mathcal{H}_r(\Sigma)$ the Banach manifold of C^r hyperbolic metrics on Σ, with the Whitney C^r topology, and by $\operatorname{Diff}_r^+(\Sigma)$ the group of orientation-preserving C^r diffeomorphisms of Σ. The orbit space $\mathcal{T}(\Sigma) = \mathcal{H}_r(\Sigma)/\operatorname{Diff}_r^+(\Sigma)$, called the *Teichmüller space* of Σ, is independent of r and homeomorphic to $\mathbb{R}^{6\,\mathrm{gen}(\Sigma)-6}$ (see [**26**]). Given any smooth hyperbolic metric $g_0 \in \mathcal{H}_r(\Sigma)$, there exists a $(6\,\mathrm{gen}(\Sigma)-6)$-dimensional submanifold $\mathcal{S}$ of $\mathcal{H}_r(\Sigma)$, consisting of smooth hyperbolic metrics, such that the restriction to $\mathcal{S}$ of the quotient map $\pi\colon \mathcal{H}_r(\Sigma) \to \mathcal{T}(\Sigma)$ is a smooth diffeomorphism onto a neighborhood of $\pi(g_0) \in \mathcal{T}(\Sigma)$. Thus, results on Banach manifolds can also be applied to $\mathcal{H}(\Sigma)$, provided they are local and invariant under diffeomorphisms.

2.1. Small eigenvalues. One of the cornerstones of the proof of our Main Theorem is the behavior of eigenvalues of the Laplacian of hyperbolic surfaces near $\frac{1}{4}$. To illustrate the peculiar nature of these eigenvalues, we recall that classical estimates imply that, for any $\mathrm{gen}(\Sigma) \ge 2$, there exist hyperbolic metrics on Σ whose first $2\,\mathrm{gen}(\Sigma) - 3$ eigenvalues are arbitrarily close to zero. On the other hand, the long-standing conjecture that at most $2\,\mathrm{gen}(\Sigma) - 2$ eigenvalues can be $\le \frac{1}{4}$ for any hyperbolic metric on Σ was recently proved by Otal and Rosas [**20**]. A well-known fact related to the above is that arbitrarily many eigenvalues can lie in the interval $[\frac{1}{4}, \frac{1}{4} + \varepsilon]$; see [**6**, Theorem 8.1.2] or [**7**, Theorem 2]. More precisely, the following holds:

Proposition 2.2. *Let Σ be a closed oriented surface of genus $\mathrm{gen}(\Sigma) \ge 2$. For all $\varepsilon > 0$ and $k \in \mathbb{N}$, there exists $h \in \mathcal{H}(\Sigma)$ such that $\lambda_k(\Sigma, h) < \frac{1}{4} + \varepsilon$.*

The above implies that, up to deforming the hyperbolic metric $h \in \mathcal{H}(\Sigma)$,

$$n_a(\Sigma, h) := \max\big\{k \in \mathbb{N} : \lambda_k(\Sigma, h) < a\big\}$$

can be made arbitrarily large for any given $a > \frac{1}{4}$. More precisely, we have the following:

Corollary 2.3. *Let (Σ, h_0) be a hyperbolic surface. For any $a > \frac{1}{4}$ and any nonnegative integer d, there exists a real-analytic path of hyperbolic metrics $h_t \in \mathcal{H}(\Sigma)$, $t \in [0, 1]$, such that $n_a(\Sigma, h_1) > n_a(\Sigma, h_0) + d$.*

Proof. Let $K = \min\{k : \lambda_k(\Sigma, h_0) \geq a\}$. Since $a > \frac{1}{4}$, by Proposition 2.2, there exists $h_1 \in \mathcal{H}(\Sigma)$ such that $\lambda_{K+d}(\Sigma, h_1) < a$. Thus $n_a(\Sigma, h_0) = K - 1$ and $n_a(\Sigma, h_1) \geq K + d$. Since $\mathcal{H}(\Sigma)$ is a real-analytic path-connected manifold, the conclusion follows. q.e.d.

2.2. Avoiding eigenvalues. In our applications, a crucial step is to avoid a given real number $\lambda > \frac{1}{4}$ in $\operatorname{spec}(\Sigma, h)$, up to perturbing the hyperbolic metric $h \in \mathcal{H}(\Sigma)$. In this context, it is natural to consider the effect of real-analytic deformations of hyperbolic metrics on the corresponding eigenvalues of the Laplacian.

Given a real-analytic path $h_\ell \in \mathcal{H}(\Sigma)$, the corresponding Laplacians Δ_{h_ℓ} form a real-analytic path of symmetric unbounded discrete operators. By the Kato Selection Theorem [**12**], their eigenvalues are real-analytic functions of ℓ, up to relabeling. More precisely, given $\lambda \in \operatorname{spec}(\Sigma, h_{\ell_0})$, there are real-analytic functions $\lambda(\ell)$, called *eigenvalue branches* through λ, such that $\lambda(\ell_0) = \lambda$ and $\lambda(\ell) \in \operatorname{spec}(\Sigma, h_\ell)$. For ℓ near ℓ_0, these are the only elements of $\operatorname{spec}(\Sigma, h_\ell)$ near λ. Notice that, since $\lambda(\ell)$ are real-analytic, they can only attain the value λ countably many times.

In this framework, we now prove the desired avoidance principle.

Proposition 2.4. *Let Σ be a closed oriented surface of genus* $\operatorname{gen}(\Sigma) \geq 2$, *and fix $\lambda > \frac{1}{4}$. Then the subset* $\mathcal{H}_\lambda(\Sigma) := \big\{h \in \mathcal{H}(\Sigma) : \lambda \notin \operatorname{spec}(\Sigma, h)\big\}$ *is open and dense.*

Proof. The condition $\lambda \notin \operatorname{spec}(\Sigma, g)$ is open in the space $\operatorname{Met}(\Sigma)$ of *all* smooth Riemannian metrics g on Σ, and hence also in $\mathcal{H}(\Sigma)$. In order to prove density of this condition, suppose $\lambda \in \operatorname{spec}(\Sigma, h)$, and let $h_\ell \in \mathcal{H}(\Sigma)$ be a real-analytic path of *pinching* hyperbolic metrics through h. In other words, $h = h_{\ell_0}$ for some $\ell_0 > 0$, and (Σ, h_ℓ) have shortest closed geodesics of length ℓ that are pinched in the limit $\ell \searrow 0$. The existence of such paths is proved by Wolpert [**29**, §2.5]. Denote by $\lambda(\ell)$ the corresponding eigenvalue branches through λ, which are real-analytic functions of ℓ, as described above. If none of the $\lambda(\ell)$ are constant functions, then we are done, as $\lambda \notin \operatorname{spec}(\Sigma, h_\ell)$ for any $\ell \neq \ell_0$ near ℓ_0. Otherwise, there exists a constant eigenvalue branch $\lambda(\ell) \equiv \lambda$; in particular, $\lim_{\ell \searrow 0} \lambda(\ell) = \lambda > \frac{1}{4}$. According to a deep result of Wolpert [**28**, Theorem 5.14], the only eigenvalue branches for degenerating hyperbolic surfaces as above whose limit is $> \frac{1}{4}$ are *nonconstant*. This contradiction implies that this latter case cannot happen. q.e.d.

Remark 2.5. Given $\lambda > \frac{1}{4}$, it is a hard problem to find an *explicit* hyperbolic surface (Σ, h) with $\lambda \notin \operatorname{spec}(\Sigma, h)$. For small λ, this is related to finding hyperbolic surfaces with large first eigenvalue. Recall that $\lambda_1(\Sigma, h) \leq 2(\operatorname{gen}(\Sigma) + 1)/(\operatorname{gen}(\Sigma) - 1)$, by an estimate of Yang and Yau [**30**]. In particular, the larger $\operatorname{gen}(\Sigma)$ is, the smaller the upper bound on $\lambda_1(\Sigma, h)$. A well-studied hyperbolic surface of genus 2 is the *Bolza* *surface* (Σ_B, h_B), which has the largest systole (shortest noncontractible closed geodesic) and the largest conformal group among such surfaces. Numeric estimates yield $\lambda_1(\Sigma_B, h_B) \cong 3.838$ (see [**25**, §5.3]) and hence provide an explicit example with $\lambda \notin \mathrm{spec}(\Sigma_B, h_B)$ for any $\lambda < 3.8$.

3. Bifurcations from constant scalar curvature product metrics

Let (N, g_N) be a compact Riemannian manifold with $\dim N = n$ and constant scalar curvature $\mathrm{scal}_N \in \mathbb{R}$, and let (Σ, h) be a hyperbolic surface. Denote by $g = g_N \oplus h$ the product metric on $N \times \Sigma$. The product manifold $(N \times \Sigma, g)$ has

$$\begin{aligned} \mathrm{scal}_g &= \mathrm{scal}_N - 2, \\ \mathrm{Vol}(N \times \Sigma, g) &= 4\pi(\mathrm{gen}(\Sigma) - 1)\,\mathrm{Vol}(N, g_N). \end{aligned} \tag{3.1}$$

In this section, we discuss the variational approach to the problem of finding constant scalar curvature metrics on $N \times \Sigma$ and establish our core results on bifurcation of solutions issuing from families of product metrics.

3.1. Variational setup. Consider the Sobolev space $H^1(N \times \Sigma)$ and the Lebesgue space $L^p(N \times \Sigma, \mathrm{vol}_g)$, where vol_g is the volume density of the product metric g. By the Gagliardo–Nirenberg–Sobolev inequality, there is a continuous inclusion $H^1(N \times \Sigma) \hookrightarrow L^{\frac{2(n+2)}{n}}(N \times \Sigma, \mathrm{vol}_g)$, and the subset

$$[g]_{\mathrm{v}} := \left\{ \begin{array}{r} \phi \in H^1(N \times \Sigma) : \int_{N \times \Sigma} \phi^{\frac{2(n+2)}{n}} \mathrm{vol}_g = \mathrm{Vol}(N \times \Sigma, g) \\ \text{and } \phi > 0 \text{ a.e.} \end{array} \right\}$$

is a smooth Hilbert submanifold of $H^1(N \times \Sigma)$. The map $[g]_{\mathrm{v}} \ni \phi \mapsto g_\phi = \phi^{\frac{4}{n}} g$ gives an identification between $[g]_{\mathrm{v}}$ and the set of Sobolev H^1 metrics in the conformal class of g that have the same volume as g. The constant map $1 \in [g]_{\mathrm{v}}$ clearly corresponds to the original metric g, and the tangent space to $[g]_{\mathrm{v}}$ at this point is

$$T_1[g]_{\mathrm{v}} = \left\{ \psi \in H^1(N \times \Sigma, g) : \int_{N \times \Sigma} \psi \, \mathrm{vol}_g = 0 \right\}. \tag{3.2}$$

It is well known that $\phi \in [g]_{\mathrm{v}}$ is a critical point of the *Hilbert–Einstein functional* $\mathcal{A}\colon [g]_{\mathrm{v}} \to \mathbb{R}$, defined by

$$\mathcal{A}(\phi) := \int_{N \times \Sigma} \left(4\tfrac{n+1}{n} \, |\nabla \phi|_g^2 + (\mathrm{scal}_N - 2)\, \phi^2 \right) \mathrm{vol}_g, \tag{3.3}$$

if and only if $\phi \in C^\infty(N \times \Sigma)$ and $g_\phi = \phi^{\frac{4}{n}} g$ has constant scalar curvature; see, e.g., [**13**, **22**, **27**]. In particular, since the product metric g has constant scalar curvature, the constant function $1 \in [g]_{\mathrm{v}}$ is a critical point of $\mathcal{A}$. The second variation of $\mathcal{A}$ at this critical point is the bilinear symmetric form on $T_1[g]_{\mathrm{v}}$ given by

$$\mathrm{d}^2\mathcal{A}(1)(\psi_1,\psi_2) = \frac{n(n+1)}{2}\int_{N\times\Sigma}\Big(g(\nabla\psi_1,\nabla\psi_2) - \frac{\mathrm{scal}_N - 2}{n+1}\psi_1\psi_2\Big)\mathrm{vol}_g\,.$$

Using the compactness of $H^1(N\times\Sigma)\hookrightarrow L^2(N\times\Sigma)$, we have the existence of a self-adjoint operator $F_g\colon T_1[g]_{\mathrm{v}}\to T_1[g]_{\mathrm{v}}$, given by a compact perturbation of the identity, such that

$$\mathrm{d}^2\mathcal{A}(1)(\psi_1,\psi_2) = \tfrac{n(n+1)}{2}\big\langle F_g\psi_1,\psi_2\big\rangle_{H^1},\qquad \text{for all } \psi_1,\psi_2\in T_1[g]_{\mathrm{v}}. \tag{3.4}$$

In particular, F_g is an essentially positive Fredholm operator of index 0. The dimension of $\ker F_g$ and the number (counted with multiplicity) of negative eigenvalues of F_g are, respectively, the *nullity* and the *Morse index* of 1 as a critical point of $\mathcal{A}$ in $[g]_{\mathrm{v}}$. As customary, the second variation of $\mathcal{A}$ is better understood using an L^2-pairing, rather than an H^1-pairing, in the space of functions on $N\times\Sigma$ with zero average. Replacing (3.2) with

$$L^2_0(N\times\Sigma,g) := \left\{\psi\in L^2(N\times\Sigma,\mathrm{vol}_g) : \int_{N\times\Sigma}\psi\;\mathrm{vol}_g = 0\right\},$$

we can describe $\mathrm{d}^2\mathcal{A}(1)$ in terms of an unbounded symmetric Fredholm operator $J_g\colon L^2_0(N\times\Sigma,g)\to L^2_0(N\times\Sigma,g)$, by means of

$$\mathrm{d}^2\mathcal{A}(1)(\psi_1,\psi_2) = \tfrac{n(n+1)}{2}\big\langle J_g\psi_1,\psi_2\big\rangle_{L^2},\qquad \text{for all } \psi_1,\psi_2\in L^2_0(N\times\Sigma,g).$$

The operator J_g, called the *Jacobi operator*, is a self-adjoint elliptic operator that can be explicitly computed as

$$J_g = \Delta_g - \frac{\mathrm{scal}_N - 2}{n+1}.$$

The kernel and the number of negative eigenvalues of F_g and J_g coincide, so the nullity and Morse index of critical points of the Hilbert–Einstein function can be computed using the spectrum of J_g. The latter is given by the (positive) spectrum of the Laplacian Δ_g, shifted to the left by $\frac{\mathrm{scal}_N-2}{n+1}$; i.e., the eigenvalues of J_g are

$$\lambda_1(N\times\Sigma,g) - \tfrac{\mathrm{scal}_N-2}{n+1} \leq \dots \leq \lambda_k(N\times\Sigma,g) - \tfrac{\mathrm{scal}_N-2}{n+1} \leq \dots \nearrow +\infty,$$

where the above are repeated according to multiplicity. This proves the following:

Lemma 3.1. *The* Morse index *and* nullity *of* $1\in[g]_{\mathrm{v}}$ *as a critical point of* $\mathcal{A}\colon[g]_{\mathrm{v}}\to\mathbb{R}$ *are, respectively,*

$$i(g) = \max\left\{k : \lambda_k(N\times\Sigma,g) < \tfrac{\mathrm{scal}_N-2}{n+1}\right\}\quad \textit{and}\quad \nu(g) = \dim\ker J_g.$$

REMARK 3.2. Note that $-\frac{\mathrm{scal}_N - 2}{n+1}$ is *not* in the spectrum of J_g, since the only constant function on $L^2_0(N \times \Sigma, g)$ is identically zero. We also recall the well-known fact that eigenfunctions of J_g are smooth and form an orthonormal basis of $L^2_0(N \times \Sigma, g)$.

3.2. Bifurcation. Instead of having a fixed hyperbolic metric on Σ, let $h_t \in \mathcal{H}(\Sigma)$, $t \in [a,b]$, be a *path* of hyperbolic metrics. Then we have a corresponding path

$$g_t = g_N \oplus h_t \in \mathrm{Met}(N \times \Sigma), \quad t \in [a,b], \tag{3.5}$$

of product metrics on $N \times \Sigma$ satisfying (3.1). We say that $t_* \in [a,b]$ is a *bifurcation instant* for g_t if there exist sequences $\{t_q\}_{q\in\mathbb{N}}$ in $[a,b]$ converging to t_* and $\{g_q\}_{q\in\mathbb{N}}$ in $\mathrm{Met}(N \times \Sigma)$ converging to g_{t_*}, called a *bifurcating branch*, such that

(i) each g_q has constant scalar curvature;
(ii) g_q is conformal to g_{t_q}, but $g_q \neq g_{t_q}$;
(iii) $\mathrm{Vol}(N \times \Sigma, g_q) = 4\pi(\mathrm{gen}(\Sigma) - 1)\,\mathrm{Vol}(N, g_N)$.

In other words, t_* is a bifurcation instant if local uniqueness of g_t as a solution to (3.1) fails around g_{t_*}. That is, for any open neighborhood of $g_{t_*} \in \mathrm{Met}(N \times \Sigma)$, there are *other* constant scalar curvature metrics (with normalized volume) in the conformal class of some g_t, with t near t_*.

We now establish the key result used in our applications, namely, the existence of paths of product metrics on $N \times \Sigma$ along which the Morse index has arbitrarily large variation.

Proposition 3.3. *Assume that the following inequalities hold:*

$$\frac{1}{4} < \frac{\mathrm{scal}_N - 2}{n+1} < \lambda_1(N). \tag{3.6}$$

Then, for any fixed (Σ, h_0) and any nonnegative integer d, there exists a real-analytic path $h_t \in \mathcal{H}(\Sigma)$ such that, setting $g_t = g_N \oplus h_t$, we have $i(g_1) > i(g_0) + d$.

Proof. Let $a \in \left[\frac{\mathrm{scal}_N - 2}{n+1}, \lambda_1(N)\right]$. From Corollary 2.3, there exists a real-analytic path $g_t = g_N \oplus h_t$, such that $n_a(\Sigma, h_1) > n_a(\Sigma, h_0) + d$. Note that $i(g_1) \geq n_a(\Sigma, h_1)$. Since $\lambda_1(N) \geq a$, the only eigenvalues of the Laplacian of $(N \times \Sigma, g_0)$ that are strictly smaller than a are of the form $\lambda_0(N) + \lambda_k(\Sigma, h_0) \in \mathrm{spec}(\Sigma, h_0)$, with $\lambda_k(\Sigma, h_0) < a$. In particular, $i(g_0) = n_a(\Sigma, h_0)$. Altogether, we have $i(g_1) \geq n_a(\Sigma, h_1) > n_a(\Sigma, h_0) + d = i(g_0) + d$. q.e.d.

Theorem 3.4. *Let Σ be a hyperbolic surface and N be a closed Riemannian manifold such that (3.6) holds. For any $h_0 \in \mathcal{H}(\Sigma)$, there exist $h'_0, h'_1 \in \mathcal{H}(\Sigma)$ with h'_0 arbitrarily close to h_0, such that the following holds: for any continuous path $h'_t \in \mathcal{H}(\Sigma)$ joining h'_0 to h'_1, there exists* *at least one bifurcation instant $t_* \in [0,1]$ for the path of constant scalar curvature metrics $g_t = g_N \oplus h'_t$ on $N \times \Sigma$.*

Proof. In order to prove the above result, we apply a standard variational bifurcation criterion, adapted to a variable domain framework as in [**8**, Appendix A]. This criterion states that, under a local Palais–Smale condition, if g_0 and g_1 are nondegenerate critical points of different Morse index, then any path of critical points joining g_0 to g_1 has at least one bifurcation instant; see [**4**, **8**, **24**] for details. In what follows, we describe how to verify each of these conditions in the above context of constant scalar curvature product metrics on $N \times \Sigma$.

First, the local Palais–Smale condition is an easy consequence of Fredholmness. Any path of hyperbolic metrics on Σ induces a path (3.5) of product metrics on $N \times \Sigma$ and a path $\mathcal{A}_t \colon [g_t]_{\mathrm{v}} \to \mathbb{R}$ of smooth functionals given by (3.3). We can assume that the constant function $1_t \in [g_t]_{\mathrm{v}}$ is an isolated critical point for $\mathcal{A}_t$ for all t, otherwise bifurcation trivially holds. Since $\mathrm{d}^2\mathcal{A}_t(1_t)$ is represented by a Fredholm operator, the first derivative $\mathrm{d}\mathcal{A}$ is a *nonlinear Fredholm map* near 1_t. This implies that each $\mathcal{A}_t$ satisfies a local Palais–Smale condition around 1_t. More precisely, for any fixed t_*, there are $\delta > 0$ and a neighborhood $\mathcal{U}$ of 1_{t_*} in $H^1(N \times \Sigma)$ such that $\mathcal{A}_t$ satisfies the Palais–Smale condition on $\mathcal{U} \cap [g_t]_{\mathrm{v}}$ for all $t \in [t_* - \delta, t_* + \delta]$. This follows from a classical argument (see [**15**]) that uses the local representation for C^1 maps having Fredholm derivative as given in [**1**, Theorem 1.7].

Second, let us describe how to verify nondegeneracy of endpoints, up to small perturbations. For any chosen $h_0 \in \mathcal{H}(\Sigma)$ and nonnegative integer d, let h_t be the path given by Proposition 3.3. In particular, this defines $h_1 \in \mathcal{H}(\Sigma)$. Then Proposition 2.4, combined with (3.6), ensures that there exist h'_0 and h'_1, arbitrarily close to h_0 and h_1, such that $\frac{\mathrm{scal}_N - 2}{n+1} \notin \mathrm{spec}(\Sigma, h'_i)$, $i = 0,1$. Consider the metrics

$$g_i := g_N \oplus h'_i, \quad i = 0,1. \tag{3.7}$$

From (3.6), the only eigenvalues of the Laplacian of g_i that are $\leq \frac{\mathrm{scal}_N - 2}{n+1}$ are of the form $\lambda_0(N) + \lambda_k(\Sigma, h'_i) \in \mathrm{spec}(\Sigma, h'_i)$; hence $\frac{\mathrm{scal}_N - 2}{n+1} \notin \mathrm{spec}(N \times \Sigma, g_i)$, $i = 0,1$. By Lemma 3.1, we have that $\nu(g_i) = 0$, i.e., $\ker \mathrm{d}^2\mathcal{A}_t(g_i)$ is trivial, so (3.7) are nondegenerate.

Finally, we verify that the Morse index of any path $g_t = g_N \oplus h'_t$ joining g_0 to g_1 is nonconstant. Again, by (3.6) and Lemma 3.1, the Morse index $i(g_t)$ of 1_t as a critical point of $\mathcal{A}_t$ is given by the number (with multiplicity) of eigenvalues of the Laplacian of h'_t that are strictly smaller than $\frac{\mathrm{scal}_N - 2}{n+1}$. By Proposition 3.3, and continuity of the spectrum, we have that $i(g_1) > i(g_0) + d$, provided that h'_0 and h'_1 are chosen sufficiently close to h_0 and h_1. q.e.d.

REMARK 3.5. It is not difficult to show that the sequence of metrics that bifurcate from g_{t_*} in Theorem 3.4 converges to g_{t_*} in the C^r-topology, for any $r \geq 2$. More generally, suppose g_q is a sequence of smooth metrics on a compact manifold M and $u_q \colon M \to \mathbb{R}_+$ is a sequence of smooth positive functions, such that

- g_q has unit volume and constant scalar curvature for all q;
- $g_q \to g_\infty$ in the C^s-topology, with $s \geq 2$;
- $u_q\, g_q$ is a unit volume constant scalar curvature metric for all q;
- $u_q \to 1$ in the Sobolev space $H^1(M)$.

Then, using L^p-estimates for solutions of second-order elliptic equations (see [**11**, Theorem 9.14]), it follows that $\mathrm{scal}(u_q\, g_q) \to \mathrm{scal}(g_\infty)$ and $u_q \to 1$ in the Sobolev space $W^{s+1,p}(M)$, where $p = \frac{2m}{m+2}$ and $m = \dim M$. In particular, if $s > r + \frac{m}{2}$, then $u_q \to 1$ in $C^r(M)$. This applies to the sequence of bifurcating metrics in Theorem 3.4, since we are assuming that the path $h'_t \in \mathcal{H}(\Sigma)$ is continuous with respect to the C^r-topology for all r.

4. Proof of Main Theorem

The proof of the Main Theorem in the Introduction is a direct application of Theorem 3.4 to the case $(N, g_N) = (\mathbb{S}^{m-2}, g_{\mathrm{round}})$, $m \geq 5$. In order to verify that (3.6) holds, notice that $n = \dim N = m - 2$, and $\mathrm{scal}(\mathbb{S}^{m-2}) = (m-2)(m-3)$, and hence $\frac{\mathrm{scal}_N - 2}{n+1} = m - 4$. Furthermore, $\lambda_1(N) = \lambda_1(\mathbb{S}^{m-2}) = m - 2$. Thus, (3.6) is satisfied for all $m \geq 5$.

Applying Theorem 3.4 with arbitrary choices of $h_0 \in \mathcal{H}(\Sigma)$ and continuous paths $h'_t \in \mathcal{H}(\Sigma)$ joining h'_0 to h'_1, one obtains the existence of uncountably many bifurcating branches $\{g_q\}_{q\in\mathbb{N}}$ of constant scalar curvature metrics on $\mathbb{S}^{m-2} \times \Sigma$ that have fixed volume and are conformal to a product metric $g_{\mathrm{round}} \oplus h'_{t_q}$. These branches consist of metrics with *positive* constant scalar curvature, since the solution to the Yamabe problem (with volume normalization) is unique in conformal classes with nonpositive conformal Yamabe energy [**2**, p. 175]. From Remark 3.5, the values of these scalar curvatures converge to $\mathrm{scal}_{m,1} = (m-4)(m-1)$.

Furthermore, each metric g_q in a bifurcating branch $\{g_q\}_{q\in\mathbb{N}}$ is conformal, but not equal, to $g_{\mathrm{round}} \oplus h'_{t_q}$. This follows from the fact that any two product metrics on a product manifold are conformal if and only if they are homothetic. In particular, two distinct product metrics with the same volume cannot be conformal. Therefore, g_q cannot be product metrics on $\mathbb{S}^{m-2} \times \Sigma$. Thus, the pull-backs of g_q to $\mathbb{S}^{m-2} \times \mathbb{H}^2$ are complete constant scalar curvature metrics that are conformal, but not equal, to the product metric g_{prod}. This proves the Main Theorem in the Introduction. q.e.d.

Remark 4.1. The usual notion of *multiple* solutions to the (singular) Yamabe problem in a conformal class $[g] = \{\phi\, g : \phi\colon M \to \mathbb{R}_+\}$ is that there exist distinct functions $\phi\colon M \to \mathbb{R}_+$ such that $\phi\, g$ has constant scalar curvature. Under this notion of multiplicity, the proof of our Main Theorem guarantees that each bifurcating branch above contains (countably many) pairwise distinct solutions to the Yamabe problem on $\mathbb{S}^{m-2} \times \Sigma$. Since there are continuous families of paths of metrics on $\mathbb{S}^{m-2} \times \Sigma$ that have bifurcating branches, we obtain uncountably many distinct solutions to the Singular Yamabe problem on $\mathbb{S}^m \setminus \mathbb{S}^1$.

In principle, some of these solutions are conformal factors that may give rise to *isometric* metrics. Recall that two distinct metrics in the same conformal class may be isometric, via pull-back by a conformal diffeomorphism. For instance, unlike any other Einstein manifold, the round sphere $(\mathbb{S}^m, g_{\text{round}})$ has uncountably many metrics of constant scalar curvature equal to $\text{scal}(g_{\text{round}}) = m(m-1)$ that form a noncompact $(m+1)$-dimensional manifold. However, all such metrics are isometric. Recall that a metric in $[g_{\text{round}}]$ has constant scalar curvature if and only if it is the pull-back of g_{round} by a conformal diffeomorphism of (S^m, g_{round}) [**22**, §2]. In particular, the moduli space of solutions is diffeomorphic to $\text{Conf}(\mathbb{S}^m, g_{\text{round}})/\text{Iso}(\mathbb{S}^m, g_{\text{round}}) \cong \mathsf{SO}(m+1,1)_0/\mathsf{SO}(m+1)$. Thus, it is natural to ask whether our result implies the existence of infinitely many pairwise *nonisometric* solutions to the Singular Yamabe problem on $\mathbb{S}^m \setminus \mathbb{S}^1$. It is not hard to show that the answer is affirmative (within each bifurcating branch), since the conformal group of the product $(\mathbb{S}^{m-2} \times \mathbb{H}^2, g_{\text{prod}})$ coincides with its isometry group. Therefore, there are uncountably many pairwise nonisometric (even nonhomothetic) periodic solutions to the Singular Yamabe problem on $\mathbb{S}^m \setminus \mathbb{S}^1$. Finally, notice that these solutions can be further chosen to be periodic with respect to infinitely many different cocompact lattices Γ—for instance, using the infinitely many possible choices of $\text{gen}(\Sigma) = \text{gen}(\mathbb{H}^2/\Gamma) \geq 2$.

Remark 4.2. A somewhat weaker nonuniqueness result (independent of bifurcation theoretic methods) follows as a by-product of the above proof and the solution to the classical Yamabe problem. Namely, for any $m \geq 5$, we obtain uncountably many hyperbolic metrics $h \in \mathcal{H}(\Sigma)$ such that the Morse index $i(g_{\text{round}} \oplus h)$ is arbitrarily large; see Proposition 3.3. Hence, the corresponding *Yamabe metric* $g_{\text{Y}}(h)$ on $\mathbb{S}^{m-2} \times \Sigma$ must be a different constant scalar curvature metric conformal to $g_{\text{round}} \oplus h$. Arguing as above, the pull-back of $g_{\text{Y}}(h)$ to $\mathbb{S}^{m-2} \times \mathbb{H}^2$ is a periodic solution with scal > 0 that is not locally isometric to g_{prod}. Note that $g_{\text{Y}}(h) \neq g_{\text{Y}}(h')$ if $h \neq h'$. Thus, there are uncountably many distinct periodic solutions to the singular Yamabe problem on $\mathbb{S}^m \setminus \mathbb{S}^1$.

References

[1] R. Abraham & J. Robbin, *Transversal mappings and flows*, W.A. Benjamin Inc., 1967, MR 0240836, Zbl 0171.44404.

[2] T. Aubin, *Some nonlinear problems in Riemannian geometry*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 1998, MR 1636569, Zbl 0896.53003.

[3] P. Aviles & R. McOwen, *Complete conformal metrics with negative scalar curvature in compact Riemannian manifolds*, Duke Math. J. **56** (1988), no. 2, 395–398, MR 0932852, Zbl 0645.53023.

[4] R.G. Bettiol & P. Piccione, *Bifurcation and local rigidity of homogeneous solutions to the Yamabe problem on spheres,* Calc. Var. Partial Differential Equations **47** (2013), no. 3–4, 789–807, MR 3070564, Zbl 1272.53042.

[5] R.G. Bettiol & P. Piccione, *Multiplicity of solutions to the Yamabe problem on collapsing Riemannian submersions,* Pacific J. Math. **266** (2013), no. 1, 1–21, MR 3105774, Zbl 1287.53030.

[6] P. Buser, *Geometry and Spectra of Compact Riemann Surfaces,* reprint of the 1992 edition, Modern Birkhäuser Classics, Birkhäuser Boston, Inc., Boston, MA, 2010, MR 2742784, Zbl 1239.32001.

[7] I. Chavel, *Eigenvalues in Riemannian geometry*, Pure and Applied Mathematics, Volume 115, Academic Press, Inc., Orlando, FL, 1984, MR 0768584, Zbl 0551.53001.

[8] L.L. de Lima, P. Piccione & M. Zedda, *On bifurcation of solutions of the Yamabe problem in product manifolds,* Ann. Inst. H. Poincaré Anal. Non Linéaire **29** (2012), no. 2, 261–277, MR 2901197, Zbl 1239.58005.

[9] D. Finn, *Behavior of positive solutions to* $\Delta_g u = u^q + Su$ *with prescribed singularities*. Indiana Univ. Math. J. **49** (2000), no. 1, 177–219, MR 1777033, Zbl 0972.35034.

[10] D. Finn, *On the negative case of the singular Yamabe problem*. J. Geom. Anal. **9** (1999), no. 1, 73–92, MR 1760721, Zbl 1003.58020.

[11] D. Gilbarg & N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Reprint of the 1998 edition, Classics in Mathematics, Springer–Verlag, Berlin, 2001, MR 1814364, Zbl 1042.35002.

[12] T. Kato, *Perturbation theory for linear operators*, Reprint of the 1980 edition. Classics in Mathematics. Springer–Verlag, Berlin, 1995, MR 1335452, Zbl 0836.47009.

[13] J. Lee & T. Parker, *The Yamabe problem,* Bull. Amer. Math. Soc. (N.S.) **17** (1987), no. 1, 37–91, MR 0888880, Zbl 0633.53062.

[14] C. Loewner & L. Nirenberg, *Partial differential equations invariant under conformal or projective transformations*, Contributions to analysis (a collection of papers dedicated to Lipman Bers), pp. 245–272, Academic Press, New York, 1974, MR 0358078, Zbl 0298.35018.

[15] A. Marino & G. Prodi, *Metodi perturbativi nella teoria di Morse*, Boll. Un. Mat. Ital. (4) **11** (1975), no. 3, suppl., 1–32, MR 0418150, Zbl 0311.58006.

[16] R. Mazzeo, *Regularity for the singular Yamabe problem,* Indiana Univ. Math. J. **40** (1991), no. 4, 1277–1299, MR 1142715, Zbl 0770.53032.

[17] R. Mazzeo & F. Pacard, *A construction of singular solutions for a semilinear elliptic equation using asymptotic analysis*, J. Differential Geom. **44** (1996), no. 2, 331–370, MR 1425579, Zbl 0869.35040.

[18] R. Mazzeo & F. Pacard, *Constant scalar curvature metrics with isolated singularities*, Duke Math. J. **99** (1999), no. 3, 353–418, MR 1712628, Zbl 0945.53024.

[19] R. Mazzeo & N. Smale, *Conformally flat metrics of constant positive scalar curvature on subdomains of the sphere,* J. Differential Geom. **34** (1991), no. 3, 581–621, MR 1139641, Zbl 0759.53029.

[20] J.-P. Otal & E. Rosas, *Pour toute surface hyperbolique de genre g,* $\lambda_{2g-2} > 1/4$*,* Duke Math. J. **150** (2009), no. 1, 101–115, MR 2560109, Zbl 1179.30041.

[21] R. Schoen, *The existence of weak solutions with prescribed singular behavior for a conformally invariant scalar equation,* Comm. Pure Appl. Math. **41** (1988), no. 3, 317–392, MR 0929283, Zbl 0674.35027.

[22] R. Schoen, *Variational theory for the total scalar curvature functional for Riemannian metrics and related topics*, Topics in calculus of variations (Montecatini Terme, 1987), 120–154, Lecture Notes in Math., 1365, Springer, Berlin, 1989, MR 0994021, Zbl 0702.49038.

[23] R. Schoen & S.-T. Yau, *Conformally flat manifolds, Kleinian groups and scalar curvature,* Invent. Math. **92** (1988), no. 1, 47–71, MR 0931204, Zbl 0658.53038.

[24] J. Smoller & A. Wasserman, *Bifurcation and symmetry-breaking*, Invent. Math. **100** (1990), 63–95, MR 1037143, Zbl 0721.58011.

[25] A. Strohmaier & V. Uski, *An algorithm for the computation of eigenvalues, spectral zeta functions and zeta-determinants on hyperbolic surfaces*, Comm. Math. Phys. **317** (2013), no. 3, 827–869, MR 3009726, Zbl 1261.65113.

[26] A.J. Tromba, *Teichmüller Theory in Riemannian Geometry*, Lecture notes prepared by Jochen Denzler, Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 1992, MR 1164870, Zbl 0785.53001.

[27] N.S. Trudinger, *Remarks concerning the conformal deformation of Riemannian structures on compact manifolds*, Ann. Scuola Norm. Sup. Pisa (3) **22** (1968), MR 0240748, Zbl 0159.23801.

[28] S. Wolpert, *Disappearance of cusp forms in special families*, Ann. of Math. (2) **139** (1994), no. 2, 239–291, MR 1274093, Zbl 0826.11024.

[29] S. Wolpert, *Spectral limits for hyperbolic surfaces, II,* Invent. Math. **108** (1992), no. 1, 91–129, MR 1156387, Zbl 0772.11017.

[30] P.C. Yang & S.-T. Yau, *Eigenvalues of the Laplacian of compact Riemann surfaces and minimal submanifolds,* Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **7** (1980), no. 1, 55–63, MR 0577325, Zbl 0446.58017.

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