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BIFURCATION PROBLEMS FOR EQUATIONS OF FLUID DYNAMICS AND COMPUTER ASSISTED PROOF

Takaaki Nishida, Hideaki Yoshihara, Kazunori Kumagai, and Yoshiaki Teramoto

Dedicated to Professor Fon-Che Liu on his sixtieth birthday

Abstract. We consider Couette-Taylor problems of the perturbation to Couette flow between two rotating cylinders, and show that the stationary bifurcation or Hopf bifurcation occurs when Taylor number increases. We make precise analysis of the eigenvalue problems.

1. Introduction

Some problems of stability and bifurcation of solutions for various equations of fluid dynamics can be reduced to the eigenvalue problems of systems of ordinary differential equations including physical parameters. For basic examples we have Bénard-Marangoni model for heat convections with the upper free surface and Taylor problem for fluid flows between two rotating coaxial cylinders. It is, however, difficult to analyze how the important eigenvalue of the system depends on parameters, since these systems are not self-adjoint and have variable coefficients in general.

We have developed a method to analyze this, which will be explained here. Namely, taking as a concrete example Taylor problem for viscous incompressible fluid flow between two rotating coaxial cylinders, we study the stability of the stationary laminar flow (Couette flow) when its Reynolds number or Taylor number increases. When the two cylinders rotate in the same direction, it

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is known that the Couette flow becomes unstable at the critical Taylor number for the axially symmetric perturbation and bifurcates to the stationary Taylor vortex if the relevant eigenvalue is "simple"; cf. [6, 2]. However there is no analysis for not axially symmetric perturbations when the cylinders rotate in the opposite directions. We will see that, at certain critical Taylor number, this Couette flow becomes unstable and the stationary or Hopf bifurcation occurs. These will be proved by showing how the eigenvalue of this system behaves as the parameters change. We explain the analysis using a computer assisted proof and the bifurcation theory.

2. Bifurcation of Couette Flow for Viscous Fluids between two Rotating Cylinders

The viscous fluid between two coaxial rotating cylinders has the equilibrium state named Couette flow in the cylindrical coordinates :

$$U = 0,\ V = V_0\, r,\ V_0 = A + \frac{B}{r^2},\ W = 0,\ P = \int {V_0}^2\, r\, dr.$$

Here

$$A = \frac{\omega_2 {r_2}^2 - \omega_1 {r_1}^2}{{r_2}^2 - {r_1}^2}, \quad B = -\frac{(\omega_2 - \omega_1){r_2}^2{r_1}^2}{{r_2}^2 - {r_1}^2}.$$

The equations for the perturbations around this Couette flow are given by the following semilinear system:

$$\begin{aligned}
\frac{D\,u}{D\,t} - \frac{v^2}{r} + V_0 \frac{\partial u}{\partial \theta} - 2\, V_0\, v + \frac{\partial p}{\partial r} &= \frac{1}{\mathcal{R}}(\Delta u - \frac{u}{r^2} - \frac{2}{r^2}\frac{\partial v}{\partial \theta}),\\
\frac{D\,v}{D\,t} + \frac{uv}{r} + V_0 \frac{\partial v}{\partial \theta} + 2\, A\, u + \frac{1}{r}\frac{\partial p}{\partial \theta} &= \frac{1}{\mathcal{R}}(\Delta v - \frac{v}{r^2} + \frac{2}{r^2}\frac{\partial u}{\partial \theta}),\\
\frac{D\,w}{D\,t} + V_0 \frac{\partial w}{\partial \theta} + \frac{\partial p}{\partial z} &= \frac{1}{\mathcal{R}}\Delta w,\\
\frac{1}{r}\frac{\partial (ru)}{\partial r} + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} &= 0,
\end{aligned}$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial r} + \frac{v}{r}\frac{\partial}{\partial \theta} + w\frac{\partial}{\partial z},$$

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial^2},$$

$$u = v = w = 0 \qquad \text{on} \quad r = r_1,\ r_2\ .$$

Here $\mathcal{R} = 1/\nu$, the Taylor number is defined by $T = -4A\omega_1 d^4 \mathcal{R}^2$, ω_1 is the inner angular velocity, $d = r_2 - r_1$ is the difference of the radii of the two cylinders, and $\eta = r_1/r_2$ is the ratio of the cylinder radii.

The linearized equations for the perturbations around the Couette flow are given by the system in the form:

$$\begin{aligned}
\frac{\partial u}{\partial t} + V_0 \frac{\partial u}{\partial \theta} - 2\, V_0\, v + \frac{\partial p}{\partial r} &= \frac{1}{\mathcal{R}}(\Delta u - \frac{u}{r^2} - \frac{2}{r^2}\frac{\partial v}{\partial \theta}),\\
\frac{\partial v}{\partial t} + V_0 \frac{\partial v}{\partial \theta} + 2\, A\, u + \frac{1}{r}\frac{\partial p}{\partial \theta} &= \frac{1}{\mathcal{R}}\,(\Delta v - \frac{v}{r^2} + \frac{2}{r^2}\frac{\partial u}{\partial \theta}),\\
\frac{\partial w}{\partial t} + V_0 \frac{\partial w}{\partial \theta} + \frac{\partial p}{\partial z} &= \frac{1}{\mathcal{R}}\Delta w,\\
\frac{1}{r}\frac{\partial (ru)}{\partial r} + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} &= 0,\\
u = v = w = 0 \qquad \text{on} \quad r &= r_1,\ r_2.
\end{aligned}$$

Since we are concerned with disturbances periodic with respect to z and θ and since the coefficients in the equations depend only on r, we can consider the solution of the linearized system in the form

$$\begin{aligned}
&q = q(r)\exp(\lambda t + \mathrm{i} n\theta + \mathrm{i} m z),\\
&n = 0, \pm 1, \pm 2, \cdots, \quad m \in R.
\end{aligned}$$

By this formulation, the original problem of stability is now reduced to the eigenvalue problems for the system of ordinary differential equations with the parameters n, m, $\mathcal{R}$, etc., and to investigating the behavior of the real part of the eigenvalue λ when the parameters vary. After changing $\lambda\mathcal{R} \to \lambda,\ p\mathcal{R} \to p$, we have the eigenvalue problem :

$$\begin{aligned}
\lambda u + \mathcal{R} V_0\, \mathrm{i}\, n\, u - 2\,\mathcal{R} V_0 v + \frac{\partial p}{\partial r} &= \Delta u - \frac{u}{r^2} - \frac{2}{r^2}\mathrm{i}\, n\, v,\\
\lambda v + \mathcal{R} V_0 \mathrm{i}\, n\, v + 2\mathcal{R}\, A\, u + \frac{1}{r}\mathrm{i}\, n\, p &= \Delta v - \frac{v}{r^2} + \frac{2}{r^2}\mathrm{i}\, n\, u,\\
\lambda w + \mathcal{R} V_0 \mathrm{i}\, n\, w + m\, p &= \Delta w,\\
\frac{1}{r}\frac{\partial (ru)}{\partial r} + \frac{1}{r}\mathrm{i}\, n\, v - m w &= 0,\\
u = v = w = 0 \qquad \text{on} \quad r &= r_1,\ r_2,
\end{aligned}$$

where

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - (\frac{1}{r^2}n^2 + m^2).$$

Let us put

$$U = {}^t(u, v, w, p).$$

Then the linear eigenvalue problem can be written in the form :

$$\lambda DU + \mathcal{R}MU - LU = 0,$$

where the matrix

$$D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

the operator L is the principal part of the linearized system, and the operator M is the linear part with multiplication by $\mathcal{R}$.

For our present concern, the problem is to find $\mathcal{R} = \mathcal{R}_c$ at which λ becomes $\pm\mathrm{i}\,\omega$ ($\omega \in R$) for certain periodicity in z, m fixed, and to show the simplicity of the eigenvalue, and further to show

$$\mathrm{Re}\left.\frac{\partial\lambda}{\partial\mathcal{R}}\right|_{\mathcal{R}=\mathcal{R}_c} > 0.$$

Since the eigenvalue problem is a boundary value problem for a system of first order ordinary differential equations for $W = {}^t(u, v, w, p, dv/dr, dw/dr)$, we use the shooting method to obtain the eigenvalue and eigenfunction and so we compute the fundamental solution of the initial value problem by the fourth order Taylor finite difference scheme. Then we use the Newton method to get a critical value $\mathcal{R}_0$ for $\mathcal{R}$, where the corresponding eigenvalue $\lambda_0 = \mathrm{i}\,\omega_0$ and the eigenfunction W_0 are obtained. In order to prove that in a small neighbourhood of these computed values there exist the true $\mathcal{R}_c$, eigenvalue $\lambda_c = \mathrm{i}\,\omega_c$ and eigenfunction $W_c = {}^t(u_c, v_c, w_c, p_c, dv_c/dr, dw_c/dr)$, we use a computer assisted proof, for which the theory of pseudo-trajectory of the systems of ordinary differential equations is used and the interval analysis using a computer software is necessary for the round-off error estimates; cf. [7]

For this existence proof we use the following criterion : Since there are three independent fundamental solutions (W_1, W_2, W_3) which correspond to the free values of ${}^t(p, dv/dr, dw/dr)$ at $r = r_1$, we take those values at $r = r_2$ to examine the determinant and we put

$$\mathcal{F}(\mathcal{R}, \lambda; m, n) = \det \left.\begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix}\right|_{r=r_2}.$$

Noting that this can be rewritten as

$$\mathcal{F}(\mathcal{R},\lambda)=\mathcal{F}(\mathcal{R}_0,\lambda_0)+\frac{\partial\mathcal{F}}{\partial\mathcal{R}}(\mathcal{R}-\mathcal{R}_0)+\frac{\partial\mathcal{F}}{\partial\lambda}(\lambda-\lambda_0)=0,$$

we can state our criterion for the existence of the critical eigenvalue based on the simplified Newton method as follows.

Theorem. *Suppose, for small $\epsilon>0$, there exist $\mathcal{R}_0$ and λ_0 such that*

$$\|\mathcal{F}(\mathcal{R}_0,\ \lambda_0)\|<\epsilon.$$

Put

$$L_0\equiv\left(\overline{\frac{\partial\mathcal{F}}{\partial\mathcal{R}}}(\mathcal{R}_0,\lambda_0)\,,\ \overline{\frac{\partial\mathcal{F}}{\partial\lambda}}(\mathcal{R}_0,\lambda_0)\right),$$

where "$\overline{\quad\quad}$" means an approximate value for the expression. Suppose further that, for small $\delta>0$, there is a $\rho>0$, such that the estimate

$$\|D\mathcal{F}\,(\mathcal{R}\,\lambda)-L_0\|<\delta$$

holds for any $(\mathcal{R},\ \lambda)$ such that

$$(\mathcal{R}-\mathcal{R}_0)^2+|\lambda-\lambda_0|^2<\rho^2.$$

For ϵ, ρ, δ and L_0 as above, if it holds that

$$\|{L_0}^{-1}\|\left(\frac{\epsilon}{\rho}+\delta\right)\leq 1,$$

then there exist some $\mathcal{R}_c$ and λ_c in the ρ-neighborhood of $\mathcal{R}_0$ and λ_0 satisfying

$$\mathcal{F}(\mathcal{R}_c,\ \lambda_c)=0.$$

Next we consider the derivative of λ with respect to $\mathcal{R}$ using the eigenfunction $\Phi={}^t(u_c,v_c,w_c,p_c)$:

$$\left.\frac{\partial\lambda}{\partial\mathcal{R}}\right|_{\mathcal{R}=\mathcal{R}_c}=-\frac{(M\Phi,\ \Psi)}{(\Phi,\ \Psi)},$$

where Ψ is the eigenfunction of the linear adjoint system at $\mathcal{R}=\mathcal{R}_c$, and $(\cdot,\cdot)$ means the L^2 inner product with respect to the measure $r\,dr$. This can be computed and verified by the following explicit formula:

$$\left.\frac{\partial\lambda}{\partial\mathcal{R}}\right|_{\mathcal{R}=\mathcal{R}_c}=\frac{2(V_0\,v,\ u^*)-2(A\,u,\ v^*)-in\{(V_0\,u,\ u^*)+(V_0\,v,\ v^*)+(V_0\,w,\ w^*)\}}{(u,\ u^*)+(v,\ v^*)+(w,\ w^*)}.$$

In these computations, especially when $(M\Phi,\ \Psi) \neq 0$, we can show the "simplicity" of the eigenvalue. For details, see [7] and [5].

By these properties of the eigenvalues and by the fact that the evolution problem for the above linearized system generates an analytic semigroup and that the full system is semilinear, we see that a sufficient condition for the occurrence of the stationary or Hopf bifurcation holds; cf. [2, 3]. Hence, we see that the stationary or Hopf bifurcation occurs at $\mathcal{R} = \mathcal{R}_c$ according to $\omega_c = 0$ or $\omega_c \neq 0$ respectively, some of the latter of which corresponds to the wavy Taylor vortex.

Also we can see that the bifurcation is supercritical, i.e., the bifurcation occurs for $\mathcal{R} > \mathcal{R}_c$ as follows. The system for the stationary problem in the case of stationary bifurcation can be written in the form :

$$F \equiv (L - \mathcal{R}M)\,U - \mathcal{R}H[U,U] = 0,$$

where the operator H is the bilinear part of the original semilinear system of equations for the perturbation. By differentiation of the bifurcation equation F with respect to the bifurcation parameter s, we have at $s = 0$

$$(L - \mathcal{R}(0)M)\dot{U}(0) = 0, \quad \mathcal{R}(0) = \mathcal{R}_c,\ \dot{U}(0) = \Phi.$$

Since $(M\Phi,\ \Psi) \neq 0$, we have, by further differentiations,

$$\begin{aligned}
&\dot{\mathcal{R}}(0) = 0, \quad \ddot{U}(0) = 2(L - \mathcal{R}(0)M)^{-1}\,\mathcal{R}(0)\,H[\Phi,\Phi], \\
&\ddot{\mathcal{R}}(0) = -\frac{\mathcal{R}(0)(H[\ddot{U}(0),\Phi] + H[\Phi,\ddot{U}(0)],\ \Psi)}{(M\Phi,\ \Psi)}, \\
&\dddot{U}(0) = 3(L - \mathcal{R}(0)M)^{-1}\{\ddot{\mathcal{R}}(0)M\Phi + \mathcal{R}(0)(H[\ddot{U}(0),\Phi] + H[\Phi,\ddot{U}(0)])\}.
\end{aligned}$$

Thus we can get and verify the value $\ddot{R}(0)$ in a way similar to that of $(M\Phi, \Psi)$.

At last we notice that we do not know whether the principle of exchange of stability holds in the case that the two cylinders rotate in the same direction, i.e., we have no proof that our eigenvalue is the first one which crosses the imaginary axis as Taylor number increases (cf. [4, 8]). Since the linearized equation is sectorial, we may comb to compute the values of the determinant at $r = r_2$ at finite but "dense" points in the triangle for each R_c.

In the following tables, the numerical values of floating points are rounded off toward zero.

Example 1. We take $\omega_1 = 1$, $\omega_2 = 0$, $\eta = 0.9$ and find the minimum Taylor number by changing m for fixed n.

n	m	λ	Taylor	$\overline{\frac{\partial\lambda}{\partial\mathcal{R}}}\Big\|_{\mathcal{R}=\mathcal{R}_0}$
0	3.12875	0.0	3646.8170	1.79282
± 1	3.13952	$\mp \mathrm{i} \times 7.32110$	3684.7760	$1.78550 \mp \mathrm{i} \times 0.51834$
± 2	3.17163	$\mp \mathrm{i} \times 14.8938$	3802.5176	$1.76290 \mp \mathrm{i} \times 1.03897$

Example 2. We take $\omega_1 = 1$, $\omega_2 = 0$, $\eta = 0.8$.

n	m	λ	Taylor	$\overline{\frac{\partial\lambda}{\partial\mathcal{R}}}\Big\|_{\mathcal{R}=\mathcal{R}_0}$
0	3.13264	0.0	3988.6474	1.11005
± 1	3.16136	$\mp \mathrm{i} \times 11.1640$	4105.3297	$1.09806 \mp \mathrm{i} \times 0.48575$
± 2	3.24655	$\mp \mathrm{i} \times 23.4254$	4489.8536	$1.05903 \mp \mathrm{i} \times 0.97696$

Example 3. We take $\omega_1 = 1$, $\omega_2 = 0$, $n = 0$, and find the minimum Taylor number by changing m and n.

η	m	λ	Taylor	$\overline{\frac{\partial\lambda}{\partial\mathcal{R}}}\Big\|_{\mathcal{R}=\mathcal{R}_0}$	$\ddot{R}(0)$
0.95	3.12747	0.0	3509.7118	2.69052	0.15399
0.9	3.12875	0.0	3646.8170	1.79282	1.13015
0.85	3.13044	0.0	3804.8820	1.37327	4.09525
0.8	3.13264	0.0	3988.6474	1.11005	11.2288
0.75	3.13541	0.0	4204.3432	0.92131	26.6805
0.5	3.16247	0.0	6199.1562	0.39411	1055.93

Example 4. We take $\omega_1 = 1$, $\omega_2 = -1$, $\eta = 0.9$.

n	m	λ	Taylor	$\left.\overline{\frac{\partial \lambda}{\partial \mathcal{R}}}\right\|_{\mathcal{R}=\mathcal{R}_0}$
0	4.40932	0.0	27135.056	2.01051
± 1	4.28616	$\mp \mathrm{i} \times 10.9488$	26674.825	$1.93430 \mp \mathrm{i} \times 0.37847$
± 2	3.89621	$\mp \mathrm{i} \times 20.3854$	25162.403	$1.71116 \mp \mathrm{i} \times 0.78599$
± 3	3.72174	$\mp \mathrm{i} \times 26.8970$	23855.563	$1.58758 \mp \mathrm{i} \times 1.18245$
± 4	3.85451	$\mp \mathrm{i} \times 33.5892$	24627.356	$1.54154 \mp \mathrm{i} \times 1.55099$
± 5	4.13825	$\mp \mathrm{i} \times 43.5337$	28185.177	$1.48282 \mp \mathrm{i} \times 1.97787$

Example 5. We take $\omega_1 = 1$, $\omega_2 = -1$, $\eta = 0.8$.

n	m	λ	Taylor	$\left.\overline{\frac{\partial \lambda}{\partial \mathcal{R}}}\right\|_{\mathcal{R}=\mathcal{R}_0}$
0	4.85722	0.0	41597.501	1.36396
± 1	4.69327	$\mp \mathrm{i} \times 18.7056$	40867.442	$1.29530 \mp \mathrm{i} \times 0.36412$
± 2	3.83507	$\mp \mathrm{i} \times 33.0079$	36750.755	$0.98253 \mp \mathrm{i} \times 0.74996$
± 3	4.08956	$\mp \mathrm{i} \times 43.1478$	37653.258	$0.94596 \mp \mathrm{i} \times 1.12139$
± 4	4.78469	$\mp \mathrm{i} \times 66.4693$	51173.553	$0.93091 \mp \mathrm{i} \times 1.61034$

Example 6. We take $\omega_1 = 1$, $\omega_2 = -1$, and find the minimum Taylor number by changing n and m.

η	n	m	λ	Taylor	$\left.\overline{\frac{\partial \lambda}{\partial \mathcal{R}}}\right\|_{\mathcal{R}=\mathcal{R}_0}$
0.95	± 4	3.68019	$\mp \mathrm{i} \times 23.3577$	20067.846	$2.40958 \mp \mathrm{i} \times 1.62474$
0.9	± 3	3.72174	$\mp \mathrm{i} \times 26.8970$	23855.563	$1.58758 \mp \mathrm{i} \times 1.18245$
0.85	± 3	3.84706	$\mp \mathrm{i} \times 34.3330$	29123.243	$1.18525 \mp \mathrm{i} \times 1.14219$
0.8	± 2	3.83507	$\mp \mathrm{i} \times 33.0079$	36750.755	$0.98253 \mp \mathrm{i} \times 0.74996$
0.75	± 2	3.87341	$\mp \mathrm{i} \times 40.0852$	46220.514	$0.79271 \mp \mathrm{i} \times 0.73340$
0.7	± 2	3.98525	$\mp \mathrm{i} \times 48.4747$	60065.149	$0.65772 \mp \mathrm{i} \times 0.71741$
0.65	± 2	4.17848	$\mp \mathrm{i} \times 58.9820$	81027.435	$0.55859 \mp \mathrm{i} \times 0.70212$
0.6	± 2	4.45812	$\mp \mathrm{i} \times 72.6491$	113959.34	$0.48340 \mp \mathrm{i} \times 0.68720$

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Takaaki Nishida, Hideaki Yoshihara, and Kazunori Kumagai
Kyoto University, Department of Mathematics, Sakyo-ku, Kyoto, Japan

Yoshiaki Teramoto
Setsunan University, Faculty of Engineering, Neyagawa-city, Japan