# 13. Bifurcation of Stable Stationary Solutions from Symmetric Modes 

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Introduction. We consider the following semilinear parabolic system of equations:

$$
\begin{align*}
& U_{t}=D(\sigma) U_{x x}+B U+F(U), \quad(t, x) \in(0,+\infty) \times(0, L)  \tag{P-1}\\
& U(t, 0)=U(t, L)=0,
\end{align*}
$$

where $U={ }^{t}(u(t, x), v(t, x)), D(\sigma)=\left(D_{u}(\sigma), D_{v}(\sigma)\right)$ and $\sigma$ is a real parameter, $B=\binom{a, b}{c, d}$ is a real constant matrix and $F(U)=^{t}\left(f_{1}(u, v), f_{2}(u, v)\right)$ is a smooth autonomous nonlinear operator which satisfies

$$
\begin{equation*}
F(0)=F_{U}(0)=0 . \tag{0-1}
\end{equation*}
$$

We assume that $B$ satisfies either of the following conditions:

$$
\begin{array}{ll}
\operatorname{det} B>0, a>0, & d=0, \\
\operatorname{det} B>0, a>0, & a+d \leqq 0 . \tag{0-3}
\end{array}
$$

Our main purpose is to show the existence of bifurcation of stable stationary solutions of (P-1) as $D(\sigma)$ varies. Stationary problem of (P-1) and its linearized system of equations at $U=0$ are given as follows :

$$
\begin{align*}
& D(\sigma) U_{x x}+B U+F(U)=0, \\
& U(0)=U(L)=0,  \tag{P-2}\\
& D(\sigma) U_{x x}+B U=0, \\
& U(0)=U(L)=0 . \tag{P-3}
\end{align*}
$$

Section 1 deals with the spectrum of ( $\mathrm{P}-3$ ) and the existence of bifurcation of stationary solutions from any mode of the eigenfunction of (P-3) under the appropriate conditions of $D(\sigma)$ and $B$. Section 2 deals with the asymptotic stability of the bifurcating solutions from symmetric modes. In section 3 we give some examples of biological system to which our theorems can apply.
§ 1. Existence. Using the Fourier series expansion of $U$,

$$
U=\sum_{n=1}^{\infty} U_{n} \sin \frac{n \pi}{L} x=\sum_{n=1}^{\infty}\binom{u_{n}}{v_{n}} \sin \frac{n \pi}{L} x,
$$

we obtain the infinite system of linear equations of $\left\{U_{n}\right\}_{n \in N}$ :

$$
M_{n} U_{n}=0, \quad M_{n}=\left(\begin{array}{lr}
-D_{u}\left(\frac{\pi}{L}\right)^{2} n^{2}+a, & b \\
c, & -D_{v}\left(\frac{\pi}{L}\right)^{2} n^{2}+d
\end{array}\right), \quad n \in N .
$$

The roots $\left\{\alpha_{n}^{i}\right\}_{i=1,2}\left(\operatorname{Re} \alpha_{n}^{1} \geqq \operatorname{Re} \alpha_{n}^{2}\right)$ of the characteristic equation
$\operatorname{det}\left(M_{n}-\alpha I\right)=0$ are the eigenvalues of (P-3) which correspond to the $\sin n \pi / L$-mode. We consider the following condition of the spectrum of (P-3) :

$$
\alpha_{n_{0}}^{1}=0, \operatorname{Re} \alpha_{n}^{i}<0 \text { for all }(i, n) \in\{1,2\} \times N \text { except }(i, n)=\left(1, n_{0}\right) .\left(S_{n_{0}}\right)
$$ The corresponding eigenfunction to $\alpha_{n_{0}}^{1}$ is denoted by $U_{n_{0}} \sin \left(n_{0} \pi / L\right) x$. The necessary and sufficient conditions of $D(\sigma)$ and $B$ to realize the condition ( $S_{n_{0}}$ ) are given in the following lemma.

For simplicity we write $D$ instead of $D(\sigma)$. We introduce the following curves in $D^{+}=\left\{\left(D_{u}, D_{v}\right) ; D_{u}>0, D_{v}>0\right\}$-plane:

$$
\begin{aligned}
& H_{n}: D_{v}=\frac{b c}{\left(\gamma n^{2}\right)^{2}} \cdot \frac{1}{D_{u}-a / \gamma n^{2}}+\frac{d}{\gamma n^{2}}, \quad \gamma=\left(\frac{\pi}{L}\right)^{2}, \quad n \in N, \\
& L: D_{u}+D_{v}=\frac{a}{\gamma}, \\
& P^{n}=\left(P_{u}^{n}, P_{v}^{n}\right) \text { is a cross point of } H_{n} \text { and } H_{n+1} \text { and } \\
& L^{n}=\left(L_{u}^{n}, L_{v}^{n}\right) \text { is a cross point of } L \text { and } H_{n} .
\end{aligned}
$$

Note that $P_{u}^{n}$ and $P_{v}^{n}$ are strictly decreasing with respect to $n$.
Lemma 1. $\mathrm{B}_{1}$ ) Suppose that $B$ satisfies (0-2). Then $S_{1}$ holds if and only if $D \in H_{1}$ and $D_{u}>P_{u}^{1}$, and for $n_{0} \geqq 2, S_{n_{0}}$ holds if and only if $D \in H_{n_{0}}$, $\max \left\{P_{u}^{n_{0}}, L_{u}^{n_{0}}\right\}<D_{u}<P_{u}^{n_{0}-1}$ and

$$
\begin{equation*}
-\frac{b c}{a^{2}}>I\left(n_{0}\right)=\frac{2 n_{0}^{3}\left(n_{0}-1\right)^{3}}{\left\{n_{0}^{2}+\left(n_{0}-1\right)^{2}\right\}^{2}} . \tag{1-1}
\end{equation*}
$$

$\mathrm{B}_{2}$ ) Suppose that $B$ satisfies (0-3). Then for each $n_{0} \in N, S_{n_{0}}$ holds if and only if $D \in H_{n_{0}}$ and $P_{u}^{n_{0}}<D_{u}<P_{u}^{n_{0}-1}\left(P_{u}^{0}=+\infty\right.$ for convention $)$.

In the following we consider the bifurcation problem of (P-2) as $D(\sigma)$ crosses the bifurcation curve stated in Lemma 1 . We assume that $D(\sigma)$ satisfies the following two conditions:

1) $D(\sigma)$ is a smooth vector-valued function of $\sigma$ defined in the neighborhood of $\sigma=0$ and $D_{0}=D(0)$ is on the bifurcation curve in Lemma 1, i.e., there exists an $n_{0} \in N$ and $D_{0} \in H_{n_{0}}$.
2) $\left.(d / d \sigma) D(\sigma)\right|_{\sigma=0}=D^{\prime}(0) \neq 0$ and the vector $D^{\prime}(0)$ intersects transversally with the curve $H_{n_{0}}$ at $D_{0}$.

Using the Theorem 2.4 of [1], we obtain the next theorem.
Theorem 1. Suppose that (0-1), (0-2) (or (0-3)), (1-2) and (1-3) hold and that in case $\left(\mathrm{B}_{1}\right) B$ satisfies the inequality (1-1) besides (0-2). Then there exists a unique one-parameter family of nontrivial classical solutions $(D(\sigma(s)), U(s))$ of (P-2) for $|s|^{{ }^{\beth} s_{0}}$ such that $\sigma(s)$ and $U(s)$ are smooth with respect to $s$ and

$$
U(s)=s U_{n_{0}} \sin \frac{n_{0} \pi}{L} x+o(s) \quad \text { as } s \rightarrow 0
$$

and

$$
\sigma(0)=0 .
$$

§ 2. Nonlinear stability. For simplicity we assume that $F(U)$ is real analytic in this section, i.e., $f_{i}(u, v)$ is a real analytic function with respect to $u$ and $v, i=1,2$.

The linearized stability of the bifurcating solution $U(s)$ is determined by the bifurcation direction, i.e., the form of $\sigma(s)$ near $s=0$ (cf. [2]). In Lemma 2 we give a simple criterion of the bifurcation direction when $n_{0}$ is an odd number. (Note that $U_{n_{0}} \sin \left(n_{0} \pi / L\right) x$ is symmetric with respect to $x$ when $n_{0}$ is odd.)

Using the methods of [3] and [4], we can prove the nonlinear stability or instability of $U(s)$ bifurcating from symmetric modes.

Lemma 2. Suppose that the assumptions of Theorem 1 hold and let $Q(U)$ be a quadratic part of $F(U)$ and let $U_{n_{0}}^{*} \sin \left(n_{0} \pi / L\right) x$ be an eigenfunction of the adjoint equation of (P-3) which corresponds to the zero eigenvalue. Then if $n_{0}$ is odd, $\dot{\sigma}(0) \neq 0(\cdot=d / d s)$ if and only if

$$
\begin{equation*}
\int_{0}^{L}\left(Q\left(U_{n_{0}} \sin \frac{n_{0} \pi}{L} x\right), U_{n_{0}}^{*} \sin \frac{n_{0} \pi}{L} x\right) d x \neq 0 \tag{C}
\end{equation*}
$$

Here (, ) denotes the usual inner product in $R^{2}$.
Remark 1. If $n_{0}$ is even, the bifurcating solution $U(s)$ in Theorem 1 is in general unstable. We shall study about this in a forthcoming paper.

We note that the criterion (C) in Lemma 2 is a fairly general condition and is satisfied by almost all the nonlinear operators.

From the relation between bifurcation direction and a critical eigenvalue in Theorem 1.16 of [2], we obtain the following lemma about linearized stability.

Lemma 3. Let $n_{0}$ be odd and assume that the criterion (C) holds. Then the bifurcation occurs on both sides of the bifurcation curve $H_{n_{0}}$, i.e., $D(\sigma(s))$ intersects transversally with $H_{n_{0}}$ as $s$ moves in $\left(-s_{0}, s_{0}\right)$. (Therefore the curve $D(\sigma(s)),|s|<s_{0}$ is divided into two parts, i.e., one is on the upper side of $H_{n_{0}}$ and another is on the lower side of it.) And the upper side bifurcating solutions are stable and the lower side bifurcating ones are unstable in a linearized sense.

The perturbed system of equations from $U(s)$ is obtained by inserting $U=U(s)+W$ into (P-1) as follows:

$$
\begin{align*}
& W_{t}=D(\sigma(s)) W_{x x}+B W+F_{U}(U(s)) W+G(W ; U(s)), \\
& W(t, 0)=W(t, L)=0  \tag{P-4}\\
& W(0, x)=W_{0}
\end{align*}
$$

where

$$
G(W ; U(s))=F(U(s)+W)-F(U(s))-F_{U}(U(s)) W
$$

Let us define the following two linear operators in $E=\left(L^{2}(0, L)\right)^{2}$ with norm $\|\cdot\|$ :

$$
A=-D(\sigma(s)) \frac{\partial^{2}}{\partial x^{2}}, \quad D(A)=\left(H^{2}(0, L)\right)^{2} \cap\left(H_{0}^{1}(0, L)\right)^{2}
$$

$$
\tilde{A}=A-B-F_{U}(U(s)), \quad D(\tilde{A})=D(A) .
$$

Using the results of [4], we conclude from Lemma 3:
Theorem 2. Let the assumptions of Theorem 1 and Lemma 3 hold. Then the upper side bifurcating solutions $U(s)$ are asymptotically stable in the topology of $D\left(A^{\alpha}\right)(1 / 2 \leqq \alpha<1)$, i.e., for any $\varepsilon>0$ there exists a positive number $\delta(\varepsilon)$ and if $\left\|A^{\alpha} W_{0}\right\|<\delta(\varepsilon)$, (P-4) has a global strict solution and we have

$$
\left\|A^{\alpha} W(t)\right\| \leqq \varepsilon e^{-b t}, \quad t \in[0,+\infty)
$$

The value $b>0$ is determined by the spectrum of $\tilde{A}$, i.e., $0<b<\operatorname{Re}(\tilde{A})$.
As for the lower side bifurcating solutions, they are unstable in the topology of $E$.
§3. Examples. 1) We consider the following system of equations (cf. [5]) :

$$
\begin{align*}
& u_{t}=D_{u} u_{x x}+\left(2+u-u^{2}\right) u-u v \\
& v_{t}=D_{v} v_{x x}-g v+u v,  \tag{3-1}\\
& u(t, 0)=u(t, L)=u_{0}, \quad v(t, 0)=v(t, L)=v_{0}
\end{align*}
$$

where $g$ is a constant such that $0<g<1 / 2$ and ( $u_{0}, v_{0}$ ) is a unique positive constant solution of (3-1).

Applying the following transformation to (3-1),

$$
\begin{equation*}
\hat{u}=u-u_{0}, \quad \hat{v}=v-v_{0} \tag{3-2}
\end{equation*}
$$

we obtain the system of equations:

$$
\begin{align*}
& \hat{u}_{t}=D_{u} \hat{u}_{x x}+(1-2 g) g \hat{u}-g \hat{v}-\hat{u}^{3}+(1-3 g) \hat{u}^{2}-\hat{u} \hat{v} \\
& \hat{v}_{t}=D_{v} \hat{v}_{x x}+v_{0} \hat{u}+\hat{u} \hat{v},  \tag{3-3}\\
& \hat{u}(t, 0)=\hat{u}(t, L)=0, \quad \hat{v}(t, 0)=\hat{v}(t, L)=0 .
\end{align*}
$$

It is easy to see that this system corresponds to the case (0-2), and we can apply Theorems 1 and 2 to (3-3).
2) (M. Mimura's patchiness model.) Next we consider the following system of equations:

$$
\begin{align*}
& u_{t}=D_{u} u_{x x}+\left(\frac{1}{9}\left(-u^{2}+16 u+35\right)-v\right) u \\
& v_{t}=D_{v} v_{x x}+\left(-\left(1+\frac{2}{5} v\right)+u\right) v  \tag{3-4}\\
& u(t, 0)=u(t, L)=5, \quad v(t, 0)=v(t, L)=10
\end{align*}
$$

where ( 5,10 ) is a unique positive constant solution of (3-4). Applying the same procedure to (3-4) as in 1), we get a system of equations which corresponds to the case ( $0-3$ ) and to which we can apply Theorems 1 and 2.

## References

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