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BIFURCATION PHENOMENA ASSOCIATED TO THE p-LAPLACE OPERATOR

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ABSTRACT. We determine the structure of the set of the solutions u of $-(|u_x|^{p-2}u_x)_x + f(u) = \lambda|u|^{p-2}u$ on $(0,1)$ such that $u(0) = u(1) = 0$, where $p > 1$ and $\lambda \in \mathbf{R}$. We prove that the solutions with k zeros are unique when $1 < p \leq 2$ but may not be so when $p > 2$.

0. Introduction. In this article we study the structure of the set E_λ of the solutions of the following nonlinear eigenvalue problem

$$\begin{cases} -(|u_x|^{p-2}u_x)_x + f(u) = \lambda|u|^{p-2}u & \text{in } (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{0.1}$$

where $p > 1$, λ is a real number and f is a C^1 real-valued odd function such that

$$r \mapsto g(r) = f(r)/(|r|^{p-2}r) \tag{0.2}$$

is increasing on $(0, +\infty)$ with limits 0 at 0 and $+\infty$ at infinity. We first investigate the unperturbed eigenvalue problem

$$\begin{cases} -(|v_x|^{p-2}v_x)_x = \lambda|v|^{p-2}v & \text{in } (0,1), \\ v(0) = v(1) = 0. \end{cases} \tag{0.3}$$

By means of an elementary integration process we prove that (0.3) admits a nontrivial solution if and only if

$$\lambda = \lambda_k = k^p(p-1)\left[2\int_0^1 \frac{dt}{(1-t^p)^{1/p}}\right]^p, \qquad k \in \mathbf{N}^*. \tag{0.4}$$

Moreover to each λ_k is associated a one-dimensional eigenspace generated by a function ω_k with exactly $k-1$ zeros in (0,1). Concerning the equation (0.1) we prove that each λ_k is a point of bifurcation as in the semilinear case ($p = 2$). More precisely we define for $k \in \mathbf{N}^*$

$$S_k = \{\varphi \in C \colon \varphi \text{ has exactly } k-1 \text{ simple zeros in } (0,1)\}, \tag{0.5}$$

where $C = \{\varphi \in C^1([0,1]) \colon \varphi(0) = \varphi(1) = 0\}$ and

$$S_k^+ = \{\varphi \in S_k \colon \varphi_x(0) > 0\}, \qquad S_k^- = -S_k^+. \tag{0.6}$$

As λ_1 is defined as the best Poincaré constant in $W_0^{1,p}(0,1)$, that is,

$$\operatorname{Inf}\left\{\int_0^1 |v_x|^p\,dx \colon v \in W_0^{1,p}(0,1), \int_0^1 |v|^p\,dx = 1\right\}, \tag{0.7}$$

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it is clear that E_λ is reduced to the zero function when $\lambda \leq \lambda_1$.

When $1 < p \leq 2$ we prove that the configuration of E_λ is exactly the same as in the case $p = 2$ [**1**], that is,

$$E_\lambda = \{0, \pm u_l, l = 1, \ldots, k\colon u_l \in S_l^+\}. \tag{0.8}$$

When $p > 2$ the structure of E_λ can be quite a bit more complicated for large values of λ. Let h be the inverse function of g and $F(r) = \int_0^r f(s)\,ds$; we define

$$\alpha(\lambda) = \left(\frac{\lambda}{p-1} h^p(\lambda) - \frac{p}{p-1} F(h(\lambda))\right)^{1/p} \tag{0.9}$$

and

$$x(\lambda) = \int_0^{h(\lambda)} \frac{ds}{(\alpha^p(\lambda) + pF(s)/(p-1) - \lambda s^p/(p-1))^{1/p}}; \tag{0.10}$$

and $\lambda \mapsto x(\lambda)$ is a decreasing positive function defined on $(0, +\infty)$. If $\lambda_k < \lambda \leq \lambda_{k+1}$ we then have

$$E_\lambda = \{0\} \cup \{\pm u_1\} \bigcup_{p=2}^{k} \{\pm E_\lambda^l\}, \tag{0.11}$$

where $u_1 \in S_1^+$ and $E_\lambda^l \subset S_l^+$ such that

(i) E_λ^l contains only one element if $2lx(\lambda) \geq 1$,

(ii) E_λ^l is diffeomorphic to $[0,1]^{l-1}$ if $0 < 2lx(\lambda) < 1$. In case (ii) the elements of E_λ^l are constant with value $(-1)^{j+1}h(\lambda)$ on l closed and disconnected subintervals $I_j \subset (0,1)$, $j = 1, \ldots, l$, with total length $1 - 2lx(\lambda)$.

1. The eigenvalue problem. For $p > 1$ we consider the following eigenvalue problem

$$\begin{cases} -(|v_x|^{p-2}v_x)_x = \lambda |v|^{p-2} v & \text{in } (0,1), \\ v(0) = v(1) = 0 \end{cases} \tag{1.1}$$

and let S be the subset of $W_0^{1,p}(0,1) \times \mathbf{R}$ of all the (v, λ), $v \neq 0$, satisfying (1.1).

THEOREM 1.1. *There exists a unique sequence of functions $v_k \in S_k^+$, $k \in \mathbf{N}^*$, with maximal value* 1 *on* $(0,1)$ *such that*

$$S = \{(\mu v_k, \lambda_k)\colon k \in \mathbf{N}^*\}, \tag{1.2}$$

where μ is any nonzero real number and

$$\lambda_k = k^p \lambda_1 = k^p (p-1) \left[2 \int_0^1 \frac{dt}{(1-t^p)^{1/p}}\right]^p. \tag{1.3}$$

Moreover the following holds for $m = 0, \ldots, k-1$:

$$v_k(x) = (-1)^m v_1(kx - m), \qquad m/k \leq x \leq (m+1)/k. \tag{1.4}$$

Before giving the proof it must be noticed that this result is partially contained in [**5**], in particular formula (1.4).

PROOF. It is clear from (1.1) and $v \in C^0([0,1])$ and then $v \in C^1([0,1])$ when $p > 2$ or $v \in C^2([0,1])$ when $1 < p \leq 2$ (the complete regularity, due to Otani [**5**], will be given in Remark 1.1).

Step 1. If $(v,\lambda)\in S$ then $v_x(0)\neq 0$ and $\lambda>0$. Multiplying (1.1) by v and integrating over $(0,1)$ yields

$$\int_0^1 |v_x|^p\,dx=\lambda\int_0^1 v^p\,dx. \tag{1.5}$$

Hence necessarily $\lambda>0$. Multiplying (1.1) by v_x and integrating over $(0,x)$, $0<x<1$, yields the energy estimate

$$(p-1)|v_x(x)|^p+\lambda|v(x)|^p=(p-1)|v_x(0)|^p+\lambda|v(0)|^p. \tag{1.6}$$

As $v(0)=0$ we need $v_x(0)\neq 0$ in order to have a nonzero v.

Step 2. *The explicit construction.* Assume v is a nonzero solution with $v_x(0)=\alpha>0$ for example. Then $v_x>0$ on $[0,x_0)$ for some $x_0\in(0,1)$ and

$$v_x(x)=\left(\alpha^p-\frac{\lambda}{p-1}(v(x))^p\right)^{1/p} \tag{1.7}$$

on $[0,x_0]$, from (1.6), which gives

$$x=\int_0^{v(x)}\frac{dt}{(\alpha^p-\lambda t^p/(p-1))^{1/p}}. \tag{1.8}$$

Moreover this formula remains valid as long as $v(x)$ remains smaller than the first positive zero of the function

$$r\mapsto\varphi(\alpha,r)=\alpha^p-\lambda r^p/(p-1) \tag{1.9}$$

which is $S(\alpha)=((p-1)/\lambda)^{1/p}\alpha$. As $S(\alpha)$ is simple we define $\theta(\alpha)$ by

$$\theta(\alpha)=\int_0^{S(\alpha)}\frac{dt}{(\alpha^p-\lambda t^p/(p-1))^{1/p}}. \tag{1.10}$$

Moreover $v(\theta(\alpha))=S(\alpha)$ and $v_x(\theta(\alpha))=0$. As $\alpha^p=\lambda S^p(\alpha)/(p-1)$ we get

$$\theta(\alpha)=\theta_\lambda=C\left(\frac{p-1}{\lambda}\right)^{1/p},\quad C=\int_0^1\frac{ds}{(1-s^p)^{1/p}}. \tag{1.11}$$

From (1.6) the function v is decreasing on some interval $[\theta_\lambda,\Theta)$, so we get

$$x-\theta_\lambda=-\int_{v(x)}^{S(\alpha)}\frac{dt}{[(\lambda/(p-1))(S^p(\alpha)-t^p)]^{1/p}}, \tag{1.12}$$

or

$$x-\theta_\lambda=-\int_{v(x)}^{S(\alpha)}\frac{dt}{(\alpha^p-\lambda t^p/(p-1))^{1/p}};$$

and this formula remains valid as long as v is decreasing, in particular as long as v is positive. If $x_1\in(0,\theta_\lambda)$ and $x_2=2\theta_\lambda-x_1$ then

$$x_1=\int_0^{v(x_1)}\frac{dt}{(\varphi(\alpha,t))^{1/p}},\quad \theta_\lambda-x_1=-\int_{v(x_2)}^{S(\alpha)}\frac{dt}{(\varphi(\alpha,t))^{1/p}}$$

and $v(x_1)=v(x_2)$. As a consequence $x=\theta_\lambda$ is an axis of symmetry for the restriction of v to $[0,2\theta_\lambda]$ and $x=2\theta_\lambda$ is a center of symmetry for the restriction of

v to $[0, 4\theta_\lambda]$. Hence the function v is $4\theta_\lambda$-periodic on $[0, +\infty)$. The necessary and sufficient condition for the restriction of v to $[0,1]$ to be a solution of (1.1) is then

$$1/2\theta_\lambda \in \mathbf{N}^*, \tag{1.13}$$

which means (1.3). As for the number of zeros of v in $(0,1)$ it is given by $1/2\theta_\lambda - 1$. Using the homogeneity of (1.1) we get the desired result as the uniqueness is a consequence of the construction of v.

REMARK 1.1. Existence and uniqueness of the first positive normalized eigenfunction of $-\operatorname{div}(|D.|^{p-2}D.)$ in $W_0^{1,p}(\Omega)$ have been obtained by De Thelin in the radial case when Ω is a ball [**7**] and Guedda-Veron for general Ω with a connected C^2 boundary [**4**].

As for the regularity of v we have

$$v \in C^\alpha([0,1]) \cap C^{\langle p \rangle}([0,1] \backslash Z) \tag{1.14}$$

where $Z = \{x \in (0,1)\colon v_x(x) = 0\}$, $\alpha = \min(\langle (2-p)/(p-1) \rangle + 1, \langle p \rangle)$ and $\langle r \rangle = +\infty$ if $r \in 2\mathbf{N}^*$ or $\langle r \rangle = \min\{n\colon n \in \mathbf{N}^*,\, n \geq r\}$ if not.

REMARK 1.2. We have the following Poincaré type relation

$$\lambda_1 = \operatorname{Inf}\left\{\int_0^1 |u_x|^p\,dx \Big/ \int_0^1 |u|^p\,dx\colon u \in W_0^{1,p}(0,1)\backslash\{0\}\right\} \tag{1.15}$$

and the infimum is achieved for $u = v_1$.

2. The bifurcation phenomena.

In this section we consider the following equation

$$\begin{cases} -(|u_x|^{p-2}u_x)_x + f(u) = \lambda |u|^{p-2}u & \text{in } (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{2.1}$$

where $p > 1$ and $\lambda \in \mathbf{R}$. As for f we first assume that

$$f \text{ is a } C^1 \text{ odd function}, \tag{2.2}$$

$$s \mapsto f(s)/s^{p-1} \text{ is strictly increasing on } (0, +\infty) \text{ with limit } 0 \text{ at } 0, \tag{2.3}$$

$$\lim_{s \to +\infty} f(s)/s^{p-1} = +\infty. \tag{2.4}$$

We then define

$$h \text{ is the inverse function of the restriction of } f(s)/s^{p-1} \text{ to } (0, +\infty), \tag{2.5}$$

$$H(s) = \lambda s^p - pF(s), \tag{2.6}$$

where $F(s) = \int_0^s f(t)\,dt$. For $\lambda > 0$ we shall also consider the following hypothesis:

$$(p-1)(H'(s))^2 - pH(s)H''(s) \geq 0 \quad \text{for any } s \in [0, h(\lambda)]. \tag{2.7}$$

Let E_λ be the set of all the solutions of (2.1) in $W_0^{1,p}(0,1)$ and λ_k be defined by (1.3). When $1 < p \leq 2$ the structure of E_λ is exactly the same as in the case $p = 2$.

THEOREM 2.1. *Assume* $1 < p \leq 2$ *and* (2.2)–(2.7). *Then*
(i) *if* $\lambda \leq \lambda_1$, $E_\lambda = \{0\}$, *and*
(ii) *if* $\lambda_k < \lambda \leq \lambda_{k+1}$ *for some* $k \in \mathbf{N}^*$

$$E_\lambda = \{0, \pm u_1, \dots, \pm u_k\}, \tag{2.8}$$

where $u_l \in S_l^+$ *for* $l = 1, \dots, k$.

REMARK 2.1. The assumption (2.7), which is equivalent to the fact that $s \mapsto H^{p-1}(s)/H'^p(s)$ is nondecreasing on $[0, h(\lambda)]$, is essential for uniqueness but not for existence. In the particular case where $f(r) = |r|^{q-1}r$ with $q > p-1$ then $h(\lambda) = \lambda^{1/(q+1-p)}$, $H(s) = \lambda s^p - ps^{q+1}/(q+1)$ and (2.7) is satisfied.

PROOF OF THEOREM 2.1. As in Theorem 1.1 it is clear that any solution of (2.1) in $W_0^{1,p}(0,1)$ is continuous and at least C^2 (remember that $1 < p \leq 2$). Multiplying the equation by u yields

$$\int_0^1 |u_x|^p \, dx + \int_0^1 uf(u) \, dx = \lambda \int_0^1 |u|^p \, dx. \tag{2.9}$$

From Remark 1.2 a nonzero solution of (2.1) can exist only if $\lambda > \lambda_1$, which will be assumed in the sequel.

Step 1. If u is a nonzero solution of (2.1) then $u_x(0) \neq 0$. Although it is a consequence of a general result due to Franchi, Lanconelli and Serrin, we give here a direct proof which also works when $p > 2$. Multiplying (2.1) by u_x yields the energy relation

$$\begin{aligned} &-\frac{p-1}{p}|u_x(x)|^p + F(u(x)) - \frac{\lambda}{p}|u(x)|^p \\ &\quad = -\frac{p-1}{p}|u_x(0)| + F(u(0)) - \frac{\lambda}{p}|u(0)|^p. \end{aligned} \tag{2.10}$$

If we assume that $u_x(0) = 0$ we get

$$|u_x(x)|^p = \frac{1}{p-1}(pF(u(x)) - \lambda|u(x)|^p). \tag{2.11}$$

As the function $x \to pF(x) - \lambda|x|^p$ is negative on $(-\rho, \rho)\backslash\{0\}$, u_x is always 0 and $u \equiv 0$.

Step 2. The explicit construction. Without any loss of generality we assume $u_x(0) = \alpha > 0$. Hence u is increasing on some interval $[0, x_0]$ and from (2.10) we get

$$u_x^p(x) = \alpha^p + \frac{p}{p-1}F(u(x)) - \frac{\lambda}{p-1}u^p(x) \tag{2.12}$$

which gives u as the inverse function of a p-elliptic integral

$$x = \int_0^{u(x)} \frac{dt}{(\alpha^p + pF(t)/(p-1) - \lambda t^p/(p-1))^{1/p}} \tag{2.13}$$

on $[0, x_0]$. Moreover this formula remains valid as long as $u(x)$ is smaller than the first positive zero of

$$r \mapsto \Psi(\alpha, r) = \alpha^p + \frac{p}{p-1}F(r) - \frac{\lambda}{p-1}|r|^p. \tag{2.14}$$

But the function $\Psi(\alpha, \cdot)$ is decreasing in $[0, h(\lambda)]$ and increasing on $[h(\lambda), +\infty)$; hence there are three possibilities.

Case 1. $\alpha^p > \lambda h^p(\lambda)/(p-1) - pF(h(\lambda))/(p-1) = \alpha^p(\lambda)$.

In that case the function $r \mapsto \int_0^r ds/(\Psi(\alpha, s))^{1/p}$ is an increasing C^2 diffeomorphism from $\mathbf{R}^+$ onto $\mathbf{R}^+$ and it is the same with u defined by (2.13) which cannot belong to E_λ.

Case 2. $\alpha^p = \alpha^p(\lambda)$.

In that case $h(\lambda)$ is a double zero for $\Psi(\alpha, \cdot)$, and as $1 < p \le 2$

$$\int_0^{h(\lambda)} ds/(\Psi(\alpha, s))^{1/p} = +\infty.$$

As in Case 1 the function $r \mapsto \int_0^r ds/(\Psi(\alpha, s))^{1/p}$ is a C^2 diffeomorphism from $[0, h(\lambda))$ onto $\mathbf{R}^+$ and u cannot belong to E_λ.

Case 3. $\alpha^p < \alpha^p(\lambda)$.

In that case $\Psi(\alpha, \cdot)$ admits a simple zero $S(\alpha)$ in $(0, h(\lambda))$. As $(\partial\Psi/\partial r)(\alpha, S(\alpha)) \neq 0$, $r \mapsto (\Psi(\alpha, r))^{-1/p}$ is integrable on $(0, S(\alpha))$ and we define

$$\theta(\alpha) = \int_0^{S(\alpha)} \frac{ds}{(\Psi(\alpha, s))^{1/p}}. \tag{2.15}$$

Relation (2.13) remains valid on $[0, \theta(\alpha)]$ and we have

$$u(\theta(\alpha)) = S(\alpha), \quad u_x(\theta(\alpha)) = 0. \tag{2.16}$$

Using the energy relation at $\theta(\alpha)$ we have

$$\frac{p-1}{p}|u_x(x)|^p = \frac{\lambda}{p} S^p(\alpha) - F(S(\alpha)) - \left(\frac{\lambda}{p} u^p(x) - F(u(x))\right) \tag{2.17}$$

or

$$|u_x(x)|^p = \alpha^p + \frac{p}{p-1} F(u(x)) - \frac{\lambda}{p-1} u^p(x).$$

Hence u is decreasing on some interval $[\theta(\alpha), \Theta]$ and we have

$$x - \theta(\alpha) = -\int_{u(x)}^{S(\alpha)} \frac{ds}{(\Psi(\alpha, s))^{1/p}}. \tag{2.18}$$

This formula remains valid as long as u is decreasing, and as in §1 $x = \theta(\alpha)$ is an axis of symmetry for the restriction of u to $[0, 2\theta(\alpha)]$ and $x = 2\theta(\alpha)$ is a center of symmetry for the restriction of u to $[0, 4\theta(\alpha)]$; the necessary and sufficient condition for u to be a solution of (2.1) is that

$$1/2\theta(\alpha) \in \mathbf{N}^*. \tag{2.19}$$

Step 3. The function $\alpha \mapsto S(\alpha)$ is convex, increasing on $[0, \alpha(\lambda))$. We have $\Psi(\alpha, S(\alpha)) = 0$ and $(\partial\Psi/\partial r)(\alpha, S(\alpha)) \neq 0$. By the implicit function theorem $\alpha \mapsto S(\alpha)$ is C^2. We also have

$$\frac{d}{d\alpha}(\Psi(\alpha, S(\alpha))) = \frac{\partial\Psi}{\partial\alpha}(\alpha, S(\alpha)) + \frac{\partial\Psi}{\partial r}(\alpha, S(\alpha))\frac{dS}{d\alpha}(\alpha)$$

which gives

$$\frac{dS}{d\alpha}(\alpha) = \frac{(p-1)\alpha^{p-1}}{\lambda S^{p-1}(\alpha) - f(S(\alpha))} = \frac{p(p-1)\alpha^{p-1}}{H'(S(\alpha))}. \tag{2.20}$$

As $S(\alpha) < h(\lambda)$, $\alpha \mapsto S(\alpha)$ is increasing on $[0, \alpha(\lambda))$. Moreover

$$\frac{d^2S}{d\alpha^2}(\alpha) = p(p-1)\frac{(p-1)\alpha^{p-2}H'(S(\alpha)) - \alpha^{p-1}H''(S(\alpha))dS/d\alpha}{(H'(S(\alpha)))^2}.$$

Using (2.20) and the definition of $S(\alpha)$ and H we get

$$\text{(2.21)} \qquad \frac{d^2S}{d\alpha^2}(\alpha) = p(p-1)\alpha^{p-2}\frac{(p-1)(H'(S(\alpha)))^2 - pH(S(\alpha))H''(S(\alpha))}{(H'(S(\alpha)))^3}.$$

From (2.7) we deduce $d^2S(\alpha)/d\alpha^2 \geq 0$.

Step 4. The function $\alpha \mapsto \theta(\alpha)$ is continuous increasing on $[0, \alpha(\lambda))$. For $t \in [0, \alpha]$ the function $s \mapsto \Psi(t, s)$ admits a first positive zero at $S(t)$ which means

$$t^p + \frac{p}{p-1}F(S(t)) - \frac{\lambda}{p-1}S^p(t) = 0 \quad \text{and} \quad \Psi(\alpha, S(t)) = \alpha^p - t^p.$$

Taking t as a new variable in (2.15) we get

$$\text{(2.22)} \qquad \theta(\alpha) = \int_0^\alpha \frac{dS}{dt}(t)\frac{dt}{(\alpha^p - t^p)^{1/p}}$$

or

$$\text{(2.23)} \qquad \theta(\alpha) = \int_0^1 \frac{dS}{dt}(\alpha\sigma)\frac{d\sigma}{(1 - \sigma^p)^{1/p}}.$$

As ds/dt is increasing and C^1 on $[0, \alpha(\lambda))$, it is the same with $\alpha \mapsto \theta(\alpha)$.

Step 5. *End of the proof.* As $\lim_{\alpha\downarrow 0} S(\alpha) = 0$ and $\lim_{\alpha\downarrow 0} F(S(\alpha))/S^p(\alpha) = 0$ we get

$$\text{(2.24)} \qquad S(\alpha) \underset{\alpha\downarrow 0}{\sim} \alpha\left(\frac{p-1}{\lambda}\right)^{1/p}$$

which implies

$$\lim_{\alpha\downarrow 0}\frac{dS}{d\alpha}(\alpha) = \left(\frac{p-1}{\lambda}\right)^{1/p}$$

and

$$\text{(2.25)} \qquad \lim_{\alpha\downarrow 0}\theta(\alpha) = \left(\frac{p-1}{\lambda}\right)^{1/p}\int_0^1 \frac{d\sigma}{(1-\sigma^p)^{1/p}} = \frac{1}{2}\left(\frac{\lambda_1}{\lambda}\right)^{1/p}.$$

For the other bound we have $\lim_{\alpha\downarrow\alpha(\lambda)} S(\alpha) = h(\lambda)$. As $h(\lambda)$ is just a double zero for $\Psi(\alpha(\lambda), r)$, there exists a continuous and bounded function φ on $[0, \alpha(\lambda)]$ such that

$$\Psi(\alpha(\lambda), r) = (h(\lambda) - r)^2\varphi(r).$$

Moreover

$$\int_0^{S(\alpha)}(\Psi(\alpha, t))^{-1/p}dt > \int_0^{S(\alpha)}(\Psi(\alpha(\lambda), t))^{-1/p}\,dt$$
$$= \int_0^{S(\alpha)}(h(\lambda) - t)^{-2/p}(\varphi(t))^{-1/p}\,dt.$$

As $1 < p \leq 2$ we get

$$\text{(2.26)} \qquad \lim_{\alpha\uparrow\alpha(\lambda)}\theta(\alpha) = \int_0^{h(\lambda)}(h(\lambda) - t)^{-2/p}(\varphi(t))^{-1/p}\,dt = +\infty.$$

As a consequence $\alpha \mapsto \theta(\alpha)$ is an increasing diffeomorphism from $(0, \alpha(\lambda))$ onto $(\frac{1}{2}(\lambda_1/\lambda)^{1/p}, +\infty)$ and $1/2\theta(\alpha)$ a decreasing diffeomorphism from $(0, \alpha(\lambda))$ onto $(0, (\lambda/\lambda_1)^{1/p})$. If we assume that $\lambda_k < \lambda \leq \lambda_{k+1}$ for some $k \in \mathbf{N}^*$ there exist exactly k integers $l = 1, \dots, k$ and k positive real numbers α_l such that $1/2\theta(\alpha_l) = l$. If u_l is the solution of the initial value problem

$$\begin{cases} -(|u_{lx}|^{p-2}u_{lx})_x + f(u_l) = \lambda |u_l|^{p-2}u_l \quad \text{on } (0,1), \\ u_l(0) = 0, \quad u_{lx}(0) = \alpha_l, \end{cases} \tag{2.27}$$

then $u_l(1) = 0$, $u_l \in S_l^+$. We get the result in considering $-u_l$, $l = 1, \dots, k$.

REMARK 2.2. If we represent the bifurcation diagram (λ, u_λ) then there exists no secondary bifurcation along the branches of solutions in $S_k^\pm$ issuing from λ_k.

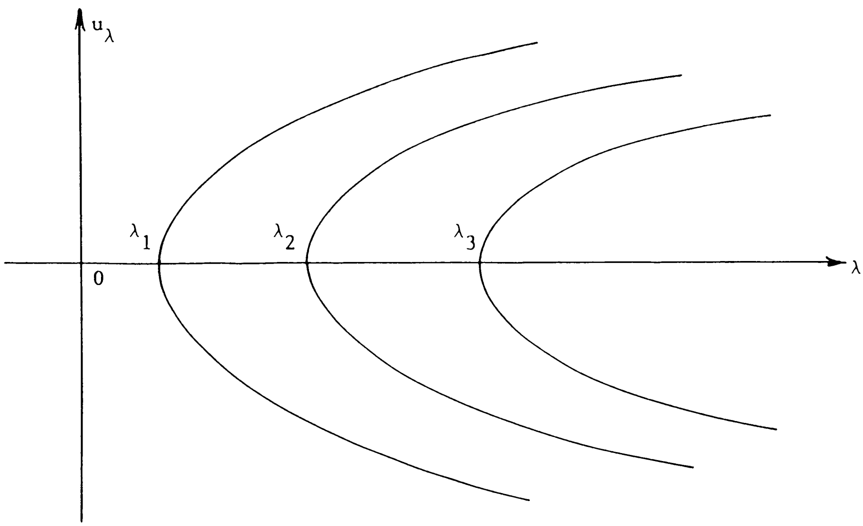

FIGURE 1

In the case $p > 2$ the main difference will come from the fact that the following integral

$$x(\lambda) = \int_0^{h(\lambda)} \frac{ds}{(\alpha^p(\lambda) + \frac{p}{p-1}F(s) - \frac{\lambda}{p-1}s^p)^{1/p}} \tag{2.28}$$

is finite as $h(\lambda)$ is a double zero of $\Psi(\alpha(\lambda), r)$.

THEOREM 2.2. *Assume* $p > 2$ *and* (2.2)–(2.7). *Then*
(i) *if* $\lambda \leq \lambda_1$ $E_\lambda = \{0\}$,
(ii) *if* $\lambda_k < \lambda \leq \lambda_{k+1}$ *for some* $k \in \mathbf{N}^*$

$$E_\lambda = \{0\} \cup \{\pm u_1\} \bigcup_{l=2}^{k} \{\pm E_\lambda^l\}, \tag{2.29}$$

where $u_1 \in S_1^+$ *and* $E_\lambda^l \subset S_l^+$, $l = 2, \dots, k$, *and*
E_λ^l *is reduced to a single element if* $2lx(\lambda) \geq 1$,
E_λ^l *is diffeomorphic to* $[0,1]^{l-1}$ *if* $0 < 2px(\lambda) < 1$.[1]

PROOF. The idea is essentially the same as in Theorem 2.1 except that in Step 2, Case 2 (that is, if $\alpha^p = \alpha^p(\lambda)$) gives rise to solutions of (2.1) with maximum value $h(\lambda)$, and in that case Serrin and Veron's existence and uniqueness result does not apply; moreover the value $u = h(\lambda)$ is a bifurcation value for (2.1).

Step 1. Assume $2x(\lambda) \geq 1$. Then the construction of Theorem 2.1 works: the function $\alpha \mapsto 1/2\theta(\alpha)$ is a decreasing diffeomorphism from $(0, \alpha(\lambda)]$ onto $[1/2x(\lambda), (\lambda/\lambda_1)^{1/p})$. As $\lambda_k < \lambda \leq \lambda_{k+1}$ there exist exactly k integers $1, 2, \dots, k$ and k positive real numbers $\alpha_1, \dots, \alpha_k$ such that $1/2\theta(\alpha_l) = l \in [1/2x(\lambda), (\lambda/\lambda_1)^{1/p})$, $l = 1, \dots, k$. and we get the corresponding solutions $u_l \in S_l^+$ by (2.26).

Step 2. Assume $4x(\lambda) \geq 1 > 2x(\lambda)$. All the elements $u_l = 2, \dots, k$ in S_l^+ are constructed as in Step 1. As for the element $u_1 \in S_1^+$ it has necessarily the following form as the initial slope must be $\alpha(\lambda)$:

for $0 \leq x \leq x(\lambda)$

$$x = \int_0^{u_1(x)} \frac{dt}{(\Psi(\alpha(\lambda), t))^{1/p}}, \tag{2.30}$$

for $x(\lambda) \leq x \leq 1 - x(\lambda)$

$$u_1(x) = h(\lambda), \tag{2.31}$$

for $1 - x(\lambda) \leq x \leq 1$

$$x - (1 - x(\lambda)) = -\int_{u_1(x)}^{h(\lambda)} \frac{dt}{(\Psi(\alpha(\lambda), t))^{1/p}}. \tag{2.32}$$

Step 3. Assume $0 < 2lx(\lambda) < 1$ for some $l \in \{2, \dots, k\}$. We can construct all the elements of $E_\lambda \cap S_l^+$ in the following way as their initial slope is necessarily $\alpha(\lambda)$:

for $0 \leq x \leq x(\lambda)$

$$x = \int_0^{u_l(x)} \frac{dt}{(\Psi(\alpha(\lambda), t))^{1/p}}, \tag{2.33}$$

for $x(\lambda) \leq x \leq x_1$ where $x_1 \in (x(\lambda), 1)$ and

$$x_1 - x(\lambda) \leq 1 - 2lx(\lambda) \tag{2.34}$$

then $u_l(x) = h(\lambda)$,

for $x_1 \leq x \leq 2x(\lambda) + x_1$

$$x - x_1 = -\int_{u_l(x)}^{h(\lambda)} \frac{dt}{(\Psi(\alpha(\lambda), t))^{1/p}}, \tag{2.35}$$

for $x_1 + 2x(\lambda) \leq x \leq x_2$ where $x_2 \in (x_1 + 2x(\lambda), 1)$ and

$$x_2 - (x_1 + 2x(\lambda)) + x_1 - x(\lambda) \leq 1 - 2lx(\lambda) \tag{2.36}$$

then $u_l(x) = -h(\lambda)$.

---

[1]And more naturally to the set $K_l = \{x = (x^1, \dots, x^l),\ x^j \geq 0,\ \sum_{j=1}^l x^j = 1 - 2lx(\lambda)\}$.

Continuing this procedure any solution $u_l \in S_l^+$ is defined by the intervals $I_j = [x_{j-1} + 2x(\lambda), x_j]$, $j = 1, \dots, l$, and $x_0 = -x(\lambda)$ where it takes the constant value $(-1)^{j+1}h(\lambda)$ and the intervals $[x_{j-1}, x_{j-1} + 2x(\lambda)]$ where it is defined by

$$x - x_{j-1} = -\int_{u_l(x)}^{h(\lambda)} \frac{dt}{(\Psi(\alpha(\lambda), t))^{1/p}} \tag{2.37}$$

if j is even or

$$x - x_{j-1} = \int_{h(\lambda)}^{u_l(x)} \frac{dt}{(\Psi(\alpha(\lambda), t))^{1/p}} \tag{2.38}$$

if j is odd.

From the above construction the total length of the I_j is $1 - 2lx(\lambda)$ and the set E_λ^l of the u_l is diffeomorphic to the $(l-1)$-dimensional cube.

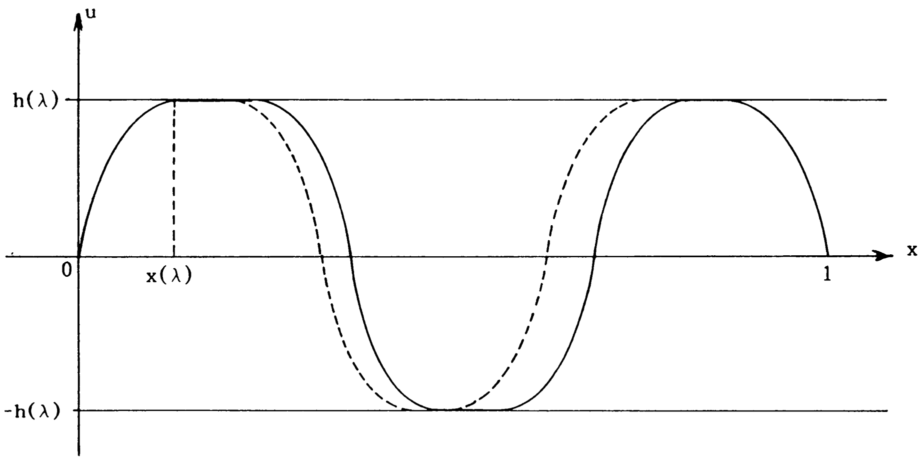

FIGURE 2. Example of construction of E_λ^3

REMARK 2.3. It is important to notice that this type of secondary bifurcation along the branch of solutions issuing from λ_k, $k \geq 2$, always appears if we have

$$\lim_{\lambda \to +\infty} x(\lambda) = 0. \tag{2.39}$$

This is in particular the case if $f(r) \underset{r \to +\infty}{\sim} |r|^{q-1} r$ which implies

(2.40)

$$x(\lambda) \underset{\lambda \to +\infty}{\sim} \lambda^{-1/p} \int_0^1 \left(\frac{q+1-p}{(p-1)(q+1)} + \frac{p}{(p-1)(q+1)} \sigma^{q+1} - \frac{\sigma^p}{p-1} \right)^{-1/p} d\sigma.$$

However this is not always the case under conditions (2.2)–(2.7), for example, with $f(r) = (|r|^{p-2} \operatorname{Log} |r|) r$ for $|r| \geq 2$, where we get

$$\lim_{\lambda \to +\infty} x(\lambda) = \int_0^1 \left(\frac{1}{p(p-1)} (1 - \sigma^p) + \frac{1}{p-1} \sigma^p \operatorname{Log} \sigma \right)^{-1/p} d\sigma. \tag{2.41}$$

We finally have the following exclusion principle.

THEOREM 2.3. *Assume* $p > 1$, (2.2)–(2.7), g *is a continuous even function increasing on* $\mathbf{R}^+$ *and* u_1 *and* u_2 *are two solutions of* (2.1); *then*

(i) *if* u_1 *and* u_2 *have the same number of zeros*

$$\int_0^1 g(u_1(x))\,dx = \int_0^1 g(u_2(x))\,dx; \tag{2.42}$$

(ii) *if* u_1 *and* u_2 *do not have the same number of zeros*

$$\int_0^1 g(u_1(x))\,dx \neq \int_0^1 g(u_2(x))\,dx. \tag{2.43}$$

PROOF. It is clear that for any function $\int_0^1 g(u(x))\,dx$ is equal to $\int_0^1 g(-u(x))\,dx$. When $p > 2$ we have only to consider two solutions of E_λ with the same number of zeros and belonging to some E_λ^l, $l \geq 2$, in the case $2lx(\lambda) < 1$. In that case u_1 and u_2 take the value $\pm h(\lambda)$ on l intervals I_j^1 and I_j^2, $j = 1, \ldots, l$, which are disconnected and have the same total length which gives

$$\int_{\bigcup_j I_j^1} g(u_1(x))\,dx = \int_{\bigcup_j I_j^2} g(u_2(x))\,dx = (1 - 2lx(\lambda))g(h(\lambda)). \tag{2.44}$$

On $(0,1)\backslash\{\bigcup_j I_j^1\}$ or $(0,1)\backslash\{\bigcup_j I_j^2\}$ u_1 and u_2 are defined by the same types of formula ((2.32) or (2.30)) and the integral of $g(u_i)$ over these sets is

$$2l \int_0^{x(\lambda)} g(u_1(x))\,dx.$$

Hence, for $i = 1, 2$, we get

$$\int_0^1 g(u_i(x))\,dx = (1 - 2lx(\lambda))g(h(\lambda)) + 2l \int_0^{x(\lambda)} g(u_i(x))\,dx \tag{2.45}$$

which proves (i).

For proving (ii) we shall assume either $1 < p \leq 2$ or $p > 2$ but u_1 and u_2 are not constant on any subinterval of $(0,1)$ (the other case is essentially the same). If u_1 and u_2 do not have the same number of zeros in $(0,1)$ we can assume $u_{1x}(0) = \alpha$, $u_{2x}(0) = \beta$, $0 < \alpha < \beta$; u_1 is $4\theta(\alpha)$-periodic, u_2 is $4\theta(\beta)$-periodic and $0 < \theta(\alpha) < \theta(\beta)$. Moreover

$$\frac{1}{2\theta(\alpha)} = k_1, \quad \frac{1}{2\theta(\beta)} = k_2, \qquad k_1, k_2 \in \mathbf{N}^*,\ k_1 > k_2. \tag{2.46}$$

Step 1. For $0 < x < \theta(\alpha)$ we have $0 < u_1(x) < u_2(x)$. On a right neighbourhood of 0 we have $u_1 < u_2$, and u_1 and u_2 are increasing on $[0, \theta(\alpha)]$. If we assume the existence of some $x_0 \in [0, \theta(\alpha)]$ such that $u_1(x_0) = u_2(x_0)$, we can always suppose that $u_1 < u_2$ in $(0, x_0)$ and then $u_{1x}(x_0) \geq u_{2x}(x_0)$. The energy relation implies

$$\begin{aligned} &\alpha^p + \frac{p}{p-1}F(u_1(x_0)) - \frac{\lambda}{p-1}u_1^p(x_0) \\ &\geq \beta^p + \frac{p}{p-1}F(u_2(x_0)) - \frac{\lambda}{p-1}u_2^p(x_0) \end{aligned} \tag{2.47}$$

and $\alpha \geq \beta$ which is impossible.

Step 2. *End of the proof.* From Step 1: $0 < u_1(x) < u_2(x')$ for $0 < x < \theta(\alpha)$ and $0 < x \le x' < \theta(\beta)$. Set φ the lowest common multiple to k_1 and k_2. There exist n_1 and $n_2 \in \mathbf{N}^*$ such that $n_1 k_1 = n_2 k_2 = \varphi$ and

$$n_1/\theta(\alpha) = n_2/\theta(\beta), \qquad 0 < n_1 < n_2. \tag{2.48}$$

Then

$$\int_0^1 g(u_1(x))\,dx = \frac{1}{\theta(\alpha)} \int_0^{\theta(\alpha)} g(u_1(x))\,dx, \tag{2.49}$$

$$\int_0^1 g(u_2(x))\,dx = \frac{1}{\theta(\beta)} \int_0^{\theta(\beta)} g(u_2(x))\,dx. \tag{2.50}$$

Setting $T = n_2\theta(\alpha) = n_1\theta(\beta)$, we have

$$\begin{aligned} \frac{1}{\theta(\alpha)} \int_0^{\theta(\alpha)} g(u_1(x))\,dx &= \frac{1}{n_2\theta(\alpha)} \int_0^{n_2\theta(\alpha)} g\left(u_1\left(\frac{\sigma}{n_2}\right)\right) d\sigma \\ &= \frac{1}{T} \int_0^T g\left(u_1\left(\frac{\sigma}{n_2}\right)\right) d\sigma \end{aligned}$$

and

$$\frac{1}{\theta(\beta)} \int_0^{\theta(\beta)} g(u_2(x))\,dx = \frac{1}{T} \int_0^T g\left(u_2\left(\frac{\sigma}{n_1}\right)\right) d\sigma,$$

which implies

$$\int_0^1 g(u_1(x))\,dx < \int_0^1 g(u_2(x))\,dx. \tag{2.51}$$

REMARK 2.4. As a consequence there exist $k+1$ different critical values for the energy functional

$$J(\omega) = \frac{1}{p} \int_0^1 |\omega_x|^p\,dx + \int_0^1 F(\omega)\,dx - \frac{\lambda}{p} \int_0^p |\omega|^p\,dx \tag{2.52}$$

defined in $W_0^{1,p}(0,1)$, for $\lambda_k < \lambda \le \lambda_{k+1}$; those critical values only depend on the set S_l, $l = 1, \ldots, k$, the critical points of (2.52) belong to. This is an immediate consequence of Theorem 2.3 and the fact that

$$J(u) = \int_0^1 \left(F(u) - \frac{1}{p} u f(u)\right) dx \tag{2.53}$$

for $u \in E_\lambda$.

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