Bifurcation, Perturbation of Simple Eigenvalues, and Linearized Stability

MICHAEL G. CRANDALL & PAUL H. RABINOWITZ

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This paper is mainly concerned with estimating the eigenvalue of minimum modulus in the spectrum of the Fréchet derivative of a nonlinear operator along a bifurcating curve of zeros of the operator. In order to be more precise, a few preliminaries are in order. Let X and Y be Banach spaces and let $F: \mathbb{R} \times X \to Y$ be twice continuously differentiable. Suppose that $F(\lambda, 0)=0$ for $\lambda \in \mathscr{I}$, where $\mathscr{I}$ is an open interval containing λ_0, and that every neighborhood of $(\lambda_0, 0)$ contains zeroes of F not lying on the curve $\mathscr{C}=\{(\lambda, 0): \lambda \in \mathscr{I}\}$. Then $(\lambda_0, 0)$ is said to be a bifurcation point of F with respect to $\mathscr{C}$. In an earlier paper [8] the authors gave fairly general criteria for $(\lambda_0, 0)$ to be a bifurcation point when 0 is a "simple eigenvalue" (in a sense made precise later) of the Fréchet derivative $F_x(\lambda_0, 0)$ of F with respect to its second argument. These sufficient conditions imply that there is a bifurcating curve $\hat{\mathscr{C}}=\{(\lambda(s), x(s)): |s|<s_0\}$ of zeroes of F intersecting $\mathscr{C}$ only at $(\lambda_0, 0)$.

It will be shown that the zero eigenvalue of $F_x(\lambda_0, 0)$ corresponds to small real eigenvalues $\gamma(\lambda)$ of $F_x(\lambda, 0)$ and $\mu(s)$ of $F_x(\lambda(s), x(s))$, where $\gamma(\lambda_0)=0=\mu(0)$. Our main interest here is in studying the relationship between $\lambda(s)$, $\gamma(\lambda)$ and $\mu(s)$. We will prove that if $F_x(\lambda_0, 0)$ satisfies appropriate simplicity conditions, then $\mu(s)$ and $-s\lambda'(s)\gamma'(\lambda_0)$ have the same zeroes and, where $\lambda'(s) \neq 0$, the same sign.

Such qualitative information about $\mu(s)$ is important in problems where $F(\lambda, x)=0$ is the equilibrium form of an evolution equation

$$\frac{dx}{dt}=F(\lambda, x). \tag{1}$$

In this case one is interested in the stability of equilibrium solutions of (1). Determination of the sign of $\mu(s)$ provides a first step in answering the stability question for the equilibrium solution $x(s)$ of (1) (for $\lambda=\lambda(s)$). Indeed, in problems in the applied mathematics literature, $x(s)$ is frequently called stable or unstable (in a linearized sense) depending on whether $\mu(s)$ is positive or negative. The stability question for (1) will not be pursued here beyond the study of $\mu(s)$. In a forthcoming paper, however, we shall study the relationship between $\mu(s)$ and the full nonlinear problem (1).

It is implicit in the fluid dynamics literature that there is a connection between $\lambda(s)$, $\mu(s)$, and $\gamma(\lambda)$ in certain important cases. No explicit mention of the relationship is made, but it is apparent for example in the formal calculations of [6] and [15], as well as in the arguments of [9] and [13] where analytic perturbation theory is employed. More recently, SATTINGER [19] studied a special case of $F(\lambda, x)=0$ which could be converted to an equivalent problem involving compact operators, for which various technical conditions were required. SATTINGER used a topological degree of mapping argument to relate the sign of $\mu(s)$ to that of $\lambda(s)$. Our results substantially generalize and sharpen those of [19]. The methods used here involve only elementary analytical arguments, and the use of topological degree is supplanted by a rather precise and simple estimate.

Section 1 is devoted to the precise formulation and proof of our main result. A number of applications are given in Section 2. In Section 3 it is shown how the ideas already developed can be translated to treat situations in which there is no bifurcation. These results apply, in particular, to problems involving positive solutions of operator equations. Section 4 contains examples of this kind.

1. Bifurcation and Linearized Stability

In this section we will establish our main result concerning the eigenvalue of minimum modulus of the linearized operator produced by bifurcation. Several preliminaries are required, particularly the notion of a K-simple eigenvalue and its perturbation properties.

Throughout this section X and Y are real Banach spaces, V is an open neighborhood of 0 in X, $\mathscr{I}=(a, b)\subset \mathbb{R}$ is an open interval, and $F: \mathscr{I}\times V \to Y$ is a twice continuously Fréchet differentiable mapping. Let F_λ, F_x, $F_{\lambda x}$, etc., denote the various derivatives of F with respect to $\lambda\in\mathscr{I}$ and $x\in X$. The null space and range of a linear mapping A are denoted by $N(A)$ and $R(A)$. Let dim and codim denote, respectively, dimension and codimension. A linear map $A: X\to Y$ is called nonsingular if A is a homeomorphism of X onto Y. Otherwise A is called singular.

The following result is contained in Theorem 1.7 of [8]:

Lemma 1.1. *Suppose that $\lambda_0\in\mathscr{I}$ and also that*

(i) $F(\lambda, 0)=0$ *for* $\lambda\in\mathscr{I}$,

(ii) $\dim N(F_x(\lambda_0, 0))=\operatorname{codim} R(F_x(\lambda_0, 0))=1$,

(iii) $F_{\lambda x}(\lambda_0, 0)x_0\notin R(F_x(\lambda_0, 0))$ *where* $x_0\in X$ *spans* $N(F_x(\lambda_0, 0))$.

Let Z be any complement of span $\{x_0\}$ in X. Then there exists an open interval $\hat{\mathscr{I}}$ containing 0 and continuously differentiable functions $\lambda: \hat{\mathscr{I}}\to\mathbb{R}$ and $\psi: \hat{\mathscr{I}}\to Z$ such that $\lambda(0)=\lambda_0$, $\psi(0)=0$, and, if $x(s)=sx_0+s\psi(s)$, then $F(\lambda(s), x(s))=0$. Moreover, $F^{-1}(\{0\})$ near $(\lambda_0, 0)$ consists precisely of the curves $x=0$ and $(\lambda(s), x(s))$, $s\in\hat{\mathscr{I}}$.

The assumptions (ii), (iii) of Lemma 1.1 imply that $F_x(\lambda_0, 0)$ has 0 as a simple eigenvalue in a sense made precise next.

Let $B(X, Y)$ denote the set of bounded linear maps of X into Y.

Definition 1.2. Let $T, K \in B(X, Y)$. Then $\mu \in \mathbb{R}$ is a K-simple eigenvalue of T if

(i) $$\dim N(T-\mu K) = \operatorname{codim} R(T-\mu K) = 1$$

and, if $N(T-\mu K) = \operatorname{span}\{x_0\}$,

(ii) $$K x_0 \notin R(T-\mu K).$$

Thus (ii), (iii) of Lemma 1.1 may be restated as: $F_x(\lambda_0, 0)$ has 0 as a $F_{\lambda x}(\lambda_0, 0)$-simple eigenvalue. This terminology is motivated by the case where $X = Y$, $K = I$ and T is a compact operator. In this case $\mu \neq 0$ is a I-simple eigenvalue of T if and only if μ is an algebraically simple eigenvalue of T.

We will show that if $F_x(\lambda_0, 0)$ has 0 as a K-simple (as well as a $F_{\lambda x}(\lambda_0, 0)$-simple) eigenvalue, then, for s near zero, $F_x(\lambda(s), x(s))$ has a unique small K-simple eigenvalue $\mu(s)$ and the sign of $\mu(s)$ is governed by that of $s\lambda'(s)$. The most important property of K-simple eigenvalues needed in the sequel is given in the following perturbation result.

Lemma 1.3. *Let $T_0, K \in B(X, Y)$ and assume that r_0 is a K-simple eigenvalue of T_0. Then there exists a value $\delta > 0$ such that whenever $T \in B(X, Y)$ and $\|T - T_0\| < \delta$, there is a unique $r(T) \in \mathbb{R}$ satisfying $|r(T) - r_0| < \delta$ for which $T - r(T)K$ is singular. The map $T \to r(T)$ is analytic and $r(T)$ is a K-simple eigenvalue of T. Finally, if $N(T_0 - r_0 K) = \operatorname{span}\{x_0\}$ and Z is a complement of span $\{x_0\}$ in X, there is a unique null vector $x(T)$ of $T - r(T)K$ satisfying $x(T) - x_0 \in Z$. The map $T \to x(T)$ is also analytic.*

Proof. As in the proof of Theorem 1.7 of [8], we use the implicit function theorem to verify the existence assertions. Uniqueness then follows from the uniqueness inherent in the implicit function theorem and an auxiliary estimate. We assume $r_0 = 0$ without loss of generality. Let x_0, Z be as in the statement of the lemma and define

$$f: B(X, Y) \times \mathbb{R} \times Z \to Y$$

by

(1.4) $$f(T, r, z) = T(x_0 + z) - rK(x_0 + z).$$

Then $f(T_0, 0, 0) = T_0 x_0 = 0$ by assumption, and the Fréchet derivative of f with respect to (r, z) at (T, r, z) is the linear map

$$(\hat{r}, \hat{z}) \to f_{(r,z)}(T, r, z)(\hat{r}, \hat{z}) = T\hat{z} - (rK\hat{z} + \hat{r}K(x_0 + z)).$$

Clearly $f_{(r,z)}(T_0, 0, 0)$ is an isomorphism of $\mathbb{R} \times Z$ onto Y. Since f is analytic the implicit function theorem implies the existence of analytic functions $r(T)$, $z(T)$ such that $f(T, r(T), z(T)) = 0$ and $r(T_0) = 0$, $z(T_0) = 0$. Setting $x(T) = x_0 + z(T)$, we have verified the existence assertions of the lemma. It remains to prove uniqueness and simplicity. To show the uniqueness of r near 0 such that $T - rK$ is singular, first observe that $T - rK$ has Fredholm index 0 since it is near T_0. Moreover $\dim N(T - rK) \leqq 1$ since the dimension of the null space is a lower semicontinuous function (see [11], Theorem 5.17). Thus either $T - rK$ is nonsingular or $\dim N(T - rK) = 1$. We shall prove that if $\|T - T_0\|$ and $|r|$ are sufficiently small and if $(T-rK)(\beta x_0+z)=0$, where $\beta\in\mathbb{R}$, $z\in Z$, then

$$\|z\|+|\beta r|\leqq \text{const.}\,|\beta|\,\|T-T_0\| \tag{1.5}$$

where the constant depends on T_0 and Kx_0. The uniqueness assertion will follow immediately since (1.5) shows that all null vectors of $T-rK$, are obtained by our application of the implicit function theorem to (1.4). To verify (1.5), suppose that $\beta x_0+z\in N(T-rK)$. Then

$$T_0 z-r\beta K x_0=(T_0-T)(\beta x_0+z)+rKz. \tag{1.6}$$

Since $(r,z)\to T_0 z-rKx_0$ is nonsingular, (1.6) implies

$$|\beta r|+\|z\|\leqq c\big(\|T-T_0\|(|\beta|+\|z\|)+|r|\,\|z\|\big) \tag{1.7}$$

for some constant c. If $c\|T-T_0\|$ and $c|r|<1/3$, (1.7) implies an estimate of the form (1.5).

It remains to show that $r(T)$ is a K-simple eigenvalue of T. According to the above remarks, $1=\dim\big(N(T-r(T)K)\big)=\operatorname{codim} R\big(T-r(T)K\big)$. We need only show that $Kx(T)=K\big(x_0+z(T)\big)\notin R\big(T-r(T)K\big)$. It is easily checked that $X=\operatorname{span}\{x(T)\}\oplus Z$ for T near T_0. Suppose $Kx(T)=\big(T-r(T)K\big)w$ for some $w\in X$. Writing $w=\bar\beta x(T)+\bar z$ for some $\bar\beta\in\mathbb{R}$ and $\bar z\in Z$ we find that

$$Kx(T)=\big(T-r(T)K\big)w=\big(T-r(T)K\big)\bar z. \tag{1.8}$$

Since $T_0\colon Z\to R(T_0)$ is an isomorphism, $\|T_0 z\|\geqq c_1\|z\|$ for some $c_1>0$ and $z\in Z$, which implies $\|\big(T-r(T)K\big)z\|\geqq c_2\|z\|$ where $c_2=c_1-\|T_0-T+r(T)K\|$. If T is sufficiently close to T_0, c_2 is positive and (1.8) implies $\|\bar z\|\leqq M$ for some constant M independent of T near T_0. Let y^* denote the dual space of Y and $\langle l,y\rangle$ the pairing of $l\in Y^*$ and $y\in Y$. Choose $l\in Y^*$ to satisfy $N(l)=R(T_0)$. Then (1.8) implies

$$\langle l,Kx(T)\rangle=\langle l,(T-T_0)\bar z\rangle-r(T)\langle l,K\bar z\rangle. \tag{1.9}$$

This implies

$$|\langle l,Kx(T)\rangle|\leqq\|l\|\,M\big(\|T-T_0\|+|r(T)|\,\|K\|\big).$$

The right-hand side above vanishes at $T=T_0$, while the left hand side is $|\langle l,Kx_0\rangle|\neq 0$. Hence $Kx(T)\notin R\big(T-r(T)K\big)$ for T near T_0. This completes the proof.

Remark 1.10. While we have assumed that X and Y were real Banach spaces, Definition 1.2 and Lemma 1.3 obviously carry over to complex scalars without change.

Remark 1.11. It is easy to vary K and T in the assertions of Lemma 1.3, though this is not needed later.

An interesting corollary of Lemma 1.3 is

Corollary 1.12. *Let* $\mathscr{L}=\{T\in B(X,Y)\colon \dim N(T)=\operatorname{codim} R(T)=1\}$. *Then* $\mathscr{L}$ *is an analytic submanifold of* $B(X,Y)$ *of codimension* 1.

This corollary is also not needed later; its proof is postponed until the end of this section.

In the next two results F, Z, x_0 are as in Lemma 1.1, and $\lambda(s)$, $x(s)$ are the functions provided by that lemma. We write $F_x(s)=F_x(\lambda(s), x(s))$, $F_\lambda(s)=F_\lambda(\lambda(s), x(s))$, etc.

Corollary 1.13. *Let $K\in B(X, Y)$ and let 0 be a K-simple eigenvalue of $F_x(\lambda_0, 0)$. Then there exist open intervals $\hat{\mathscr{I}}$, $\hat{\mathscr{J}}$ with $\lambda_0\in\hat{\mathscr{I}}$, $0\in\hat{\mathscr{J}}$ and continuously differentiable functions $\gamma: \hat{\mathscr{I}}\to\mathbb{R}$, $\mu: \hat{\mathscr{J}}\to\mathbb{R}$, $u: \hat{\mathscr{I}}\to X$, $w: \hat{\mathscr{J}}\to X$ such that*

$$(1.14)\quad \begin{aligned} &\text{(i)} && F_x(\lambda, 0)\,u(\lambda)=\gamma(\lambda)\,K u(\lambda) \quad \textit{for } \lambda\in\tilde{\mathscr{I}},\\ &\text{(ii)} && F_x(s)\,w(s)=\mu(s)\,K w(s) \quad \textit{for } s\in\tilde{\mathscr{J}}. \end{aligned}$$

Moreover

$$\gamma(\lambda_0)=\mu(0)=0, \quad u(\lambda_0)=x_0=w(0), \quad \textit{and} \quad u(\lambda)-x_0\in Z, \quad w(s)-x_0\in Z.$$

Proof. The corollary follows at once from Lemma 1.3, that is, if $r(T)$ is the function provided by Lemma 1.3 for $T_0=F_x(0)$ and $r_0=0$, then $\mu(s)=r(F_x(s))$, etc. The differentiability assertions follow from the observation that $F_x(\lambda, 0)$ and $F_x(s)$ are continuously differentiable with respect to λ and s respectively.

Remark 1.15. In many applications of interest $F(\lambda, x)=0$ is the equilibrium form of (1), and $X\subset Y$, and the injection, which we call i, is continuous. In this case one is interested in the stability of equilibrium solutions. Here the natural choice of K is $K=i$, for then equations (1.14) become the usual equations encountered in the study of linearized stability. Several examples in Section 2 will be of this kind.

The main result relating $\mu(s)$, $\lambda(s)$ and $\gamma(\lambda)$ is

Theorem 1.16. *Let the assumptions of Corollary 1.13 hold and let γ, μ be the functions provided by the corollary. Then $\gamma'(\lambda_0)\neq 0$, and near $s=0$ the functions $\mu(s)$ and $-s\lambda'(s)\gamma'(\lambda_0)$ have the same zeroes, and, whenever $\mu(s)\neq 0$, the same sign. More precisely*

$$(1.17)\qquad \lim_{\substack{s\to 0\\ \mu(s)\neq 0}} \frac{-s\lambda'(s)\gamma'(\lambda_0)}{\mu(s)}=1.$$

Moreover, there is a constant C such that

$$(1.18)\qquad \|x'(s)-w(s)\|\leqq C\min\{|s\lambda'(s)|, |\mu(s)|\}$$

near $s=0$.

Proof. Differentiating (1.14, (i)), we obtain

$$F_{\lambda x}(\lambda, 0)\,u(\lambda)+F_x(\lambda, 0)\,u'(\lambda)=\gamma(\lambda)\,K u'(\lambda)+\gamma'(\lambda)\,K u(\lambda).$$

In particular, setting $\lambda=\lambda_0$, we get

$$(1.19)\qquad F_{\lambda x}(\lambda_0, 0)\,x_0+F_x(\lambda_0, 0)\,u'(\lambda_0)=\gamma'(\lambda_0)\,K x_0.$$

Since $F_{\lambda x}(\lambda_0, 0)x_0\notin R(F_x(\lambda_0, 0))$, (1.19) implies that $\gamma'(\lambda_0)\neq 0$. The proof now proceeds by establishing that

$$(1.20)\qquad \|x'(s)-w(s)\|\leqq c(|s\lambda'(s)|+|\mu(s)|)$$

for some c, and then showing that

$$|s\lambda'(s)\gamma'(\lambda_0)+\mu(s)|\leqq o(1)(|s\lambda'(s)|+|\mu(s)|) \quad \text{as} \quad s\to 0. \tag{1.21}$$

The remaining assertions of the theorem follow from (1.20), (1.21) and the observation that if $a, b\in\mathbb{R}$, $\theta\in(0, 1)$, and $|a+b|\leqq\theta(|a|+|b|)$, then

$$ab\leqq 0 \quad \text{and} \quad (1-\theta)(1+\theta)^{-1}|b|\leqq|a|\leqq(1+\theta)(1-\theta)^{-1}|b|. \tag{1.22}$$

Since $F(\lambda(s), x(s))=0$

$$\lambda'(s)F_\lambda(s)+F_x(s)x'(s)=0. \tag{1.23}$$

Subtracting (1.14 (ii)) from (1.23) yields

$$F_x(s)(x'(s)-w(s))+\lambda'(s)F_\lambda(s)+\mu(s)Kw(s)=0. \tag{1.24}$$

By Taylor's theorem, Lemma 1.1, and Corollary 1.13

$$\begin{aligned} &\text{(i)} \quad F_\lambda(s)=F_\lambda(0)+s(F_{\lambda x}(0)x'(0)+\lambda'(0)F_{\lambda\lambda}(0))+o(s)=sF_{\lambda x}(0)x_0+o(s),\\ &\text{(ii)} \quad F_x(s)=F_x(0)+o(1),\\ &\text{(iii)} \quad w(s)=x_0+o(1) \end{aligned} \tag{1.25}$$

as $s\to 0$. Putting (1.25) in (1.24) yields

$$\begin{aligned} &F_x(0)(x'(s)-w(s))+o(1)(x'(s)-w(s))\\ &\qquad+s\lambda'(s)F_{\lambda x}(0)x_0+\lambda'(s)o(s)+\mu(s)Kx_0+\mu(s)Ko(1)=0 \end{aligned} \tag{1.26}$$

as $s\to 0$. Since $x'(s)-x_0\in Z$ (Lemma 1.1) and $w(s)-x_0\in Z$ (Corollary 1.13), it follows that $x'(s)-w(s)\in Z$. Since $F_x(0)$ is an isomorphism of Z onto $R(F_x(0))$, there is a number $c>0$ such that $\|F_x(0)z\|\geqq c\|z\|$ for $z\in Z$. This with (1.26) yields

$$\begin{aligned} \|x'(s)-w(s)\|\leqq c_1 &\|s\lambda'(s)F_{\lambda x}(0)x_0+\mu(s)Kx_0\|\\ &+o(1)(|s\lambda'(s)|+|\mu(s)|)\leqq c_2(|s\lambda'(s)|+|\mu(s)|) \end{aligned}$$

for suitable constants c_1, c_2 and sufficiently small $|s|$. Thus (1.20) is established. Next let $l\in Y^*$ satisfy $N(l)=R(F_x(0))$. Applying l to (1.19), we obtain

$$\langle l, F_{\lambda x}(0)x_0\rangle=\gamma'(\lambda_0)\langle l, Kx_0\rangle, \tag{1.27}$$

while applying l to (1.26) shows that

$$\begin{aligned} &|s\lambda'(s)\langle l, F_{\lambda x}(0)x_0\rangle+\mu(s)\langle l, Kx_0\rangle|\\ &\qquad\leqq o(1)|\langle l, x'(s)-w(s)\rangle|+o(1)(|s\lambda'(s)|+|\mu(s)|). \end{aligned} \tag{1.28}$$

Using the identity (1.27) and the estimate (1.20) in conjunction with (1.28) immediately yields (1.21), and the proof is complete.

Remark 1.29. The relation (1.19) holds provided only that functions $u(\gamma)$, $\gamma(\lambda)$ possessing the properties asserted by Corollary 1.13 exist. It follows from (1.19), however, that 0 is a K-simple eigenvalue of $F_x(\lambda_0, 0)$ whenever 0 is an $F_{\lambda x}(\lambda_0, 0)$-simple eigenvalue of $F_x(\lambda_0, 0)$. This is a partial converse of Corollary 1.13. Note finally that continuous functions $u(\lambda)$, $\gamma(\lambda)$ may exist possessing all the required properties, save differentiability, even if 0 is not a K-simple eigenvalue of $F_x(\lambda_0, 0)$. See Example (1.35).

It is usually the case in applications that F is a nonlinear differential operator. A specialization of Lemma 1.1 tailored to this situation was given in [8] (Theorem 2.4); we will give a simpler analogue for Theorem 1.16.

Example 1.30. Consider the problem

$$(L-\lambda B)u+H(\lambda, u)=0, \qquad \lambda\in\mathbb{R},\ u\in D(L) \tag{1.31}$$

where $D(L)$ is the domain of L. In (1.31), L and B are closed linear operators with domains and ranges in a Banach space U. In addition, $D(L)\subset D(B)$ and $H\colon \mathbb{R}\times D(L)\to U$. To put (1.31) into the framework of Theorem 1.16, let $X=D(L)$ (with the graph topology) and $Y=U$. Define $F\colon R\times X\to Y$ by $F(\lambda, x)=(L-\lambda B)x+H(\lambda, x)$. We assume H (and hence F) is a twice continuously differentiable map of $\mathbb{R}\times X$ into Y and $H(\lambda, 0)=0$, $H_x(\lambda, 0)=0$. Then $F_x(\lambda, 0)=(L-\lambda B)$ and $F_{\lambda x}(\lambda, 0)=-B$. The further assumption that 0 is a B-simple eigenvalue of $(L-\lambda_0 B)$ allows us to invoke Lemma 1.1 to obtain a curve of nontrivial solutions $(\lambda(s), x(s))$ of (1.26) bifurcating from $(\lambda_0, 0)$. Let i denote the injection of X into Y. Suppose that 0 is also an i-simple eigenvalue of $L-\lambda_0 B$ (see Example 1.34 below in this regard). Then Theorem 1.16 is applicable here with $K=i$; consequently $\mu(s)$ and $-s\lambda'(s)\gamma'(\lambda_0)$ have the same zeroes and, wherever $\mu(s)\neq 0$, the same sign. In this case, (1.27) becomes $-\langle l, Bx_0\rangle=\gamma'(\lambda_0)\langle l, x_0\rangle$, which will be used later.

Remark 1.32. Equation (1.31) was discussed in [19] by SATTINGER. In addition to hypotheses similar to those above, he assumes that L possesses a compact inverse which can be used to convert (1.31) into an equivalent equation of the form

$$(I-\lambda T)u+N(\lambda, u)=0, \qquad \lambda\in\mathbb{R},\ u\in U \tag{1.33}$$

where T and N are compact mappings. (The B-simple assumption was used but not explicitly mentioned in [19]. See also Example 1.34 below.) It was also assumed in [19] that $\gamma(\lambda)<0$ for $\lambda_0<\lambda$, $\gamma(\lambda_0)=0$, $\gamma(\lambda)>0$ for $\lambda<\lambda_0$, and that $\lambda'(s)\neq 0$ for $s\neq 0$. The main result of [19], whose proof involved the use of Leray-Schauder degree, was that $\mu(s)$ and $s\lambda'(s)$ have the same sign for $s\neq 0$. This assertion follows immediately from Theorem 1.16 (without several superfluous assumptions) since the fact that $\gamma(\lambda)>0$ for $\lambda<\lambda_0$ implies $\gamma'(\lambda_0)<0$.

Example 1.34. Here we show, in the framework of Example 1.30, that if 0 is an i-simple but not a B-simple eigenvalue of $L-\lambda_0 B$, then there may be no bifurcating curve $(\lambda(s), u(s))$, while if 0 is a B-simple but not an i-simple eigenvalue of $L-\lambda_0 B$, then the real eigenvalue of $L-\lambda_0 B$ may not continue as such for $L-\lambda B$ or $L-\lambda(s)B+H_u(\lambda(s), u(s))$. Let $X=Y=\mathbb{R}^2$ and write $x=(x_1, x_2)\in\mathbb{R}^2$. Set $L(x_1, x_2)=(x_1+x_2, 0)$, $B(x_1, x_2)=(x_2, 0)$, and $H(\lambda, x)=(0, x_1^2+x_2^2)$. Then, with $F(\lambda, x)=(L-\lambda B)x+H(\lambda, x)$, 0 is an I-simple eigenvalue of $F_x(1, 0)$. However, 0 is not a B-simple eigenvalue of $F_x(1, 0)$, and the only solutions of $F(\lambda, x)=0$ are $x=0$.

Next define $L(x_1, x_2)=(x_2, x_1)$, $B(x_1, x_2)=(0, x_1)$, $H(\lambda, x)=(0, -x_1^2)$, and $F(\lambda, x)=(L-\lambda B)x+H(\lambda, x)$. Then 0 is a B-simple eigenvalue of $F_x(1,0)$, but 0 is not an I-simple eigenvalue of $F_x(1,0)$. The curve of Lemma 1.1 through $(1,0)$ is $(\lambda(s), x(s))=(1+s, (s,0))$. The eigenvalues of $F_x(\lambda, 0)$ $(F_x(\lambda(s), x(s)))$ are $\pm\sqrt{1-\lambda}$ $(\pm\sqrt{s})$, which are complex for $\lambda>1$ $(s<0)$.

Example 1.35. This is the example referred to in Remark 1.29. Let $X=Y=\mathbb{R}^3$, $L(x_1, x_2, x_3)=(x_2, x_3, x_1)$, $B(x_1, x_2, x_3)=(0, 0, x_1)$, and $F(\lambda, x)=(L-\lambda B)x$. Then 0 is an $F_{\lambda x}(1,0)$-simple eigenvalue of $F_x(1,0)$. However $x_0=(1,0,0)\in R(F_x(1,0))$, while $F_x(\lambda, 0)u(\lambda)=\gamma(\lambda)u(\lambda)$ for $\gamma(\lambda)=(1-\lambda)^{1/3}$, $u(\lambda)=(1, \gamma(\lambda), \gamma(\lambda)^2)$.

We conclude this section with the proof of Corollary 1.12.

Proof of Corollary 1.12. Given $T_0\in\mathscr{L}$, choose any $K\in B(X, Y)$ such that 0 is a K-simple eigenvalue of T_0. Let $r(T)$ be given by Lemma 1.3 (here $r_0=0$). By Lemma 1.3, if T is near T_0 then $T\in\mathscr{L}$ if and only if $r(T)=0$. The result follows from the analyticity of r and the fact that $r'(T_0)\neq 0$. We give details for completeness. Let $\mathscr{M}$ be any complement of span $\{K\}$ in $B(X, Y)$. Then any T near T_0 has the form $T=T(\gamma, M)=T_0+\gamma K+M$, where $\gamma\in\mathbb{R}$ and $M\in\mathscr{M}$ are small. Consider $r(T)=r(T_0+\gamma K+M)=h(\gamma, M)$. Now $h(0,0)=0$, and (with the notation of Lemma 1.3) differentiating the relation

$$(1.36)\qquad (T_0+\gamma K+M)(x_0+z(T(\gamma, M)))=h(\gamma, M)\,K(x_0+z(T(\gamma, M)))$$

with respect to γ at $\gamma=0$, $M=0$, gives

$$(1.37)\qquad (Kx_0-h_\gamma(0,0)\,Kx_0)\in R(T_0).$$

Thus $h_\gamma(0,0)=1$, so the equation $h(\gamma, M)=r(T_0+\gamma K+M)=0$ is uniquely solvable for $\gamma(M)$ near $\gamma=0$, $M=0$, and the map $M\to\gamma(M)K+M$ and its inverse are analytic near $M=0$.

2. Applications of Theorem 1.16

This section consists of several applications of Theorem 1.16.

Example 2.1. Let $F(\lambda, u)=-u''+h(u^2+(u')^2)u-\lambda u$. Here $u\in X=C_0^2([0,\pi])=\{u\in C^2([0,\pi]): u(0)=u(\pi)=0\}$, $Y=C([0,\pi])$, $h\colon \mathbb{R}\to\mathbb{R}$ is twice continuously differentiable, and $h(0)=0$. If $\lambda_0=1$, the assumptions of Lemma 1.1 are satisfied and the bifurcating curve is given by $\lambda(s)=1+h(s^2)$, $u(s)=s\sin x$ (as is easy to check, or see [8, pg. 331]). If, e.g., $h(\tau)=\exp(\tau^{-1})\sin(\tau^{-1})$ for $\tau>0$, $h(0)=0$, $\lambda'(s)$ changes sign in every one-sided neighborhood of $s=0$. Letting K in Theorem 1.16 be the natural injection of X into Y, we find (clearly $\gamma(\lambda)=1-\lambda$) that $\mu(s)$ has the same sign and same zeros as $s\lambda'(s)$ near $s=0$. It does not seem to be a simple matter to verify this directly from the equation. Moreover, (1.17) and (1.18) provide estimates on $\mu(s)$ and the corresponding eigenfunction.

Remark 2.2. It is interesting to note that while it is impractical to determine the sign of $\mu(s)$ in Example 2.1 by computing a topological degree, one can easily do the converse. More precisely, the condition $F(\lambda, u)=0$ for this case may be rewritten as $\Phi(\lambda, u) = u - T(\lambda, u) = 0$, where T: $\mathbb{R} \times E \to E$,

$$E = \{u \in C^1([0, \pi]) : u(0) = u(\pi) = 0\}$$

and

$$T(\lambda, u) = \int_0^\pi g(x, y)[\lambda u(y) - h((u(y)^2 + (u'(y))^2) u(y)] \, dy$$

with

$$g(x, y) = g(y, x) = y(1 - (x/\pi)) \quad \text{for } 0 \leqq y \leqq x \leqq \pi.$$

Since $\Phi_u(\lambda(s), u(s))$ has a nontrivial null space if and only if $F_u(\lambda(s), u(s))$ has a nontrivial null space, Theorem 1.16 implies that $\Phi_u(\lambda(s), u(s))$ is invertible whenever $\lambda'(s) \neq 0$. Therefore the Leray-Schauder index of $u(s)$ as a solution of $\Phi(\lambda(s), \cdot) = 0$ is defined and equal to $(-1)^\beta$, where β is the sum of the multiplicities of the characteristic values of $T_u(\lambda(s), u(s))$ in (0, 1). The characteristic values of $T_u(0, 0)$ are $\nu_j = j^2$, $j = 1, 2, \ldots$, and each has multiplicity 1. Hence the compactness of $T_u(\lambda(s), u(s))$ implies that for small s, $T_u(\lambda(s), u(s))$ has at most one characteristic value in (0, 1) and, if there is one such, it is simple. Applying Theorem 1.16, with K being the identity of E, shows that the eigenvalue $\mu(s)$ of $\Phi_u(\lambda(s), u(s))$ near zero has the same sign as $s\lambda'(s)$. Hence the characteristic value of $T_u(\lambda(s), u(s))$ near 1 (namely $(1 - \mu(s))^{-1}$) lies in (0, 1) if and only if $s\lambda'(s) < 0$. Thus the Leray-Schauder index is determined.

Example 2.3. Let Ω be a bounded open subset of $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$. Set $X = C_0^{2,\alpha}(\bar{\Omega}) = \{u \in C^{2,\alpha}(\bar{\Omega}) \mid u = 0 \text{ on } \partial\Omega\}$ and $Y = C^\alpha(\bar{\Omega})$. Here $\alpha \in (0, 1)$ and $C^{k,\alpha}(\bar{\Omega})$ is the space of k-times continuously differentiable functions on $\bar{\Omega}$ whose k-th order derivatives are Hölder continuous with exponent α, equipped with the usual norm. Let F: $\mathbb{R} \times X \to Y$ be given by

$$\begin{aligned} F(\lambda, u) = &- \sum_{i,j=1}^n a_{ij}(x) u_{x_i x_j} + \sum_{i=1}^n b_i(x) u_{x_i} + c(x) u \\ &- (\lambda a(x) u + h(\lambda, x, u, Du, D^2 u)) \end{aligned} \tag{2.4}$$

where Du and $D^2 u$ represent vectors whose components are the first and second order partial derivatives of u. We assume that $a_{ij}, b_i, c, a \in C^\alpha(\bar{\Omega})$ for $1 \leqq i, j \leqq n$, $c \geqq 0$ and $a > 0$ in $\bar{\Omega}$,

$$\sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j > 0 \quad \text{for } x \in \bar{\Omega} \text{ and } \xi \in \mathbb{R}^n,$$

h is three times continuously differentiable in its arguments, and

$$h(\lambda, x, 0, 0, 0) = 0, \quad h_{D^\sigma u}(\lambda, x, 0, 0, 0) = 0 \quad \text{for } |\sigma| \leqq 2.$$

As is well known [7],

$$Lu = - \sum_{i,j=1}^n a_{ij} u_{x_i x_j} + \sum_{i=1}^n b_i(x) u_{x_i} + c(x) u$$

defines an isomorphism L of X onto Y. Moreover, under our assumptions F is twice continuously differentiable and either $F_u(\lambda, 0)$ is an isomorphism or

$N(F_u(\lambda, 0))$ is of finite positive dimension. Now $F_u(\lambda, 0)v=0$ if and only if

$$v=\lambda L^{-1} a v. \tag{2.5}$$

It is a consequence (see, for example [17]) of the KREIN-RUTMAN theorem [14, Theorem 6.3] that if $W=C_0^{1,\alpha}(\bar{\Omega})$, then the restriction of $L^{-1}a$ to W has a positive ($I-$) simple characteristic value λ_0 with a corresponding eigenvector u_0 which is strictly positive in Ω. Since the null space of $I-\lambda_0 L^{-1}a$ is the same in Y and X as in W and $L^{-1}a$: $Y\to Y$ is compact, 0 is an I-simple eigenvalue of $I-\lambda_0 L^{-1}a$ in X. This is equivalent to the statement that 0 is an $F_{\lambda u}(\lambda_0, 0)=a$-simple eigenvalue of $F_u(\lambda_0, 0)=L-\lambda_0 a$. (Observe that $Lu_0\notin R(F_u(\lambda_0, 0))$ if and only if $au_0\notin R(F_u(\lambda_0, 0))$ whenever $(L-\lambda_0 a)u_0=0$.) It follows that Lemma 1.1 is applicable here; hence there is a curve $(\lambda(s), u(s))$ of zeroes of F bifurcating from $(\lambda_0, 0)$.

Below we verify that 0 is also an i-simple eigenvalue of $L-\lambda_0 a$, where i is the injection of X into Y, so that Theorem 1.16 can be applied with $K=i$. In the process it is verified that $\gamma'(\lambda_0)<0$, whence $\mu(s)$ and $s\lambda'(s)$ have the same zeroes and the same sign. Indeed, the KREIN-RUTMAN theorem implies the existence of a null vector $l_0\in W^*\subset X^*$ of $(I-\lambda_0 L^{-1}a)^*|_{W^*}$ such that $\langle l_0, \varphi\rangle>0$ for all nonzero $\varphi\in W$ such that $\varphi>0$ in the ordering defined in [17]. If 0 is not an i-simple eigenvalue of $L-\lambda_0 a$, there exists an element $u\in X$ such that $iu_0=u_0=(L-\lambda_0 a)u$. Then $L^{-1}u_0=(I-\lambda_0 L^{-1}a)u_0$ and $\langle l_0, L^{-1}u_0\rangle=\langle l_0, (I-\lambda_0 L^{-1}a)u\rangle=0$. However, $L^{-1}u_0>0$ in Ω by the maximum principle, and this is a contradiction. Finally we may use the functional $v\to\langle l_0, L^{-1}v\rangle$ as l in (1.27). Thus $-\langle l_0, L^{-1}au_0\rangle=\gamma'(\lambda_0)\langle l_0, L^{-1}u_0\rangle$ and $\gamma'(\lambda_0)<0$.

Remark 2.6. With the notation of Example 2.3, assume that the coefficients of L are sufficiently smooth so that the formal adjoint (with respect to the $L^2(\Omega)$ inner-product) L^* of L is defined and has continuous coefficients. If $L=L^*$, then 0 is an i-simple and an a-simple eigenvalue of $F_u(\lambda_0, 0)$ whenever $\dim N(F_u(\lambda_0, 0))=1$, this being independent of the sign of c. Indeed, the Fredholm alternative holds here, so that

$$\operatorname{codim} R(F_u(\lambda_0, 0))=1 \quad \text{and} \quad R(F_u(\lambda_0, 0))=\{v\in Y\colon \int_\Omega v u_0\, dx=0\}$$

where u_0 is the null vector of $F_u(\lambda_0, 0)$. In particular, u_0 and au_0 do not lie in $R(F_u(\lambda_0, 0))$.

Similarly, in the context of Example 1.30, symmetry properties for L and B can be used to help verify K-simplicity for various choices of K.

As a final example in this section, we indicate a precise sense in which Theorem 1.16 applies to the Bénard problem.

Example 2.7. (The Bénard problem.) This has been precisely formulated in [8]; we refer the reader to this reference for the definitions of X, Y, etc., below. Note however that the problem has the form

$$F(\lambda, u)=L_1 u-\lambda L_2 u-N_1(u)=0 \tag{2.8}$$

where the operators L_i: $X\to Y$ are linear and N_1 is the diagonal of a continuous bilinear map on $X\times X$. Suppose λ_0 is an L_2-simple eigenvalue of $L_1-\lambda_0 L_2$.

Then $R(L_1-\lambda_0 L_2)$ was characterized in [8] by

$$R(L_1-\lambda_0 L_2)=\{(f,\varphi)\in Y \mid \int_C (\bar{U}_0\cdot f+\lambda_0\theta_0\varphi)\,dx=0\}$$

where (U_0, θ_0, p_0) is the null vector of $L_1-\lambda_0 L_2$. Let $K(\bar{U},\theta,p)=(\bar{U},\theta)$. Theorem 1.16 is applicable to this K since clearly $(U_0,\theta_0)\notin R(L_1-\lambda_0 L_2)$. If λ_0 is the smallest L_2-simple eigenvalue of L_1, a uniqueness result of Joseph [10] states that (2.8) has no nontrivial solutions for $\lambda<\lambda_0$. A slight extension of this result shows this statement in fact holds for $\lambda\leqq\lambda_0$. Thus $\lambda(s)>\lambda_0$ for $s\neq 0$. Due to the analyticity of the present case, this implies that $s\lambda'(s)>0$ for small nonzero s. It is easy to see that $\gamma'(\lambda_0)<0$, whence $\mu(s)>0$ for small nonzero s.

3. Positivity and Linearized Stability

By way of motivation, consider the following situation. Let X and Y be Banach spaces and let F: $\mathbb{R}\times X\to Y$ be continuously differentiable. Suppose that $F(0,0)=0$ and that $F_x(0,0)$ is an isomorphism of X onto Y. Then, by the implicit function theorem, the solutions of

$$F(\lambda,x)=0 \tag{3.1}$$

near (0, 0) are given by a continuously differentiable curve $(\lambda, x(\lambda))$ with $x(0)=0$. Elementary arguments show that there is a number $\bar{\lambda}$, $0<\bar{\lambda}\leqq\infty$, which is maximal with respect to the existence of a continuous function x: $[0,\bar{\lambda})\to X$ for which $F(\lambda,x(\lambda))=0$ and $F_x(\lambda,x(\lambda))$ is nonsingular for $0\leqq\lambda<\bar{\lambda}$. Moreover, $\bar{\lambda}$ and $x(\lambda)$ are unique. Assuming that $\bar{\lambda}<\infty$, that bounded subsets of $F^{-1}(\{0\})$ are precompact in $\mathbb{R}\times X$, and that $\liminf_{\lambda\uparrow\bar{\lambda}}\|x(\lambda)\|<\infty$, then there is at least one $\bar{x}\in X$ such that $F(\bar{\lambda},\bar{x})=0$ and $\bar{x}=\lim_{n\to\infty} x(\lambda_n)$ for some sequence $\lambda_n\uparrow\bar{\lambda}$. The maximality of $\bar{\lambda}$ then implies that $F_x(\bar{\lambda},\bar{x})$ is singular.

Using the ideas of [8] and Section 1, one can precisely describe the solutions of (3.1) near $(\bar{\lambda},\bar{x})$ under certain assumptions, and in fact can continue the curve of solutions uniquely and smoothly through $(\bar{\lambda},\bar{x})$ (although parameterization by λ is not generally possible). Moreover, the shape of the curve and the eigenvalue of F_x near zero along the curve are related. This type of situation is encountered in particular when there is some kind of positivity inherent in the problem. This will be illustrated in some examples in Section 4. First the required analogues of Lemma 1.1 and Theorem 1.16 will be given.

Theorem 3.2. *Let $(\bar{\lambda},\bar{x})\in\mathbb{R}\times X$ and let F be a continuously differentiable mapping of an open neighborhood of $(\bar{\lambda},\bar{x})$ into Y. Let $N(F_x(\bar{\lambda},\bar{x}))=\operatorname{span}\{x_0\}$ be one dimensional and* $\operatorname{codim} R(F_x(\bar{\lambda},\bar{x}))=1$. *Let $F_\lambda(\bar{\lambda},\bar{x})\notin R(F_x(\bar{\lambda},\bar{x}))$. If Z is a complement of* $\operatorname{span}\{x_0\}$ *in X, then the solutions of $F(\lambda,x)=F(\bar{\lambda},\bar{x})$ near $(\bar{\lambda},\bar{x})$ form a curve $(\lambda(s),x(s))=(\bar{\lambda}+\tau(s),\bar{x}+sx_0+z(s))$, where $s\to(\tau(s),z(s))\in\mathbb{R}\times Z$ is a continuously differentiable function near $s=0$ and $\tau(0)=\tau'(0)=0$, $z(0)=z'(0)=0$. Moreover, if F is k-times continuously differentiable (analytic), so are $\tau(s)$, $z(s)$.*

Proof. Define f: $\mathbb{R}\times\mathbb{R}\times Z\to Y$ by

$$f(s,\tau,z)=F(\bar{\lambda}+\tau,\bar{x}+sx_0+z)-F(\bar{\lambda},\bar{x}).$$

The assertions of the theorem follow at once from the implicit function theorem and the observations that $f(0,0,0)=0$, that $f_{(r,z)}(0,0,0)$ is an isomorphism, and that $f_s(0,0,0)=0$.

Remark 3.3. For use in Section 4, observe that if F depends smoothly on some parameter ε, and if the assumptions of Theorem 3.2 are satisfied at $\varepsilon=0$, then the same proof shows that the zeroes of $F(\lambda,x,\varepsilon)=0$ near $(\bar{\lambda},\bar{x},0)$ are given by smooth functions $\lambda(s,\varepsilon)$, $x(s,\varepsilon)$, where s has the same meaning as before.

Remark 3.4. Theorem 3.2 is of course independent of the preliminary discussion in this section. However, when $\bar{x}=\lim_{n\to\infty}x(\lambda_n)$ as in these remarks, Theorem 3.2 implies that $\bar{x}=\lim_{\lambda\uparrow\bar{\lambda}}x(\lambda)$ and that the curve $(\lambda,x(\lambda))$ can be continued smoothly through $(\bar{\lambda},\bar{x})$ via parametrization by s. A result very closely related to Theorem 3.2 was given by ANSELONE and MOORE [4, Lemma 6.1].

If $K\in B(X,Y)$ and 0 is a K-simple eigenvalue of $F_x(\bar{\lambda},\bar{x})$, Lemma 1.3 (see the proof of Corollary 1.13) provides continuous functions $\mu(s)\in\mathbb{R}$, $w(s)\in X$ defined near $s=0$ such that

$$\begin{aligned}&\text{(i)}\quad F_x(s)w(s)=\mu(s)Kw(s),\\ \text{(3.5)}\qquad&\text{(ii)}\quad w(s)-x_0\in Z,\\ &\text{(iii)}\quad w(0)=x_0,\qquad \mu(0)=0,\end{aligned}$$

where $F_x(s)=F_x(\lambda(s),x(s))$. The analogue of Theorem 1.16 is

Theorem 3.6. *Let the assumptions of Theorem 3.2 hold and let $\mu(s)$, $w(s)$ be as in (3.5) where 0 is a K-simple eigenvalue of $F_x(\bar{\lambda},\bar{x})$. Let $l\in Y^*$ satisfy $N(l)=R(F_x(\bar{\lambda},\bar{x}))$. Then, near $s=0$, $\langle l,Kx_0\rangle\mu(s)$ and $-\lambda'(s)\langle l,F_\lambda(\bar{\lambda},\bar{x})\rangle$ have the same zeroes and, whenever $\lambda'(s)\neq 0$, the same sign. Moreover,*

$$\lim_{s\to 0}\frac{\mu(s)}{\lambda'(s)}=-\frac{\langle l,F_\lambda(\bar{\lambda},\bar{x})\rangle}{\langle l,Kx_0\rangle}.$$

Proof. The proof is similar to that of Theorem 1.16 and is only sketched. First, the relation $\lambda'(s)F_\lambda(s)+F_x(s)x'(s)=0$ together with (3.5) and the fact that $w(s)-x'(s)\in Z$ implies

$$\|x'(s)-w(s)\|\leqq c(|\lambda'(s)|+|\mu(s)|)\tag{3.7}$$

for some constant c. Then applying l to the equation

$$\begin{aligned}\mu(s)Kx_0+\lambda'(s)F_\lambda(0)&=F_x(0)(w(s)-x'(s))\\ &\quad+(F_x(s)-F_x(0))(w(s)-x'(s))+\mu(s)K(x_0-w(s))+\lambda'(s)(F_\lambda(0)-F_\lambda(s))\end{aligned}$$

and using (3.7) and (3.5) yields

(3.8) $$|\langle l, Kx_0\rangle \mu(s)+\lambda'(s)\langle l, F_\lambda(0)\rangle| \leqq o(1)\big(|\mu(s)|+|\lambda'(s)|\big) \quad \text{as } s\to 0,$$

and the proof is completed as in Theorem 1.16.

4. Some Application to Problems Involving Positivity

In this section Theorems 3.2 and 3.6, as well as the preliminary remarks concerning continuation, are applied to some ordinary and partial differential equations. The first application is to a family of ordinary differential equations.

Example 4.1. Consider the problem

(4.2) $$F(\lambda, u)=-\big(u''+\lambda g(u)\big)=0$$

where F: $\mathbb{R}\times C_0^2([0,\pi])\to C([0,\pi])$, $C_0^2([0,\pi])$ is as in Example 2.1, and g satisfies the conditions

(Hg)
- (i) g: $\mathbb{R}\to\mathbb{R}$ is twice continuously differentiable,
- (ii) $g(0)>0$,
- (iii) $g'(0)>0$ and $g''(\eta)>0$ if $\eta>0$,
- (iv) $\lim\limits_{\eta\to\infty}\dfrac{g(\eta)}{\eta}=\infty$.

Clearly $F(0,0)=0$ and $F_u(0,0)=-d^2/dx^2$ is an isomorphism of $C_0^2([0,\pi])$ onto $C([0,\pi])$. The introductory remarks of Section 3 show that there is a curve $\big(\lambda, u(\lambda)\big)$ of solutions of (4.2) on a maximal interval $[0,\bar\lambda)$ such that $F_u\big(\lambda, u(\lambda)\big)$ is nonsingular. The maximum principle and (Hg) (ii) imply that $u(\lambda)(x)>0$ for $x\in(0,\pi)$ and small positive λ, and it is easy to see (again, by the maximum principle) that this holds for all $\lambda\in(0,\bar\lambda)$ (see for example [18]). We will show that $\bar\lambda<\infty$ and that (4.2) has a solution $(\bar\lambda,\bar u)$ through which the curve $\big(\lambda, u(\lambda)\big)$ can be smoothly extended. Moreover, the least eigenvalue of F_u changes sign at $(\bar\lambda,\bar u)$ along the curve.

Clearly, bounded subsets of $F^{-1}(\{0\})$ are compact. The next lemmas establish that $\bar\lambda<\infty$ and $\limsup\limits_{\lambda\uparrow\bar\lambda}\|u(\lambda)\|<\infty$, so that (as in the introductory remarks) $\lim\limits_{n\to\infty}u(\lambda_n)=\bar u$ exists for some sequence $\{\lambda_n\}$, $\lambda_n\uparrow\bar\lambda$. The next result, proved here for completeness, is a special case of the results of [12].

Lemma 4.3. $\bar\lambda\leqq 1/g'(0)$.

Proof. Let $\kappa(\lambda)$ denote the smallest value of κ for which

(4.4) $$v''+\kappa g'\big(u(\lambda)\big)v=0, \qquad v\in C_0^2([0,\pi]),$$

has a nontrivial solution. Then $\kappa(\lambda)$ is a continuous function of λ on $[0,\bar\lambda)$. Clearly $\kappa(0)=1/g'(0)$. The hypothesis (Hg) (iii) and the fact that $u(\lambda)>0$ on $(0,\pi)$ if $\lambda>0$ imply that $\kappa(\lambda)<1/g'(0)$ for $\lambda\in(0,\bar\lambda)$. If $\bar\lambda>1/g'(0)$, then $\kappa(\lambda)-\lambda$ must have a zero $\lambda^* \in (0, 1/g'(0))$. But then $F_u(\lambda^*, u(\lambda^*))$ is singular, contradicting the definition of $\bar{\lambda}$. Therefore $\bar{\lambda} \leqq 1/g'(0)$.

Lemma 4.5. *Let $\varepsilon > 0$. Then there is a number M such that if $(\lambda, u) \in \mathbb{R} \times C_0^2([0, \pi])$ satisfies* (4.2), *$\lambda \geqq \varepsilon$, and $u > 0$ on $(0, \pi)$, then*

$$\|u\|_{C_0^2([0,\pi])} \leqq M.$$

Proof. Let $\varepsilon > 0$ and (λ, u) be as above. Clearly it suffices to show that $u \leqq M_1$ for some M_1. Define

$$I(c) = \{x \in (0, \pi) : u(x) > c\} = (a(c), b(c))$$

where $0 \leqq a(c) \leqq b(c) \leqq \pi$. Since u satisfies (4.2), (Hg) (ii) and (iii) imply that u is concave and $I(c)$ is an interval. If $c > 0$ and $I(c)$ is not empty, then $-u'' = \lambda \rho_c u$ on $I(c)$, where $\rho_c(x) = g(u(x))/u(x)$. Due to (Hg) (iii), $\rho_c(x) \geqq (g(c) - g(0))/c$ on $I(c)$. Since u has no zeroes on $I(c)$, it follows from the Sturm comparison theorem that

$$b(c) - a(c) < \frac{\pi}{\sqrt{\lambda(g(c) - g(0))/c}}.$$

Choose c_0 so that $\pi/\sqrt{\varepsilon(g(c_0) - g(0))/c_0} < \pi/3$, fraction bar (which is possible by (Hg) (iv)). Then either $a(c_0) > \pi/3$ or $b(c_0) < 2\pi/3$. If we assume the former for definiteness, then (4.2) and the facts that $u \leqq c_0$ and $g(u) \leqq g(c_0)$ on $[0, a(c_0))$ imply

$$u'(a(c_0)) \geqq u'(0) - \lambda g(c_0) a(c_0) \tag{4.6}$$

while (since u is concave)

$$c_0 = u(a(c_0)) = \int_0^{a(c_0)} u'(x)\,dx \geqq a(c_0) u'(a(c_0)) \geqq (\pi/3) u'(a(c_0)). \tag{4.7}$$

Combining (4.6) and (4.7) gives

$$(3/\pi) c_0 + \lambda g(c_0) a(c_0) \geqq u'(0).$$

Lastly, since $a(c_0) \leqq \pi$ and u is concave,

$$\pi((3/\pi) c_0 + \lambda g(c_0) \pi) \geqq u'(0) x \geqq u(x) \qquad \text{for } x \in [0, \pi].$$

The above preliminaries now yield

Theorem 4.8. *Let F be given by* (4.2), *where g satisfies the hypothesis* (Hg). *If u: $[0, \bar{\lambda}) \to C_0^2([0, \pi])$ is as above, then $\lim_{\lambda \uparrow \bar{\lambda}} u(\lambda) = \bar{u}$ exists, and $F_u(\bar{\lambda}, \bar{u})$ has a null-vector u_0 which is positive on $(0, \pi)$. The zeroes of F near $(\bar{\lambda}, \bar{u})$ form a smooth curve $(\lambda(s), \tilde{u}(s))$ for which $(\lambda(0), \tilde{u}(0)) = (\bar{\lambda}, \bar{u})$, $\lambda'(0) = 0$, $\lambda''(0) < 0$. If $\mu(s)$ is the least value of μ for which the equation*

$$-(w'' + \lambda(s) g'(\tilde{u}(s)) w) = \mu w, \qquad w \in C_0^\pi([0, \pi])$$

has a nontrivial solution, then $\mu(s)$ *and* $\lambda'(s)$ *have the same sign for* $s \neq 0$. *More precisely,*

$$\mu(s) = \frac{s\int_0^\pi \bar{\lambda} g''(\bar{u}) u_0^3\,dx}{\int_0^\pi u_0^2\,dx} + O(s^2)$$

as $s \to 0$.

Proof. According to what was shown above, there is a sequence $\{\lambda_n\}$ and an element $\bar{u} \in C_0^2([0,\pi])$ such that $\lambda_n \uparrow \bar{\lambda}$, $u(\lambda_n) \to \bar{u}$, and $F_u(\bar{\lambda}, \bar{u})$ is singular.

Let $r(\lambda)$, $0 \leqq \lambda \leqq \bar{\lambda}$, be the least number such that the equation

$$F_u(\lambda, u(\lambda))\,w = -\big(w'' + \lambda g'(u(\lambda))\,w\big) = r(\lambda)\,w \tag{4.9}$$

has a nontrivial solution $w \in C_0^2([0,\pi])$, where $u(\bar{\lambda}) = \bar{u}$ by definition. Then $r(\lambda)$ is continuous on $[0, \bar{\lambda})$, $r(\lambda_n) \to r(\bar{\lambda})$, and $r(0) = 1$. Since $F(\lambda, u(\lambda))$ is nonsingular on $[0, \bar{\lambda})$, we have $r(\lambda) > 0$ on $[0, \bar{\lambda})$. This implies that $r(\bar{\lambda}) \geqq 0$. However, $F(\bar{\lambda}, \bar{u}(\bar{\lambda})) = F(\bar{\lambda}, \bar{u})$ is singular, so that $r(\bar{\lambda}) \leqq 0$; it follows that $r(\bar{\lambda}) = 0$. Since 0 is the least eigenvalue of $F_u(\bar{\lambda}, \bar{u})$, the corresponding null vector is of one sign on $(0, \pi)$. This establishes the assertions concerning u_0. The other assumptions of Theorem 3.2 clearly being satisfied, only the condition $F_\lambda(\bar{\lambda}, \bar{u}) = -g(\bar{u}) \notin R(F_u(\bar{\lambda}, \bar{u}))$ need be checked. Since

$$\int_0^\pi g(\bar{u})\,u_0\,dx > 0, \tag{4.10}$$

this condition is satisfied. But then the structure of $F=0$ near $(\bar{\lambda}, \bar{u})$ is determined by Theorem 4.2, and this clearly implies $\lim_{\lambda \uparrow \bar{\lambda}} u(\lambda) = \bar{u}$. Let $(\lambda(s), \tilde{u}(s))$ be the curve provided by Theorem 3.2. A direct calculation shows that

$$\lambda''(0)\,g(\bar{u}) + g''(\bar{u})\,u_0^2 \in R(F_u(\bar{\lambda}, \bar{u})) \tag{4.11}$$

hence

$$\lambda''(0) = \frac{-\int_0^\pi g''(\bar{u})\,u_0^3\,dx}{\int_0^\pi u_0\,g(\bar{u})\,dx} < 0 \tag{4.12}$$

determines $\lambda''(0)$ $(\lambda'(0) = 0$ by Theorem 3.2). The assertions concerning $\mu(s)$ follow from (4.12) upon using $x_0 = u_0$, $\langle l, u \rangle = \int_0^\pi u_0 u\,dx$, and K=injection of $C_0^2([0,\pi])$ into $C([0,\pi])$ in Theorem 3.6.

Remark 4.13. With minor modifications g can also be permitted to depend on x. Equation (4.2) was also treated by LAETSCH [16]. Since $\lambda''(0) < 0$, it follows that for each $\mu < \bar{\lambda}$ but sufficiently close to $\bar{\lambda}$ there exist numbers s_1, s_2, $s_1 < 0 < s_2$, such that $\lambda(s_1) = \mu = \lambda(s_2)$. Moreover, the normalization $u_0 > 0$ and the form of $\tilde{u}(s)$ implies that $\tilde{u}(s)(x)$ is monotone increasing in s for $x \in (0, \pi)$. This shows that $u(\lambda(s)) = \tilde{u}(s)$ for small negative s (as opposed to small positive s), since $u(\lambda)(x)$ is monotone increasing in λ for $x \in (0, \pi)$, as is easy to see [12].

It is natural to attempt to generalize Example 4.1 in the following way: Replace $(0, \pi)$ by a smooth bounded domain $\Omega \subseteq \mathbb{R}^n$, $-d^2/dx^2$ by a self-adjoint second order elliptic differential operator for which the maximum principle is valid, and $C_0^2([0, \pi])$ by $C_0^{2,\alpha}(\bar{\Omega})$ (with the notation of Example 2.3). Indeed, with the notable exception of Lemma 4.5, the arguments just used carry over to the resulting problem. The question of bounds to replace Lemma 4.5 is a delicate one in the theory of partial differential equations, but the next example is one where such bounds may be obtained.

Example 4.14. Set $X = C_0^{2,\alpha}(\bar{\Omega})$ and $Y = C^\alpha(\bar{\Omega})$ as in Example 2.3, where $\alpha \in (0, 1)$ and $\Omega \subseteq \mathbb{R}^3$ is a smoothly bounded open set. Consider the problem

$$F(\lambda, u) = -(\Delta u + \lambda(1 + u + u^2)) = 0. \tag{4.15}$$

Let $\bar{\lambda}$, $u(\lambda)$, $0 \leqq \lambda < \bar{\lambda}$, be defined as before. The following analogue of Lemma 4.3 is proved as before.

Lemma 4.16. *We have $\bar{\lambda} \leqq \lambda_1$, where λ_1 is the smallest eigenvalue of $-\Delta$ under Dirichlet boundary conditions.*

Using the next lemma in place of Lemma 4.5, we can easily establish an analogue (which will not be stated) of Theorem 4.8 for (4.15).

Lemma 4.17. *$\|u(\lambda)\|_X$ is uniformly bounded in λ, $\lambda \in [0, \bar{\lambda})$.*

Proof. Let $r(\lambda)$ be the least eigenvalue of

$$-\Delta w - \lambda(1 + 2u(\lambda))w = r(\lambda)w, \qquad w \in C_0^{2,\alpha}(\bar{\Omega}). \tag{4.18}$$

As before, $r(0) = \lambda_1$ and $0 < r(\lambda) \leqq \lambda_1$ for $0 \leqq \lambda < \bar{\lambda}$. The variational characterization of $r(\lambda)$ [7] implies that

$$0 \leqq r(\lambda) \int_\Omega v^2\, dx \leqq \int_\Omega (|\nabla v|^2 - \lambda(1 + 2u(\lambda))v^2)\, dx \tag{4.19}$$

for $v \in C_0^{2,\alpha}(\bar{\Omega})$. Further, since $F(\lambda, u(\lambda)) = 0$ we have

$$\int_\Omega |\nabla u(\lambda)|^2\, dx = \lambda \int_\Omega (u(\lambda) + u(\lambda)^2 + u(\lambda)^3)\, dx. \tag{4.20}$$

Setting $v = u(\lambda)$ in (4.19) and using (4.20) implies that

$$\int_\Omega [(u(\lambda) + u(\lambda)^2 + u(\lambda)^3) - (u(\lambda)^2 + 2u(\lambda)^3)]\, dx = \int_\Omega (u(\lambda) - u(\lambda)^3)\, dx \geqq 0. \tag{4.21}$$

The Hölder inequality, the fact that $u(\lambda) \geqq 0$, and (4.21) together yield

$$\int_\Omega u(\lambda)^3\, dx \leqq \text{measure of } \Omega = c_1. \tag{4.22}$$

Then (4.20) implies

$$\int_\Omega |\nabla u(\lambda)|^2\, dx \leqq \lambda c_2 \tag{4.23}$$

for some c_2 and, in turn, (4.23) in conjunction with the Sobolev inequalities for $n = 3$ shows that

$$\int_\Omega u(\lambda)^6\, dx \leqq \lambda c_3 \tag{4.24}$$

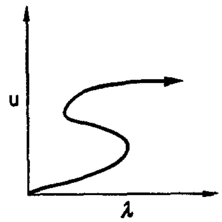

Fig. 1

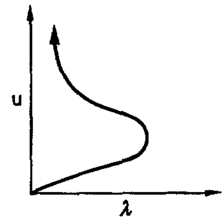

Fig. 2

for some c_3. If $f(\lambda)=\lambda(1+u(\lambda)+u(\lambda)^2)$, (4.24) implies $\int\limits_{\Omega} f(\lambda)^3\,dx \leqq c_4$, where c_4 is independent of λ for $0\leqq\lambda\leqq\bar{\lambda}\leqq\lambda_1$. Since $-\Delta u(\lambda)=f(\lambda)$, the L^p theory of the Dirichlet problem [1] shows that $\|u(\lambda)\|_{2,3}\leqq c_5$ for $0\leqq\lambda\leqq\bar{\lambda}$, where $\|\cdot\|_{2,3}$ denotes the norm in the Sobolev space $W_0^{2,3}$. Thus $u(\lambda)$ (and hence $f(\lambda)$) is bounded in $C^\alpha(\bar{\Omega})$ by the Sobolev inequalities. Since $-\Delta u(\lambda)=f(\lambda)$, $u(\lambda)$ remains bounded in $C_0^{2,\alpha}(\bar{\Omega})$ and the proof is complete.

Remark 4.25. One can obviously adapt the above analysis to treat more general elliptic operators and nonlinearities. However, we do not know whether nonlinearities $g(u)$ which grow more rapidly than the "Sobolev cut-off" power can be handled. It may be pointed out that the above estimates differ from Lemma 4.5 in more than technique. A bound depending on λ is not obtained for all positive solutions of $F(\lambda, u)=0$, but only for those positive solutions for which $F_u(\lambda, u)$ has a nonnegative smallest eigenvalue.

By perturbing Examples 4.14 or 4.1 an example of another type can be obtained. We will work within the framework of Example 4.14. The point of the present example is that while the main features of Example 4.14 are preserved, it is also possible to obtain a bound on u independent of $\lambda\geqq 0$. It then follows from other arguments that the problem has a positive solution u for every $\lambda>0$. Heuristically one pictures a curve of solutions like Figure 1 below in the λ, u "plane". This contrasts with the heuristic picture (Figure 2) of the curve of solutions for Examples 4.1 and 4.14.

Example 4.26. Consider the problem

$$F(\lambda, u, \varepsilon)=-(\Delta u+\lambda(1+u+u^2-\varepsilon u^3))=0 \tag{4.27}$$

where X, Y are as in the preceeding example and $\varepsilon\in\mathbb{R}$ is a real parameter. First note that if (4.27) holds, then

$$\varepsilon u^3\leqq 1+u+u^2 \text{(respectively, } 1+u+u^2\leqq\varepsilon u^3) \tag{4.28}$$

at a maximum (respectively, minimum) of u. Hence, if $\varepsilon>0$, then $\sup\limits_{\Omega}|u|<M(\varepsilon)$ for some M depending on ε. Then, as in Example 4.14, it follows from L^p estimates and the Sobolev embedding theorems that $\|u\|_X\leqq K(\varepsilon)$, for some K depending on $\varepsilon>0$.

The discussion proceeds by treating the equation $F(\lambda, u, \varepsilon)=0$ as a perturbation of $F(\lambda, u, 0)=0$. Let $\bar{\lambda}$, $u(\lambda)$, $0\leqq\lambda<\bar{\lambda}$, be defined by the problem $F(\lambda, u, 0)=0$

as before. To begin with, observe that for each δ_1, $0<\delta_1<\bar{\lambda}$, the curve $(\lambda, u(\lambda))$, $0\leqq\lambda\leqq\bar{\lambda}-\delta_1$ is compact and $F_u(\lambda, u(\lambda), 0)$ is nonsingular on this curve. Hence there is a neighborhood N_1 of $\{(\lambda, u(\lambda)): 0\leqq\lambda\leqq\bar{\lambda}-\delta_1\}$ and an $\varepsilon_1>0$ such that $F(\lambda, u, \varepsilon)=0$ has a unique solution $(\lambda, u(\lambda, \varepsilon), \varepsilon)$ subject to

$$(\lambda, u(\lambda, \varepsilon), \varepsilon)\in N_1\times[0, \varepsilon_1].$$

Moreover $u(\lambda, \varepsilon)$ is a smooth function of its arguments and it can be assumed that $F_u(\lambda, u(\lambda, \varepsilon), \varepsilon)$ is nonsingular.

Next, $\lim_{\lambda\uparrow\bar{\lambda}} u(\lambda)=\bar{u}$ exists by the analogue of Theorem 4.8 for (4.15), and the solutions of $F(\lambda, u, 0)=0$ near $(\bar{\lambda}, \bar{u})$ form a smooth curve $(\lambda(s), \tilde{u}(s))$ such that $(\lambda(0), \tilde{u}(0))=(\bar{\lambda}, \bar{u})$, $\lambda'(0)=0$, $\lambda''(0)<0$, and $u(\lambda(s))=\tilde{u}(s)$ for small $s<0$ (see Remark 4.13). By Remark 3.3, there are functions $\lambda(s, \varepsilon)$, $\tilde{u}(s, \varepsilon)$ defined on some square $S=\{(s, \varepsilon): |s|, |\varepsilon|\leqq\delta_2\}$, where $\delta_2>0$, such that $F(\lambda(s, \varepsilon), \tilde{u}(s, \varepsilon))=0$, and an open neighborhood N_2 of $(\bar{\lambda}, \bar{u})$ such that if $F(\lambda, u, \varepsilon)=0$, $(\lambda, u)\in N_2$, and $|\varepsilon|\leqq\delta_2$, then $(\lambda, u)=(\lambda(s, \varepsilon), \tilde{u}(s, \varepsilon))$ for exactly one $(s, \varepsilon)\in S$. Since $\lambda(s, 0)=\lambda(s)$, $\tilde{u}(s, 0)=\tilde{u}(s)$, we can assume that $(\partial/\partial s)\lambda(s, \varepsilon)$ has a unique zero $s(\varepsilon)$, $|s(\varepsilon)|<\delta_2$ for $|\varepsilon|\leqq\delta_2$ and that $(\partial^2/\partial s^2)\lambda(s, \varepsilon)>0$ on S.

The next step is to relate $(\lambda, u(\lambda, \varepsilon))$ and $(\lambda(s, \varepsilon), \tilde{u}(s, \varepsilon))$. Choose δ_1 sufficiently small to guarantee that $(\bar{\lambda}-\delta_1, u(\bar{\lambda}-\delta_1))\in N_2$. Then $(\bar{\lambda}-\delta_1, u(\bar{\lambda}-\delta_1, \varepsilon))\in N_2$ for, say, $0\leqq\varepsilon\leqq\delta_3<\min(\varepsilon_1, \delta_2)$. It follows that $(\lambda, u(\lambda, \varepsilon))=(\lambda(s, \varepsilon), \tilde{u}(s, \varepsilon))$ has a unique and smooth solution (by the meaning of s) $s(\lambda, \varepsilon)$ for $0\leqq\varepsilon\leqq\delta_3$ and λ near $\bar{\lambda}-\delta_1$. In this way, a smooth curve of zeroes of $F(\lambda, u, \varepsilon)=0$ is obtained by piecing together the curves $(\lambda, u(\lambda, \varepsilon))$ and $(\lambda(s, \varepsilon), \tilde{u}(s, \varepsilon))$. At the point on this curve corresponding to $s=s(\varepsilon)$ (where $(\partial/\partial s)\lambda(s, \varepsilon)=0$), F_u is singular since each λ close to but less than $\lambda(s(\varepsilon), \varepsilon)$ corresponds to two values of s, while $s\to\tilde{u}(s, \varepsilon)$ is one-to-one. It follows that $\bar{\lambda}(\varepsilon)$, determined by (4.27) with ε fixed, is finite.

Lastly, we shall show that even though $\bar{\lambda}(\varepsilon)<\infty$, equation (4.27) has solutions (λ, u) with $u>0$ in Ω for every $\lambda>0$, $\varepsilon>0$. This follows from [18, Theorems 6.2 and 6.6]. Indeed these results imply that there is an unbounded continuum $\mathscr{C}_\varepsilon$ of solutions (λ, u) of (4.27) in $[0, \infty)\times C_0^{2,\alpha}(\bar{\Omega})$ such that $u>0$ in Ω. For $\varepsilon>0$ the solutions of (4.27) are bounded in $C_0^{2,\alpha}(\bar{\Omega})$ uniformly for $\lambda\geqq0$, and the result follows.

Remark 4.29. Whenever one can obtain a priori bounds for positive solutions of equations such as (4.27), monotone iteration schemes can be constructed which converge to maximal and minimal positive solutions of the equation (see, for example, [7], [3], [20]). It is fairly easy to see that for each $\lambda\in[0, \bar{\lambda}(\varepsilon))$ the corresponding $u(\lambda, \varepsilon)$ is the minimal positive solution of (4.27). This, the results of [2], and what was proved above imply that there is an $\eta(\varepsilon)>0$ such that $F(\lambda, u, \varepsilon)=0$ has at least three distinct positive solutions u for $\bar{\lambda}(\varepsilon)-\eta(\varepsilon)<\lambda<\bar{\lambda}(\varepsilon)$, which is consistent with Figure 1.

Remark 4.30. (The chemical reaction problem.) It has been conjectured that for some values of the parameters α, $\beta>0$ one gets a qualitative picture similar to Figure 1 for the solutions of the equation

$$F(\lambda, u)=-(\Delta u+\lambda(\alpha-u)\exp(-\beta/(1+u))=0 \tag{4.31}$$

where $u \in C_0^{2,\alpha}(\bar{\Omega})$, etc., as before. This equation arises in chemical reactor theory. As above, it is simple to check that $0 \leqq u(x) < \alpha$ for $x \in \Omega$ whenever u is a non-negative solution of (4.31) and $\lambda \geqq 0$. It has been shown by AMANN [2] that if for some α, β the corresponding maximal and minimal solutions of (4.31) are distinct, then there is a third positive solution lying between them. Aside from some numerical indications (e.g. [5]) that these solutions are distinct for some α, β, this has not been verified mathematically. In attempting to treat (4.31) by the arguments of this section, we find two difficulties. First, we are unable to show that $\bar{\lambda} < \infty$. Assuming this, however, then $\lim_{\lambda \uparrow \bar{\lambda}} u(\lambda) = \bar{u}$ exists and Theorem 3.2 applies at $(\bar{\lambda}, \bar{u})$. To see this, let $\bar{u}$ be defined first through a sequence $\{u(\lambda_n)\}$. Then $F_u(\bar{\lambda}, \bar{u})$ has 0 as its smallest eigenvalue, the smallest eigenvalue is simple, and the corresponding null-vector u_0 is of one sign on Ω. All this is proved as in Example 4.2. Since $F_\lambda(\bar{\lambda}, \bar{u}) = (\alpha - \bar{u}) \exp(-\beta/(1+\bar{u})) > 0$ in Ω and $R(F_u(\bar{\lambda}, \bar{u})) = \{v \in Y: \int_\Omega v u_0 \, dx = 0\}$, we see that $F_\lambda(\bar{\lambda}, \bar{u}) \notin R(F_u(\bar{\lambda}, \bar{u}))$. However, we are unable to determine the first nonvanishing derivative of $\lambda(s)$ (and its sign) at $s = 0$.

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Department of Mathematics
University of California
Los Angeles

and

Department of Mathematics
University of Wisconsin
Madison

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