Bifurcation Problems with $\mathbb{O}(2) \oplus \mathbb{Z}_2$ Symmetry and the Buckling of a Cylindrical Shell (*).

ERNESTO BUZANO - ALESSANDRO RUSSO

Summary. – *In this paper we employ equivariant singularity theory to study the post-buckling behavior of a cylindrical shell under axial compression, obtaining some results about the existence of secondary bifurcations and how they are connected to each other. The basic idea, first employed by Bauer, Keller and Reiss in* [1], *and then coupled with singularity theory by Schaeffer and Golubitsky in* [16] *and* [17] *and by Buzano in* [4], *consists in unfolding a multiple eigenvalue, obtained by forcing two eigenvalues to coalesce by varying the geometric parameters of the shell. This approah is made possible by a general analysis of bifurcation problems invariant with respect to the symmetries of the cylinder i.e. with respect to the group* $\mathbb{O}(2) \oplus \mathbb{Z}_2$.

Introduction.

The buckling of a complete (1) thin cylindrical shell has been the subject of a vast number of investigations since the beginning of this century and has always presented great difficulties. For example experimental results show that the buckling can occur long before then it is theoretically expected (even with a 60 percent error). This disagreement between theory and experiment has been explained for the first time by VON KÁRMÁN and TSIEN [10], by studying the post-buckling behavior by means of suitable nonlinear equations. These results have been both inserted in the framework of a general theory of elastic stability and improved by KOITER, who also carried out an analysis of imperfection-sensitivity [11]. All these studies together with their subsequent generalizations and improvements have been carried out in a heuristic framework and in any case no theoretical results have been obtained about the existence of possible secondary bifurcations.

The interested reader may consult the up-to-date books by DIKMEN [6] on the general theory of thin shells and by YAMAKI [19] on cylindrical shells.

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Indirizzo degli AA.: E. BUZANO: Dipartimento di Matematica, Università di Torino, Via Carlo Alberto 10, 10123 Torino, Italia; A. RUSSO: Istituto di Analisi Numerica (IAN) del Consiglio Nazionale delle Ricerche, Corso Carlo Alberto 5, 27100 Pavia, Italia.

(1) We seize the opportunity of distinguishing between complete (closed) cylindrical shells and cylindrical panels which are only a part of a cylinder and yield much simpler buckling equations.

In this paper we employ equivariant singularity theory to study the post-buckling behavior of a cylindrical shell under axial compression, obtaining some results about the existence of secondary bifurcations and how they are connected to each other (see Theorem 7.3 and Figure 2 of Section 7). The basic idea, first employed by BAUER, KELLER and REISS in [1], and then coupled with singularity theory by SCHAEFFER and GOLUBITSKY in [16] and [17] and by BUZANO in [4], consists in unfolding a multiple eigenvalue, obtained by forcing two eigenvalues to coalesce by varying the geometric parameters of the shell. This approach is made possible by a general analysis of bifurcation problems invariant with respect to the symmetry group of the cylinder i.e. $\mathbb{O}(2)\oplus\mathbb{Z}_2$.

Before describing the content of this paper, we would like to remark that for the first time we state our results in a precise analytic way without resorting to the germ formalism which is not suitable to describe the solution set of a bifurcation equation.

In Section 1 a short account of the non-linear model of Donnell is given. In Sections 2 and 3 we state the variational problem in a functional analysis framework and show how to reduce it to a finite dimensional one by the method of Lyapunov-Schmidt. In Section 4 we investigate how the symmetries of the cylinder are inherited by the energy functional and the reduced bifurcation equations. In Section 5 we compute the first eigenvalue of the linearized equation and the relevant eigenfunctions. In Section 6 we fix some general notation and terminology concerning bifurcation diagrams. In Section 7 we state our results, which are also illustrated by some diagrams. The proofs are given in Section 8, 10 and 11 while in Section 9 we recall some general results on equivariant singularity theory.

1. - The model.

We begin with a description of the mechanical model employed in this paper. Consider a thin circular cylinder of *radius R*, *length l* and *thickness h*, made of elastic material and subject along its edges to uniform axial compression given by a *dead-load* λ per unit of circumference. Let X, Y, Z be orthogonal coordinates fixed in space. We specify the cylinder by the following vector function

$$\boldsymbol{r}(\theta, \zeta) = \left(R\cos\theta,\, R\sin\theta,\, \frac{l}{\pi}\zeta\right) \tag{1.1}$$

where $(\theta, \zeta) \in [0, 2\pi]\times[0, \pi]$ are cylindrical coordinates and $\boldsymbol{r}$ is the *position-vector* joining the origin O with a point P on the cylinder, see Figure 1.

Under suitable simplifying assumptions (the so called *shallow buckling modes*), the study of the post-buckling behavior of a thin shell under dead-loading on the edges reduces to the problem of finding the critical points of the following energy

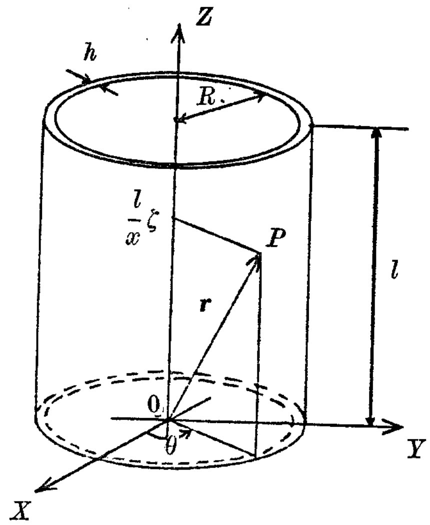

Figure 1.

functional (see [13], (2.14) to (2.16)):

$$\mathcal{F} = \int_S \left[\frac{1}{2} N^{\alpha\beta} w_{,\alpha} w_{,\beta} + \frac{1}{2} h E^{\alpha\beta\lambda\mu} \gamma_{\alpha\beta}\gamma_{\lambda\mu} + \frac{1}{24} h^3 E^{\alpha\beta\lambda\mu} \varrho_{\alpha\beta}\varrho_{\lambda\mu}\right] dS \tag{1.2}$$

where:

a) a comma followed by a subscript indicates partial differentiation;

b) Greek indices take over the values 0 and 1;

c) summation convention for a repeated index is employed;

d) S is the middle surface of the shell;

e) $N^{\alpha\beta}$ is the middle surface stress tensor in the fundamental (pre-buckled) state;

f) $\gamma_{\alpha\beta}$ and $\varrho_{\alpha\beta}$ are the tensors of the strains and of the changes of curvature of the middle surface respectively. They are given by

$$\gamma_{\alpha\beta} = \tfrac{1}{2}(u_{\alpha|\beta} + u_{\beta|\alpha}) - b_{\alpha\beta} w + \tfrac{1}{2} w_{,\alpha} w_{,\beta} \quad \text{and} \quad \varrho_{\alpha\beta} = w_{|\alpha\beta} \tag{1.3}$$

where the vertical stroke indicates covariant middle surface differentiation, $b_{\alpha\beta}$ is the second fundamental tensor of the middle surface and u_1, u_2, w are the components of the displacements with respect to the coordinates on the middle surface;

g) $E^{\alpha\beta\lambda\mu}$ is the elastic moduli tensor and is given by

$$E^{\alpha\beta\lambda\mu} = \frac{E}{2(1-\nu^2)}[(1-\nu)(a^{\alpha\lambda}a^{\beta\mu} + a^{\alpha\mu}a^{\beta\lambda}) + 2\nu a^{\alpha\beta}a^{\lambda\mu}] \tag{1.4}$$

where $a^{\alpha\beta}$ is the first fundamental tensor of the middle surface and E, ν are the Young modulus and the Poisson ratio respectively. We have $E > 0$ and $0 < \nu < \frac{1}{2}$.

Now we compute the functional $\mathcal{T}$ when S is the cylinder defined by (1.1). First of all we have to evaluate the tensor $N^{\alpha\beta}$ in the fundamental state. It is usual, as a first approximation, to assume that the fundamental state, to which the buckled state is referred, is obtained by pure expansion and compression, so that the shell mantains its cylindrical shape (see [6], Section 11.4 (*a*), page 114). More precisely we assume that the displacement field of the fundamental state is given by

$$u_1 = 0\,, \qquad u_2 = \frac{1}{Eh}\left(\frac{l}{\pi}\right)^2 \lambda \left(\frac{\pi}{2} - \zeta\right), \qquad w = \frac{\nu R}{Eh}\,. \tag{1.5}$$

Now the constitutive equations for a thin shell are ([12], (8.9))

$$N^{\alpha\beta} = hE^{\alpha\beta\lambda\mu}\gamma_{\lambda\mu}\,,$$

thus by making use of the cylindrical coordinates (1.1) and of (1.3), (1.4) and (1.5) we have

$$N^{11} = 0\,, \qquad N^{12} = 0\,, \qquad N^{22} = -\left(\frac{\pi}{l}\right)^2 \lambda\,. \tag{1.6}$$

Then, by substituting (1.6) in (1.2) and using cylindrical coordinates (1.1), one obtains the following *energy functional* (obtained for the first time by Donnell in 1934):

$$\begin{aligned} (1.7)\qquad f(u, v, w, \lambda, h, R, l) &= \frac{(1-\nu^2)\pi R^3}{Ehl}\,\mathcal{T}(u, v, w, \lambda, h, R, l) = \\ &= \frac{1}{2}\int_\Omega \Bigg\{\left(Ru_{,\theta} + Rw + \frac{1}{2}w_{,\theta}^2\right)^2 + 2\nu\left(\frac{\pi R}{l}\right)^2\left(Ru_{,\theta} + Rw + \frac{1}{2}w_{,\theta}^2\right)\left(\frac{l}{\pi}v_{,\zeta} + \frac{1}{2}w_{,\zeta}^2\right) + \\ &\quad + \frac{1}{2}(1-\nu)\left(\frac{\pi R}{l}\right)^2\left(Ru_{,\zeta} + \frac{l}{\pi}v_{,\theta} + w_{,\theta}w_{,\zeta}\right)^2 + \left(\frac{\pi R}{l}\right)^4\left(\frac{l}{\pi}v_{,\zeta} + \frac{1}{2}w_{,\zeta}^2\right)^2 + \\ &\quad + \frac{h^2}{12}\left[w_{,\theta\theta}^2 + 2\nu\left(\frac{\pi R}{l}\right)^2 w_{,\theta\theta}w_{,\zeta\zeta} + 2(1-\nu)\left(\frac{\pi R}{l}\right)^2 w_{,\theta\zeta}^2 + \left(\frac{\pi R}{l}\right)^4 w_{,\zeta\zeta}^2\right] - \\ &\quad - \frac{(1-\nu^2)R^2}{Eh}\left(\frac{\pi R}{l}\right)^2 w_{,\zeta}^2\Bigg\}\,d\theta\,d\zeta \end{aligned}$$

where

$$\Omega = (0, 2\pi)\times(0, \pi) \tag{1.8}$$

and u, v, w are the *physical components of the displacements* (i.e. referred to an orthonormal basis), that is

$$u = \frac{1}{R} u_1 , \qquad v = \frac{\pi}{l} u_2$$

(w is already a physical component).

2. – The bifurcation problem.

The energy functional f given by (1.7) is defined on the cylinder, thus the displacements u, v and w are periodic functions of the variable θ. Moreover they must satisfy suitable boundary conditions imposed when $\zeta = 0, \pi$. We assume that the shell is simply-supported along the edges:

$$u(\theta, 0) = u(\theta, \pi) = 0 , \tag{2.1a}$$

$$v_{,\zeta}(\theta, 0) = v_{,\zeta}(\theta, \pi) = 0 , \tag{2.1b}$$

$$w(\theta, 0) = w(\theta, \pi) = 0 , \tag{2.1c}$$

$$w_{,\zeta\zeta}(\theta, 0) = w_{,\zeta\zeta}(\theta, \pi) = 0 \tag{2.1d}$$

for each $\theta \in R$. Remark that (2.1b) specify v up to a constant. This means that the position of the shell is specified up translations along Z-axis. In order to avoid this indeterminateness we impose the further constraint

$$\int_\Omega v \, d\theta \, d\zeta = 0 . \tag{2.2}$$

Now set

$$\tilde{\Omega} = \mathbb{R} \times (0, 2\pi)$$

and consider the following Sobolev Space

$$H^k_\#(\tilde{\Omega}) = \left\{ g \colon \tilde{\Omega} \to \mathbb{R} \colon \left(\frac{\partial^{i+j} g}{\partial \theta^i \, \partial \zeta^j} \right)_{|\Omega} \in L^2(\Omega) \quad \text{for all } i, j \geq 0, \ i + j \leq k \right.$$
$$\left. \text{and } g(\theta, \zeta) = g(\theta + 2\pi, \zeta) \text{ a.e. in } \tilde{\Omega} \right\}$$

where Ω is defined in (1.8). $H^k_\#(\tilde{\Omega})$ is a Hilbert space with respect to the scalar product

$$(g, h)_k \overset{=}{_{\text{def}}} \sum_{\substack{i,j \geq 0 \\ i+j \leq k}} \int_\Omega \frac{\partial^{i+j} g}{\partial \theta^i \, \partial \zeta^j} \frac{\partial^{i+j} h}{\partial \theta^i \, \partial \zeta^j} \, d\theta \, d\zeta .$$

Set

$H = \{(u, v, w) \in H^1_\#(\tilde{\Omega}) \times H^1_\#(\tilde{\Omega}) \times H^2_\#(\tilde{\Omega})$: u and w satisfy (2.1a) and (2.1c) and v satisfies (2.2)$\}$.

H is a Hilbert space because it is a closed subspace of $H^1_\#(\tilde{\Omega}) \times H^1_\#(\tilde{\Omega}) \times H^2_\#(\tilde{\Omega})$. Let $\|\cdot\|_H$ and $(\cdot,\cdot)_H$ denote respectively the norm and the scalar product of H. Then f is a nonlinear functional defined on $H \times \mathbb{R} \times (\mathbb{R}^*_+)^3$, where

$$\mathbb{R}^*_+ = \{x \in \mathbb{R}: x > 0\}.$$

For the sake of brevity we set

$$\xi = (u, v, w) \quad \text{and} \quad \tau = (h, R, l)$$

and write f as

$$f(\xi, \lambda, \tau) = \tfrac{1}{2}[\mathcal{A}(\xi, \xi, \tau) - \lambda \mathcal{B}(\xi, \xi, \tau) + \mathcal{C}(\xi, \tau)]$$

where

$$\begin{aligned}(2.3)\quad \mathcal{A}(\xi, \hat{\xi}, \tau) = \int_\Omega \Big\{ & R^2(u_{,\theta} + w)(\hat{u}_{,\theta} + \hat{w}) + \nu R^2 \frac{\pi R}{l}[(u_{,\theta} + w)\hat{v}_{,\zeta} + (\hat{u}_{,\theta} + \hat{w})v_{,\zeta}] + \\ & + \frac{1}{2}(1-\nu)\left(\frac{\pi R}{l}\right)^2 \left(Ru_{,\zeta} + \frac{l}{\pi} v_{,\theta}\right)\left(R\hat{u}_{,\zeta} + \frac{l}{\pi}\hat{v}_{,\theta}\right) + R^2 \left(\frac{\pi R}{l}\right)^2 v_{,\zeta}\hat{v}_{,\zeta} + \\ & + \frac{h^2}{12}\Big[w_{,\theta\theta}\hat{w}_{,\theta\theta} + \nu\left(\frac{\pi R}{l}\right)^2 (w_{,\theta\theta}\hat{w}_{,\zeta\zeta} + \hat{w}_{,\theta\theta}w_{,\zeta\zeta}) + \\ & + 2(1-\nu)\left(\frac{\pi R}{l}\right)^2 w_{,\theta\zeta}\hat{w}_{,\theta\zeta} + \left(\frac{\pi R}{l}\right)^4 w_{,\zeta\zeta}\hat{w}_{,\zeta\zeta}\Big]\Big\} d\theta\, d\zeta,\end{aligned}$$

$$(2.4)\quad \mathcal{B}(\xi, \hat{\xi}, \tau) = \frac{(1-\nu^2)R}{Eh}\left(\frac{\pi R}{l}\right)^2 \int_\Omega w_{,\zeta}\hat{w}_{,\zeta}\, d\theta\, d\zeta.$$

and $\mathcal{C}$ is the remainder. Resorting to the Sobolev embedding $H^1_\#(\tilde{\Omega}) \hookrightarrow L^q(\Omega)$ $(q \geq 2)$, one can easily prove the following

PROPOSITION 2.1. – For each $\tau \in (\mathbb{R}^*_+)^3$, $\mathcal{A}$ and $\mathcal{B}$ are bilinear symmetric forms; moreover there exists a positive constant $c(\tau)$ such that

$$\mathcal{A}(\xi, \xi, \tau) \leq c(\tau)\|\xi\|^2_H \quad \text{and} \quad \mathcal{B}(\xi, \xi, \tau) \leq c(\tau)\|\xi\|^2_H \quad \text{for each } \xi \in H$$

and

$$\mathcal{C}(\xi, \tau) = O(\|\xi\|_H^3) \quad \text{for } \|\xi\|_H \to 0 .$$

Finally we have $f \in C^\infty(H \times \mathbb{R} \times (\mathbb{R}_+^*)^3)$. ■

COROLLARY 2.2.

$$D_\xi f(0, \lambda, \tau) = 0 \quad \text{for each } (\lambda, \tau) \in \mathbb{R} \times (\mathbb{R}_+^*)^3$$

and

$$D_\xi^2 f(0, \lambda, \tau)[\xi, \hat{\xi}] = \mathcal{A}(\xi, \hat{\xi}, \tau) - \lambda \mathcal{B}(\xi, \hat{\xi}, \tau)$$

for each $(\xi, \hat{\xi}, \lambda, \tau) \in H \times H \times \mathbb{R} \times (\mathbb{R}_+^*)^3$. ■

REMARK 2.3. – $D_\xi^k f(\xi, \lambda, \tau)[\xi_1, \dots, \xi_k]$ denotes the value the Fréchet partial derivative $D_\xi^k f(\xi, \lambda, \tau)$ takes on $(\xi_1, \dots, \xi_k) \in H^k$.

Now define the C^∞ map

$$\mathcal{F}\colon H \times \mathbb{R} \times (\mathbb{R}_+^*)^3 \to H$$

by

$$(2.5) \qquad (\mathcal{F}(\xi, \lambda, \tau), \hat{\xi})_H = D_\xi f(\xi, \lambda, \tau)[\hat{\xi}] \quad \text{for each } \hat{\xi} \in H .$$

$\mathcal{F}$ is the *gradient* of the energy functional f, thus the critical points of f (which yield the buckled states) are the solutions to the *bifurcation equation*:

$$(2.6) \qquad \mathcal{F}(\xi, \lambda, \tau) = 0 .$$

For each $\tau \in (\mathbb{R}_+^*)^3$ set

$$(2.7) \qquad \mathcal{S}_\tau = \{(\xi, \lambda) \in H \times \mathbb{R}\colon \mathcal{F}(\xi, \lambda, \tau) = 0\} .$$

Observe that

$$\{0\} \times \mathbb{R} \subset \mathcal{S}_\tau \quad \text{for each } \tau \in (\mathbb{R}_+^*)_3$$

by Corollary 2.2. The elements of $\{0\} \times \mathbb{R}$ are called *trivial solutions* and all of them correspond to the fundamental state. We want to study $\mathcal{S}_\tau$ near $\{0\} \times \mathbb{R}$.

To this purpose define for each $\tau \in (\mathbb{R}_+^*)^3$ two linear operators $A_\tau, B_\tau^3\colon H \to H$ such that

$$(A_\tau \xi, \hat{\xi})_H = \mathcal{A}(\xi, \hat{\xi}, \tau) \quad \text{and} \quad (B_\tau \xi, \hat{\xi})_H = \mathcal{B}(\xi, \hat{\xi}, \tau)$$

for each $\xi, \hat{\xi} \in H$. By Proposition 2.1 we have that A_τ and B_τ are bounded and self-adjoint, moreover from Corollary 2.2 it follows that

$$D_\xi \mathcal{F}(0, \lambda, \tau)[\xi] = (A_\tau - \lambda B_\tau)\xi . \tag{2.8}$$

PROPOSITION 2.4. – Consider the equation

$$(A_\tau - \lambda B_\tau)\xi = 0 . \tag{2.9}$$

The eigenvalues of (2.9) make an unbounded increasing sequence of positive real numbers $\lambda_n(\tau)$. Whenever λ is not an eigenvalue, $A_\tau - \lambda B_\tau$ is an isomorphism. On the other hand if $\lambda = \lambda_n(\tau)$, then $A_\tau - \lambda_n(\tau) B_\tau$ is a Fredholm operator with index 0, that is $N_n(\tau) = \ker\big(A_\tau - \lambda_n(\tau) B_\tau\big)$ has finite dimension and $\big(A_\tau - \lambda_n(\tau) B_n\big)|_{E_n(\tau)}$, where $E_n(\tau) = \big(N_n(\tau)\big)^\perp$, is an isomorphism on $E_n(\tau)$.

PROOF. – Exactly in the same way as in [2], Théorème 6.1-1 and Lemme 3.4-2, one proves that $\mathcal{A}(\xi, \xi, \tau)$, given by (2.3), is coercive for all $\tau \in (\mathbb{R}_+^*)^3$ (observe that Théorème 5.1-1 of [2] (Rigid Motion Theorem) does not apply to our situation, but its Corollaire 5.2-1 still holds as one can easily check). It follows that A_τ has a bounded inverse for each $\tau \in (\mathbb{R}_+^*)^3$. On the other hand it follows from Rellich-Kondrachov Theorem that B_τ is a compact operator. Therefore we can rewrite equation (2.9) as $(I - \lambda A_\tau^{-1} B_\tau)\xi = 0$, where $A_\tau^{-1} B_\tau$ is a compact self-adjoint operator with respect to the scalar product $\big((\xi, \hat{\xi})\big) = (A_\tau \xi, \hat{\xi})_H$, which is equivalent to $(\xi, \hat{\xi})_H$ because $\mathcal{A}$ is coercive. Consequently the statement follows from the spectral theory of Hilbert-Schmidt. ■

Recall now the following easy consequence of the Implicit Function Theorem:

PROPOSITION 2.5. – If $D_\xi \mathcal{F}(0, \lambda, \tau)$ is an isomorphism on H then there exists a neighborhood $\mathcal{U}$ of $(0, \lambda) \in H \times \mathbb{R}$ such that $\mathcal{S}_\tau \cap \mathcal{U} = \big(\{0\} \times \mathbb{R}\big) \cap \mathcal{U}$. ■

From (2.8) and Propositions 2.4 and 2.5 we have that if λ is smaller than the first eigenvalue $\lambda_0(\tau)$, the trivial solution $(0, \lambda)$ is isolated. Hence $\big(0, \lambda_0(\tau)\big)$ is the first possible bifurcation point which coincides with the critical load of the shell. *So our bifurcation problem consists in studying* $\mathcal{S}_\tau$ *near* $\big(0, \lambda_0(\tau)\big)$.

3. – The reduced bifurcation equation.

In this Section we obtain a new bifurcation equation defined on a finite dimensional space.

Fix a value $\tau = \tau_*$ around which we want to study our problem and set

$$\lambda_* = \lambda_0(\tau_*) , \tag{3.1a}$$

(3.1*b*) $$N = N_0(\tau_*) = \mathrm{Ker}\,(A_{\tau_*} - \lambda_* B_{\tau_*})\,,$$

(3.1*c*) $$E = N^{\perp} = \mathrm{Im}\,(A_{\tau_*} - \lambda_* B_{\tau_*})\,.$$

Of course we have $H = N \oplus E$. Denote by P_N and P_E the projections onto N and E respectively. It is clear that the bifurcation equation (2.6) is equivalent to the system

$$\begin{cases} P_N \mathcal{F}(\xi, \lambda, \tau) = 0 \\ P_E \mathcal{F}(\xi, \lambda, \tau) = 0\,. \end{cases}$$

Denote by z and ω the elements of N and E respectively. From (2.8) and Proposition 2.4 we have that

$$D_\omega P_E \mathcal{F}(0, \lambda_*, \tau_*) = (A_{\tau_*} - \lambda_* B_{\tau_*})_{|E}$$

is an isomorphism on E. Therefore by the Implicit Function Theorem it is easy to prove the following

PROPOSITION 3.1. – There exist open connected neighborhoods

(3.2*a*) $\mathcal{U}$ of 0 in N,

(3.2*b*) $\mathcal{I}$ of 0 in $\mathbb{R}$,

(3.2*c*) $\mathcal{T}$ of 0 in $(\mathbb{R}_+^*)^3$,

(3.2*d*) $\mathcal{V}$ of 0 in E

and a C^∞ map

$$\omega^{\#}\colon \mathcal{U} \times \mathcal{I} \times \mathcal{T} \to \mathcal{V}$$

such that for each $\tau \in \mathcal{T}$ we have

(3.3) $$\mathcal{S}_\tau \cap (\mathcal{U} \times \overline{\mathcal{V}} \times \mathcal{I}) = \{(\xi, \lambda) \in H \times \mathcal{I}\colon \text{there exists } z \in \mathcal{U} \text{ such that } \xi = z \oplus \omega^{\#}(z, \lambda, \tau) \text{ and } F(z, \lambda, \tau) = 0\}$$

where $\overline{\mathcal{V}}$ is the closure of $\mathcal{V}$ and F is given by

(3.4) $$F(z, \lambda, \tau) \underset{\mathrm{def}}{=} P_N \mathcal{F}\big(z \oplus \omega^{\#}(z, \lambda, \tau), \lambda, \tau\big)\,. \quad \blacksquare$$

(3.5) $$F(z, \lambda, \tau) = 0$$

is called the *reduced bifurcation equation*. It is easy to verify that $F \in C^\infty(\mathcal{U}\times\mathfrak{J}\times\mathfrak{T}, N)$ and that

$$F(0, \lambda, \tau) = 0 \quad \text{for each } (\lambda, \tau)\in\mathfrak{J}\times\mathfrak{T} \quad \text{and} \quad D_z F(0, \lambda_*, \tau_*) = 0\,. \tag{3.6}$$

4. – The symmetries of the problem.

The study of our problem can be simplified substantially by making use of the symmetries of the cylinder which are inherited by the energy functional. The cylinder is invariant with respect to the compact Lie group $\Gamma = \mathbb{O}(2)\oplus\mathbb{Z}_2$. Denote the elements of Γ by $(\varphi_\varepsilon, \delta)$ where

$$\varphi_\varepsilon = \begin{bmatrix}\cos\varphi & -\varepsilon\sin\varphi\\ \sin\varphi & \varepsilon\cos\varphi\end{bmatrix},$$

$\varphi\in\mathbb{R}$ and $\varepsilon, \delta = \pm 1$. With this notation the multiplication of $\mathbb{O}(2)$ becomes

$$\varphi_\varepsilon\cdot\psi_\delta = (\varphi+\varepsilon\psi)_{\varepsilon\delta}\,.$$

Of course φ_1 is a pure rotation, 0_1 is the identity matrix and 0_{-1} is the reflection $\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$. Consider the action $\varrho = \varrho_H\colon H\times\Gamma\to H$, defined by

$$\varrho_{(\varphi_\varepsilon,1)}\cdot\begin{bmatrix}u(\theta,\zeta)\\ v(\theta,\zeta)\\ w(\theta,\zeta)\end{bmatrix} = \begin{bmatrix}\varepsilon u(\varepsilon\theta+\varepsilon\varphi,\zeta)\\ v(\varepsilon\theta+\varepsilon\varphi,\zeta)\\ w(\varepsilon\theta+\varepsilon\varphi,\zeta)\end{bmatrix} \tag{4.1}$$

and

$$\varrho_{(0_1,-1)}\cdot\begin{bmatrix}u(\theta,\zeta)\\ v(\theta,\zeta)\\ w(\theta,\zeta)\end{bmatrix} = \begin{bmatrix}u(\theta,\pi-\zeta)\\ -v(\theta,\pi-\zeta)\\ w(\theta,\pi-\zeta)\end{bmatrix}. \tag{4.2}$$

It is easy to verify that ϱ is *orthogonal* with respect to the scalar product of H and that *the energy functional f is Γ-invariant*, that is

$$f(\varrho_\gamma\cdot\xi, \lambda, \tau) = f(\xi, \lambda, \tau) \quad \text{for each } \gamma\in\Gamma\,.$$

It follows that the gradient $\mathcal{F}$ of f defined by (2.5) is Γ equivariant, that is

$$\mathcal{F}(\varrho\cdot_\gamma\xi, \lambda, \tau) = \varrho_\gamma\cdot\mathcal{F}(\xi, \lambda, \tau) \quad \text{for each } \gamma\in\Gamma\,.$$

Therefore the solutions to $\mathcal{F} = 0$ are *orbits* of the action ϱ that is if ξ is a solution also $\varrho_\gamma \cdot \xi$ is for all $\gamma \in \Gamma$. Moreover the reduced bifurcation equation $F = 0$ is Γ-equivariant as proved in the following

PROPOSITION 4.1. – (i) N and E defined in (3.1*b*) and (3.1*c*) are Γ invariant, that is $\varrho_\gamma \cdot N \subset N$ and $\varrho_\gamma \cdot E \subset E$ for all $\gamma \in \Gamma$.

(ii) One can choose neighborhoods (3.2) such that $\mathfrak{U}$ and $\mathfrak{V}$ are Γ-invariant and the conclusions of Proposition 3.1 hold.

(iii) Provided that $\mathfrak{U}$ and $\mathfrak{V}$ are Γ-invariant, we have that $\omega^\#$ and F defined in Proposition 3.1 are Γ-equivariant.

PROOF. – Because $\mathcal{F}$ is Γ-equivariant also $A_\tau - \lambda B_\tau = D_\xi \mathcal{F}(0, \lambda, \tau)$ is. Then (i) follows from the fact that ϱ is orthogonal.

(ii) Because Γ is compact there exist open connected Γ-invariant neighborhoods $\widehat{\mathfrak{U}}$ of 0 in N and $\widehat{\mathfrak{V}}$ of 0 in E such that $\widehat{\mathfrak{V}} \subset \mathfrak{V}$ and $\widehat{\mathfrak{U}} \times \mathfrak{J} \times \mathfrak{T} \subset (\omega^\#)^{-1}(\widehat{\mathfrak{V}}) \subset \subset \mathfrak{U} \times \mathfrak{J} \times \mathfrak{T}$, where $\mathfrak{U}$, $\mathfrak{J}$, $\mathfrak{T}$ and $\mathfrak{V}$ are neighborhoods (3.2). Thus it suffices to choose $\widehat{\mathfrak{U}}$ and $\widehat{\mathfrak{V}}$ instead of $\mathfrak{U}$ and $\mathfrak{V}$.

(iii) $\omega^\#$ is Γ-equivariant by uniqueness, so F is Γ-equivariant by (ii) and (3.4). ■

5. – Computation of the first eigenvalue and of the relevant eigenfunctions.

Now we devote ourselves to computing the first eigenvalue and the relevant eigenspace of equation (2.9). It is easy to check that

$$\varphi^1_{mp} = \begin{bmatrix} \cos m\theta \quad \sin p\zeta \\ 0 \\ 0 \end{bmatrix}, \quad \varphi^2_{mp} = \begin{bmatrix} 0 \\ \cos m\theta \quad \cos p\zeta \\ 0 \end{bmatrix}, \quad \varphi^3_{mp} = \begin{bmatrix} 0 \\ 0 \\ \cos m\theta \quad \sin p\zeta \end{bmatrix},$$

$$\psi^1_{mp} = \begin{bmatrix} \sin m\theta \quad \sin p\zeta \\ 0 \\ 0 \end{bmatrix}, \quad \psi^2_{mp} = \begin{bmatrix} 0 \\ \sin m\theta \quad \cos p\zeta \\ 0 \end{bmatrix}, \quad \psi^3_{mp} = \begin{bmatrix} 0 \\ 0 \\ \sin m\theta \quad \sin p\zeta \end{bmatrix}$$

where $m, p \in \mathbb{N}$ and $(m, p) \neq (0, 0)$, form a complete orthogonal set for H. It follows that each element $\xi \in H$ can be written as

$$\xi = \sum_{j=1}^{3} \sum_{\substack{m, p \geq 0 \\ (m,p) \neq (0,0)}} (a^j_{mp} \varphi^j_{mp} + b^j_{mp} \psi^j_{mp}) \,. \tag{5.1}$$

Now from (2.8) we have that ξ is a solution to (2.9) if and only if

$$\mathcal{A}(\xi, \hat{\xi}, \tau) - \lambda \mathcal{B}(\xi, \hat{\xi}, \tau) = 0 \quad \text{for all } \hat{\xi} \in H . \tag{5.2}$$

Thus, by substituting the series (5.1) in (5.2) and using Proposition (2.1), we obtain that ξ is a solution to (2.9) if and only if

$$\sum_{j=1}^{3} \sum_{\substack{m,p \geq 0 \\ (m,p)\neq(0,0)}} \{a^j_{mp}[\mathcal{A}(\varphi^j_{mp}, \hat{\xi}, \tau) - \lambda \mathcal{B}(\varphi^j_{mp}, \hat{\xi}, \tau)] + b^j_{mp}[\mathcal{A}(\psi^j_{mp}, \hat{\xi}, \tau) - \lambda \mathcal{B}(\psi^j_{mp}, \hat{\xi}, \tau)]\} = 0 \tag{5.3}$$

for all $\hat{\xi} \in H$. Now φ^j_{mp} and ψ^j_{mp} are smooth functions, thus, by integrating by parts (5.3) and making use of (2.1), (2.3) and (2.4), a long but straightforward computation yields the following couple of systems for each $m, p \in \mathbb{N}$:

$$\tag{5.4a} \begin{cases} -\left(m^2 + \dfrac{1-\nu}{2}\left(\dfrac{\pi R}{l}\right)^2 p^2\right) a^1_{mp} + \dfrac{1+\nu}{2}\dfrac{\pi R}{l} mp b^2_{mp} - m^3 b^3_{mp} = 0 , \\ \dfrac{1+\nu}{2}\dfrac{\pi R}{l} mp a^1_{mp} - \left(\dfrac{1-\nu}{2} m^2 + \left(\dfrac{\pi R}{l}\right)^2 p^2\right) b^2_{mp} + \nu \dfrac{\pi R}{l} p b^3_{mp} = 0 , \\ R^2 m a^1_{mp} - \nu R^2 \dfrac{\pi R}{l} p b^2_{mp} + \\ \qquad + \left[R^2 + \dfrac{h^2}{12}\left(m^2 + \left(\dfrac{\pi R}{l}\right)^2 p^2\right)^2 - \lambda \dfrac{(1-\nu^2) R^2}{Eh}\left(\dfrac{\pi R}{l}\right)^2 p^2\right] b^3_{mp} = 0 , \end{cases}$$

$$\tag{5.4b} \begin{cases} -\left(m^2 + \dfrac{1-\nu}{2}\left(\dfrac{\pi R}{l}\right)^2 p^2\right) b^1_{mp} - \dfrac{1+\nu}{2}\dfrac{\pi R}{l} mp a^2_{mp} + m a^3_{mp} = 0 , \\ -\dfrac{1+\nu}{2}\dfrac{\pi R}{l} mp b^1_{mp} - \left(\dfrac{1-\nu}{2} m^2 + \left(\dfrac{\pi R}{l}\right)^2 p^2\right) a^2_{mp} + \nu \dfrac{\pi R}{l} p a^3_{mp} = 0 , \\ - R^2 m b^1_{mp} - \nu R^2 \dfrac{\pi R}{l} p a^2_{mp} + \\ \qquad + \left[R^2 + \dfrac{h^2}{12}\left(m^2 + \left(\dfrac{\pi R}{l}\right)^2 p^2\right)^2 - \lambda \dfrac{(1-\nu^2) R^2}{Eh}\left(\dfrac{\pi R}{l}\right)^2 p^2\right] a^3_{mp} = 0 \end{cases}$$

where a^j_{mp} and b^j_{mp} are the unkkowns. An elementary computation shows that both systems (5.4) have non-trivial solutions if and only if

$$\frac{h^2}{12}\left(m^2 + \left(\frac{\pi R}{l}\right)^2 p^2\right)^4 + (1-\nu^2) R^2 \left(\frac{\pi R}{l}\right)^4 p^4 - \lambda \frac{(1-\nu^2) R^2}{Eh}\left(m^2 + \left(\frac{\pi R}{l}\right)^2 p^2\right)^2 \left(\frac{\pi R}{l}\right)^2 p^2 = 0 . \tag{5.5}$$

Now notice that (5.5) has no solution if $p = 0$. It follows that *the eigenvalues of* (2.9) *are given by*

$$\Lambda_{mp}(\tau) = Eh\left[\frac{h^2}{12(1-\nu^2)R^2}\,\frac{(m^2+(\pi R/l)^2p^2)^2}{(\pi R/l)^2p^2}+\frac{(\pi R/l)^2p^2}{(m^2+(\pi R/l)^2p^2)^2}\right] \tag{5.6}$$

where $m \geq 0$ and $p > 0$. Once we got the eigenvalues we can immediately compute the eigenfunctions of (2.9) by solving (5.4) with $\lambda = \Lambda_{mp}$. For each (m, p) we obtain this way two linearly independent eigenfunctions which reduce to one if $m = 0$:

$$\Phi_{mp} = \begin{bmatrix} -\alpha_{mp}\sin m\theta & \sin p\zeta \\ \beta_{mp}\cos m\theta & \cos p\zeta \\ \cos m\theta & \sin p\zeta \end{bmatrix} = -\alpha_{mp}\varphi^1_{mp} + \beta_{mp}\psi^2_{mp} + \psi^3_{mp}\,, \tag{5.7a}$$

$$\Psi_{mp} = \begin{bmatrix} \alpha_{mp}\cos m\theta & \sin p\zeta \\ \beta_{mp}\sin m\theta & \cos p\zeta \\ \sin m\theta & \sin p\zeta \end{bmatrix} = \alpha_{mp}\psi^1_{mp} + \beta_{mp}\varphi^2_{mp} + \varphi^3_{mp} \tag{5.7b}$$

where

$$\alpha_{mp} = \frac{m(m^2+(2+\nu)(\pi R/l)^2p^2)}{(m^2+(\pi R/l)^2p^2)^2}\,,\qquad \beta_{mp} = \frac{(\pi R/l)p(\nu(\pi R/l)^2p^2-m^2)}{(m^2+(\pi R/l)^2p^2)^2}\,.$$

In particular *the first eigenvalue of* (2.9) *and its relevant eigenspace are given by*

$$\lambda_0(\tau) = \min\{\Lambda_{mp}(\tau)\colon m \geq 0, p \geq 1\}\,, \tag{5.8a}$$

$$N_0(\tau) = \ker\left(A_\tau - \lambda_0(\tau)B_\tau\right) = \bigcup_{\Lambda_{mp}=\lambda_0} N_{mp}(\tau) \tag{5.8b}$$

where

$$N_{mp}(\tau) = \{x\Phi_{mp} + y\Psi_{mp}\colon x, y \in \mathbb{R}\}\,. \tag{5.9}$$

We notice that the eigenvalue $\lambda_0(\tau)$ we have found coincides with the well-known critical load of a cylindrical shell, see [6], (11.52). Moreover we remark that in the engineering terminology the eigenfunctions belonging to N_0 are called *buckling modes*, while the integers m, p such that $\Lambda_{mp} = \lambda_0$ are the *wave numbers* of the buckling modes.

6. – Regular bifurcation diagrams.

We give some general definitions which turn out to be useful afterwards.

DEFINITION 6.1. – Given a topological space X, a *bifurcation diagram* is a pair $(\mathcal{S}, \mathcal{U})$, where $\mathcal{U}$ is an open connected subset of $X \times \mathbb{R}$ and $\mathcal{S}$ is a continuum (i.e. closed and connected) subset of the closure $\overline{\mathcal{U}}$ of $\mathcal{U}$.

DEFINITION 6.2. – Two bifurcation diagrams $(\mathcal{S}_1, \mathcal{U}_1)$ and $(\mathcal{S}_2, \mathcal{U}_2)$ with $\mathcal{U}_j \subset X_j \times \mathbb{R}$ are *isomorphic* if there exists a homeomorphism $\psi: \overline{\mathcal{U}}_1 \to \overline{\mathcal{U}}_2$ given by $\psi(x, \lambda) = (X(x, \lambda), \Lambda(\lambda))$ such that Λ is monotone increasing and $\psi(\mathcal{S}_1) = \mathcal{S}_2$.

The next definition follows that of voisinage adapté of RABIER [15], Définition 2.1-1, page 181 and CIARLET and RABIER [5], page 142.

DEFINITION 6.3. – Given a bifurcation diagram $(\mathcal{S}, \mathcal{U})$ with $\mathcal{U} \subset X \times \mathbb{R}$, we say that $\mathcal{U}$ is *distinguished* if there exist a closed interval $\mathcal{J} \subset \mathbb{R}$ and a family $\{U_\lambda\}_{\lambda \in \mathcal{J}}$ of open connected subsets of X such that

$$\overline{\mathcal{U}} = \bigcup_{\lambda \in \mathcal{J}} (\overline{\mathcal{U}}_\lambda \times \{\lambda\}) \quad \text{and} \quad \partial \mathcal{U}_\lambda \cap \mathcal{S}_\lambda = \emptyset \quad \text{for each } \lambda \in \mathcal{J}$$

where $\partial \mathcal{U}_\lambda$ is the boundary of $\mathcal{U}_\lambda$ and

$$\mathcal{S}_\lambda = \{x \in X: (x, \lambda) \in \mathcal{S}\} .$$

Recall now that an arc in $X \times \mathbb{R}$ is a subspace homeomorphic to $[0, 1] \subset \mathbb{R}$. We define as endpoints of an arc the images of 0 and 1.

DEFINITION 6.4. – A bifurcation diagram $(\mathcal{S}, \mathcal{U})$ is *regular* if:

(i) $\mathcal{U}$ is distinguished.

(ii) The set $\mathcal{S}_\lambda$ is finite for each $\lambda \in \mathcal{J}$.

(iii) $\mathcal{S}$ is a finite union of arcs which may intersect at most in a finite number of points.

(iv) Each arc ends on another arc or on the boundary of $\mathcal{U}$.

DEFINITION 6.5. – Let $(\mathcal{S}, \mathcal{U})$ be a regular bifurcation diagram, then we say that:

(i) $(x_0, \lambda_0) \in \mathcal{S}$ is a *bifurcation point* if it lies on two (or more) arcs but it is not the endpoint of two arcs only.

(ii) $(x_0, \lambda_0) \in \mathcal{S}$ is a *limit point* if there exists a neighborhood $\mathcal{V}$ of (x_0, λ_0) in $X \times \mathbb{R}$, such that $\mathcal{S}_\lambda \cap \mathcal{V} = \emptyset$ either for each $\lambda < \lambda_0$ or for each $\lambda > \lambda_0$.

(iii) $(x_0, \lambda_0) \in \mathcal{S}$ is *subcritic* (*supercritic*) with respect to $\lambda_1 \in \mathbb{R}$ if $\lambda_0 < \lambda_1$ $(\lambda_0 > \lambda_1)$.

The following proposition is almost obvious and is left to the reader.

PROPOSITION 6.6. – Let $(\mathcal{S}_1, \mathcal{U}_1)$ and $(\mathcal{S}_2, \mathcal{U}_2)$ be two isomorphic bifurcation diagrams. Then $(\mathcal{S}_1, \mathcal{U}_1)$ is regular if and only if $(\mathcal{S}_2, \mathcal{U}_2)$ is regular. Moreover the isomorphism induces a bijection between bifurcation points, limit points and subcritical and supercritical points. ■

7. – Statement of the results.

Now we can state our results, referring to next sections for the proofs.

We employ the notations of Sections 3 and 5.

The dimension of N_0 is the (geometric) multiplicity of λ_0. We say that λ_0 has *order* $k \geqq 1$ if there exist k distinct pairs of integers $(m_1, p_1), \ldots, (m_k, p_k)$, such that $\lambda_0 = \Lambda_{m_1 p_1} = \ldots = \Lambda_{m_k p_k}$. It is easy to show by numerical examples that when τ varies $\lambda_0(\tau)$ may have every multiplicity and order. Now from (5.6), (5.7), (5.8) and (5.9) we have that λ_0 has odd multiplicity if and only if N_0 contains eigenfunctions which do not depend on θ, i.e. axisymmetric. Now $\mathbf{O}(2)$ acts *trivially* on axisymmetric functions, thus we exclude this case from our analysis.

Among eigenvalues with even multiplicity, those of order 1 give rise to bifurcation problems with circular symmetry which have already been studied, see for example [9], Section 5. Consequently *from now on we make the following assumption*:

(A) for $\tau = \tau_*$, the first eigenvalue $\lambda_* = \lambda_0(\tau_*)$ has order 2 and multiplicity 4.

Therefore there exist integers m, n, p and q such that

$$\lambda_* = \Lambda_{mp} = \Lambda_{nq} \tag{7.1a}$$

and

$$m, n, p, q \geqq 1 \quad \text{and} \quad (m, p) \neq (n, q)\,. \tag{7.1b}$$

Moreover $N = N_0(\tau_*)$ can be identified with $\mathbf{C}^2$ by the bijection

$$(z_1, z_2) \to z = x_1 \varphi_{mp} + y_1 \psi_{mp} + x_2 \varphi_{nq} + y_2 \psi_{nq} \tag{7.2}$$

where $z_j = x_j + i y_j$ $(i = \sqrt{-1})$. This implies that the restriction of the action ϱ to N becomes

$$\varrho_{(\varphi_1, 1)} \cdot (z_1, z_2) = \left(\exp[im\varphi] z_1, \exp[in\varphi] z_2\right), \tag{7.3a}$$

$$\varrho_{(0_{-1}, 1)} \cdot (z_1, z_2) = (\bar{z}_1, \bar{z}_2)\,, \tag{7.3b}$$

$$\varrho_{(0_1, -1)} \cdot (z_1, z_2) = \left((-1)^{p+1} z_1, (-1)^{q+1} z_2\right) \tag{7.3c}$$

where the overbar indicates complex conjugation.

Now we give the reduced bifurcation equation an invariant form. Set

$$\sigma_j(z) = z_j \bar{z}_j\,, \qquad \sigma = (\sigma_1, \sigma_2)\,, \qquad \eta(z) = z_1^{\hat{n}} \bar{z}_2^{\hat{m}} + \bar{z}_1^{\hat{n}} z_2^{\hat{m}} \tag{7.4}$$

and

$$m = t\hat{m}\,, \qquad n = t\hat{n}\,, \qquad t = GCD(m, n)\,. \tag{7.5}$$

PROPOSITION 7.1. – One can choose neighborhoods (3.2) such that the conclusions of Proposition 3.1 and 4.1 hold and there exist four C^∞ functions $P_j, Q_j\colon \mathbb{R}^3\times\mathbb{R}\times\times\mathbb{R}^3\to\mathbb{R}$ $(j=1,2)$ such that:

(i) if $\hat{n}(p+1)+\hat{m}(q+1)$ is even

$$F(z,\lambda,\tau)=\begin{bmatrix}P_1(\sigma(z),\eta(z),\lambda-\lambda_*,\tau)z_1+Q_1(\sigma(z),\eta(z),\lambda-\lambda_*,\tau)\bar{z}_1^{\hat{n}-1}z_2^{\hat{m}}\\ P_2(\sigma(z),\eta(z),\lambda-\lambda_*,\tau)z_2+Q_2(\sigma(z),\eta(z),\lambda-\lambda_*,\tau)z_1^{\hat{n}}z_2^{\hat{m}-1}\end{bmatrix}\tag{7.6a}$$

(ii) if $\hat{n}(p+1)+\hat{m}(q+1)$ is odd

$$F(z,\lambda,\tau)=\begin{bmatrix}P_1(\sigma(z),\eta^2(z),\lambda-\lambda_*,\tau)z_1+Q_1(\sigma(z),\eta^2(z),\lambda-\lambda_*,\tau)\eta(z)\bar{z}_1^{\hat{n}-1}z_2^{\hat{m}}\\ P_2(\sigma(z),\eta^2(z),\lambda-\lambda_*,\tau)z_2+Q_2(\sigma(z),\eta^2(z),\lambda-\lambda_*,\tau)\eta(z)z_1^{\hat{n}}\bar{z}_2^{\hat{m}-1}\end{bmatrix}\tag{7.6b}$$

for all $(z,\lambda,\tau)\in\mathfrak{U}\times\mathfrak{J}\times\mathfrak{T}$.

Moreover the Taylor expansion at the origin of the functions P_j and Q_j are uniquely determined by F.

PROOF. – The proof is given in Section 10. ■

REMARK 7.2. – For the sake of brevity we limited ourselves to investigate the case where $\hat{m},\hat{n}>3$ and $\hat{n}(p+1)+\hat{m}(q+1)$ is even. Of course also the other case can be studied in a similar though more cumbersome way.

As we already said in Section 4, the bifurcation equation $\mathcal{F}=0$ defined by (2.5) is Γ-equivariant. In particular *the solutions to* $\mathcal{F}=0$ *are orbits of the action* ϱ. Denote by H/Γ the *orbit space* endowed with the quotient topology and by $\xi^*\in H/\Gamma$ the orbit generated by $\xi\in H$. Moreover if $A\subset H\times\mathbb{R}$, then we set

$$A^*=(\pi\times\mathrm{id}_{\mathbb{R}})(A),$$

where $\pi\colon H\to H/\Gamma$ is the natural map taking $\xi\in H$ into its orbit ξ^*.

THEOREM 7.3. – Assume that

$$\hat{m},\hat{n}>3,\qquad \hat{n}(p+1)+\hat{m}(q+1)\quad\text{is even}\tag{7.7}$$

so that (7.6*a*) holds. Consider the following Taylor expansion

$$P_1(\sigma,\eta,\lambda-\lambda_*,\tau)=a_0(\tau)+a_1(\tau)\sigma_1+a_2(\tau)\sigma_2+a_3(\tau)(\lambda-\lambda_*)+ + O(\sigma_1^2,\sigma_2^2,\eta,(\lambda-\lambda_*)^2),\tag{7.8a}$$

$$Q_1(\sigma,\eta,\lambda-\lambda_*,\tau)=b_0(\tau)+O(\sigma_1,\sigma_2,\eta,\lambda-\lambda_*),\tag{7.8b}$$

$$(7.8c)\quad P_2(\sigma, \eta, \lambda - \lambda_*, \tau) = c_0(\tau) + c_1(\tau)\sigma_1 + c_2(\tau)\sigma_2 + c_3(\tau)(\lambda - \lambda_*) + O(\sigma_1^2, \sigma_2^2, \eta, (\lambda - \lambda_*)^2)\,,$$

$$(7.8d)\quad Q_2(\sigma, \eta, \lambda - \lambda_*, \tau) = d_0(\tau) + O(\sigma_1, \sigma_2, \eta, \lambda - \lambda_*)$$

where $a_j(\tau)$, $b_0(\tau)$, $c_j(\tau)$, $d_0(\tau)$ are C^∞ functions of τ and *are uniquely determined by* F, defined by (3.4), *and therefore by the energy functional* f. By (3.6) we have

$$(7.9)\qquad a_0(\tau_*) = c_0(\tau_*) = 0\,.$$

Assume the following non-degeneracy hypotheses:

$$(7.10)\qquad a_1(\tau_*),\, a_3(\tau_*),\, b_0(\tau_*),\, c_2(\tau_*),\, c_3(\tau_*),\, d_0(\tau_*) \neq 0$$

and

$$(7.11a)\qquad A_1,\, A_2,\, A_3 \neq 0$$

where

$$(7.11b)\quad A_1 = \begin{vmatrix} a_1(\tau_*) & a_2(\tau_*) \\ c_1(\tau_*) & c_2(\tau_*) \end{vmatrix}, \quad A_2 = \begin{vmatrix} a_1(\tau_*) & a_3(\tau_*) \\ c_1(\tau_*) & c_3(\tau_*) \end{vmatrix}, \quad A_3 = \begin{vmatrix} a_2(\tau_*) & a_3(\tau_*) \\ c_2(\tau_*) & c_3(\tau_*) \end{vmatrix}.$$

Then we may, if that is the case, shrink neighborhood (3.2c) so that there exists a family $\{\mathcal{W}_\tau\}_{\tau\in\mathcal{T}}$ of open connected Γ-invariant neighborhoods of 0 in $H\times\mathbb{R}$ such that for each $\tau\in\mathcal{T}$ we have:

(i) $\overline{\mathcal{W}}_\tau^* \subset H/\Gamma\times\mathbb{R}$ is distinguished and $(\mathcal{S}_\tau^* \cap \overline{\mathcal{W}}_\tau^*, \mathcal{W}_\tau^*)$ (see 2.7) is a regular bifurcation diagram.

(ii) $\mathcal{S}_\tau \cap \overline{\mathcal{W}}_\tau^*$ is the union of five arcs $\mathcal{C}_0$, $\mathcal{C}_{mp}$, $\mathcal{C}_{nq}$, $\mathcal{C}_t^+$, $\mathcal{C}_t^-$ at the most (notation will turn out clear later on).

Every orbit in $\mathcal{S}_\tau \cap \overline{\mathcal{W}}_\tau$ is generated by the action of the subgroup $\mathbf{SO}(2)\oplus\{1\}$ of Γ (i.e. by pure rotations) starting from each one of its elements.

Recall now that the isotropy subgroup of $\xi\in H$ is defined by

$$\Gamma_\xi = \{\gamma\in\Gamma\colon \varrho_\gamma\cdot\xi = \xi\}\,.$$

Of course Γ_{ξ_1} and Γ_{ξ_2} are conjugate whenever $\xi_1^* = \xi_2^*$. Moreover we have that when (ξ_1, λ_1) and (ξ_2, λ_2) belong to one and the same of the five arcs listed above, there exist representatives of the orbits ξ_1^* and ξ_2^* with the same isotropy subgroup. Thus we can attach (up to conjugacy) to each arc an isotropy subgroup according to the following table (refer to Section 4 for notation):

Arc	Isotropy subgroup	Generators
$\mathcal{C}_0$	Γ_0	$\mathbf{O}(2)\oplus \mathbb{Z}_2$
$\mathcal{C}_{mp}$	Γ_{mp}	$\left(\left(\frac{2\pi}{m}\right)_1, 1\right),\ (0_{-1}, 1), \left(\left(\frac{p+1}{m}\pi\right)_1, -1\right)$
$\mathcal{C}_{nq}$	Γ_{nq}	$\left(\left(\frac{2\pi}{n}\right)_1, 1\right),\ (0_{-1}, 1), \left(\left(\frac{q+1}{n}\pi\right)_1, -1\right)$
$\mathcal{C}_t^+$	Γ_t^+	$\left(\left(\frac{2\pi}{t}\right)_1, 1\right),\ (0_{-1}, 1), \left(\left(\frac{2q'-p-1}{m}\pi\right)_1, -1\right)$
$\mathcal{C}_t^-$	Γ_t^-	$\left(\left(\frac{2\pi}{t}\right)_1, 1\right), \left(\left(\frac{2m'\pi}{m}\right)_{-1}, 1\right), \left(\left(\frac{2q'-p-1}{m}\pi\right)_1, -1\right)$

where p', q' are integers such that $2(\hat{n}p'-\hat{m}q')=\hat{n}(p+1)-\hat{m}(q+1)$ (remark that $\hat{n}(p+1)-\hat{m}(q+1)$ is even by (7.7)) and m', n' are integers such that $\hat{n}m'-\hat{m}n'=1$. In particular we have that Γ_{mp}, Γ_{nq} and $\Gamma_t^{\pm}$ are isomorphic to the dihedral groups $\mathbb{D}_m$, $\mathbb{D}_n$, and $\mathbb{D}_t$ respectively, thus solutions with orbit on $\mathcal{C}_{mp}$, $\mathcal{C}_{nq}$ and $\mathcal{C}_t^{\pm}$ are periodic of period $2\pi/m$, $2\pi/n$ and $2\pi/t$ respectively.

(iii) Set

(7.12) $$A_0(\tau)=c_3(\tau_*)\,D_\tau a_0(\tau_*)[\tau-\tau_*]-a_3(\tau_*)\,D_\tau c_0(\tau_*)[\tau-\tau_*]\,,$$

then we have the following facts:

(*a*) $\mathcal{S}_\tau^*\cap\overline{\mathcal{W}}_\tau^*$ has no limit points.

(*b*) Orbit-solutions on $\mathcal{C}_0$ are the trivial ones: $(\xi^*,\lambda)=(0,\lambda)$.

(*c*) $\mathcal{C}_{mp}$ and $\mathcal{C}_{nq}$ have one endpoint on $\mathcal{C}_0$ and the other on the boundary of $\mathcal{W}_\tau^*$.

(*d*) There are two (possibly coincident) bifurcation points on $\mathcal{C}_0$. They are given by $\big(0, \Lambda_{mp}(\tau)\in\mathcal{C}_0\cap\mathcal{C}_{mp}$ and $\big(0, \Lambda_{nq}(\tau)\big)\in\mathcal{C}_0\cap\mathcal{C}_{nq}$. Moreover we have that $\min\{\Lambda_{mp}(\tau), \Lambda_{nq}(\tau)\}=\lambda_0(\tau)$ and that $\Lambda_{mp}(\tau)\lesseqgtr\Lambda_{nq}(\tau)$ if $A_0(\tau)\cdot a_3(\tau_*)\cdot c_3(\tau_*)\gtreqless 0$, while $\tau=\tau_*$ (so that $A_0(\tau_*)=0$) implies $\Lambda_{mp}(\tau_*)=\Lambda_{nq}(\tau_*)=\lambda_*$.

(*e*) $\mathcal{C}_{mp}$ and $\mathcal{C}_{nq}$ have no points in common except possibly for $(0,\lambda_*)$, for example when $\tau=\tau_*$.

(*f*) $\mathcal{C}_{mp}$ ($\mathcal{C}_{nq}$) is subcritic or supercritic with respect to $\Lambda_{mp}(\tau)$ $\big(\Lambda_{nq}(\tau)\big)$ according as $a_1(\tau_*)\cdot a_3(\tau_*)$ $\big(c_2(\tau_*)\cdot c_3(\tau_*)\big)$ is positive or negative.

(*g*) If $A_2A_3>0$ and $A_0(\tau)A_2<0$, $\mathcal{C}_t^+$ and $\mathcal{C}_t^-$ connect $\mathcal{C}_{mp}$ and $\mathcal{C}_{nq}$, with the endpoints in common on $\mathcal{C}_{mp}$ and $\mathcal{C}_{nq}$ respectively and with no other points in common between them or with $\mathcal{C}_{mp}$ or $\mathcal{C}_{nq}$.

(*h*) If $A_2A_3 > 0$ and $\tau = \tau_*$ (so that $A_0(\tau_*) = 0$) or $A_0(\tau)A_2 > 0$ there are no orbit solutions $\mathcal{C}_t^{\pm}$.

(*l*) If $A_2A_3 < 0$, $\mathcal{C}_t^-$ and $\mathcal{C}_t^+$ have one endpoint in common either on $\mathcal{C}_{mp}$ or on $\mathcal{C}_{nq}$ and the other on the boundary of $\mathcal{W}_\tau^*$. Moreover there are no other points in common between $\mathcal{C}_t^+$ and $\mathcal{C}_t^-$ or with $\mathcal{C}_{mp}$ or $\mathcal{C}_{nq}$. Let (ξ_t^*, λ_t) be the endpoint in common, then (ξ_t^*, λ_t) is nontrivial whenever $A_0(\tau) \neq 0$ and lies on $\mathcal{C}_{mp}$ or $\mathcal{C}_{nq}$ according as $A_0(\tau)A_2$ is negative or positive. Finally we have $(\xi_t^*, \lambda_t) = (0, \lambda_*) \in \mathcal{C}_0 \cap \mathcal{C}_{mp} \cap \mathcal{C}_{nq}$ whenever $\tau = \tau_*$.

(*m*) $\mathcal{C}_t^{\pm}$ are subcritic or supercritic with respect to λ_t according as A_1A_2 is positive or negative.

PROOF. – The proof is given in Section 8. ■

REMARK 7.4. – With some more work one can ascertain in case (*g*) if the arcs $\mathcal{C}_t^{\pm}$ meet first $\mathcal{C}_{mp}$ or $\mathcal{C}_{nq}$.

REMARK 7.5. – Several of the various hypotheses we made at point (iii) involve in particular the non-degeneracy assumption

$$A_0(\tau) \neq 0 \quad \text{for } \tau \neq \tau_* . \tag{7.13}$$

Now as we said at point (*d*), (7.13) implies $\Lambda_{mp}(\tau) \neq \Lambda_{nq}(\tau)$ and this inequality is generically satisfied for τ near τ_*, as it follows easily from (5.6). In particular this means that assumption (7.13) is consistent with our problem.

From Theorem 7.3 it turns out that the system may show quite a different behavior according to the sign of the coefficients that appear in the non-degeneracy hypotheses (7.10) and (7.11*a*). In order to decide which of the possible cases actually occurs, one should compute the coefficients a_j, b_0, c_j, d_0 of the expansion (7.8) in terms of the geometric parameters h, R, l and of the elastic moduli E, ν. Now this computation, we do not perform here, is quite arduous. However it can be done, at least in principle, as follows. First one has to compute the Taylor expansion of $\omega^\#$, defined in Proposition 3.1, by solving a sequence of linear partial differential equations obtained by differentiating with respect to z the identity $P_E\mathcal{F}(z \oplus \omega^\#(z, \lambda, \tau), \lambda, \tau) = 0$. Then one has to substitute this expansion in (3.4) and write F in invariant form (7.6*a*).

Comparing with the engineering literature on the buckling of cylindrical shells (starting from the pioneering paper by von KÁRMÁN and TSIEN [10] up to the most recent results contained in the book of YAMAKI [19]) suggests that $\mathcal{C}_{mp}$ and $\mathcal{C}_{nq}$ are subcritic, that is that

$$a_1(\tau_*)a_3(\tau_*) > 0 \quad \text{and} \quad c_2(\tau_*)c_3(\tau_*) > 0 \tag{7.14}$$

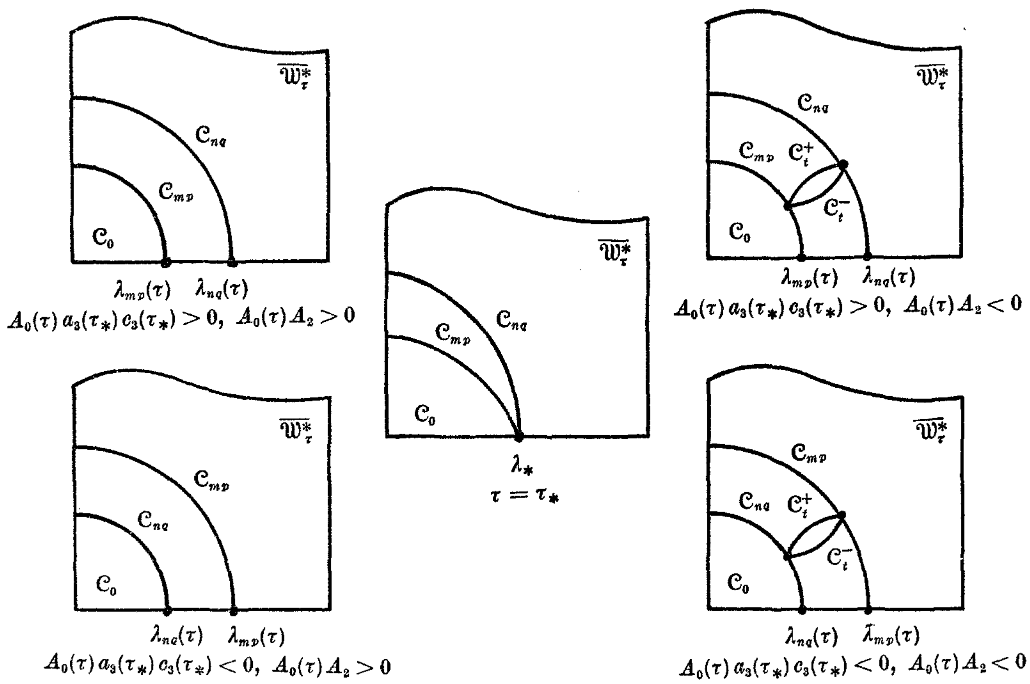

Figure 2 *a*). - $A_2 A_3 > 0$.

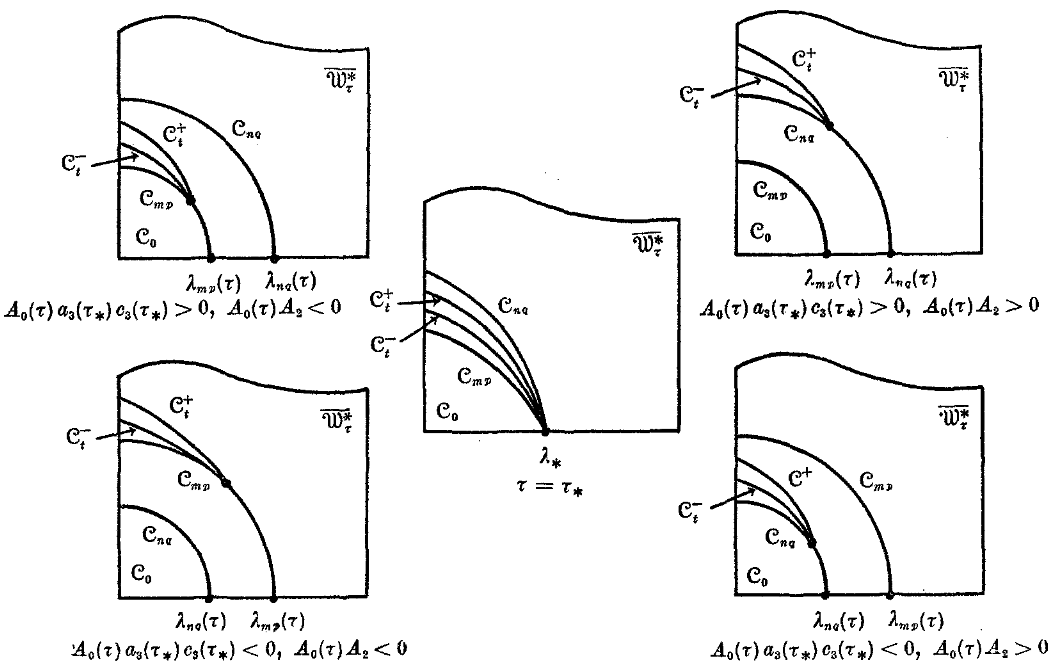

Figure 2 *b*). - $A_2 A_3 < 0,\ A_1 A_2 > 0$.

(see Theorem 7.3 (iii), (f)). Therefore we end this section by illustrating, by means of schematic diagrams, the possible cases corresponding to hypotheses (7.14). See Figures 2a, b, c.

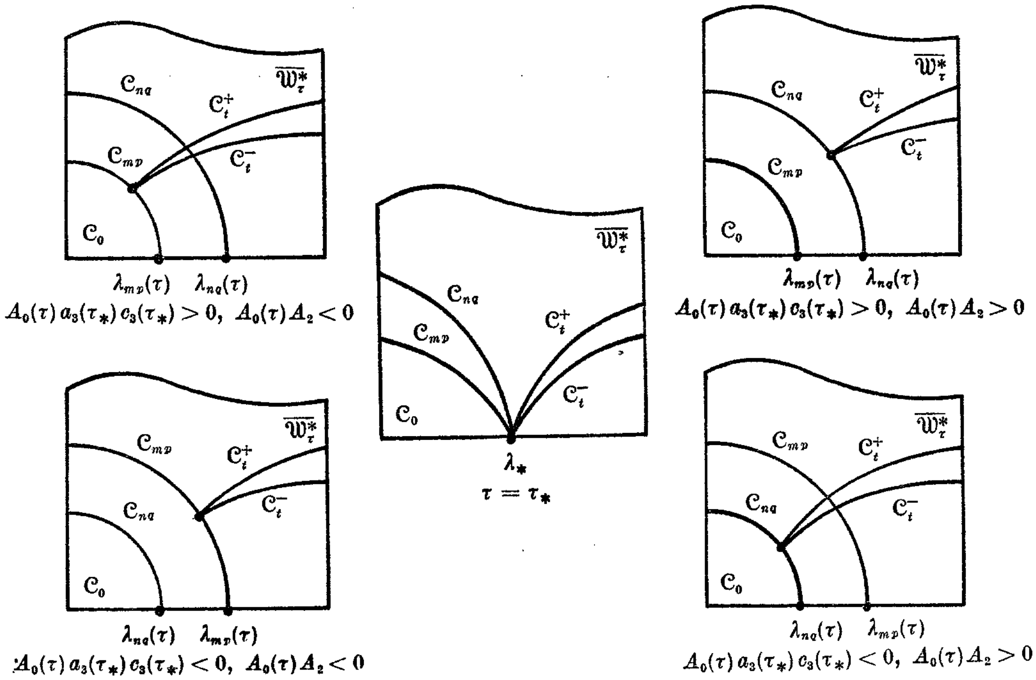

Figure 2 c). – $A_2A_3 < 0$, $A_1A_2 < 0$.

8. – Proof of Theorem 7.3.

By Proposition 4.1 $N = N_0(\tau_*)$ is Γ-invariant. It follows that $N/\Gamma \subset H/\Gamma$ and that the orthogonal projection P_N induces a surjection

$$P_N^* \times \mathrm{id}_{\mathbf{R}}\colon (\mathcal{U} \oplus \overline{\mathcal{V}})^* \times \mathcal{J} \to \mathcal{U}^* \times \mathcal{J}$$

where $\mathcal{U}$, $\mathcal{J}$ and $\mathcal{V}$ are neighborhoods (3.2a), (3.2b) and (3.2d). For each $\tau \in \mathcal{T}$ (see (3.2c)), define

$$S_\tau^* = \{(z^*, \lambda) \in \mathcal{U}^* \times \mathcal{J}\colon F(\hat{z}, \lambda, \tau) = 0 \text{ for all } \hat{z} \in z^*\}$$

where F is defined by (3.4).

Proposition 8.1. – For each closed neighborhood $\mathcal{X}$ of 0 in $N \times \mathbf{R}$, such that $\mathcal{X} \subset \mathcal{U} \times \mathcal{J}$, we have that

(i) The restriction of $P_N^* \times \mathrm{id}_{\mathbf{R}}$ to $\mathcal{S}_\tau^* \cap (\mathcal{X} \oplus \overline{\mathcal{V}})^*$ (see (3.3)) is a bijection between $\mathcal{S}_\tau^* \cap (\mathcal{X} \oplus \overline{\mathcal{V}})^*$ and $S_\tau^* \cap \mathcal{X}^*$.

(ii) The bifurcation diagram $\mathcal{S}_\tau^* \cap (\mathfrak{X} \oplus \overline{\mathfrak{V}})^*$ is regular if and only if $\mathcal{S}_\tau^* \cap \mathfrak{X}^*$ is regular.

(iii) The bijection of point (i) preserves bifurcation points, limit points and subcritic and supercritic solutions.

PROOF. – It is a simple consequence of (3.3), Proposition 4.1 and of the fact that $\pi\colon H \to H/\Gamma$ is closed ([3], Theorem 3.1). ■

REMARK 8.2. – The bijection of point (i) is *not* an isomorphism in the sense of Definition 6.2.

PROPOSITION 8.3. – Under the assumption (A) of Section 7 and the hypotheses of Theorem 7.3 one can choose neighborhoods (3.2) in such a way as the conclusions of Propositions 3.1, 4.1 and 7.1 hold and there exist C^∞ maps

$$K\colon \mathbb{C}^2 \times \mathfrak{U} \times \mathfrak{J} \times \mathfrak{T} \to \mathbb{C}^2$$

$$Z\colon \mathfrak{U} \times \mathfrak{J} \times \mathfrak{T} \to \mathbb{C}^2$$

$$\Delta\colon \mathfrak{J} \times \mathfrak{T} \to \mathbb{R}$$

$$T\colon \mathfrak{T} \to \mathbb{R}^4$$

such that

(i) $$Z(0, \lambda_*, \tau_*) = 0\,, \qquad \Delta(\lambda_*, \tau_*) = 0\,, \qquad T(\tau_*) = 0\,.$$

(ii) For each $(z, \lambda, \tau) \in \mathfrak{U} \times \mathfrak{J} \times \mathfrak{T}$ the map $\mathbb{C}^2 \to \mathbb{C}^2$

$$\chi \mapsto K(\chi, z, \lambda, \tau)$$

is $\mathbb{R}$-linear and invertible.

(iii) For each $\tau \in \mathfrak{T}$

$$(z, \lambda) \mapsto \big(Z(z, \lambda), \Delta(\lambda)\big)$$

is a diffeomorphism defined on $\mathfrak{U} \times \mathfrak{J}$ and

$$\lambda \mapsto \Delta(\lambda, \tau)$$

is monotonic increasing.

(iv) $$K(\varrho_\gamma \cdot \chi, \varrho_\gamma \cdot z, \lambda, \tau) = \varrho_\gamma \cdot K(\chi, z, \lambda, \tau)$$

$$Z(\varrho_\gamma \cdot z, \lambda, \tau) = \varrho_\gamma \cdot Z(z, \lambda, \tau)$$

for each $\gamma \in \Gamma$.

(v) $$F(z, \lambda, \tau) = K\Big(\mathfrak{G}\big(Z(z, \lambda, \tau), \Delta(\lambda, \tau), T(\tau)\big), z, \lambda, \tau\Big)$$

for each $(z, \lambda, \tau) \in \mathcal{U} \times \mathfrak{J} \times \mathcal{T}$, where F is defined by (3.4),

(8.1) $$\mathcal{G}(z, \delta, \alpha, \gamma) = \\ = \begin{bmatrix} [\alpha_0 + a_1(\tau_*)\sigma_1(z) + (a_2(\tau_*) + \alpha_2)\sigma_2(z) + a_3(\tau_*)\delta] + b_0(\tau_*)\bar{z}_1^{\hat{n}-1} z_2^{\hat{m}} \\ [(c_1(\tau_*) + \gamma_1)\sigma_1(z) + c_2(\tau_*)\sigma_2(z) + (c_3(\tau_*) + \gamma_3)\delta] + d_0(\tau_*) z_1^{\hat{n}} \bar{z}_2^{\hat{m}-1} \end{bmatrix},$$

$$\alpha = (\alpha_0, \alpha_2), \qquad \gamma = (\gamma_1, \gamma_3),$$

$$T(\tau) = (\alpha_0(\tau), \alpha_2(\tau), \gamma_1(\tau), \gamma_3(\tau))$$

and σ_j and $\hat{m}$, $\hat{n}$ are given by (7.4) and (7.5).

(vi) Moreover we have

(8.2) $$\alpha_0(\tau_*) = \alpha_2(\tau_*) = \gamma_1(\tau_*) = \gamma_3(\tau_*) = 0$$

and

(8.3) $$\alpha_0(\tau) = \frac{A_0(\tau)}{c_3(\tau_*)} + O(\|\tau - \tau_*\|^2)$$

where $A_0(\tau)$ is given by (7.12).

PROOF. – The proof is given in Section 11. ∎

Set now

(8.4) $$G(z, \delta, \tau) = \mathcal{G}(z, \delta, T(\tau))$$

and denote by $\Theta: \mathbf{C}^2 \to \mathbf{R}^3$ the polynomial map

$$z \mapsto (\sigma_1(z), \sigma_2(z), \eta(z))$$

where σ_j and η are given by (7.4). Of course Θ induces a continuous map $\Theta^*: \mathbf{C}^2/\Gamma \to \mathbf{R}^3$.

PROPOSITION 8.4. – $\Theta^*: \mathbf{C}^2/\Gamma \to \Theta(\mathbf{C}^2)$ is a homeomorphism.

PROOF. – See [14], Proposition 1, Chapitre II. ∎

It easy to see that (see Figure 3):

$$\Theta(\mathbf{C}^2) = \{x = (x_1, x_2, x_3) \in \mathbf{R}^3 : x_1 \geq 0,\ x_2 \geq 0,\ x_3^2 \leq 4x_1^{\hat{n}} x_2^{\hat{m}}\}.$$

Because $G(z, \delta, \tau)$ is Γ-equivariant with respect to z, it induces a continuous map

$$G^*: \mathbf{C}^2/\Gamma \times \mathbf{R} \times \mathcal{T} \to \mathbf{C}^2/\Gamma.$$

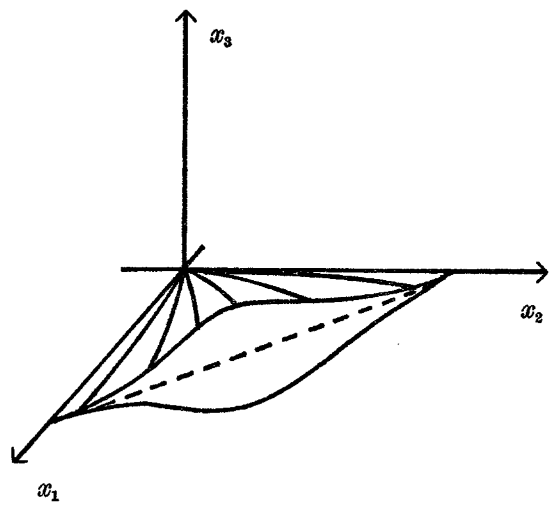

Figure 3.

Set

$$g = \Theta^* \circ G^* \circ (\Theta^{*-1} \times \mathrm{id}_{\mathbf{R}} \times \mathrm{id}_{\mathbf{R}^2}) \colon \Theta(\mathbf{C}^2) \times \mathbf{R} \times \mathcal{T} \to \Theta(\mathbf{C}^2)\,.$$

Now we explicit the map g. Set

(8.5a) $$p_1(x, \delta, \tau) = \alpha_0(\tau) + a_1(\tau_*)x_1 + \big(a_2(\tau_*) + \alpha_2(\tau)\big)x_2 + a_3(\tau_*)\delta\,,$$

(8.5b) $$p_2(x, \delta, \tau) = \big(c_1(\tau_*) + \lambda_1(\tau)\big)x_1 + c_2(\tau_*)x_2 + \big(c_3(\tau_*) + \gamma_3(\tau)\big)\delta\,,$$

(8.5c) $$q_1 = b_0(\tau_*)\,, \qquad q_2 = d_0(\tau_*)$$

where $\alpha_j(\tau)$ and $\gamma_j(\tau)$ are defined at point (v) of Proposition 8.3. From (8.1) and (8.4) we obtain

$$G(z, \delta, \tau) = \begin{bmatrix} p_1(\sigma(z), \eta(z), \delta, \tau)z_1 + q_1\bar{z}_1^{\hat{n}-1}z_2^{\hat{m}} \\ p_2(\sigma(z), \eta(z), \delta, \tau)z_2 + q_2 z_1^{\hat{n}}\bar{z}_2^{\hat{m}-1} \end{bmatrix},$$

thus it is easy to see that the components g_1, g_2, g_3 of g are given by

(8.6a) $$g_1(x, \delta, \tau) = p_1^2(x, \delta, \tau)x_1 + p_1(x, \delta, \tau)q_1x_3 + q_1^2x_1^{\hat{n}-1}x_2^{\hat{m}}\,,$$

(8.6b) $$g_2(x, \delta, \tau) = p_2^2(x, \delta, \tau)x_2 + p_2(x, \delta, \tau)q_2x_3 + q_2^2x_1^{\hat{n}}x_2^{\hat{m}-1}\,,$$

(8.6c) $$g_3\big(\sigma(z), \eta(z), \delta, \tau\big) =$$
$$= 2\,\mathrm{Re}\Big\{\big(p_1(\mathrm{g}(z), \eta(z), \delta, \tau)\,z_1 + q_1\bar{z}_1^{\hat{n}-1}z_2^{\hat{m}}\big)^{\hat{n}}\big(p_2(\sigma(z), \eta(z), \delta, \tau)\,\bar{z}_2 + q_2\bar{z}_1^{\hat{n}}z_2^{\hat{m}-1}\big)^{\hat{m}}\Big\}.$$

PROPOSITION 8.5. – $x \in \Theta(\mathbf{C}^2)$ is a solution to $g = 0$ if and only if is a solution to one of the following systems (which are obtained either by taking sign $+$ or $-$ in all the equations below):

$$x_1^{\frac{1}{2}}(p_1 \pm q_1 x_1^{(\hat{n}-2)/2} x_2^{\hat{m}/2}) = 0 , \tag{8.7a}$$

$$x_2^{\frac{1}{2}}(p_2 \pm q_2 x_1^{\hat{n}/2} x_2^{(\hat{m}-2)/2}) = 0 , \tag{8.7b}$$

$$x_3 = \pm x_1^{\hat{n}/2} x_2^{\hat{m}/2} . \tag{8.7c}$$

PROOF. – First we prove that if $x \in \Theta(\mathbb{C}^2)$ is a solution to $g_1 = 0$, than it is also a solution to $g_3 = 0$. In fact if $x \in \Theta(\mathbb{C}^2)$, then there exists $z \in \mathbb{C}^2$ such that $x_j = \sigma_j(z)$ and $x_3 = \eta(z)$. Consequently we have

$$0 = g_1(\sigma(z), \eta(z), \delta, \tau) =$$
$$= [p_1(\sigma(z), \eta(z), \delta, \tau) z_1 + q_1 \bar{z}_1^{\hat{n}-1} z_2^{\hat{m}}][p_1(\sigma(z), \eta(z), \delta, \tau) z_1 + q_1 z_1^{\hat{n}-1} \bar{z}_2^{\hat{m}}] ,$$

whence it follows immediately that

$$g_3(x, \delta, \tau) = g_3(\sigma(z), \eta(z), \delta, \tau) = 0 .$$

Therefore the equation $g_3 = 0$ is a consequence of the equations $g_1 = 0$ and $g_2 = 0$ and we can disregard it.

On the other hand $g_1 = 0$ is equivalent to

$$p_1^2 x_1 + q_1^2 x_1^{\hat{n}-1} x_2^{\hat{m}} = - p_1 q_1 x_3 ,$$

whence by squaring and adding to both sides $- 4p_1^2 q_1^2 x_1^{\hat{n}} x_2^{\hat{m}}$ one gets

$$(p_1^2 x_1 - q_1^2 x_1^{\hat{n}-1} x_2^{\hat{m}})^2 = p_1^2 q_1^2 x_3^2 - 4 x_1^{\hat{n}} x_2^{\hat{m}} . \tag{8.8}$$

Since $x \in \Theta(\mathbb{C}^2)$ implies $x_3^2 \leq 4 x_1^{\hat{n}} x_2^{\hat{m}}$ equation (8.8) can be satisfied only when both sides vanish. In particular we must have

$$p_1 q_1 = 0 \quad \text{or} \quad x_3^2 = 4 x_1^{\hat{n}} x_2^{\hat{m}} .$$

Now, because $q_1 \neq 0$ by (8.5*c*) and (7.10) and $x_1, x_2 \geq 0$ for $x \in \Theta(\mathbb{C}^2)$, one easily obtains equations (8.7). ∎

Observe now that equation (8.7) are no longer polynomial. To avoid difficulties due to absence of smoothness at the origin, consider the further homeomorphism $\Psi: \mathbb{R}_+^2 \times \mathbb{R} \to \mathbb{R}_+^2 \times \mathbb{R}$ $(\mathbb{R}_+ = \{x \in \mathbb{R}: x \geq 0\})$ defined as:

$$\Psi: (x_1, x_2, x_3) \mapsto (x_1^2, x_2^2, x_3)$$

and set

$$\Sigma = \Psi^{-1}\circ\Theta .$$

Of course Σ induces the homeomorphism

$$\Sigma^*\colon \mathbf{C}^2/\Gamma \to \Sigma(\mathbf{C}^2)$$

where

$$\Sigma(\mathbf{C}^2) = \{x\in\mathbf{R}^2\colon x_1,\, x_2\geq 0 \text{ and } |x_3|\leq 2x_1^{\hat n}x_2^{\hat m}\} .$$

Now define $H\colon \Sigma(\mathbf{C}^2)\times\mathbf{R}\times\mathcal{T}\to\Theta(\mathbf{C}^2)$ as:

$$h = g\circ(\Psi\times\mathrm{id}_{\mathbf{R}}\times\mathrm{id}_{\mathbf{R}^s}) .$$

According to Proposition 8.5 we have that $x\in\Theta(\mathbf{C}^2)$ is a solution to $h=0$ if and only if is a solution to the following systems (where one has to take sign $+$ or $-$ in all the equations):

(8.9*a*) $$x_1\big(p_1(x_1^2, x_2^2, x_3, \delta, \tau)\pm q_1x_1^{\hat n-2}x_2^{\hat m}\big) = 0 ,$$

(8.9*b*) $$x_2\big(p_2(x_1^2, x_2^2, x_3, \delta, \tau)\pm q_2x_1^{\hat n}x_2^{\hat m-2}\big) = 0 ,$$

(8.9*c*) $$x_3 = \pm 2x_1^{\hat n}x_2^{\hat m} .$$

For each $\tau\in\mathcal{T}$ define

$$s_\tau = \{(x, \delta)\in\Sigma(\mathbf{C}^2)\times\mathbf{R}\colon h(x, \delta, \tau) = 0\} .$$

Denote by (Z_τ, Δ_τ) the diffeomorphism, defined in Proposition 8.3 (iii), for fixed $\tau\in\mathcal{T}$ and by Z_τ^* the factorization of $Z_\tau^{\natural}$ through ϱ. The following proposition is an immediate consequence of Propositions 8.3 and 8.4.

PROPOSITION 8.6. – For each open neighborhood $\mathcal{O}\times\mathcal{J}$ of 0 in $\Sigma(\mathbf{C}^2)\times\mathbf{R}$ such that

(8.10) $$\mathcal{O} \text{ is connected}, \quad \mathcal{J} \text{ is an interval}$$

and

(8.11) $$\overline{\mathcal{O}\times\mathcal{J}}\subset(\Sigma^*\times id_R)\circ(Z_\tau^*\times\Delta_\tau^{\natural})(\mathcal{U}^*\times\mathfrak{I}) \quad \text{for all } \tau\in\mathcal{T}$$

where $\mathcal{U}$, $\mathfrak{I}$ and $\mathcal{T}$ are neighborhoods (3.2*a*), (3.2*b*) and (3.2*c*), we have that $(\Sigma^*\times\times\mathrm{id}_{\mathbf{R}})\circ(Z_\tau^*\times\Delta_\tau^{\natural})$ is an isomorphism between the bifurcation diagrams $S_\tau^*\cap\mathcal{X}_\tau^*$ and

$\mathcal{S}_\tau \cap \overline{\mathcal{O}\times\mathcal{J}}$, where

$$\mathfrak{X}^*_\tau = [(\Sigma^* \times id_R)\circ(Z^*\times\Delta_\tau)]^{-1}(\overline{\mathcal{O}\times\mathcal{J}}) . \quad \blacksquare$$

We reduced this way to study the bifurcation equation $h = 0$ on $\Sigma(\mathbf{C}^2)\times\mathbf{R}$. Now it is easy to see that equations (8.9) are equivalent to the following

(8.12) $x = 0$,

(8.13a) $x_1 \neq 0$, $x_2 = 0$, $x_3 = 0$,

(8.13b) $\alpha_0(\tau) + a_1(\tau_*)x_1^2 + a_3(\tau_*)\delta = 0$,

(8.14a) $x_1 = 0$, $x_2 \neq 0$, $x_3 = 0$,

(8.14b) $c_2(\tau_*)x_2^2 + \big(c_3(\tau_*) + \gamma_3(\tau)\big)\delta = 0$,

(8.15a) $x_3 \neq 0$, $x_3 = \pm 2x_1^{\hat n}x_2^{\hat m}$,

(8.15b) $\alpha_0(\tau) + a_1(\tau_*)x_1^2 + \big(a_2(\tau_*) + \alpha_2(\tau)\big)x_2^2 + a_3(\tau_*)\delta \pm b_0(\tau_*)x_1^{\hat n-2}x_2^{\hat m} = 0$,

(8.15c) $\big(c_1(\tau_*) + \gamma_1(\tau)\big)x_1^2 + c_2(\tau_*)x_2^2 + \big(c_3(\tau_*) + \gamma_3(\tau)\big)\delta \pm d_0(\tau_*)x_1^{\hat n}x_2^{\hat m-2} = 0$.

Denote by $C_0(\tau)$, $C_{mp}(\tau)$, $C_{nq}(\tau)$, $C_t^\pm(\tau)$ the solution sets of systems (8.12) to (8.15) respectively.

PROPOSITION 8.7. – Under the assumption (A) of Section 7 and the hypotheses of Theorem 7.3 we can choose neighborhood (3.2c) in such a way as there exists an open neighborhood $\mathcal{O}\times\mathcal{J}$ of 0 in $\Sigma(\mathbf{C}^2)\times\mathbf{R}$ satisfying (8.10) and (8.11) and such that

$$\mathcal{S}_\tau \cap \overline{\mathcal{O}\times\mathcal{J}} = (C_0 \cup C_{mp} \cup C_{nq} \cup C_t^+ \cup C_t^-) \cap \overline{\mathcal{O}\times\mathcal{J}}$$

for all $\tau \in \mathcal{C}$. Let

$$\tilde C_0 = C_0 \cap \overline{\mathcal{O}\times\mathcal{J}} , \quad \tilde C_{mp} = C_{mp} \cap \overline{\mathcal{O}\times\mathcal{J}} , \quad \tilde C_{nq} = C_{nq} \cap \overline{\mathcal{O}\times\mathcal{J}} , \quad \tilde C_t^\pm = C_t^\pm \cap \overline{\mathcal{O}\times\mathcal{J}} ,$$

then for each $\tau \in \mathcal{C}$ we have that:

(a) $\mathcal{S}_\tau \cap \overline{\mathcal{O}\times\mathcal{J}}$ has no limit point.

(b) $\tilde C_0$ is made of trivial solutions $(0, \lambda)$, for $\lambda \in \mathcal{J}$.

(c) $\tilde C_{mp}$ and $\tilde C_{nq}$ are arcs with one endpoint on $\tilde C_0$ and the other on $\mathcal{O}\times\partial\mathcal{J}$ (where $\partial\mathcal{J}$ is the boundary of $\mathcal{J}$).

(d) Let $\big(0, \delta_{mp}(\tau)\big)$ and $\big(0, \delta_{nq}(\tau)\big)$ be respectively the endpoints of $\tilde C_{mp}$ and $\tilde C_{nq}$ which lie on $\tilde C_0$, then we have $\delta_{nq}(\tau) = 0$ for all $\tau \in \mathcal{C}$ and $\delta_{mp}(\tau) \lesseqgtr \delta_{nq}(\tau) = 0$ according as $\alpha_0(\tau)\,a_3(\tau_*) \lesseqgtr 0$. Finally $\tau = \tau_*$ implies $\delta_{mp}(\tau_*) = 0$.

(*e*) $\tilde{C}_{mp}$ and $\tilde{C}_{nq}$ have no point in common except possibly for $(0, 0)$, for example when $\tau = \tau_*$.

(*f*) $\tilde{C}_{mp}$ $(\tilde{C}_{nq})$ is subcritic or supercritic with respect to $\delta_{mp}(\tau)$ $\big(\delta_{nq}(\tau)\big)$ according as $a_1(\tau_*)\cdot a_2(\tau_*)$ $\big(c_2(\tau_*)\cdot c_3(\tau_*)\big)$ is positive or negative.

(*g*) If $A_2A_3 > 0$ (see (7.4)) and $\alpha_0(\tau)A_2 < 0$, $\tilde{C}_t^+$ and $\tilde{C}_t^-$ are arcs connecting $\tilde{C}_{mp}$ and $\tilde{C}_{nq}$ with the endpoints in common on $\tilde{C}_{mp}$ and $\tilde{C}_{nq}$ respectively and with no other point in common between them or with $\tilde{C}_{mp}$ or $\tilde{C}_{nq}$.

(*h*) If $A_2A_3 > 0$ and $\alpha_0(\tau)A_2 \geq 0$, $\tilde{C}_t^\pm$ are empty.

(*l*) If $A_2A_3 < 0$, $\tilde{C}_t^+$ and $\tilde{C}_t^-$ have one endpoint in common either on $\tilde{C}_{mp}$ or on $\tilde{C}_{nq}$ and the other on $\mathcal{O}\times\partial\mathcal{J}$. Moreover there is no other point in common between $\tilde{C}_t^+$ and $\tilde{C}_t^-$ or with $\tilde{C}_{mp}$ or $\tilde{C}_{nq}$. Let (x_t, δ_t) be the endpoint in common, then (x_t, δ_t) is non-trivial whenever $\alpha_0(\tau) \neq 0$ and lies on $\tilde{C}_{mp}$ or on $\tilde{C}_{nq}$ according as $\alpha_0(\tau)A_2 \lessgtr 0$. Finally we have $(x_t, \delta_t) = (0, 0)$ when $\alpha_0(\tau) = 0$, for example when $\tau = \tau_*$.

(*m*) $\tilde{C}_t^\pm$ are subcritic or supercritic with respect to δ_t according as $A_1A_2 \gtrless 0$.

Before proving this proposition, *we prove points* (i) *and* (iii) *of Theorem* 7.3. For each $\tau \in \mathcal{T}$, consider the map $\Xi_\tau^{\,}\colon (\mathfrak{U}\times\overline{\mathfrak{V}})^*\times\mathfrak{I} \to \Sigma(\mathbb{C}^2)\times\mathbb{R}$ defined as

$$\Xi_\tau = (\Sigma^*\times\mathrm{id}_{\mathbb{R}})\circ(Z_\tau^*\times\Delta_\tau)\circ(P_N^*\times\mathrm{id}_{\mathbb{R}})\,.$$

Set

$$\mathfrak{W}_\tau^* = \Xi_\tau^{-1}(\overline{\mathcal{O}\times\mathcal{J}})\,.$$

By Propositions 8.1 and 8.6 we have that, for each $\tau \in \mathcal{T}$, Ξ_τ is a bijection between $\mathfrak{S}_\tau^* \cap \mathfrak{W}_\tau^*$ and $\sigma_\tau \cap \overline{\mathcal{O}\times\mathcal{J}}$, hence, by defining $\mathfrak{C}_0$, $\mathfrak{C}_{mp}$, $\mathfrak{C}_{nq}$, $\mathfrak{C}_t^\pm$ and (ξ_t^*, λ_t) respectively as inverse-image of $\tilde{C}_0$, $\tilde{C}_{mp}$, $\tilde{C}_{nq}$, $\tilde{C}_t^\pm$ and (x_t, δ_t) through $\Xi_\tau|_{\mathfrak{S}_\tau^*}$, we have that:

(i) By Proposition 2.5 eigenvalues $\big(0, \Lambda_{mp}(\tau)\big)$ and $\big(0, \Lambda_{nq}(\tau)\big)$ are inverse-images of $\big(0, \delta_{mp}(\tau)\big)$ and $\big(0, \delta_{nq}(\tau)\big)$ respectively.

(ii) The bifurcation diagram $\sigma_\tau \cap \overline{\mathcal{O}\times\mathcal{J}}$ is regular.

(iii) Points (i) and (iii) of Theorem 7.3 follow from Propositions 6.6, 8.1, 8.6 and 8.7. ∎

In order to prove Proposition 8.7, we need the following

LEMMA 8.8. – Let $U\times V$ be an open neighborhood of 0 in $\mathbb{R}^n\times\mathbb{R}^k$ and $f\colon U\times V \to \mathbb{R}^n$ a continuous map. Assume that the map $f_v\colon u \mapsto f(u, v)$ is injective for all $v \in V$, then, for each open ball $B(0, r) = \{u \in U\colon \|u\| < r\}$, there exists $\varepsilon > 0$ such that $\bigcap_{\|v\|<\varepsilon} f_v\big(B(0, r)\big)$ has non-empty interior.

PROOF. – By Domain Invariance Theorem ([18], Corollary 3.22), because f_v is injective and continuous, we have that f_v is an open map. In particular $f_v\colon U \to f_v(U)$ is a homeomorphism for each $v \in V$.

Let $w_0 = f(0, 0) \in f_0(B(0, r))$. Because $f_0(B(0, r))$ is open in $\mathbb{R}^n$, there exists $\nu > 0$ such that $B(w_0, \nu) \subset f_0(B(0, r))$ and the ν-neighborhood $\mathcal{U}_\nu$ of the boundary $\partial f_0(B(0, r))$ does not intersect $B(w_0, \nu)$. Because $\overline{B(0, r)}$ is compact and f is continuous, there exists $\varepsilon > 0$ such that $f_v(\partial B(0, r)) \subset \mathcal{U}_\nu$ for each $v \in V$ such that $\|v\| < \varepsilon$. In particular $f_v(\partial B(0, r))$ does not intersect $B(w_0, \nu)$ for each v such that $\|v\| < \varepsilon$. On the other hand, by Jordan Theorem ([18], Theorem 3.21), $\mathbb{R}^n \setminus f_v(\partial B(0, r))$ has two connected components, one of which is of course $f_v(B(0, r))$: in fact $f_v(B(0, r))$ is connected and its boundary coincides with the boundary of the components. Now $B(w_0, \nu)$ is contained in $\mathbb{R}^n \setminus f_v(\partial B(0, r))$ as we have seen above, hence it is contained in one of the two connected components because it is connected. On the other hand $\lim_{v\to 0} f(0, v) = f(0, 0) = w_0$, thus $B(w_0, \nu) \subset f(B(0, r))$ for each v such that $\|v\| < \varepsilon$ and the proof is complete. ■

PROOF OF PROPOSITION 8.7. – By Lemma 8.8 we can choose $\mathcal{T}$ in such a way as $\bigcap_{\tau\in\mathcal{T}} (Z_\tau(\mathcal{U}\times\mathfrak{I})\times\Delta_\tau(\mathfrak{I}))$ has non-empty interior. Now because the natural surjection $\pi\colon H \to H/\Gamma$ is open by definition it follows immediately that there exists a neighborhood $\mathcal{O}\times\mathcal{J}$ of O in $\Sigma(\mathbb{C}^2)\times\mathbb{R}$ satisfying (8.10) and (8.11). Of course we can always choose

$$\mathcal{O} = \mathcal{O}(r) = \{x \in \Sigma(\mathbb{C}^2)\colon x_1^2 + x_2^2 < r^2\} \quad \text{and} \quad \mathcal{J} = \mathcal{J}_\varepsilon = (-\varepsilon, \varepsilon) \subset \mathbb{R}\,.$$

Now point (*b*) of Proposition 8.7 is trivial. Moreover from (8.13) and (8.14) one sees immediately that solutions C_{mp} and C_{nq} are parabolas, hence it is straightforward to determine $\mathcal{T}$ and ε in such a way as points (*c*) and (*d*) hold. Now observe that equations (8.15*b*) and (8.15*c*) do not contain the variable x_3, thus it suffices to study them on $\mathcal{B}(r) \cap \mathbb{R}^2_+$ where

$$\mathcal{B}(r) = \{(x_1, x_2) \in \mathbb{R}^2\colon x_1^2 + x_2^2 < r^2\}\,.$$

Moreover from (8.15*a*) we have that C_t^+ and C_t^- do not meet but at the endpoints. Finally we observe that from now on we can limit ourselves to investigate case $+$, the other being perfectly analogous. We need the following

LEMMA 8.9. – One can choose r, and consequently $\mathcal{T}$ and ε in the proof of points (*b*) to (*f*) of Proposition 8.7, in such a way as there exist a diffeomorphism Φ from an open neighborhood $\mathcal{A}$ of 0 in $\mathbb{R}^2$ onto $\mathcal{B}(r)$ and a C^∞ map $M\colon \mathbb{R}^2\times\mathbb{R}\times\mathcal{A} \to \mathbb{R}^2$ such that

(8.16*a*) $\quad \Phi(0) = 0\,, \quad M(\cdot\,, x_1, x_2) \quad$ is linear and invertible for all $(x_1, x_2) \in \mathcal{A}$

and

$$(8.16b)\quad M\big(K(\Phi(x_1, x_2), \delta, \tau), x_1, x_2\big) = \\ = \begin{bmatrix} \alpha_0(\tau) + a_1(\tau_*)x_1^2 + \big(a_2(\tau_*) + \alpha_2(\tau)\big)x_2^2 + a_3(\tau_*)\delta \\ \big(c_1(\tau_*) + \gamma_1(\tau)\big)x_1^2 + c_2(\tau_*)x_2^2 + \big(c_3(\tau_*) + \gamma_3(\tau)\big)\delta \end{bmatrix}$$

where $K = (k_1, k_2)$ and $k_1 = 0$, $k_2 = 0$ are the equations (8.15b) and (8.15c) respectively.

Before proving this lemma, we end the outstanding proof. Under the hypotheses (7.11) one can easily see that it is possible to choose $\mathcal{T}$ in such a way as for each $\tau \in \mathcal{T}$ we have that:

(i) If $A_2A_3 > 0$, curve (8.16b) exists (is real) if and only if $\alpha_0(\tau) \geq 0$. Moreover, when $\alpha_0(\tau) > 0$ it is closed and has only two limit points, while it degenerates to a point whenever $\alpha_0(\tau) = 0$, for example when $\tau = \tau_*$.

(ii) If $A_2A_3 < 0$, curve (8.16b) always exists and is made of two connected components, each one with a unique limit point and with no point in common except for the origin when $\alpha_0(\tau) = 0$, for example when $\tau = \tau_*$.

Moreover it is elementary to verify that $\mathcal{T}$ and ε can be chosen in such a way as the closed curve of case (i) is all contained in the neighborhood $\mathcal{A} \times \mathfrak{J}_\varepsilon$, while in case (ii) the two components have non-empty intersection with $\mathcal{A} \times \mathfrak{J}_\varepsilon$ and are made of two arcs contained respectively in $\mathcal{A} \times [-\varepsilon, 0]$ and $\mathcal{A} \times [0, \varepsilon]$, with both endpoints on $\mathcal{A} \times \{-\varepsilon\}$ and $\mathcal{A} \times \{\varepsilon\}$ respectively (see Figure 4).

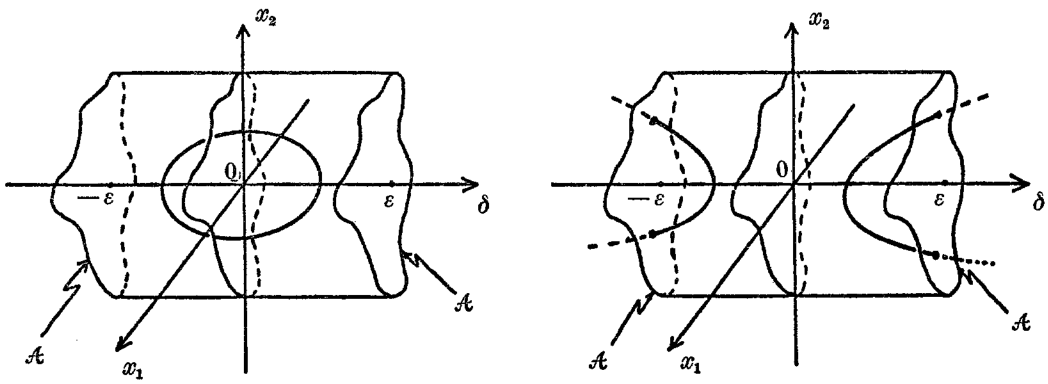

Figure 4.

Observe now that a direct computation shows that:

1) In case (i) the curve C_t^+ intersects transversally the coordinate half-planes $x_1 = 0$, $x_2 \geq 0$ and $x_1 \geq 0$, $x_2 = 0$ exactly in two points, which lie on C_{mp} and C_{nq} respectively.

2) In case (ii) the curve C_t^+ intersects transversally one and only one of the

coordinate hals-planes $x_1 = 0$, $x_2 \geq 0$ and $x_1 \geq 0$, $x_2 = 0$ exactly in one point, which either lies on C_{mp} or C_{nq}.

Thus, by resorting to Lemma 8.9 and to (8.3) and by studying directly the curve of solutions to (8.15*b*) and (8.15*c*) on a neighborhood of the intersection points with C_{mp} and C_{nq}, one easily completes the proof of Proposition 8.7. ∎

PROOF OF LEMMA 8.9. – The proof of this lemma rests upon singularity theory of smooth maps, for an account of which we refer to GIBSON [7].

Recall first of all that two smooth germs $f, g\colon \mathbb{R}^n, 0 \to \mathbb{R}^n, 0$ are $\mathcal{K}$*-equivalent* ([7], Chapter IV, Section 2, page 143) if there exist two smooth germs $\varphi\colon \mathbb{R}^n, 0 \to \mathbb{R}^n, 0$ and $\mu\colon \mathbb{R}^n \times \mathbb{R}^n, 0 \to \mathbb{R}^n, 0$ such that φ is a diffeomorphism, $v \mapsto \mu(v, u)$ is linear and invertible and

$$g(u) = \mu\big(f(\varphi(u)), u\big) .$$

A smooth germ f is $\mathcal{K}$*-k-determined* ([7], Chapter V, Section 2, page 191) if every other smooth germ with the same Taylor polynomial to order k is $\mathcal{K}$-equivalent to f.

It is clear that it suffices to prove that there exists $\mathcal{T}$ such that the germ at 0 of the following map from $\mathbb{R}^2$ into $\mathbb{R}^2$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \begin{bmatrix} a_1(\tau_*)x_1^2 + \big(a_2(\tau_*) + \alpha_2(\tau)\big)x_2^2 + b_0(\tau_*)x_1^{\hat{n}-2}x_2^{\hat{m}} \\ \big(c_1(\tau_*) + \gamma_1(\tau)\big)x_1^2 + c_2(\tau_*)x_2^2 + \big(c_3(\tau) + \gamma_3(\tau)\big)x_1^{\hat{n}}x_2^{\hat{m}-2} \end{bmatrix} \tag{8.17}$$

is $\mathcal{K}$-2-determined for each $\tau \in \mathcal{T}$, in that it is then $\mathcal{K}$-equivalent to the terms of order two only. This means that there exist M and Φ such that (8.16) hold on a suitable neighborhood of the origin. Now it is clear that we can choose r such small as $\mathcal{B}(r)$ is contained in the image of Φ. Consequently we can take $\mathcal{A} = \Phi^{-1}(\mathcal{B}(r))$.

Therefore it remains to verify that (8.17) is $\mathcal{K}$-2-determined. To this purpose, recall the definition of $\mathcal{K}$*-tangent space* ([7], Chapter V, Section 2, page 152). Denote by $\mathcal{E}_n$ the ring of germs at 0 of smooth functions $\mathbb{R}^n \to \mathbb{R}$ and by $\boldsymbol{E}_n$ the $\mathcal{E}_n$-module of germs at 0 of smooth maps $\mathbb{R}^n \to \mathbb{R}^n$. Given a smooth germ $f\colon \mathbb{R}^n, 0 \to \mathbb{R}^n, 0$ we define the $\mathcal{K}$-tangent space to f as the following submodule of $\boldsymbol{E}_n$:

$$T(f) = J_f + I_f \boldsymbol{E}_n$$

where I_f is the ideal of $\mathcal{E}_n$ generated by the components $f_1, \ldots, f_n$ of f and J_f is the submodule of $\boldsymbol{E}_n$ generated by the maps $\partial f/\partial u_1, \ldots, \partial f/\partial u_n$, where $u_1, \ldots, u_n$ are the coordinates of $\mathbb{R}^n$. Denote by $\mathcal{M}_n$ the maximal ideal of $\mathcal{E}_n$, then we have that

$$\mathcal{M}_n^{k+1} \boldsymbol{E}_n \subset T(f) \tag{8.18}$$

implies that f is $\mathcal{K}$-k-determined ([7], Proposition 6.1, page 191). In our case we have to show that (8.18) is satisfied with $k = 2$ and f given by (8.17). By Nakayama's Lemma ([7], Proposition 2.6, page 102), it suffices to show that

$$\mathcal{M}_2^3 E_3 \subset T + \mathcal{M}_2^4 E_3 \,. \tag{8.19}$$

Now by writing out the generators of the tangent space T modulo $\mathcal{M}_2^4 E_2$ and using (7.7), (7.10) and (8.2) one easily obtains that (8.19) holds for τ near enough τ_*. ■

It remains to give the *proof of point* (ii) *of Theorem* 7.3. First of all observe that from Propositions 8.1 and 8.3 it follows that $(Z_\tau \times \Delta_\tau)\circ(P_N \times \mathrm{id}_{\mathbf{R}})$ induces for each $\tau \in \mathcal{T}$ a Γ-equivariant bijection between $\mathcal{S}_\tau \cap \overline{\mathcal{W}}_\tau$ and $\Sigma^{-1}(\mathfrak{s}_\tau \cap \overline{\mathcal{O}\times\mathcal{J}})$. In particular, points which are in correspondence have the same isotropy subgroup. Therefore to prove point (ii) it suffices to consider a representative of each orbit in $\Sigma^{-1}(\mathfrak{s}_\tau \cap \cap \overline{\mathcal{O}\times\mathcal{J}})$ and compute its relevant isotropy subgroup. The following map from $\Sigma(\mathbf{C}^2)$ into $\mathbf{C}^2$ associates in a natural way to each orbit in $\Sigma(\mathbf{C}^2)$ one of its representative:

$$x \mapsto \begin{cases} (x_1, 0) & \text{if } x_2 = 0 \\ (0, x_2) & \text{if } x_1 = 0 \\ (x_1, x_2 \exp(\varphi/\hat{m})) & \text{if } x_1, x_2 \neq 0 \end{cases} \tag{8.20}$$

where $\varphi = \arccos(x_3/2x_1^{\hat{n}} x_2^{\hat{m}})$. Images through map (8.20) of the arcs $\tilde{C}_0$, $\tilde{C}_{mp}$, $\tilde{C}_{nq}$, $\tilde{C}_t^+$ and $\tilde{C}_t^-$ satisfy respectively to the following relations:

$$z_1 = z_2 = 0\,, \tag{8.21a}$$

$$z_1 \in \mathbf{R}_+^*\,, \quad z_2 = 0\,, \tag{8.21b}$$

$$z_1 = 0\,, \quad z_2 \in \mathbf{R}_+^* \tag{8.21c}$$

$$z_1, z_2 \in \mathbf{R}_+^* \tag{8.21d}$$

$$z_1 \in \mathbf{R}_+^*\,, \quad z_2 \in \{r \exp(i\pi/\hat{m}) : r \in \mathbf{R}_+^*\}\,. \tag{8.21e}$$

Thus point (ii) of Theorem 7.3 follows from

PROPOSITION 8.10. – (i) Γ_z equals Γ_0, Γ_{mp}, Γ_{nq}, Γ_t^+, Γ_t^- (see the table of point (ii) of Theorem 7.3) according as $z \in \mathbf{C}^2$ satisfies (8.21*a*) to (8.21*e*) respectively.

(ii) The orbit of a point $z \in \mathbf{C}^2$ satisfying one of the relations (8.21) is generated by the subgroup $\mathbf{SO}(2) \oplus \{1\}$ of Γ.

PROOF. – (i) Is verified by a direct computation we leave to the reader.

(ii) It suffices to observe that for each isotropy subgroup Γ_z of the table at point (ii) of Theorem 7.3 there exist θ_{-1}, $\psi_\varepsilon \in \mathbf{O}(2)$ such that $(\theta_{-1}, -1)$, $(\psi_\varepsilon, -1) \in \in \Gamma_z$. ■

9. – Equivariant singularity theory.

In this section we recall some general results of equivariant singularity theory, we need to prove Propositions 7.1 and 8.3. We assume that the reader is familiar with the papers of POÈNARU [14] and GOLUBITSKY and SCHAEFFER [8] and [9].

Let $\varrho\colon \Gamma\times\mathbb{R}^n\to\mathbb{R}^n$ be an *orthogonal* action of a *compact* Lie group Γ on $\mathbb{R}^n$. Denote by $\mathcal{E}_n^\Gamma$ the ring of germs at 0 of Γ-invariant C^∞ functions $\mathbb{R}^n\to\mathbb{R}$ and by E_n^Γ the $\mathcal{E}_n^\Gamma$-module of germs at 0 of Γ-equivariant C^∞ maps $\mathbb{R}^n\to\mathbb{R}^n$. Futhermore, let $\mathcal{I}_n^\Gamma\subset\mathcal{E}_n^\Gamma$ and $P_n^\Gamma\subset E_n^\Gamma$ be the ring of Γ-invariant polynomials and of Γ-equivariant polynomial maps respectively.

THEOREM 9.1. – $\mathcal{I}_n^\Gamma$ is an $\mathbb{R}$-algebra of finite type.

PROOF. – [14], Théorème 1, page 6. ■

THEOREM 9.2. – $\mathcal{E}_n^\Gamma$ is an $\mathbb{R}$-algebra of finite type with the same generators of $\mathcal{I}_n^\Gamma$.

PROOF. – [14], Corollaire au Théorème Fondamental, page 22. ■

THEOREM 9.3. – P_n^Γ is a $\mathcal{I}_n^\Gamma$-module of finite type and E_n^Γ is an $\mathcal{E}_n^\Gamma$-module of finite type generated by the same generators of P_n^Γ.

PROOF. – [14], Lemme 1.4.1, page 106. ■

Denote by $x=(x_1,\dots,x_n)$ the elements of $\mathbb{R}^n$. Let $\sigma_1(x),\dots,\sigma_n(x)\in\mathcal{I}_n^\Gamma$ be a set of generators of $\mathcal{I}_n^\Gamma$. Consider the diagonal action of Γ on $\mathbb{R}^n\times\mathbb{R}^m$ composed by the action ϱ on $\mathbb{R}^m$ and the trivial action on $\mathbb{R}^n$, then we have the following

THEOREM 9.4. – $\mathcal{I}_{n+m}^\Gamma$ is generated by $\sigma_1(x),\dots,\sigma_h(x),y_1,\dots,y_m$, where y_j are coordinate functions on $\mathbb{R}^m$.

PROOF. – [14], Théorème 1, page 34. ■

Denote by $\mathcal{E}_{h+m}$ the ring of germs at 0 of C^∞ functions $\mathbb{R}^h\times\mathbb{R}^m\to\mathbb{R}$. Let $\sigma\colon\mathbb{R}^n\to\mathbb{R}^h$ be the map

$$\sigma\colon x\mapsto\big(\sigma_1(x),\dots,\sigma_h(x)\big)\,,$$

then it follows from Theorems 9.2 and 9.4 that the transposed map $(\sigma\times\mathrm{id}_{\mathbb{R}}\,m)^T$: $\mathcal{E}_{h+m}\to\mathcal{E}_{n+m}^\Gamma$, defined as

$$(\sigma\times\mathrm{id}_{\mathbb{R}^m})^T\colon f\mapsto f\circ(\sigma\times\mathrm{id}_{\mathbb{R}^m})\,,$$

is a surjection. It is easy to prove the following

PROPOSITION 9.5. – Assume that $\mathcal{P}_n^\Gamma$ is a polynomial ring, that is that there are no non-trivial polynomial relations between $\sigma_1(x), \ldots, \sigma_h(x)$, then $\ker(\sigma \times \mathrm{id}_{\mathbb{R}^m})^* \subset \mathcal{M}_{h+m}^\infty$, where

$$\mathcal{M}_{h+m} = \{g \in \mathcal{E}_{h+m} \colon g(0, 0) = 0\}$$

and $\mathcal{M}_{h+m}^\infty = \bigcap_{r \geq 1} \mathcal{M}_{h+m}^r$. ■

Now let $E_{n+m,n}^\Gamma$ be the $\mathcal{E}_{n+m}^\Gamma$-module of germs at 0 of Γ-equivariant C^∞ maps $\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and set $E_{h+m,k} = \mathcal{E}_{h+m} \times \ldots \times \mathcal{E}_{h+m}$ (k-times).

Let $\Omega_1(x), \ldots, \Omega_k(x)$ be a system of generators of E_n^Γ over $\mathcal{E}_n^\Gamma$. By Theorems 9.2, 9.3 and 9.4 we have that the map $\Omega \colon E_{h+m,k} \to E_{n+m,n}^\Gamma$, defined as

$$\Omega \colon (g_1, \ldots, g_k) \mapsto \sum_{j=1}^{k} [g_j \circ (\sigma \times \mathrm{id}_{\mathbb{R}^m})] \Omega_j , \tag{9.1}$$

is a surjection.

PROPOSITION 9.6. – Under the hypotheses of Proposition 9.5, assume furthermore that E_n^Γ is a free module over $\mathcal{E}_n^\Gamma$, with a basis given by $\Omega_1, \ldots, \Omega_k$, then $\ker \Omega \subset \mathcal{M}_{h+m}^\infty E_{h+m,k}$.

PROOF. – It is an immediate consequence of Proposition 9.5. ■

THEOREM 9.7. – Under the hypotheses of Propositions 9.5 and 9.6, given a Γ-equivariant C^∞ map $F \colon \mathcal{U} \to R^n$, where $\mathcal{U}$ is a Γ-invariant open neighborhood of 0 in $\mathbb{R}^n \times \mathbb{R}^m$, there exist k C^∞ functions $P_j \colon \mathbb{R}^h \times \mathbb{R}^m \to \mathbb{R}$ and a Γ-invariant open connected neighborhood $\mathcal{V} \times \mathcal{W}$ of 0 in $\mathbb{R}^n \times \mathbb{R}^m$, contained in $\mathcal{U}$, such that

$$F(x, y) = \sum_{j=1}^{k} P_j\big(\sigma(x), y\big) \Omega_j(x) \qquad \text{for each } (x, y) \in \mathcal{V} \times \mathcal{W} .$$

Moreover the Taylor expansion at the origin of the functions P_j is uniquely determined by F.

PROOF. – It follows immediately from Theorems 9.2, 9.3, 9.4 and from Proposition 9.6, recalling the definition of germ and the fact that the origin has a fundamental system of open connected Γ-invariant neighborhoods because the group Γ is compact. ■

Consider now the diagonal action of Γ on $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ (trivial on the third component) and the following $\mathcal{E}_{n+1}^\Gamma$-module

$\mathcal{K}_{n+1,n}^\Gamma = \{$germs at 0 of C^∞ maps $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, which are Γ-equivariant and linear in the first set of variables$\}$.

In a similar way as for Theorem 9.3, one proves the following

PROPOSITION 9.8. – $\mathcal{K}^{\Gamma}_{n+1,n}$ is an $\mathcal{E}^{\Gamma}_{n+1}$-module of finite type. ∎

Now we define Γ-equivalence and the universal unfolding of a germ. Given $G, H \in E^{\Gamma}_{n+1,n}$ such that $G(0,0) = H(0,0) = 0$, we say that G and H are *Γ-equivalent* if there exist germs $K \in \mathcal{K}^{\Gamma}_{n+1,n}$, $X \in E^{\Gamma}_{n+1,n}$ and $\Delta \in \mathcal{E}_1$ such that

$$X(0,0) = 0\,, \qquad \Delta(0) = 0\,,$$
$$\operatorname{Det} D_y K(0,0,0) \neq 0\,, \qquad \operatorname{Det} D_x X(0,0) \neq 0\,, \qquad D_\delta \Delta(0,0) > 0\,,$$
$$H(x,\delta) = K\big(G(X(x,\delta), \Delta(\delta)), x, \delta\big)$$

where (y, x, δ) denote the coordinates of $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$.

Given $G \in E^{\Gamma}_{n+1,n}$, we call *unfolding* of G a germ at the origin of a Γ-equivariant C^∞ map $\mathcal{G}\colon \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^r \to \mathbb{R}^n$ (Γ acts diagonally on $\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^r$ and trivially on $\mathbb{R}$ and $\mathbb{R}^r$) such that $\mathcal{G}(x,\delta,0) = G(x,\delta)$. Given two unfoldings $\mathcal{G}(x,\delta,\alpha)\colon \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^r \to \mathbb{R}^n$ and $\mathcal{H}(x,\delta,\beta)\colon \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^s \to \mathbb{R}^n$ of $G \in E^{\Gamma}_{n+1,n}$ such that $G(0,0) = 0$, we say that $\mathcal{H}$ *factors through* $\mathcal{G}$ if there exist Γ-equivariant smooth germs at the origin

$$K\colon \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^s \to \mathbb{R}^n\,, \quad X\colon \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^s \to \mathbb{R}^n\,, \quad \Delta\colon \mathbb{R} \times \mathbb{R}^s \to \mathbb{R}\,, \quad A\colon \mathbb{R}^s \to \mathbb{R}^r$$

such that

$$K(y,x,\delta,0) = y\,, \quad X(x,\delta,0) = x\,, \quad \Delta(\delta,0) = \delta\,, \quad A(0) = 0\,,$$
$$y \mapsto K(y,x,\delta,\beta) \quad \text{is linear}\,,$$
$$\mathcal{H}(x,\delta,\beta) = K\big(\mathcal{G}(X(x,\delta,\beta), \Delta(\delta,\beta), A(\beta)), x, \delta, \beta\big)\,.$$

The map A is called *factoring map*. An unfolding $\mathcal{G}$ of G, such that $G(0,0) = 0$, is called *universal* if every other unfolding of G factors through $\mathcal{G}$.

Given $G \in E^{\Gamma}_{n+1,n}$ such that $G(0,0) = 0$, define the *reduced tangent space* as

$$\text{(9.2)} \qquad \tilde{T}_\Gamma(G) = \text{submodule of } E^{\Gamma}_{n+1,n} \text{ generated by } D_x G\cdot\Omega_1, \dots, D_x G\cdot\Omega_k,$$
$$K_1\big(G(x,\delta), x, \delta\big), \dots, K_l\big(G(x,\delta), x, \delta\big)$$

where $D_x G$ is the Jacobian of G with respect to x and $k_1, \dots, k_l$ are the generators of $\mathcal{K}^{n}_{n+1,n}$ (see Proposition 9.8). Then define the *tangent space* to G as

$$\text{(9.3)} \qquad T_\Gamma(G) = \tilde{T}_\Gamma(G) \oplus_{\mathbb{R}} \mathcal{E}_1 \cdot D_\delta G$$

where $\mathcal{E}_1 = \{g_{|\{0\}\times\mathbb{R}} : g \in \mathcal{E}^{\Gamma}_{n+1}\}$ and $D_\delta G$ is the Jacobian matrix of G with respect to δ. Finally we set

$$\tilde{\mathrm{T}}(G) = \Omega^{-1}(\tilde{T}_\Gamma(G)) \quad \text{and} \quad \mathrm{T}(G) = \Omega^{-1}(T_\Gamma(G)) \tag{9.4}$$

where Ω is the map (9.1).

THEOREM 9.9. – Under the hypotheses of Propositions 9.5 and 9.6, given $G \in E^{\Gamma}_{n+1,n}$ such that $G(0, 0) = 0$, assume that there exists an $\mathbb{R}$-vector space $V \subset E_{n+1,k}$ such that $\tilde{\mathrm{T}}(G + H) = \tilde{\mathrm{T}}(G)$ for each $H \in \Omega(V)$, then G is Γ-equivalent to $G + H$ for each $H \in \Omega(V)$.

PROOF. – See [9], Proposition 1.12. ■

THEOREM 9.10 – Under the hypotheses of Propositions 9.5 and 9.6, given $G \in E^{\Gamma}_{n+1,n}$ such that $G(0, 0) = 0$, assume that $\dim_{\mathbb{R}} E_{h+1,k/\tilde{\mathrm{T}}(G)} < \infty$. Then a universal unfolding of G is given by

$$\mathcal{G}(x, \delta, \alpha) = G(x, \delta) + \sum_{j=1}^{r} \alpha_j q_j(x, \delta)$$

where $\alpha_j \in \mathbb{R}$, $q_j = \Omega(Q_j)$ and $Q_1, \ldots, Q_r$ are a basis for an $\mathbb{R}$-vector space W such that $E_{h+1,k} = \mathrm{T}(G) \oplus_{\mathbb{R}} W$.

PROOF. – See [9], Theorem 1.8. ■

Finally it is easy to prove the following

PROPOSITION 9.11. – Under the hypotheses of Theorem 9.10, given an unfolding $\mathcal{H}(x, \delta, \beta): \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^s \to \mathbb{R}^n$ of G, let $A: \mathbb{R}^s \to \mathbb{R}^r$ be the factoring map of $\mathcal{H}$ through $\mathcal{G}$. Denote by $A_i(\beta)$ the components of $A(\beta)$, then we have that $(\partial A_i / \partial \beta_j)|_{\beta=0}$ are the unique real numbers such that

$$\left.\frac{\partial \hat{\mathcal{H}}}{\partial \beta_j}\right|_{\beta=0} \equiv \sum_{i=1}^{r} \left.\frac{\partial A_i}{\partial \beta_j}\right|_{\beta=0} Q_i \text{ modulo } \mathrm{T}(G), \quad \text{for } j = 1, \ldots, s,$$

where $\hat{\mathcal{H}}$ is any map $\mathbb{R}^h \times \mathbb{R} \times \mathbb{R}^s \to \mathbb{R}^k$ such that $\Omega(\hat{\mathcal{H}}) = \mathcal{H}$. ■

Now we show how to employ these results in studying bifurcation problems and in particular in proving Proposition 8.3. Under the hypotheses of Propositions 9.5 and 9.6, consider a Γ-equivariant C^∞ map $F: \mathcal{U} \to \mathbb{R}^n$, where $\mathcal{U}$ is an open Γ-invariant neighborhood of 0 in $\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^s$ (Γ acts trivially on $\mathbb{R}$ and $\mathbb{R}^s$) such that $F(0, 0, 0) = 0$. Set

$$F_0(x, \delta) = F(x, \delta, 0)$$

and assume that the germ at 0 of F_0 satisfies the hypotheses of Theorems 9.9 and 9.10. Now it is easily seen that the vector subspace $\Omega(V)$ is contained in $\tilde{T}_\Gamma(F_0)$, so it has a finite dimensional complement U:

$$E^\Gamma_{n+1,n} = U \oplus_{\mathbf{R}} \Omega(V) . \tag{9.5}$$

Let

$$\mathcal{M}^\Gamma_{n+1} = \{f \in \mathcal{E}^\Gamma_{n+1} : f(0, 0) = 0\}$$

be the maximal ideal of $\mathcal{E}^\Gamma_{n+1}$. Because $\Omega(V)$ has finite codimension with respect to $E^\Gamma_{n+1,n}$, there exists an integer l such that

$$(\mathcal{M}^\Gamma_{n+1})^l E^\Gamma_{n+1,n} \subset \Omega(V) .$$

It follows that $\Omega(V)$ has a complement of polynomial maps, thus we may choose U satisfying (9.5) as made of polynomial maps. Denote by $P_U: E^\Gamma_{n+1,n} \to E^\Gamma_{n+1,n}$ the projection onto U, then by Theorem 9.9 F_0 is Γ-equivalent to the *polynomial* map

$$G = P_U(F_0) ,$$

say through the triple $\big(K^\#(y, x, \delta), X^\#(x, \delta), \Delta^\#(\delta)\big)$.

Clearly G satisfies the hypotheses of Theorem 9.10. In particular $\mathbb{T}(G)$ has a finite dimensional complement W which we may choose as made of polynomial maps. Therefore W has a polinomial basis $Q_1, \ldots, Q_r$ and by Theorem 9.10 G has a *polynomial* universal unfolding

$$\mathcal{G}(x, \delta, \alpha) = G(x, \delta) + \sum_{j=1}^{r} \alpha_j q_j(x, \delta)$$

where $q_j = \Omega(Q_j)$.

Now we use the unfolding $\mathcal{G}$ to study the bifurcation problem $F = 0$. To this end, define

$$F^\#(x, \delta, \beta) = K^\#\big(F\big(X^\#(x, \delta), \Delta^\#(\delta), \beta\big), x, \delta\big) .$$

Of course $F^\#$ is an unfolding of G, thus it factors through $\mathcal{G}$. Denote by A the relevant factoring map, which can be computed to first order thanks to Proposition 9.11 (of course subject to computation of $(K^\#, X^\#, \Delta^\#)$ to the right order). Then, by composing this factorization with the Γ-equivalence $(K^\#, X^\#, \Delta^\#)$, one obtains there exist Γ-equivariant smooth germs $F(y, x, \delta, \beta)$, $X(x, \delta, \beta)$, $\Delta(\delta, \beta)$ such that, together with $A(\beta)$, we have

(9.6*a*) $\quad X(0, 0, 0) = 0 , \quad \Delta(0, 0) = 0 , \quad A(0) = 0 ,$

(9.6*b*) $\quad y \mapsto K(y, x, \delta, \beta) \quad$ is linear , (9.6c) $\quad \operatorname{Det} D_y K(0,0,0,0) \neq 0\,, \qquad \operatorname{Det} D_x X(0,0,0) \neq 0\,, \qquad D_\delta \Delta(0,0) > 0\,,$

(9.6d) $\quad F(x,\delta,\beta) = K\big(\mathcal{G}(X(x,\delta,\beta), \Delta(\delta,\beta), A(\beta)), x, \delta, \beta\big)\,.$

Of course identity (9.6d), obtained for germs, holds also for functions on a suitable open connected Γ-invariant neighborhood $\mathcal{V}\times\mathcal{J}\times\mathcal{B}$ of 0 in $\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^s$ and contained in $\mathcal{U}$. Moreover (9.6c) becomes

$y \mapsto K(y, x, \delta, \beta)$ is invertible for each $(x,\delta,\beta)\in\mathcal{V}\times\mathcal{J}\times\mathcal{B}$.

$(x,\delta)\mapsto\big(X(x,\delta,\beta), \Delta(\delta,\beta)\big)$ is a diffeomorphism defined on $\mathcal{V}\times\mathcal{J}$ for each $\beta\in\mathcal{B}$.

$\delta\mapsto\Delta(\delta,\beta)$ is monotonic increasing for each $\beta\in\mathcal{B}$.

10. – Proof of Proposition 7.1.

Following the lines stated in the preceding section, we begin by computing a set of generators for the $\mathbb{R}$-algebra $\mathcal{P}_4^\Gamma$ of Γ-invariant polynomials with respect to the action ϱ generated by (7.3). Employing complex notation, a polynomial $g\in\mathcal{P}_4^\Gamma$ can be written as

$$g(z) = \Sigma a_{klrs} z_1^k \bar z_1^l z_2^r \bar z_2^s$$

where $z=(z_1,z_2)\in\mathbb{C}^2$ and $a_{klrs}\in\mathbb{C}$ are such that

$$a_{klrs} = \bar a_{lksr}\,. \tag{10.1}$$

Identity (10.1) means that $g(z)=\overline{g(\bar z)}$. Moreover Γ-invariance yields

$$\Sigma a_{klrs}\, e^{i[m(k-l)+n(r-s)]\varphi} z_1^k \bar z_1^l z_2^r \bar z_2^s = \Sigma a_{klrs} z_1^k \bar z_1^l z_2^r \bar z_2^s \qquad \text{for all } \varphi\in\mathbb{R}\,,$$
$$\Sigma a_{klrs} \bar z_1^k z_1^l \bar z_2^r z_2^s = \Sigma a_{klrs} z_1^k \bar z_1^l z_2^r \bar z_2^s\,,$$
$$\Sigma a_{klrs} (-1)^{(k+l)(p+1)+(r+s)(q+1)} z_1^k \bar z_1^l z_2^r \bar z_2^s = \Sigma a_{klrs} z_1^k \bar z_1^l z_2^r \bar z_2^s\,,$$

which are equivalent to the following

$$a_{klrs} = 0 \qquad \text{unless} \qquad m(k-l)+n(r-s)=0\,, \tag{10.2}$$

$$a_{klrs} = a_{lksr}\,, \tag{10.3}$$

$$a_{klrs} = (-1)^{(k+l)(p+1)+(r+s)(q+1)} a_{klrs}\,. \tag{10.4}$$

In particular from (10.1) and (10.3) we have that

$$a_{klrs}\in\mathbb{R}\,. \tag{10.5}$$

Now we exploit (10.2). From $m(k-l)+n(r-s)=0$ we have

$$\hat{m}(k-l)=\hat{n}(s-r)$$

where $\hat{m}$ and $\hat{n}$ are given by (7.5). Now $\hat{m}$ and $\hat{n}$ have no common factor, thus $k-l=h\hat{n}$ and $s-r=h\hat{m}$ for some $h\in\mathbb{Z}$. Hence by (10.3) and (10.5) we have that $g(z)$ can be written in the form:

$$g(z)=\Sigma b_{hkl}(z_1\bar{z}_1)^h(z_2\bar{z}_2)^k(z^{l\hat{n}}\bar{z}_2^{l\hat{m}}+\bar{z}_1^{l\hat{n}}z_2^{l\hat{m}})$$

for suitable $b_{hkl}\in\mathbb{R}$. Then it is easy to see by induction on h that

$$g=\Sigma c_{hkl}\sigma_1^h\sigma_2^k\eta^l$$

for suitable $c_{hkl}\in\mathbb{R}$ and with σ_j and η given by (7.6).

It remains to consider (10.4). Obviously σ_j satisfy (10.4), while, as regards η, we have that (10.4) transforms η in $(-1)^{\hat{n}(p+1)+\hat{m}(q+1})\eta$. Thus we can conclude that

PROPOSITION 10.1. – The $\mathbb{R}$-algebra $\mathcal{I}_4^\Gamma$ is generated by

(i) σ_1, σ_2 and η (given by (5.5)) if $\hat{n}(p+1)+\hat{m}(q+1)$ is even.

(ii) σ_1, σ_2 and η^2 if $\hat{n}(p+1)+\hat{m}(q+1)$ is odd. ■

In the same way one can also prove the following

PROPOSITION 10.2. – The $\mathcal{I}_4^\Gamma$-module P_4^Γ is generated by

(i) $\Omega_1=\begin{bmatrix} z_1 \\ 0 \end{bmatrix}, \quad \Pi_1=\begin{bmatrix} \bar{z}_1^{\hat{n}-1}z_2^{\hat{m}} \\ 0 \end{bmatrix}, \quad \Omega_2=\begin{bmatrix} 0 \\ z_2 \end{bmatrix}, \quad \Pi_2=\begin{bmatrix} 0 \\ z_1^{\hat{n}}\bar{z}_2^{\hat{m}-1} \end{bmatrix}$

if $\hat{n}(p+1)+\hat{m}(q+1)$ is even.

(ii) $\Omega_1, \eta\Pi_1, \Omega_2, \eta\Pi_2$ if $\hat{n}(p+1)+\hat{m}(q+1)$ is odd. ■

After we have found the generators of $\mathcal{I}_4^\Gamma$ and P_4^Γ, let us prove that the hypotheses of Propositions 9.5 and 9.6 are satisfied.

PROPOSITION 10.3. – (i) There are no non-trivial polynomial relations between σ_1, σ_2 and η.

(ii) The generators $\Omega_1, \Pi_1, \Omega_2, \Pi_2$ are free over $\mathcal{E}_4^\Gamma$.

PROOF. – (i) Consider the map $\Theta\colon \mathbb{R}^4\to\mathbb{R}^3$ defined by

$$\Theta\colon (x_1, y_1, x_2, y_2)\mapsto\big(x_1^2+y_1^2,\ x_2^2+y_2^2,\ 2\ \mathrm{Re}\,[(x_1-iy_1)^{\hat{n}}(x_2+iy_2)^{\hat{m}}]\big)\,.$$

It is easy to verify that the Jacobian matrix $D\Theta$ has maximal rank for $x_1 = 1$, $y_1 = 0$, $x_2 = \cos 1/\hat{m}$, $y_2 = \sin 1/\hat{m}$. It follows that Θ is locally surjective, whence $\Theta(\mathbb{R}^4)$ contains an open subset. Assume now that $p\colon \mathbb{R}^3 \to \mathbb{R}$ is a polynomial such that $q(z) = p(\sigma(z), \eta(z))$ vanishes for all $z \in \mathbb{C}^2$, then p vanishes on $\Theta(\mathbb{R}^4)$ and so it must be identically zero because it is a polynomial and $\Theta(\mathbb{R}^4)$ has non-empty interior.

(ii) Let $\mathcal{A}_1(z)$, $\mathcal{B}_1(z)$, $\mathcal{A}_2(z)$, $\mathcal{B}_2(z) \in \mathcal{E}_4^{\Gamma}$ be such that

$$\sum_{j=1}^{2} [\mathcal{A}_j(z)\Omega_j(z) + \mathcal{B}_j(z)\Pi_j(z)] = 0 \tag{10.6}$$

for all $z \in \mathbb{C}^2$. Because $\mathcal{A}_j$ and $\mathcal{B}_j$ are real, we have that (10.6) is equivalent to the following systems

$$\begin{cases} \mathcal{A}_1 z_1 + \mathcal{B}_1 \bar{z}_1^{\hat{n}-1} z_2^{\hat{m}} = 0 \\ \mathcal{A}_1 \bar{z}_1 + \mathcal{B}_1 z_1^{\hat{n}-1} \bar{z}_2^{\hat{m}} = 0 \end{cases} \quad \text{and} \quad \begin{cases} \mathcal{A}_2 z_2 + \mathcal{B}_2 z_1^{\hat{n}} \bar{z}_2^{\hat{m}-1} = 0 \\ \mathcal{A}_2 \bar{z}_2 + \mathcal{B}_2 \bar{z}_1^{\hat{n}} z_2^{\hat{m}-1} = 0 \end{cases}$$

It follows that $\mathcal{A}_j$ and $\mathcal{B}_j$ must vanish on the complement of the zero-set of $z_1^{\hat{n}} \bar{z}_2^{\hat{m}} - \bar{z}_1^{\hat{n}} z_2^{\hat{m}}$, which is a dense subset of $\mathbb{C}^2$, therefore they must be identically zero by continuity. ∎

Therefore we can conclude that Proposition 7.1 follows immediately from Propositions 10.1, 10.2, 10.3 and Theorem 9.7.

11. – Proof of Proposition 8.3.

We apply the remarks stated at the end of Section 9 to our reduced bifurcation equation $F = 0$.

First of all we fix some further general notation in accordance with that of Section 9:

(i) $\langle g_1, \ldots, g_r \rangle \subset \mathcal{E}_{h+1}$ is the ideal generated by $g_1, \ldots, g_r \in \mathcal{E}_{h+1}$. In particular $\mathcal{M}_{h+1} = \langle \sigma_1, \ldots, \sigma_h, \delta \rangle$ is the maximal ideal of $\mathcal{E}_{h+1}$.

(ii) $\{G_1, \ldots, G_s\} \subset \boldsymbol{E}_{h+1,k}$ is the $\mathbb{R}$-vector subspace generated by $G_1, \ldots, G_s \in \boldsymbol{E}_{h+1,k}$.

(iii) Given k ideals $\mathfrak{J}_1, \ldots, \mathfrak{J}_k$ of $\mathcal{E}_{h+1}$, denote by $(\mathfrak{J}_1, \ldots, \mathfrak{J}_k) \subset \boldsymbol{E}_{h+1\,k}$ the $\mathcal{E}_{h+1}$-submodule $\{(f_1, \ldots, f_k) \in \boldsymbol{E}_{h+1,k}\colon f_j \in \mathfrak{J}_j \text{ for } j = 1, \ldots, k\}$. Set now

$$F_0(z, \delta) = F(z, \lambda_* + \delta, \tau_*) \tag{11.1a}$$

and

$$\mathcal{G}_0(z, \delta) = \mathcal{G}(z, \delta, 0, 0) \tag{11.1b}$$

where F is defined by (3.4), λ_* is given by (3.1*a*) and $\mathcal{G}$ is given by (8.1). In the following lemma we compute the tangent spaces.

LEMMA 11.1. – Under the Assumption (*A*) of Section 7 and the hypotheses of Theorem 7.3 we have that

(i) $\tilde{\mathbf{T}}(F_0) = \tilde{\mathbf{T}}(\mathcal{G}_0) = \{(2a_1(\tau_*)\sigma_1, (\hat{n}-2)b_0(\tau_*), 2c_1(\tau_*)\sigma_1, \hat{n}d_0(\tau_*)), (2a_2(\tau_*)\sigma_2, \hat{m}b_0(\tau_*), 2c_2(\tau_*)\sigma_2, (\hat{m}-2)d_0(\tau_*)), (a_1(\tau_*)\sigma_1 + a_2(\tau_*)\sigma_2 + a_3(\tau_*)\delta, 0, 0, 0), (0, 0, c_1(\tau_*)\sigma_1 + c_2(\tau_*)\sigma_2 + c_3(\tau_*)\delta, 0)\} \oplus_{\mathbf{R}} V$

where V is the following submodule of $E_{3+1,4}$:

$$V = (\langle\eta\rangle + \mathcal{M}^2_{3+1}, \mathcal{M}_{3+1}, \langle\eta\rangle + \mathcal{M}^2_{3+1}, \mathcal{M}_{3+1}) . \tag{11.2}$$

(ii) $\mathbf{T}(\mathcal{G}_0) = \{(a_3(\tau_*), 0, c_3(\tau_*), 0), (a_3(\tau_*)\delta, 0, c_3(\tau_*)\delta, 0)\} \oplus_{\mathbf{R}} \tilde{\mathbf{T}}(\mathcal{G}_0)$.

Before proving this lemma, we give the *proof of Proposition* 8.3. As in (9.1) define the map $\Omega\colon E_{3+1,4} \to E^r_{4+1,4}$ by

$$\Omega\colon (\mathcal{A}_1, \mathcal{B}_1, \mathcal{A}_2, \mathcal{B}_2) \mapsto \sum_{j=1}^{2} (\mathcal{A}_j\Omega_j + \mathcal{B}_j\Pi_j) \tag{11.3}$$

where Ω_j and Π_j are the generators of Proposition 10.2. On the ground of remarks at the end of Section 9, Proposition 8.3 is a straightforward consequence to the following

PROPOSITION 11.2. – (i) $\mathcal{G}(z, \delta, \alpha, \gamma)$ given by (8.1) is a universal unfolding of $\mathcal{G}_0(z, \delta)$ defined in (11.1*b*).

(ii) F_0, defined in (11.1*a*), is Γ-equivalent to $\mathcal{G}_0(z, \delta)$.

(iii) Let $(K^\#, Z^\#, \Delta^\#)$ be a Γ-equivalence between F_0 and $\mathcal{G}_0$, then $F^\#$, defined by

$$F^\#(z, \delta, \tau) = K^\#\big(F(Z^\#(z, \delta), \lambda_* + \Delta^\#(\delta), \tau), z, \delta\big) , \tag{11.4}$$

factors through $\mathcal{G}$ with factoring map

$$T(\tau) = \big(\alpha_0(\tau), \alpha_2(\tau), \gamma_1(\tau), \gamma_3(\tau)\big)$$

where

$$D_\tau\alpha_0(\tau_*) = \frac{1}{c_3(\tau_*)}[c_3(\tau_*)\, D_\tau a_0(\tau_*) - a_3(\tau_*)\, D_\tau c_0(\tau_*)] . \tag{11.5}$$

PROOF. – (i) From (7.7), (7.10) and Lemma 11.1 it follows easily that

$$E_{3+1,4} = \{(1, 0, 0, 0), (\sigma_2, 0, 0, 0), (0, 0, \sigma_1, 0), (0, 0, \delta, 0)\} \oplus_{\mathbf{R}} \mathbf{T}(\mathcal{G}_0) . \tag{11.6}$$

Thus point (i) is consequence of Theorem 9.10.

(ii) We have that $F_0 - \mathcal{G}_0 \in \Omega(V)$, where Ω and V are given by (11.3) and (11.2) respectively. Moreover, by Lemma 11.1 (i), $\tilde{\mathbb{T}}(F_0) = \tilde{\mathbb{T}}(\mathcal{G}_0)$ does not depend on V. Thus point (ii) follows from Theorem 9.9.

(iii) Because F_0 and $\mathcal{G}_0$ agree modulo $\Omega(V)$, by expanding both sides of (10.10) one can show by a long but elementary computation that

$$K^{\#}(\chi, z, \delta) = \chi + O(|z|\cdot|\chi|, \delta|\chi|) \quad \text{and} \quad Z^{\#}(z, \delta) = z + O(|z|^2, \delta|z|)$$

where $\chi \in \mathbf{C}^2$. Whence we have

$$F^{\#}(z, \delta, \tau) = \begin{bmatrix} a_0(\tau) z_1 \\ c_0(\tau) z_2 \end{bmatrix} + O(|z|^2, \delta|z|) .$$

Then from Lemma 11.1, (11.6) and Proposition 9.11 we obtain

$$(D_\tau a_0(\tau_*), 0, D\ \ c_0(\tau_*), 0) = (D_\tau \alpha_0(\tau_*), 0, 0, 0) - \frac{D_\tau c_0(\tau_*)}{c_3(\tau_*)} (a_3(\tau_*), 0, c_3(\tau_*), 0)$$

whence (11.5) follows immediately. ∎

PROOF OF LEMMA 11.1. – Following the same lines of the proof of Proposition 10.1, one easily sees that under assumptions (7.7) the module $\mathcal{K}^{\Gamma}_{4+1,4}$ is generated over $\mathcal{E}^{\Gamma}_{4+1}$ by:

$$(11.7) \quad \begin{cases} r_1 = \begin{bmatrix} \chi_1 \\ 0 \end{bmatrix}, \; r_2 = \begin{bmatrix} z_1^{\hat{n}} \bar{z}_2^{\hat{m}} \chi_1 \\ 0 \end{bmatrix}, \; r_3 = \begin{bmatrix} z_1^2 \bar{\chi}_1 \\ 0 \end{bmatrix}, \; r_4 = \begin{bmatrix} \bar{z}_1^{\hat{n}-2} z_2^{\hat{m}} \bar{\chi}_1 \\ 0 \end{bmatrix}, \; r_5 = \begin{bmatrix} z_1 \bar{z}_2 \chi_2 \\ 0 \end{bmatrix}, \\ r_6 = \begin{bmatrix} \bar{z}_1^{\hat{n}-1} z_2^{\hat{m}-1} \chi_2 \\ 0 \end{bmatrix}, \; r_7 = \begin{bmatrix} z_1^{\hat{n}+1} \bar{z}_2^{\hat{m}-1} \bar{\chi}_2 \\ 0 \end{bmatrix}, \; r_8 = \begin{bmatrix} z_1 z_2 \bar{\chi}_2 \\ 0 \end{bmatrix}, \; r_9 = \begin{bmatrix} \bar{z}_1^{\hat{n}-1} z_2^{\hat{m}+1} \bar{\chi}_2 \\ 0 \end{bmatrix}, \end{cases}$$

$$(11.8) \quad \begin{cases} s_1 = \begin{bmatrix} 0 \\ \chi_2 \end{bmatrix}, \; s_2 = \begin{bmatrix} 0 \\ \bar{z}_1^{\hat{n}} z_2^{\hat{m}} \chi_2 \end{bmatrix}, \; s_3 = \begin{bmatrix} 0 \\ z_2^2 \bar{\chi}_2 \end{bmatrix}, \; s_4 = \begin{bmatrix} 0 \\ z_1^{\hat{n}} \bar{z}_2^{\hat{m}-2} \bar{\chi}_2 \end{bmatrix}, \; s_5 = \begin{bmatrix} 0 \\ \bar{z}_1 z_2 \chi_1 \end{bmatrix}, \\ s_6 = \begin{bmatrix} 0 \\ z_1^{\hat{n}-1} \bar{z}_2^{\hat{m}-1} \chi_1 \end{bmatrix}, \; s_7 = \begin{bmatrix} 0 \\ \bar{z}_1^{\hat{n}-1} z_2^{\hat{m}+1} \bar{\chi}_1 \end{bmatrix}, \; s_8 = \begin{bmatrix} 0 \\ z_1 z_2 \bar{\chi}_1 \end{bmatrix}, \; s_9 = \begin{bmatrix} 0 \\ z_1^{\hat{n}+1} \bar{z}_2^{\hat{m}-1} \bar{\chi}_1 \end{bmatrix}, \end{cases}$$

where $\chi = (\chi_1, \chi_2) \in \mathbf{C}^2$ and $z = (z_1, z_2) \in \mathbf{C}^2$.

Employing complex notation, it is easy to see that the reduced tangent space (9.2) is given by

$$\tilde{T}_\Gamma(F_0) = \text{submodule of } E^{\Gamma}_{4+1,4} \text{ generated by } \delta F_0 \cdot \Omega_j,\ \delta F_0 \cdot \Omega_j,\ r_l(F_0(z, \delta), z),$$
$$s_l(F_0(z, \delta), z), \text{ for } j = 1, 2 \text{ and } l = 1, \ldots, 9,$$

where δF_0 acts on a germ $H \in E^{\Gamma}_{4+1,4}$ as

$$\delta F_0 \cdot H = \begin{bmatrix} F_{01,z_1} H_1 + F_{01,\bar z_1}\bar H_1 + F_{01,z_2} H_2 + F_{01,\bar z_2}\bar H_2 \\ F_{02,z_1} H_1 + F_{02,\bar z_1}\bar H_1 + F_{02,z_2} H_2 + F_{02,\bar z_2}\bar H_2 \end{bmatrix} \tag{11.9}$$

where F_{0j} and H_j are the (complex) components of F_0 and H respectively (note that $F_0, H\colon \mathbb{C}^2 \times \mathbb{R} \to \mathbb{C}^2$) and a subscript after a comma represents partial differentiation. Set

$$\mathfrak{P}_j(\sigma, \eta, \delta) = P_j(\sigma, \eta, \delta, \tau_*) \quad \text{and} \quad \mathfrak{Q}_j(\sigma, \eta, \delta) = Q_j(\sigma, \eta, \delta, \tau_*)$$

where P_j and Q_j are given by (7.4). Then we have the following

LEMMA 11.3. – Under assumptions (7.7) we have that $\widetilde{\mathbb{T}}(F_0)$ is generated by

$$K_1 = \big(\mathfrak{P}_1 + 2\sigma_1 \mathfrak{P}_{1,\sigma_1},\ (\hat n - 1)\mathfrak{Q}_1 + 2\sigma_1 \mathfrak{Q}_{1,\sigma_1},\ 2\sigma_1 \mathfrak{P}_{2,\sigma_1},\ \hat n Q_2 + 2\sigma_1 \mathfrak{Q}_{2,\sigma_1}\big)\,,$$

$$K_2 = \big(\sigma_1^{\hat n - 1}\sigma_2^{\hat m}\big((\hat n - 1)\mathfrak{Q}_1 + (\sigma_1 - 1)\mathfrak{Q}_{1,\sigma_1}\big) + \eta \mathfrak{P}_{1,\sigma_1},\ \mathfrak{P}_1 + \eta \mathfrak{Q}_{1,\sigma_1},\ \eta \mathfrak{P}_{2,\sigma_1} + {} + \hat n \sigma_1^{\hat n - 1}\sigma_2^{\hat m - 1}\mathfrak{Q}_2,\ \eta \mathfrak{Q}_{2,\sigma_1}\big)\,,$$

$$K_3 = \big(2\sigma_2 \mathfrak{P}_{1,\sigma_2},\ \hat m \mathfrak{Q}_1 + 2\sigma_2 \mathfrak{Q}_{1,\sigma_2},\ \mathfrak{P}_2 + 2\sigma_2 \mathfrak{P}_{2,\sigma_2},\ (\hat m - 1)\mathfrak{Q}_2 + 2\sigma_2 \mathfrak{Q}_{2,\sigma_2}\big)\,,$$

$$K_4 = \big(\eta \mathfrak{P}_{1,\sigma_2} + \hat m \sigma_1^{\hat n - 1}\sigma_2^{\hat m - 1}\mathfrak{Q}_1,\ \eta \mathfrak{Q}_{1,\sigma_2},\ \sigma_1^{\hat n}\sigma_2^{\hat m - 1}\big((\hat m - 1)\mathfrak{Q}_2 + (\sigma_2 - 1)\mathfrak{Q}_{2,\sigma_2}\big) + {} + \eta \mathfrak{P}_{2,\sigma_2},\ \mathfrak{P}_2 + \eta \mathfrak{Q}_{2,\sigma_2}\big)\,,$$

$$R_1 = (\mathfrak{P}_1, \mathfrak{Q}_1, 0, 0)\,,$$

$$R_2 = (\eta \mathfrak{P}_1 + \sigma_1^{\hat n - 1}\sigma_2^{\hat m}\mathfrak{Q}_1,\ -\sigma_1 \mathfrak{P}_1, 0, 0)\,,$$

$$R_3 = (\sigma_1 \mathfrak{P}_1 + \eta \mathfrak{Q}_1,\ -\sigma_1 \mathfrak{Q}_1, 0, 0)\,,$$

$$R_4 = (\sigma_1^{\hat n - 2}\sigma_2^{\hat m}\mathfrak{Q}_1,\ \mathfrak{P}_1, 0, 0)\,,$$

$$R_5 = (\sigma_2 \mathfrak{P}_2 + \eta \mathfrak{Q}_2,\ -\sigma_1 \mathfrak{Q}_2, 0, 0)\,,$$

$$R_6 = (\sigma_1^{\hat n - 1}\sigma_2^{\hat m - 1}\mathfrak{Q}_2,\ \mathfrak{P}_2, 0, 0)\,,$$

$$R_7 = (\eta \mathfrak{P}_2 + \sigma_1^{\hat n}\sigma_2^{\hat m - 1}\mathfrak{Q}_2,\ -\sigma_1 \mathfrak{P}_2, 0, 0)\,,$$

$$R_8 = (\sigma_2 \mathfrak{P}_2,\ \sigma_1 \mathfrak{Q}_2, 0, 0)\,,$$

$$R_9 = (-\sigma_1^{\hat n - 1}\sigma_2^{\hat m}\mathfrak{Q}_2,\ \sigma_2 \mathfrak{P}_2 + \eta \mathfrak{Q}_2, 0, 0)\,,$$

$$S_1 = (0, 0, \mathfrak{P}_2, \mathfrak{Q}_2)\,,$$

$$S_2 = (0, 0, \eta \mathfrak{P}_2 + \sigma_1^{\hat n}\sigma_2^{\hat m - 1}\mathfrak{Q}_2,\ -\sigma_2 \mathfrak{P}_2)\,,$$

$$S_3 = (0, 0, \sigma_2 \mathfrak{P}_2 + \eta \mathfrak{Q}_2,\ -\sigma_2 \mathfrak{Q}_2)\,,$$

$$S_4 = (0, 0, \sigma_1^{\hat n}\sigma_2^{\hat m - 2}\mathfrak{Q}_2,\ \mathfrak{P}_2)\,,$$

$$S_5 = (0, 0, \sigma_1 \mathfrak{P}_1 + \eta \mathfrak{Q}_1,\ -\sigma_2 \mathfrak{Q}_1)\,,$$

$$S_6 = (0,\, 0,\, \sigma_1^{\hat{n}-1}\sigma_2^{\hat{m}-1}\mathfrak{Q}_1,\, \mathfrak{T}_1)\,,$$

$$S_7 = (0,\, 0,\, \eta\mathfrak{T}_1 + \sigma_1^{\hat{n}-1}\sigma_2^{\hat{m}}\mathfrak{Q}_1,\, -\sigma_2\mathfrak{T}_1)\,,$$

$$S_8 = (0,\, 0,\, \sigma_1\mathfrak{T}_1,\, \sigma_2\mathfrak{Q}_1)\,,$$

$$S_9 = (0,\, 0,\, -\sigma_1^{\hat{n}}\sigma_2^{\hat{m}-1}\mathfrak{Q}_1,\, \sigma_1\mathfrak{T}_1 + \eta\mathfrak{Q}_1)\,,$$

where, as usual, a subscript after a comma represents partial differentiation.

PROOF. – From definition (9.4) it follows that the computation of the generators of $\widetilde{\mathrm{T}}(F_0)$ consists in substituting

$$F_0 = \sum_{j=1}^{2} (\mathfrak{T}_j\Omega_j + \mathfrak{Q}_j\Pi_j) \tag{11.10}$$

into the generators of $\widetilde{\mathrm{T}}_\Gamma(F_0)$, computing their components with respect to Ω_j and Π_j and choosing a suitable inverse-image of each component through the map Ω defined in (11.3). This computation is long, but it does not present difficulties. As an example we show how to compute $K_2 \in \Omega^{-1}(\delta F_0 \cdot \Omega_2)$ and $R_2 \in \Omega^{-1}(r_2(F_0, z))$.

From Proposition 10.2 (i) and (11.9) we have

$$\delta F_0 \cdot \Omega_2 = \begin{bmatrix} F_{01,z_1}\bar{z}_1^{\hat{n}-1}z_2^{\hat{m}} + F_{01,\bar{z}_1}z_1^{\hat{n}-1}\bar{z}_2^{\hat{m}} \\ F_{02,z_1}\bar{z}_1^{\hat{n}-1}z_2^{\hat{m}} + F_{02,\bar{z}_1}z_1^{\hat{n}-1}\bar{z}_2^{\hat{m}} \end{bmatrix} \tag{11.11}$$

hence from (11.10) we obtain

$$\begin{aligned} F_{01,z_1}\bar{z}_1^{\hat{n}-1}z_2^{\hat{m}} + F_{0.1\bar{z}_1}z_1^{\hat{n}-1}\bar{z}_2^{\hat{m}} &= (\mathfrak{T}_1 + \sigma_1\mathfrak{T}_{1,\sigma_1} + \bar{z}_1^{\hat{n}}z_2^{\hat{m}}\mathfrak{Q}_{1,\sigma_1})\bar{z}_1^{\hat{n}-1}z_2^{\hat{m}} + \\ &\quad + \big(z_1^2\mathfrak{T}_{1,\sigma_1} + \sigma_1\bar{z}_1^{\hat{n}-2}z_2^{\hat{m}}\mathfrak{Q}_{1,\sigma_1} + (\hat{n}-1)\bar{z}_1^{\hat{n}-2}z_2^{\hat{m}}\mathfrak{Q}_1\big)z_1^{\hat{n}-1}\bar{z}_2^{\hat{m}} = \\ &= [\sigma_1^{\hat{n}-1}\sigma_2^{\hat{m}}\big((\hat{n}-1)\mathfrak{Q}_1 - \mathfrak{Q}_{1,\sigma_1} + \sigma_1\mathfrak{Q}_{1,\sigma_1}\big) + \eta\mathfrak{T}_{1,\sigma_1}]z_1 + (\mathfrak{T}_1 + \eta\mathfrak{Q}_{1,\sigma_1})\bar{z}_1^{n-1}z_2^{\hat{m}}\,. \end{aligned}$$

In the same way one computes the second component of (11.11) and obtains K_2. As regards R_2, from Proposition 10.2, (11.7) and (11.10) we have

$$r_2(F_0, z) = \begin{bmatrix} z_1^{\hat{n}}\bar{z}_2^{\hat{m}}(\mathfrak{T}_1 z_1 + \mathfrak{Q}_1\bar{z}_1^{\hat{n}-1}z_2^{\hat{m}}) \\ 0 \end{bmatrix} = \begin{bmatrix} \mathfrak{T}_1\cdot(\eta z_1 - \sigma_1\bar{z}_1^{\hat{n}-1}z_2^{\hat{m}}) + \sigma_1^{\hat{n}-1}\sigma_2^{\hat{m}}\mathfrak{Q}_1 z_1 \\ 0 \end{bmatrix},$$

whence one gets R_2. ■

Now we show that $V \subset \widetilde{\mathrm{T}}(F_0)$, where V is the submodule (11.2). By Nakayama's Lemma ([7], Proposition 2.6, page 102) it suffices to show that

$$V \subset \widetilde{\mathrm{T}}(F_0) + \mathcal{M}_{3+1}V\,. \tag{11.12}$$

To this purpose, substitute the expansion (7.8) with $\tau = \tau_*$ into the generators of $\tilde{T}(F_0)$ given in Lemma 11.3 and consider modulo $\mathcal{M}_{3+1}V$ the following elements of $E_{3+1,4}$:

$$\text{(11.13)} \qquad \begin{cases} \sigma_1 K_1,\ \sigma_2 K_1,\ K_2,\ \sigma_1 K_3,\ \sigma_2 K_3,\ K_4\,, \\ \sigma_1 R_1,\ \sigma_2 R_1,\ \delta R_1,\ R_3,\ R_4,\ R_6,\ R_8,\ R_9\,, \\ \sigma_1 S_1,\ \sigma_2 S_1,\ \delta S_1,\ S_3,\ S_4,\ S_6,\ S_8,\ S_9\,. \end{cases}$$

Then, by using (7.7), (7.9), (7.10) and (7.11) one verifies that (11.13) span V, whence (11.12) follows.

Now consider the generators of $\tilde{T}(F_0)$ modulo V. By resorting again to (7.8) with $\tau = \tau_*$, it is easy to see that there are only four generators which are linearly independent over R module V:

$$K_1 \equiv \big(3a_1(\tau_*)\sigma_1 + a_2(\tau_*)\sigma_2 + a_3(\tau_*)\delta,\ (\hat{n}-1)b_0(\tau_*),\ 2c_1(\tau_*)\sigma_1,\ \hat{n}d_0(\tau_*)\big) \bmod V\,,$$

$$K_3 \equiv \big(2a_2(\tau_*)\sigma_2,\ \hat{m}b_0(\tau_*),\ c_1(\tau_*)\sigma_1,\ c_1(\tau_*)\sigma_1 + 3c_2(\tau_*)\sigma_2 + c_3(\tau_*)\delta,\ (\hat{m}-1)d_0(\tau_*)\big) \bmod V,$$

$$R_1 \equiv \big(a_1(\tau_*)\sigma_1 + a_2(\tau_*)\sigma_2 + a_3(\tau_*)\delta,\ b_0(\tau_*),\ 0,\ 0\big) \bmod V\,,$$

$$S_1 \equiv \big(0,\ 0,\ c_1(\tau_*)\sigma_1 + c_2(\tau_*)\sigma_2 + c_3(\tau_*)\delta,\ d_0(\tau_*)\big) \bmod V\,,$$

whence one computes $\tilde{T}(F_0)$.

Finally from (7.8) and (8.1) we have that F_0 and $\mathcal{G}_0$ coincide modulo $\Omega(V)$, whence we have $\tilde{T}(\mathcal{G}_0) = \tilde{T}(F_0)$ and the proof of point (i) is complete.

It remains to compute $T(\mathcal{G}_0)$. From (9.3) and (9.4) we have that it suffices to observe that by (8.1) and (11.7*b*)

$$D_\delta \mathcal{G}_0 = a_3(\tau_*)\Omega_1 + c_3(\tau_*)\Omega_2$$

and

$$\delta D_\delta \mathcal{G}_0 = a_3(\tau_*)\delta\Omega_1 + c_3(\tau_*)\delta\Omega_2\,. \quad \blacksquare$$

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