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BIFURCATIONS AND STABILITY OF FAMILIES OF DIFFEOMORPHISMS

by S. NEWHOUSE, J. PALIS *and* F. TAKENS

Abstract

We consider one parameter families or arcs of diffeomorphisms. For families starting with Morse-Smale diffeomorphisms we characterize various types of (structural) stability at or near the first bifurcation point. We also give a complete description of the stable arcs of diffeomorphisms whose limit sets consist of finitely many orbits. Universal models for the local unfoldings of the bifurcating periodic orbits (especially saddle-nodes) are established, as well as several results on the global dynamical structure of the bifurcating diffeomorphisms. Moduli of stability related to saddle-connections are introduced.

CONTENTS

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I. — INTRODUCTION

The study of the geometric structure of the orbits of dynamical systems (differential equations, flows, vector fields, or diffeomorphisms) defined on a manifold has been considered in many works since Poincaré and Liapunov. Two diffeomorphisms f and f' are said to have the same geometric structure if they are topologically conjugate, i.e. if there is a homeomorphism h from the domain of f to that of f' such that $hf = f'h$. Two flows or vector fields, are called topologically equivalent if there is a homeomorphism sending orbits of one system to orbits of the other; if, in addition, the homeomorphism preserves the flow parameter, we again say the systems are conjugate. In general terms, we aim at the classification of dynamical systems under conjugacy or topological equivalence. Since, however, much pathological behavior can occur, we must restrict ourselves to interesting special classes of systems. We shall be concerned here with systems having only mild recurrence; in particular, we shall frequently assume that their limit sets consist of only finitely many orbits.

The space of differentiable systems of class C^r, $r \geq 1$, has a natural topology given by uniform convergence of the first r derivatives. This is called the C^r topology. Given any equivalence relation E on the set of dynamical systems, one can define systems to be E-*stable* if they lie in the interior of their E-equivalence classes. When topological equivalence is used for E, an E-stable system is called structurally stable (or just stable). The stable diffeomorphisms and vector fields whose limit sets have finitely many orbits coincide with the Morse-Smale ones [24]. In fact, in this case the Birkhoff center [15] is finite because non-trivial recurrence implies uncountably many orbits in the limit set. Since one understands the structure of Morse-Smale systems pretty well, it is natural to consider one-parameter families of systems starting at a Morse-Smale one and to attempt to describe the structure of the elements of such families. In the present work we shall define three natural equivalence relations on these families, and we shall characterize their stable families in terms of geometric properties. In particular, we will characterize the stable one parameter families of diffeomorphisms whose elements have only finitely many orbits in their limit sets. This corresponds to the characterization of the Morse-Smale diffeomorphisms as the ones which are stable and have finitely many orbits in their limit sets. The results can, of course, be translated to certain classes of vector fields (those with global cross-sections). In [26*a*] related results are obtained for families of gradient vector fields.

We first present a preliminary description of our main results. Later, we shall give their precise statements.

In [19], [20], one studied how a one-parameter family of diffeomorphisms starting at a Morse-Smale one ceases to be stable (i.e. goes through a bifurcation) when the parameter evolves. For a generic family, the description given there is complete assuming that the diffeomorphism at the first bifurcation point has its limit set made of a finite number of orbits. It turns out that these orbits are all periodic except at most one. If the periodic orbits are all hyperbolic, then their stable and unstable manifolds meet transversally except along one orbit. In case one of the periodic orbits is not hyperbolic, this orbit must be an elementary bifurcation (a saddle-node, flip, or Hopf orbit), the other periodic orbits must be hyperbolic, and all stable and unstable manifolds meet transversally. The orbit structure of the diffeomorphism at the first bifurcation point will be basic for our results on the stability of these arcs.

Throughout this paper, M denotes a compact C^∞ manifold without boundary, and Diff(M) denotes the set of C^∞ diffeomorphisms of M. We let $\mathscr{P} = \mathscr{P}(M)$ be the space of C^∞ arcs of diffeomorphisms on M. That is, if I is the unit interval, then $\mathscr{P}(M)$ consists of C^∞ mappings $\Phi : M \times I \to M \times I$, such that $\Phi(m, \mu) = (\varphi_\mu(m), \mu)$ where $m \mapsto \varphi_\mu(m)$ is a C^∞ diffeomorphism for each $\mu \in I$. Elements of $\mathscr{P}$ will also be called one-parameter families or arcs of diffeomorphisms and will frequently be denoted by $\{\varphi_\mu\}$ or φ. We give Diff(M) and $\mathscr{P}$ the usual C^∞ topologies.

Let us consider three equivalence relations on the set $\mathscr{P}$ of one-parameter families of diffeomorphisms. We say two families are *topologically conjugate* if, modulo an orientation preserving homeomorphism of the interval I, each element of the first family is topologically conjugate to an element of the second family, and the conjugacy varies continuously with the parameter. If conjugacies exist but do not necessarily vary continuously, we say the families are *mildly conjugate.* Finally, if the elements of the families are topologically conjugate up to and including their first bifurcation points, we say they are *left conjugate.* The interiors of the equivalence classes of the preceding equivalence relations define, respectively, *stable*, *mildly stable*, and *left stable* arcs of diffeomorphisms.

For arcs $\{\varphi_\mu\}$ such that the limit set of φ_μ consists of finitely many orbits for each μ, we will give necessary and sufficient conditions for stability, and we will geometrically characterize left stability. For mild stability, our characterization is complete except for one case which we present as an interesting open question. In all cases, we exhibit necessary and sufficient conditions in terms of the orbit structures at the bifurcation points of the arcs. For arcs beginning at Morse-Smale diffeomorphisms we will first study the different types of stability for an interval in I containing the first bifurcation point in its interior.

Let us describe the results. To begin with, if, at the first bifurcation point, some stable and unstable manifolds meet non-transversally, then the arc is not even left stable. In fact, in this case we have at least a one-parameter family of different equivalence classes of arcs near the initial one. This corresponds to the existence of the modulus of stability as discussed in [27], [28]. In the other possible cases, when there is a saddle-node, a flip or a Hopf periodic orbit, the arc is left stable. When the arc goes through a Hopf orbit it is never mildly stable; this is due to the appearance of invariant circles with irrational rotations. On the other hand, we always get stability for arcs going through flip orbits. The case of a saddle-node orbit deserves a special discussion. First of all, it is much harder than in the flip case to prove the existence up to conjugacy of a universal model for its local unfolding. This is done in Chapter II. On the other hand, the restriction of such a conjugacy to the center manifold is surprisingly rigid. We discuss several applications of this fact in section 5 of Chapter III. Moreover, the strong stable and strong unstable foliations of the stable and unstable manifolds must be preserved by a conjugacy between two arcs going through saddle-nodes. Thus, criticallity or tangency of the invariant (stable, unstable) manifolds of other periodic orbits with respect to one of these foliations implies that the arc is not stable. Criticallity with respect to both foliations implies that the arc is not even mildly stable. Another crucial factor is the existence of cycles for the periodic orbits. When the arc goes through a saddle-node which is critical but not bicritical and has no cycles, then this arc is mildly stable but not stable. If there is a cycle, the arc is not stable and, we believe, it is not even mildly stable. We are able to prove this last statement for cycles of length bigger than one and for one-cycles when the saddle-node is normally attracting or repelling. This follows from the appearance of a non-transversal homoclinic orbit, which implies a non zero modulus of stability. To prove the existence of such a homoclinic orbit, we reduce the question to one-parameter families of endomorphisms of the circle and introduce a generalized notion of rotation number. Necessary conditions for the types of stability mentioned above are also established in Chapter III. The proof that, under these conditions, the arcs are stable, mildly or left stable is performed in the last chapter. There we use a suitable version of tubular families or foliations, some of them with singularities. Our constructions also provide a more elegant proof of the stability of Morse-Smale diffeomorphisms originally established in [24], [25]. Stability for arcs containing saddle-nodes was generalized by Robinson [31] to certain families starting at Axiom A diffeomorphisms.

Many of these results were announced in [21]; however, we mistakenly claimed to have characterized mild stability. As we mentioned above, it remains to prove that certain *l*-cycle cases are not mildly stable.

Let us now review some definitions and be more precise.

For $g \in \mathrm{Diff}(M)$, the *orbit* $\mathcal{O}(x)$, of a point $x \in M$, is defined as $\mathcal{O}(x) = \{g^n(x) \mid n \in \mathbf{Z}\}$. A point $y \in M$ is called a *limit point* of g if for some sequence $n_i \in \mathbf{Z}$ with $|n_i| \to \infty$, $\lim g^{n_i}(x) = y$. We denote by $L(g)$ the closure of the set of these limit points. A point $x \in M$ is a *periodic point* of g with period k if $g^k(x) = x$ and $g^\ell(x) \neq x$ for all $0 < \ell < k$; x is *hyperbolic* if $dg^k(x)$ has no eigenvalues on the unit circle. The stable, unstable sets or manifolds $W^s(x, g)$, $W^u(x, g)$, of a periodic point x are defined as

$$\{y \in M \mid \rho(g^n(y), g^n(x)) \to 0 \text{ for } n \to +\infty\}$$

and

$$\{y \in M \mid \rho(g^n(y), g^n(x)) \to 0 \text{ for } n \to -\infty\},$$

respectively, where ρ is a metric on M. If x is a hyperbolic periodic point of g, $W^s(x, g)$ and $W^u(x, g)$ are injectively immersed sub-manifolds of M. We say that a diffeomorphism $g \in \text{Diff}(M)$ with finitely many periodic orbits has a *k-cycle* if there is a sequence of periodic orbits $\mathscr{O}(p_0), \ldots, \mathscr{O}(p_k)$ with $\mathscr{O}(p_0) = \mathscr{O}(p_k)$ and $\mathscr{O}(p_{i+1}) \subset \text{closure}\,(W^u(\mathscr{O}(p_i)))$ for $0 \leq i < k$, and $\mathscr{O}(p_i) \neq \mathscr{O}(p_j)$ for $0 \leq i < j < k$, and if this sequence is maximal.

A diffeomorphism g is *Morse-Smale* if $L(g)$ is finite and hyperbolic and if all the intersections of stable and unstable manifolds are transverse. We denote the set of Morse-Smale diffeomorphisms by MS. This set is open [24] and each $g \in \text{MS}$ is stable in the sense that any g' which is C^1 near g is conjugate to g [24], [25], i.e. there is a homeomorphism $h: M \to M$ so that $g'h = hg$.

For an arc $\{\varphi_\mu\} \in \mathscr{P}$ with $\varphi_0 \in \text{MS}$, let $b = b(\varphi) = \inf\{\mu \in I \mid \varphi_\mu \notin \text{MS}\}$. We always assume that $b(\varphi) < 1$. If $\{\varphi_\mu\}, \{\varphi'_\mu\} \in \mathscr{P}$, then we say that $(h, \{H_\mu\})$ is a *conjugacy* if $h: [0, 1] \to [0, 1]$ is a homeomorphism with $h(b(\varphi)) = b(\varphi')$, $H_\mu: M \to M$ is a conjugacy between φ_μ and $\varphi'_{h(\mu)}$ for all μ in some neighborhood of $[0, b(\varphi)]$, and H_μ depends continuously on μ. If H_μ does not necessarily depend continuously on μ, we say that $(h, \{H_\mu\})$ is a *mild conjugacy*, and if H_μ is only a conjugacy for $\mu \leq b(\varphi)$, not necessarily continuous in μ, then $(h, \{H_\mu\})$ is called a *left conjugacy*. Conjugacy, mild conjugacy and left conjugacy define equivalence relations in the set of those arcs in $\mathscr{P}$ which start in MS. An arc $\{\varphi_\mu\} \in \mathscr{P}$ is called stable, mildly stable or left stable if is an interior point of its corresponding equivalence class.

Now we come to the description of the class of arcs to which our results apply.

Definition. — $\mathscr{A} \subset \mathscr{P}$ is the subset of those $\{\varphi_\mu\} \in \mathscr{P}$ such that

1. $\varphi_0 \in \text{MS}$;
2. $b = b(\varphi) = \inf\{\mu \in [0, 1] \mid \varphi_\mu \notin \text{MS}\} < 1$;
3. the limit set of φ_b has finitely many orbits.

For the arcs of diffeomorphisms in $\mathscr{A}$ it is often useful to impose certain generic conditions on the first bifurcation. In order to describe them we need some more conditions.

Let x be a fixed point of a diffeomorphism $g \in \text{Diff}(M)$. We call x *quasi-hyperbolic* if one of the following three conditions holds:

— $(dg)_x$ has one eigenvalue one, the other eigenvalues have norm different from 1 and there is a g-invariant curve α through x such that $g \mid \alpha$ has first but not second order contact with the identity at x; in this case, x is a saddle-node;
— $(dg)_x$ has one eigenvalue -1, the other eigenvalues have norm different from 1 and there is a g-invariant curve α through x such that $g^2 \mid \alpha$ has second but not third order contact with the identity at x; in this case, x is a flip.
— $(dg)_x$ has a pair $\lambda \neq \bar{\lambda}$ of eigenvalues on the unit circle, the other eigenvalues have norm different from 1 and there is a g-invariant surface α through x, tangent to the

generalized eigenspace of the pair λ, $\bar{\lambda}$ at x and such that the 3-jet of g at x makes $g \mid \alpha$ an attractor or a repeller; in this case, x is a Hopf point.

For periodic orbits there is an analogous definition of quasi-hyperbolicity.

When $\{\varphi_\mu\} \in \mathscr{A}$, we say that $\{\varphi_\mu\}$ is *elementary* at its first bifurcation value b (or φ_b is elementary) if it fulfills one of the two following conditions: (1) all periodic points of φ_b are hyperbolic, there is one orbit of non-transversal intersections of a stable and an unstable manifold, and all other intersections of stable and unstable manifolds are transversal; or (2) there is one quasi-hyperbolic orbit, the other periodic orbits are hyperbolic and all intersections of stable and unstable manifolds are transverse.

In case x is a flip or saddle-node of φ_b we also require stable and unstable manifolds to be transversal to the strong stable and unstable manifold of x; the strong stable (resp. unstable) manifold consists of the points y such that the distance from x to $\varphi_b^i(y)$ (resp. $\varphi_b^{-i}(y)$) goes exponentially to zero (see also Chapter IV).

If $\{\varphi_\mu\} \in \mathscr{A}$, φ_b is elementary, and φ_b has a quasi-hyperbolic periodic orbit, then there are generic conditions one may impose on the dependence upon μ at these quasi-hyperbolic periodic points. Such conditions are described in Chapter II, § 3 (for the saddle-node), § 4 (for the flip) and § 5 (for the Hopf orbit). If these conditions are satisfied we say that the quasi-hyperbolic orbit *unfolds generically.*

Definition. — $\mathscr{B} \subset \mathscr{A}$ is defined to be the set of those arcs $\{\varphi_\mu\}$ in $\mathscr{A}$ for which φ_b is elementary and for which the quasi-hyperbolic periodic point of φ_b, if there is any, unfolds generically.

It can be shown [4], [19], [20], [36] that there is a residual subset $\mathscr{B}'$ in $\mathscr{P}$ such that $\operatorname{int}(\mathscr{B}) = \mathscr{A} \cap \mathscr{B}'$. We want to give necessary and sufficient conditions for arcs in $\mathscr{B}$ to be stable, mildly stable or left stable. For this we need to describe the notion of criticallity.

Let $g \in \operatorname{Diff}(M)$ and let x be a saddle-node of g. Then there is a unique foliation $\mathscr{F}^{ss}$ of $W^s(x, g)$ with smooth leaves such that the boundary of $W^s(x, g)$ is a leaf and such that g maps leaves to leaves; see [12]. $\mathscr{F}^{ss}$ is called the strong stable foliation. A corresponding foliation of $W^u(x, g)$, the strong unstable foliation, is denoted by $\mathscr{F}^{uu}$. We call x *s-critical* if there is some hyperbolic periodic point y of g such that $W^u(y, g)$ intersects some leaf of $\mathscr{F}^{ss}$ non-transversally; *u-critical* is defined similarly. Now, x is called *semi-critical* if it is either s- or u-critical, x is called *bi-critical* if it is both s- and u-critical, and x is called *non-critical* if it is not semi-critical.

Theorem. — *Let* $\{\varphi_\mu\}$ *be an arc in* $\mathscr{B}$.

1. $\{\varphi_\mu\}$ *is left stable if and only if all stable and unstable manifolds of* φ_b *intersect transversally;*
2. *if* $\{\varphi_\mu\}$ *is left stable, if the quasi-hyperbolic orbit is not a bi-critical saddle-node or a Hopf orbit and if* φ_b *has no cycles, then* $\{\varphi_\mu\}$ *is mildly stable;*

2'. *if* $\{\varphi_\mu\}$ *is mildly stable, then* $\{\varphi_\mu\}$ *is left stable, the quasi-hyperbolic orbit is not a bi-critical saddle-node or a Hopf orbit,* φ_b *has no cycles of length greater than* 1, *and* φ_b *has no non-critical* 1-*cycles*;

3. $\{\varphi_\mu\}$ *is stable if and only if* $\{\varphi_\mu\}$ *is left stable, the quasi-hyperbolic orbit of* φ_b *is not a semi-critical saddle-node or a Hopf orbit and* φ_b *has no cycles.*

This theorem, together with the remark at the end of section 2, Chapter III, and Theorem (4.4) in Chapter IV implies the following characterization of stable arcs of diffeomorphisms with limit sets consisting of finitely many orbits.

Theorem. — *Let* $\{\varphi_\mu\}$, $\mu \in [0, 1]$ *be an arc of diffeomorphisms such that the limit set of each* φ_μ *consists of finitely many orbits,* $\mu \in [0, 1]$. *Then* $\{\varphi_\mu\}$ *is stable if and only if there are only finitely many bifurcation values, say* $b_1, \ldots, b_s$, *in* $(0, 1)$ *and for each* $1 \leq i \leq s$, φ_{b_i} *has the following properties*:

— *all stable, strong stable, unstable, and strong unstable manifolds intersect transversally*;

— φ_{b_i} *has no cycles and has exactly one non-hyperbolic periodic orbit which is either a flip or a non-critical saddle-node*; *this non-hyperbolic periodic orbit unfolds generically.*

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II. — LOCAL DESCRIPTION OF THE ELEMENTARY BIFURCATIONS

1. Introduction

Let M be a smooth manifold. We shall consider smooth arcs of diffeomorphisms $\varphi_\mu : \mathrm{M} \to \mathrm{M}$, $\mu \in \mathrm{R}$, i.e. such that the corresponding map $\Phi : \mathrm{M} \times \mathrm{R} \to \mathrm{M} \times \mathrm{R}$, defined by $\Phi(x, \mu) = (\varphi_\mu(x), \mu)$ is C^∞. For such an arc, there usually are points $(x, \mu) \in \mathrm{M} \times \mathrm{R}$ where φ_μ does not satisfy the Kupka-Smale conditions [13], [33] along the orbit of φ_μ through x. In this chapter we shall analyze the behavior of Φ near those points. Such a point is called an *elementary bifurcation* (point) of Φ (or φ_μ).

There are two types of elementary bifurcation points, namely those (x, μ) for which x is a non-hyperbolic periodic point of φ_μ, and those (x, μ) where x is a non-transversal point of intersection of stable and unstable manifolds of φ_μ (see [36]). Before we go into details we describe some facts concerning center manifolds in relation to the periodic elementary bifurcation points. Then a description of the types of periodic elementary bifurcations occurring in generic arcs is given. In subsequent sections all these types are analyzed. The saddle-node elementary bifurcation shows some exceptional and unexpected topological properties. In the final section, non-transversal intersections of stable and unstable manifolds are treated.

2. Center manifolds and periodic elementary bifurcations

Let $\{\varphi_\mu : \mathrm{M} \to \mathrm{M}\}$ be a smooth arc of diffeomorphisms having a periodic bifurcation at $(\bar{x}, \bar{\mu})$. We assume $\bar{x}$ to be a fixed point of $\varphi_{\bar{\mu}}$; if not we replace $\varphi_{\bar{\mu}}$ by $\varphi_{\bar{\mu}}^k$ where k is the period of $\bar{x}$. Let c be the number of eigenvalues of $(d\varphi_{\bar{\mu}})_{\bar{x}}$ of norm 1; since $(\bar{x}, \bar{\mu})$ is a bifurcation, $c \geq 1$.

From the theory of invariant manifolds [12], we conclude the existence of a "center manifold depending on μ", namely a differentiable submanifold W^c of $\mathrm{M} \times \mathbf{R}$ such that:

- $(\bar{x}, \bar{\mu}) \in \mathrm{W}^c$;
- $\Phi(\mathrm{W}^c) \cap \mathrm{W}^c$ is open in W^c (and contains $(\bar{x}, \bar{\mu})$);
- the dimension of W^c is $c + 1$ and at each point $(x, \mu) \in \mathrm{W}^c$, W^c is transversal with respect to $\mathrm{M} \times \{\mu\}$;
- $d(\varphi_{\bar{\mu}} \mid \mathrm{W}^c \cap (\mathrm{M} \times \{\bar{\mu}\}))_{\bar{x}}$ has only eigenvalues of norm 1.

For any $k < \infty$ we may assume that W^c is C^k (but as k gets bigger, it may be necessary to take W^c smaller). However if in some neighborhood of $\bar{x}$ in $\mathrm{W}^c_{\bar{\mu}} = \mathrm{W}^c \cap (\mathrm{M} \times \{\bar{\mu}\})$, every orbit $\varphi^i_{\bar\mu}(x)$ tends to $\bar x$ for $i \to \pm\infty$, then W^c can be chosen so that $W^c_{\bar\mu}$ is C^∞; even in this last case, W^c can not be made C^∞.

We may consider $\Phi \mid W^c$ as a (local) arc of diffeomorphisms; the parameter being the restriction of μ to W^c. Also from the invariant manifold theory [26], [30] it follows that Φ near $(\bar x, \bar\mu)$, up to a " local conjugacy ", is completely determined by $\Phi \mid W^c$ and the normal data. The normal data consist of the numbers of eigenvalues of $(d\varphi_{\bar\mu})_{\bar x}$ with norm > 1, resp. < 1, and of the signs of the determinants of $(d\varphi_{\bar\mu})_{\bar x}$, restricted to the maximal invariant expanding, resp. contracting subspaces of $T_{\bar x}(M)$. We say that an arc $\Phi : M \times \mathbf{R} \to M \times \mathbf{R}$ is at $(\bar x, \bar\mu)$ *locally conjugate* to the arc $\bar\Phi : \bar M \times \mathbf{R} \to \bar M \times \mathbf{R}$ at (x, μ) if there is a homeomorphism (the local conjugacy) h from a neighborhood of $(\bar x, \bar\mu)$ to a neighborhood of (x, μ) such that

— $h \circ \Phi = \bar\Phi \circ h$ wherever defined;
— there is a local homeomorphism $h_{\mathbf{R}} : \mathbf{R} \to \mathbf{R}$, defined in a neighborhood of $\bar\mu$ such that the μ-coordinate of $h(x, \mu)$ equals $h_{\mathbf{R}}(\mu)$ wherever $h(x, \mu)$ is defined.

So in order to analyze Φ up to local conjugacy at $(\bar x, \bar\mu)$ it suffices to analyze $\Phi \mid W^c$ up to local conjugacy; this will be done in the following sections.

It should be pointed out that for Φ as above, one can choose invariant manifolds W^{cs} and W^{cu}, the center-stable and the center-unstable manifolds, for Φ, containing W^c such that the tangent space at $(\bar x, \bar\mu)$ is the direct sum of the tangent space of W^c at $(\bar x, \bar\mu)$ and the maximal invariant subspace of $T_{\bar x}(M)$ on which $(d\varphi_\mu)_{\bar x}$ has only eigenvalues with norm smaller, resp. bigger, than 1. These invariant manifolds are in general not unique. In W^{cs} and W^{cu} one can choose invariant continuous foliations $\mathcal{F}^{ss}$ and $\mathcal{F}^{uu}$, the strong stable and the strong unstable foliations, such that the leaves are C^1, the tangent planes of the leaves depend continuously on the base-point in W^{cs}, resp. W^{cu}, and such that each leaf intersects W^c transversally (in W^{cs}, resp. W^{cu}) in one point. Also these foliations are not unique. For details on invariant manifolds and foliations see [12] and also Chapter IV of this paper. If the leaves of $\mathcal{F}^{ss}$, or $\mathcal{F}^{uu}$, have co-dimension one in $W^{cs}_\mu = W^{cs} \cap \{\mu\}$, or $W^{cu}_\mu = W^{cu} \cap \{\mu\}$, then, by [11], the foliation $\mathcal{F}^{ss}$, resp. $\mathcal{F}^{uu}$, is C^1.

Let Φ and $\bar\Phi$ be two arcs of diffeomorphisms with center manifolds W^c and $\overline{W}^c$, let $h : W^c \to \overline{W}^c$ be a local conjugacy between $\Phi \mid W^c$ and $\bar\Phi \mid \overline{W}^c$, and let the normal data of Φ and $\bar\Phi$ be equal. Choose invariant manifolds and foliations W^{cs}, W^{cu}, $\mathcal{F}^{ss}$ and $\mathcal{F}^{uu}$ for Φ, and $\overline{W}^{cs}$, $\overline{W}^{cu}$, $\bar{\mathcal{F}}^{ss}$ and $\bar{\mathcal{F}}^{uu}$ for $\bar\Phi$. From [26] it follows that there is an extension H of h to a local conjugacy between Φ and $\bar\Phi$ so that $H(W^{cs}) = \overline{W}^{cs}$, $H(W^{cu}) = \overline{W}^{cu}$. $H(\mathcal{F}^{ss}) = \bar{\mathcal{F}}^{ss}$ and $H(\mathcal{F}^{uu}) = \bar{\mathcal{F}}^{uu}$.

In the case of generic arcs Φ the only periodic bifurcations are those of the following three types (recall we assume $\bar x$ fixed):

1. $c = 1$, $(d\varphi_{\bar\mu})_{\bar x}$ has an eigenvalue 1 and the 2-jet of $\varphi_{\bar\mu} \mid W^c_{\bar\mu}$ at $\bar x$ is different from the 2-jet of the identity; in this case $(\bar x, \bar\mu)$ is called a saddle-node of Φ;

2. $c = 1$, $(d\varphi_{\bar{\mu}})_{\bar{x}}$ has an eigenvalue -1 and the 3-jet of $\varphi^2_{\bar{\mu}} \mid W^c_{\bar{\mu}}$ at $\bar{x}$ is different from the 3-jet of the identity; in this case we call $(\bar{x}, \bar{\mu})$ a flip of Φ;
3. $c = 2$, $(d\varphi_{\bar{\mu}})_{\bar{x}}$ has a pair of non-real complex conjugate eigenvalues on the unit circle and the 3-jet of $\varphi_{\bar{\mu}} \mid W^c_{\bar{\mu}}$ makes $\bar{x}$ an attractor or a repeller, we call these points Hopf points.

A periodic bifurcation will be called *elementary* if it is of type 1, 2, or 3 above and it unfolds generically with μ. This last condition will be explained in sections 3, 4, and 5.

Before we consider these cases in more detail, we make some more general remarks:

— in all these cases $W^c_{\bar{\mu}}$ is C^∞ (because near $\bar{x}$, each orbit $\varphi^i_{\bar{\mu}}(x)$ tends to $\bar{x}$ as $i \to +\infty$ or as $i \to -\infty$), but generically it is not possible to choose W^c to be C^∞ (we shall not use or prove this fact but see [37]).

This results described in this section, except those concerning saddle-nodes are all more or less well-known. Apart form the references in the various sections, one may consult [1], [4], [19], [20], [35], [36].

3. The saddle-node

First we consider arcs of diffeomorphisms $\{\varphi_\mu\}$ of $\mathbf{R}$ (with coordinate x) such that $\varphi_0(0) = 0$, $\dfrac{d}{dx}\varphi_0(0) = 1$, $\dfrac{d^2}{dx^2}\varphi_0(x) \neq 0$ and $\dfrac{d}{d\mu}\varphi_\mu(0) \neq 0$ (to simplify notation we took $\bar{\mu} = 0$ and $\bar{x} = 0$). Note that if $\bar{\Phi}$ is a generic arc of diffeomorphisms, then $\bar{\Phi}$, restricted to a center manifold of a saddle-node, has the above form. Without loss of generality, we shall assume that $\dfrac{d^2}{dx^2}\varphi_0(x) > 0$ and $\dfrac{d}{d\mu}\varphi_\mu(0) > 0$. We shall also assume that $\varphi_\mu(x)$, as a function of x and μ, is C^6 and that $\varphi_0(x)$ is a C^∞ function of x. Arcs of diffeomorphisms $\{\varphi_\mu\}$ satisfying the above conditions will be called (in this section) saddle-node arcs. We shall prove the following results.

Theorem **(3.1)**. — *Any two saddle-node arcs are locally conjugate near* $(0, 0)$. *Moreover the conjugacy can be chosen to be continuously differentiable off the fixed point set.*

We recall that if $\{\varphi_\mu\}$ is a saddle-node arc, there is a unique C^∞ vector field X, defined on a neighborhood of 0 in $\mathbf{R}$, such that the time 1 map X_1 of X equals φ_0 [39].

Theorem **(3.2)**. — *Let* $\{\varphi_\mu\}$ *and* $\{\bar{\varphi}_\mu\}$ *be two saddle-node arcs with corresponding vector fields* X *and* $\bar{X}$, i.e. *such that* $X_1 = \varphi_0$ *and* $\bar{X}_1 = \bar{\varphi}_0$. *Let h be a local conjugacy between* $\{\varphi_\mu\}$ *and* $\{\bar{\varphi}_\mu\}$. *Then* $h_- = h \mid \{x < 0, \mu = 0\}$ *and* $h_+ = h \mid \{x > 0, \mu = 0\}$ *are* C^∞ *and* $(h_\pm)_* X = \bar{X} \mid \{\pm x > 0\}$.

Remark **(3.3)**. — The above theorem implies that the choice of the conjugacy along $\{\mu = 0\}$ is extremely restricted. Instead of the usual freedom to fix the conjugacy arbitrarily on a fundamental domain, we are here only free to fix h in two points: one in $\{x < 0\}$ and one in $\{x > 0\}$.

Let $\{\varphi_\mu\}$ be an arc of diffeomorphisms of an n-dimensional manifold M having a saddle-node fixed point at $(\bar{x}, \bar{\mu})$. Let $\Phi(x, \mu) = (\varphi_\mu(x), \mu)$. We say that $(\bar{x}, \bar{\mu})$ *unfolds generically* if for some (or any) center manifold W^c at $(\bar{x}, \bar{\mu})$, $\Phi \mid W^c$ is a saddle-node arc. Similarly, one can define generic unfolding of periodic saddle-nodes.

Theorem **(3.4)**. — *Let Φ be a smooth arc of diffeomorphisms of* M, *and let $(\bar{x}, \bar{\mu})$ be a saddle-node of Φ which unfolds generically. Let* $W^s = \{(x, \bar{\mu}) \mid \Phi^i(x, \bar{\mu}) \to (\bar{x}, \bar{\mu})$ *as* $i \to \infty\}$ (note that $W^s \subset W^{cs}$, that $\dim(W^s) = \dim(W^{cs}) - 1$ and that W^s has a boundary containing $(\bar{x}, \bar{\mu})$). *Then the strong stable foliation $\mathcal{F}^{ss}$, restricted to* W^s, *is unique and is preserved under any conjugacy.*

The proofs of these theorems occupy the rest of this section. As part of the proof of Theorem (3.1), we need to prove the corresponding theorem for vector fields.

Consider vector fields $X = X(x, \mu)\dfrac{\partial}{\partial x}$ on $\mathbf{R}^2$ with $X(0, 0) = 0$, $\dfrac{\partial}{\partial x}X(0, 0) = 0$, $\dfrac{\partial^2}{\partial x^2}X(0, 0) > 0$, $\dfrac{\partial}{\partial \mu}X(0, 0) > 0$ and which are at least C^2. These vector fields are called saddle-node fields. A saddle-node field can of course be considered as a one-parameter family of vector fields on $\mathbf{R}$; its time one map is a saddle-node arc (except for the differentiability); see the beginning of this section. Two saddle-node fields X and $\bar{X}$ are called *locally conjugate* if there is a homeomorphism h (compare section 2) from a neighborhood of $(0, 0)$ in $\mathbf{R}^2$ to another such neighborhood such that

— $h \circ X_t = \bar{X}_t \circ h$, whenever defined ($X_t$ stands for the time t map of the vector field X);
— there is a local homeomorphism $h_{\mathbf{R}} : \mathbf{R} \to \mathbf{R}$ such that the μ-coordinate of $h(x, \mu)$ equals $h_{\mathbf{R}}(\mu)$ whenever defined.

First we shall prove:

Theorem **(3.5)**. — *Any two saddle-node fields are locally conjugate by a conjugacy which is continuously differentiable with respect to x, in the complement of the singular set.*

Proof of Theorem (3.5). — Let X and $\bar{X}$ be two saddle-node fields.

For $X = X(x, \mu)\dfrac{\partial}{\partial x}$ we choose a box $U = \{(x, \mu) \mid |x| \leq a, 0 \leq \mu \leq \varepsilon\}$ such that $X(x, \mu)$ and $\dfrac{\partial}{\partial \mu}X(x, \mu)$ are positive on $U \backslash (0, 0)$. Define $f : (0, \varepsilon] \to \mathbf{R}_+$ by $X_{f(\delta)}(-a, \delta) = (+a, \delta)$, or, equivalently, $f(\delta) = \int_{-a}^{+a}(X(x, \delta))^{-1}dx$. From this last formula it is clear that $\lim_{\delta \to 0} f(\delta) = +\infty$ and that the derivative $f'(\delta)$ is less than zero for $\delta \in (0, \varepsilon]$.

Let $\bar{U}$, $\bar{f}$, $\bar{a}$, $\bar{\varepsilon}$ be defined analogously for $\bar{X}$. Pick $0 < \alpha < \varepsilon$ so that

$f(\alpha) > \max(f(\varepsilon), \bar{f}(\bar{\varepsilon}))$ and take $0 < \bar{\alpha} < \bar{\varepsilon}$ so that $\bar{f}(\bar{\alpha}) = f(\alpha)$. Then there is a homeomorphism $\eta : [0, \alpha] \to [0, \bar{\alpha}]$ such that $f(\delta) = \bar{f}(\eta(\delta))$ for $0 < \delta \leq \alpha$. A conjugacy h from X to $\bar{X}$, restricted to $\{0 \leq \mu \leq \alpha\}$ is now determined by putting

$$h(-a, \mu) = (-\bar{a}, \eta(\mu));$$
$$h(a, \mu) = (\bar{a}, \eta(\mu));$$
$$h(0, 0) = (0, 0);$$

and requiring that $hX_t = \bar{X}_t h$. The continuity of h is automatic except at $(0, 0)$. We now prove continuity at $(0, 0)$. For any sequence (x_i, μ_i) in U, there are sequences s_i, t such that

$$X_{s_i}(a, \mu_i) = x_i \quad \text{and} \quad X_{t_i}(-a, \mu_i) = x_i.$$

$(x_i, \mu_i) \to (0, 0)$ if and only if $\mu_i \to 0$, $s_i \to -\infty$, and $t_i \to +\infty$. Hence h maps sequences converging to $(0, 0)$ to sequences converging to $(0, 0)$ which proves continuity.

The construction of $h \mid \{\mu < 0\}$ is easy. For example, take $h(-a, \mu) = (-\bar{a}, \mu)$, $h(0, \mu) = (0, \mu)$ and $h(a, \mu) = (\bar{a}, \mu)$ and extend with $hX_t = \bar{X}_t h$. On the complement of the set of singularities of X, $\dfrac{\partial h}{\partial x}$ clearly exists and equals $\bar{X}(h(x, \mu)) . (X(x, \mu))^{-1}$ which is continuous. This completes the proof of the theorem.

Let $\{\varphi_\mu\}$ be a saddle-node arc, and let X be the C^∞ vector field defined near 0 in $\mathbf{R}$ such that $X_1 = \varphi_0$. It would be useful if there were a C^∞ field $\hat{X}$ extending X to a neighborhood of $(0, 0)$ in $\mathbf{R}^2$ of the form $\hat{X}(x, \mu) = \tilde{X}(x, \mu) \dfrac{\partial}{\partial x}$ such that $\hat{X}(\cdot, \mu)_1 = \varphi_\mu(\cdot)$. Although we cannot find such an $\hat{X}$, lemma (3.6) provides us with a suitable substitute.

We say that a C^4 saddle-node field X is *adapted* to a saddle-node arc $\{\varphi_\mu\}$ if the function $g(x, \mu)$, defined by $(\varphi_\mu(x) + g(x, \mu), \mu) = X_1(x, \mu)$, vanishes along the x-axis and has at $(0, 0)$ its 4-jet equal to zero.

Lemma **(3.6)**. — *For each saddle-node arc* $\{\varphi_\mu\}$ *there is an adapted saddle-node field* X.

Proof. — φ_0 is C^∞ and has, at $x = 0$, only a finite order of contact with the identity, so by [39], φ_0 embeds in a unique C^∞-vector field $\tilde{X} = \tilde{X}(x) \dfrac{\partial}{\partial x}$. In general [14], [40] if $\Psi : (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ is a C^∞ diffeomorphism with all eigenvalues of $(d\Psi)_0$ equal to 1, there is a unique k-jet of a vector field $[Y]_k$, $k \geq 1$, such that any representative $Y \in [Y]_k$ satisfies

— the k-jets of Y_1 and Ψ agree at 0;
— the eigenvalues of dY at 0 are all equal to zero.

If we apply this to $\Phi : (x, \mu) \to (\varphi_\mu(x), \mu)$ at $(0, 0)$ we see that the corresponding 4-jet $[X]$ at $(0, 0)$ is uniquely determined and has a representative of the form $\hat{X}(x, \mu) \dfrac{\partial}{\partial x}$.

The restriction of this 4-jet to the x-axis equals the 4-jet of the previously constructed $\tilde{X}$ because of uniqueness of that 4-jet (which follows if we apply the above general statement to φ_0 at 0). X is now obtained by taking $X = (\hat{X}(x, \mu) + \tilde{X}(x) - \hat{X}(x, 0)) \frac{\partial}{\partial x}$. The proof of (3.6) is complete.

The next lemma compares high iterates of a saddle-node arc with high iterates of the time one map of an adapted saddle-node field.

Lemma (3.7). — *Let* $\{\varphi_\mu\}$ *be a saddle-node arc and* $X = X(x, \mu) \frac{\partial}{\partial x}$ *an adapted saddle-node field which is at least* C^5. *Let* $U = \{(x, \mu) \mid 0 \leq \mu \leq \bar{\mu}, -a \leq x \leq a\}$ *be so that for* $(x, \mu) \in U \backslash (0, 0)$, $X(x, \mu) > 0$, $\varphi_\mu(x) > x$, $(\varphi_\mu(-a), \mu) \in U$ *and* $|g(x, \mu)| < \varphi_\mu(x) - x$. (Here $g(x, \mu)$ is the function defined just before Lemma (3.6). The fact that the last condition is satisfied for a and μ small enough follows from $|g(x, \mu)| = o(|(x, \mu)|^4)$ and $|\varphi_\mu(x) - x| \geq k(\mu + x^2)$ on $\{\mu \geq 0\}$ for some constant k.)

Then there are constants C_1, C_2 *such that for any* $(x, \mu) \in U$, $i \in \mathbf{N}$, *with* $(\varphi_\mu^i(x), \mu) \in U$,

(i) $X_\alpha(x, \mu) = (\varphi_\mu^i(x), \mu)$ *for some* $\alpha \in \mathbf{R}$ *with* $|i - \alpha| \leq \mu . C_1$;

(ii) $|\log((d(\varphi_\mu^i)_x) X(x, \mu)) - \log(X(\varphi_\mu^i(x), \mu))| \leq \mu . C_2$.

Proof. — The above two estimates on (high) iterates of φ_μ are implied by the following two: there are constants K_1, K_2 such that for any $(x, \mu) \in U$ with $(\varphi_\mu(x), \mu) \in U$

(i)′ $X_\alpha(x, \mu) = (\varphi_\mu(x), \mu)$ for some α with

$$|1 - \alpha| \leq \mu . (\varphi_\mu(x) - x) . K_1;$$

(ii)′ $|\log((d\varphi_\mu)_x . X(x, \mu)) - \log(X(\varphi_\mu(x), \mu))| \leq \mu . (\varphi_\mu(x) - x) . K_2$.

Of course in (ii)′ we have to assume that $(x, \mu) \neq 0$, but since for $\mu = 0$ the whole lemma is trivial we shall assume, in what follows, that always $\mu > 0$. To show that (ii)′ really implies (ii) it suffices to observe that

$$\frac{d(\varphi_\mu^i)_x . (X(x, \mu))}{X(\varphi_\mu^i(x), \mu)} = \prod_{j=0}^{i-1} \frac{d(\varphi_\mu)_{\varphi_\mu^j(x)}(X(\varphi_\mu^j(x), \mu))}{X(\varphi_\mu^{j+1}(x), \mu)}.$$

In the following calculation, Δ will indicate that the formula in which it occurs is valid if Δ is replaced by some positive constant, i.e. independent of $(x, \mu) \in \{(x, \mu) \mid (x, \mu) \in U, \mu > 0, (\varphi_\mu(x), \mu) \in U\}$. From the various definitions we obtain

$$|X(x, \mu)| \geq \Delta . (x^2 + \mu^2)$$

and

$$|g(x, \mu)| \leq \Delta . \mu . (x^2 + \mu^2)^2$$

(here one uses that X is C^5). Define $h(x, \mu)$ by $X_{1+h(x,\mu)}(x, \mu) = (\varphi_\mu(x), \mu)$. Since $h(x, \mu) = \int_\alpha^\beta [X(y, \mu)]^{-1} dy$, where $\alpha = X_1(x, \mu)$ and $\beta = \varphi_\mu(x)$, the above inequalities imply

$$|h(x, \mu)| \leq \Delta . \mu . (x^2 + \mu^2).$$

Now (i)′ follows from the obvious inequality

$$|\varphi_\mu(x) - x| \geq \Delta.(x^2 + \mu^2).$$

To prove (ii)′, we observe that from previous definitions it follows that

(a) the 4-jet of $(\Phi_* X) - X$ at $(0, 0)$ is zero (since the 4-jet of X_1 and Φ agree at $(0, 0)$; see Lemma (3.6))

(b) $\Phi_*(X) - X$ is zero along $\{\mu = 0\}$.

Hence

$$|(d\varphi_\mu)_x . X(x, \mu) - X(\varphi_\mu(x), \mu)| \leq \Delta.\mu.((\varphi_\mu(x))^2 + \mu^2)^2$$

and

$$|X(\varphi_\mu(x), \mu)| \geq \Delta.((\varphi_\mu(x))^2 + \mu^2),$$

so

$$\frac{|(d\varphi_\mu)_x . X(x, \mu) - X(\varphi_\mu(x), \mu)|}{|X(\varphi_\mu(x), \mu)|} \leq \Delta.\mu.((\varphi_\mu(x))^2 + \mu^2).$$

Since $(x, \mu) \mapsto (\varphi_\mu(x), \mu)$ is a diffeomorphism, $((\varphi_\mu(x))^2 + \mu^2) \leq \Delta(x^2 + \mu^2)$. From these inequalities it easily follows that

$$|\log((d\varphi_\mu)_x . X(x, \mu)) - \log(X(\varphi_\mu(x), \mu))| \leq \Delta.\mu.(x^2 + \mu^2) \leq \Delta.\mu.(\varphi_\mu(x) - x);$$

this proves (ii)′.

Proof of Theorem (3.1). — The proof of Theorem (3.1) will be obtained by showing that if $\{\varphi_\mu\}$ is a saddle-node arc and X an adapted vector field (of class C^5) then there is a local conjugacy $H(x, \mu) = (h_\mu(x), \mu)$ from $\{\varphi_\mu\}$ to the time one map X_1 of X. We shall also construct H so that $\dfrac{\partial H}{\partial x}$ exists and is continuous on the complement of the fixed-point set of Φ. Indeed, by Theorem (3.5) and Lemma (3.6), this implies (3.1).

We take $h_0 = H | \{\mu = 0\}$ to be the identity. Let

$$U = \{(x, \mu) \mid 0 \leq \mu \leq \bar{\mu},\ -a \leq x \leq a\}$$

be as in Lemma (3.7). Take:

1. $h_\mu(-a) = -a$ for $0 \leq \mu \leq \bar{\mu}$ and hence
2. $(h_\mu(\varphi^n_\mu(-a)), \mu) = X_n(-a, \mu)$ whenever $(\varphi^n_\mu(-a), \mu) \in U$;
3. extend the definition of $h_\mu(x)$ to

$$\{(x, \mu) \mid 0 \leq \mu \leq \bar{\mu},\ -a \leq x \leq \varphi_\mu(-a)\}$$

in such a way that it is C^1 and such that the extension, defined by

$$h_\mu(\varphi_\mu(x), \mu) = X_1(h_\mu(x), \mu)$$

is also differentiable in a neighborhood of $\{(\varphi_\mu(-a), \mu) \mid 0 \leq \mu \leq \bar{\mu}\}$. Now $H | U$ is uniquely determined; the continuity of H along $\{(x, \mu) \mid x \geq 0,\ \mu = 0\}$ follows from Lemma (3.7); also the fact that $\dfrac{\partial H}{\partial x}$ is continuous on $U \backslash (0, 0)$ follows from that same lemma.

The extension of H to $\{\mu \leq 0\}$ is simple: see the proof of Theorem (3.5).

Proof of Theorem (3.2). — Let $\{\varphi_\mu\}$ be a saddle-node arc and X an adapted saddle-node field, of class C^5, defined in a box U as in Lemma (3.7). We take $-a < x < 0$ and $0 < \bar{x} < a$ and show how $\{\varphi_\mu\}$ gives rise to a canonical homeomorphism $T_{\bar{x}x}$ from a neighborhood of x (in $\mathbf{R}$) to a neighborhood of $\bar{x}$. These maps $T_{\bar{x}x}$ are called translations.

Let $\{x_i\}_{i\in\mathbf{N}}$ be any sequence converging to x; choose a sequence $\{\mu_i\}_{i\in\mathbf{N}}$ such that $\varphi^i_{\mu_i}(x_i)$ converges to $\bar{x}$ with $\mu_i \to 0$ as $i \to \infty$. To get such a sequence, choose $\mu_i \to 0$ as $i \to \infty$ such that $(X_i)(x_i, \mu_i) \to (\bar{x}, 0)$ and apply Lemma (3.7). Let x' be a point close to x; without loss of generality we may put $(x', 0) = (X_\alpha)(x, 0)$ for some $\alpha \in \mathbf{R}$. Then for any sequence $x'_i \to x'$, $\lim\limits_{i\to\infty} \varphi^i_{\mu_i}(x'_i)$ (the same μ_i's as above) exists and equals $(X_\alpha)(\bar{x}, 0)$. This can be seen as follows: for some β_i, β'_i, α_i we have

$$\begin{aligned}(\varphi^i_{\mu_i}(x_i), \mu_i) &= X_{\beta_i}(x_i, \mu_i);\\ (\varphi^i_{\mu_i}(x'_i), \mu_i) &= X_{\beta'_i}(x'_i, \mu_i);\\ X_{\alpha_i}(x_i, \mu_i) &= (x'_i, \mu_i).\end{aligned}$$

By Lemma (3.7) $|i - \beta_i| \leq \mu_i . C_1$ and $|i - \beta'_i| \leq \mu_i . C_1$ for some constant C_1; because $x_i \to x$, $x'_i \to x'$ and $X_\alpha(x, 0) = (x', 0)$, $\alpha_i \to \alpha$. From this we conclude that if $X_{\tilde{\alpha}_i}(\varphi^i_{\mu_i}(x_i), \mu_i) = (\varphi^i_{\mu_i}(x'_i), \mu_i)$ then $\tilde{\alpha}_i = \alpha_i + \beta'_i - \beta_i$ so $\lim\limits_{i\to\infty} \tilde{\alpha}_i = \alpha$. We define $T_{\bar{x}x}(x') = \lim\limits_{i\to\infty} \varphi^i_{\mu_i}(x'_i)$. It is clear that the definition of the local homeomorphism $T_{\bar{x}x}$ is independent of X: X was only used to show that $\lim\limits_{i\to\infty} \varphi^i_{\mu_i}(x'_i)$ exists. Observe that $(T_{\bar{x}x})_*(X) = X$. Now we extend the notion of translation. If x, $\bar{x}$ are on the same side of 0 and x' on the other side, we may define $T_{\bar{x}x} = T_{\bar{x}x'} \circ T_{x'x}$; this definition is independent (at least near x) of the choice of x'. Hence we see that the translations form the pseudo-group of all local diffeomorphisms of $(-a, +a)\backslash\{0\}$ which preserve $X \mid \{\mu = 0\}$.

If h is a local conjugacy between two saddle-node arcs $\{\varphi_\mu\}$ and $\{\bar{\varphi}_\mu\}$, it must also conjugate the translations defined by $\{\varphi_\mu\}$ with those of $\{\bar{\varphi}_\mu\}$. Let X, $\bar{X}$ be the smooth vector fields such that $X_1 = \varphi_0$ and $\bar{X}_1 = \bar{\varphi}_0$. For $x \neq 0$, $\bar{x} \neq 0$ and $z = X_t(x)$ near x, let $x' = h(x)$, $\bar{x}' = h(\bar{x})$, $z' = h(z)$ and $z' = \bar{X}_{t'}(x')$. Since h conjugates $T_{\bar{x}x}$ and $\bar{T}_{\bar{x}'x'}$, and $T_{\bar{x}x}(z) = X_t(\bar{x})$, $\bar{T}_{\bar{x}'x'}(z') = \bar{X}_{t'}(\bar{x}')$, we conclude that there is a (continuous) function $t' = \sigma(t)$ such that $hX_th^{-1} = \bar{X}_{\sigma(t)}$. From the group property $X_{t_1} \circ X_{t_2} = X_{t_1+t_2}$ we deduce that $\sigma(t)$ is linear in t. Since $X_1 = \varphi_0$, $\bar{X}_1 = \bar{\varphi}_0$, we have $h \circ X_1 \circ h^{-1} = \bar{X}_1$ and hence $h \circ X_t \circ h^{-1} = \bar{X}_t$ for all t. This proves the theorem.

Proof of Theorem (3.4). — From Theorem (3.1) and invariant manifold theory (see Section 2 and [26]), it follows that Φ, near $(\bar{x}, \bar{\mu})$, is locally conjugate with $\tilde{\Phi}$ near (0, 0) where

$$\tilde{\Phi}(x_1, \ldots, x_m, \mu) = \left(x_1 + x_1^2 + \mu, \pm 2x_2, \ldots, \pm 2x_h, \pm \frac{1}{2} x_{h+1}, \ldots, \pm \frac{1}{2} x_m, \mu\right).$$

The stable manifold $\widetilde{W}^s$ of $\widetilde{\varphi}_0$ is $\{(x_1, \ldots, x_m) : x_1 \leq 0\}$. We take the strong stable foliation $\widetilde{\mathscr{F}}^{ss}$ of $\widetilde{\Phi}$ near $(0, 0)$ to consist of the manifolds

$$\{(x_1, \ldots, x_m, \mu) : (x_1, \ldots, x_h, \mu) = \text{constant}\}.$$

In a neighborhood U of o, we give a dynamical characterization of the leaves of $\widetilde{\mathscr{F}}^{ss} \cap \widetilde{W}^s$: $q_1, q_2 \in \widetilde{W}^s$ belong to the same leaf of $\widetilde{\mathscr{F}}^{ss}$ if and only if there are sequences $\{\mu_i\}_{i \in N}$, $\{q_1^i\}_{i \in N}$ and $\{q_2^i\}_{i \in N}$ such that

— $(q_1^i, \mu_i) \to (q_1, 0)$ and $(q_2^i, \mu_i) \to (q_2, 0)$,
— for each i, $\{\widetilde{\varphi}_{\mu_i}^j(q_1^i)\}_{j=0}^i$ and $\{\widetilde{\varphi}_{\mu_i}^j(q_2^i)\}_{j=0}^i$ are contained in U;
— $\lim(\widetilde{\varphi}_{\mu_i}^i(q_1^i), \mu_i) = \lim(\widetilde{\varphi}_{\mu_i}^i(q_2^i), \mu_i)$ and this limit is not the saddle-node point.

This follows from the special form of $\widetilde{\Phi}$.

Let $\mathscr{F}^{ss}$ be some strong stable foliation for Φ. The local conjugacy between Φ and $\widetilde{\Phi}$ mentioned before can be chosen to map $\mathscr{F}^{ss}$ to $\widetilde{\mathscr{F}}^{ss}$. This implies that $\mathscr{F}^{ss} \cap W^s$ also satisfies the above dynamical characterization. Hence, $\mathscr{F}^{ss} \cap W^s$ is unique and is preserved under conjugacies. Theorem (3.4) is proved.

Note that a local conjugacy of $\widetilde{\varphi}_0$, or $\varphi_{\widetilde{\mu}}$, does not have to respect the above foliation. This has to do with the fact that the dynamical characterization was only possible by using Φ on a full neighborhood of $(0, 0)$ in $R^m \times R$.

4. The flip

Here we consider arcs $\{\varphi_\mu\}$ of diffeomorphisms on R such that $\varphi_0(0) = 0$, $(d\varphi_0)_0 = -1$ and such that the 3-jet of $(\varphi_0)^2$ at the origin differs from the 3-jet of the identity. Such arcs of diffeomorphisms are obtained by restricting a generic arc of diffeomorphisms, at a flip, to a center manifold. The origin is either a source or a sink of φ_0; in the following we shall assume it to be a sink of φ_0; the other case then occurs for φ_0^{-1} and is completely analogous. With a coordinate change of the form

$$\widetilde{x} = \widetilde{x}(x, \mu)$$
$$\widetilde{\mu} = \mu$$

we can put φ_μ in the form

$$\varphi_{\widetilde{\mu}}(\widetilde{x}) = -\widetilde{x} + \widetilde{x}^3 + \lambda(\widetilde{\mu}).\widetilde{x} + o(\widetilde{x}^4) + o(|\widetilde{\mu}|.\widetilde{x}^2)$$

where λ is a real function and $\lambda(0) = 0$. We shall say that the flip *unfolds generically* if $\dfrac{d\lambda}{d\widetilde{\mu}}(0) \neq 0$. This is, of course, a generic condition. In higher dimensions, we will use this terminology if $\dfrac{d\lambda}{d\widetilde{\mu}}(0) \neq 0$ on some center manifold. Similar considerations apply to periodic flips.

Now we return to the one dimensional case. We shall assume $\dfrac{d\lambda}{d\widetilde{\mu}}(0) > 0$; the other case can be reduced to this by replacing $\tilde{\mu}$ by $-\tilde{\mu}$. It is clear that the phase portrait looks as follows:

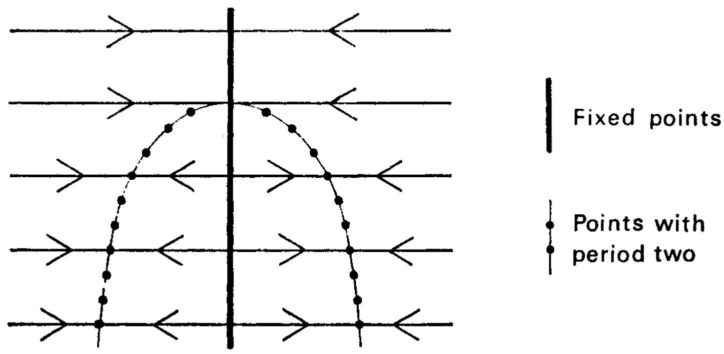

FIG. 1

The following theorem can be easily proved using the methods of proof used in Theorem (3.5) in the region $\{\mu \leq 0\}$:

Theorem (4.1). — *Any arc* $\{\varphi_\mu\}$ *of diffeomorphisms on* $\mathbf{R}$ *which is of the form* $\varphi_\mu(x) = -x + x^3 + \lambda(\mu).x + o(x^4) + o(|\mu|.x^2)$ *with* $\lambda'(0) > 0$ *is locally conjugate with* $\bar{\varphi}_\mu(x) = -x + x^3 + \mu x$.

5. The Hopf point

We consider arcs $\{\varphi_\mu\}$ of diffeomorphisms of $\mathbf{R}^2$ such that $\varphi_0(0) = 0$, $(d\varphi_0)_0$ has eigenvalues on the unit circle, but different from ± 1, and such that the 3-jet of φ_0 makes the origin an attractor or a repeller. We shall assume it to be an attractor; otherwise we consider φ_μ^{-1}. Up to a change of coordinates, the origin will be a fixed point of φ_μ if $|\mu|$ is small. Let $\lambda(\mu)$, $\bar{\lambda}(\mu)$ be the eigenvalues of $(d\varphi_\mu)_0$. For generic arcs one has $\frac{d}{d\mu}|\lambda(\mu)|_{\mu=0} \neq 0$, and we shall say, in this case, that the Hopf orbit *unfolds generically.* As for the saddle node and flip, we shall use the same terminology for periodic Hopf points if they unfold generically on center manifolds. In the following, we return to dimension two, and we shall assume that $\frac{d}{d\mu}|\lambda(\mu)|_{\mu=0} > 0$; otherwise we replace μ by $-\mu$. These arcs have been extensively studied; for references see [32].

From the fact that, for $\mu \leq 0$, the origin is an attractor of φ_μ one concludes:

Proposition (5.1). — *If* $\{\varphi_\mu\}$ *and* $\{\bar{\varphi}_\mu\}$ *are arcs of diffeomorphisms of* $\mathbf{R}^2$, *satisfying all the above requirements, then there is a continuous one parameter family of homeomorphisms* h_μ, $\mu \leq 0$, *from a neighborhood of* $0 \in \mathbf{R}^2$ *to a neighborhood of* $0 \in \mathbf{R}^2$, *such that* $h_\mu \circ \varphi_\mu = \bar{\varphi}_\mu \circ h_\mu$ *whenever defined*; i.e. $(x, \mu) \mapsto (h_\mu(x), \mu)$ *is a local conjugacy between* φ_μ *and* $\bar{\varphi}_\mu$ *on* $\mu \leq 0$.

A consequence of Proposition (5.1) is that arcs $\{\varphi_\mu\}$ as above are locally stable for $\mu \leq 0$. For $\mu > 0$ this is not the case.

Theorem **(5.2).** — *Let $\{\varphi_\mu\}$ be a C^∞ arc of diffeomorphisms as above. Then in any C^∞ neighborhood of $\{\varphi_\mu\}$ there is an arc $\{\bar\varphi_\mu\}$ such that the arcs $\{\varphi_\mu\}$, $\{\bar\varphi_\mu\}$ are not conjugate, even not mildly conjugate.*

Proof. — We can first approximate the arc $\{\varphi_\mu\}$ by an arc $\{\tilde\varphi_\mu\}$ such that the eigenvalues of $(d\varphi_0)_0$ have the form $e^{2\pi i\alpha}$ with α irrational. Then we know [40] that $\tilde\varphi_\mu$ is rotationally symmetric in the formal sense. By this we mean that there is a smooth S^1 $(=\mathbf{R}/2\pi)$ action $\tilde{R}$ on $\mathbf{R}^2 \times \mathbf{R}$, differentiably conjugate with the usual action $R : R_\alpha(x_1, x_2, \mu) = (x_1.\cos\alpha + x_2.\sin\alpha, -x_1\sin\alpha + x_2\cos\alpha, \mu)$, such that for each α, $\tilde{R}_\alpha \circ \tilde\Phi$ and $\tilde\Phi \circ \tilde{R}_\alpha$ have the same ∞-jet in the origin. So with a second perturbation one can find an arc $\{\bar\varphi'_\mu\}$ such that the corresponding map $\bar\Phi'$ defined by $\bar\Phi'(x, \mu) = \bar\varphi'_\mu(x)$ commutes with the S^1 action $\tilde{R}$, at least on a small neighborhood of $(x = 0, \mu = 0)$.

One knows [32] that $\bar\varphi'_\mu$, for $\mu > 0$, has an invariant circle, say C_μ. Because $\bar\Phi'$ commutes with the smooth S^1 action (for μ small), $\bar\varphi'_\mu \mid C_\mu$ is differentiably conjugate to a rotation. Since there is a residual subset of the diffeomorphisms of S^1 no element of which is conjugate to a rotation, there is finally an approximation $\bar\varphi_\mu$ of $\bar\varphi'_\mu$ (in the C^∞ sense) such that $\bar\varphi_\mu$ also has C_μ as invariant circle but such that for some sequence $\{\mu_i\}_{i\in I}$, $\mu_i \to 0$ as $i \to \infty$, $\bar\varphi_\mu \mid C_\mu$ is not conjugate to a rotation.

It follows from the construction that $\{\bar\varphi_\mu\}$ and $\{\bar\varphi'_\mu\}$ are not conjugate and even not mildly conjugate near $(0, 0)$. Hence φ_μ cannot be (mildly) conjugate to both $\{\bar\varphi_\mu\}$ and $\{\bar\varphi'_\mu\}$. This proves the theorem.

Remark **(5.3).** — If $\{\varphi_\mu\}$ is an arc of diffeomorphisms on a manifold M of dimension $m > 2$ which has a Hopf point at $(\bar x, \bar\mu)$ then the conclusion of Theorem (5.2) remains valid. In the construction of the C^∞ arcs $\{\bar\varphi'_\mu\}$, $\{\bar\varphi_\mu\}$, one has to add in this case a preliminary step, namely one has to modify $\{\varphi_\mu\}$ first so that it has a C^∞ centermanifold. Then the modifications, as described in the proof of (5.2) are carried out in that center manifold and extended to a neighborhood.

6. Quasi-transversal intersections

We consider a C^∞ arc of diffeomorphisms $\{\varphi_\mu : M \to M\}$ and assume that for some $\bar\mu \in \mathbf{R}$ and some $\bar x \in M$, there is a non-transversal intersection at $\bar x$ of a stable and an unstable manifold (of some periodic points) of $\varphi_{\bar\mu}$. In such a situation, if we denote the stable, resp. unstable, manifold by W^s, W^u, there is a canonical quadratic map $D : T_{\bar x}(W^s) \cap T_{\bar x}(W^u) \to T_{\bar x}(M)/T_{\bar x}(W^s) + T_{\bar x}(W^u)$, which is analogous to the intrinsic 2^{nd} order derivative [3] and is defined as follows.

Let $i : T_{\bar x}(W^s) \cap T_{\bar x}(W^u) \to W^u$ be a smooth map such that $i(0) = \bar x$ and $(di)_0$ is the canonical injection; let π be a projection (*i.e.* $\pi^2 = \pi$) of a neighborhood U of $\bar x$

to a submanifold whose dimension equals the dimension of $T_{\bar{x}}(M)/(T_{\bar{x}}(W^s) + T_{\bar{x}}(W^u))$, and such that $\pi(\bar{x}) = \bar{x}$, $(d\pi)_{\bar{x}}(T_{\bar{x}}(W^s) + T_{\bar{x}}(W^u)) = 0$ and $\pi(W^s) = \bar{x}$; let $N = \pi(U)$. Then $T_{\bar{x}}(N)$ is canonically isomorphic with $T_{\bar{x}}(M)/(T_{\bar{x}}(W^s) + T_{\bar{x}}(W^u))$.

The map $\pi \circ i$ maps 0 to $\bar{x}$, $d(\pi \circ i)_0 = 0$; hence the second derivative $d^2(\pi \circ i)_0$: $T_{\bar{x}}(W^s) \cap T_{\bar{x}}(W^u) \to T_{\bar{x}}(N)$ is a well defined quadratic map; we define D to be the composition of $d^2(\pi \circ i)_0$ with the canonical isomorphism

$$T_{\bar{x}}(N) \cong T_{\bar{x}}(M)/(T_{\bar{x}}(W^s) + T_{\bar{x}}(W^u)).$$

The maps D does not depend on the various choices.

It is not hard to show that, for generic arcs of diffeomorphisms $\{\varphi_\mu\}$, all the non-transversal intersection points $\bar{x}$ of stable and unstable manifolds of $\varphi_{\bar{\mu}}$, for $\bar{\mu} \in \mathbf{R}$, $\bar{x} \in M$, will satisfy (in the above terminology):

a) $\dim T_{\bar{x}}(M)/(T_{\bar{x}}(W^s) + T_{\bar{x}}(W^u)) = 1$;

b) D is non-degenerate.

Under these circumstances there are coordinates $x_1, \ldots, x_m$ on a neighborhood of $\bar{x}$ ($\bar{x}$ corresponding to 0) such that W^s and W^u locally have the following

Canonical form **(6.1)**

$$W^s = \{x_1 = \ldots = x_{m-s} = 0\}$$
$$W^u = \{x_{u+2} = \ldots = x_m = 0,\ x_1 = f(x_{m-s+1}, \ldots, x_{u+1})\}$$

where

$$m = \dim(M);$$
$$s = \dim(W^s);$$
$$u = \dim(W^u);$$

f is a homogeneous non-degenerate quadratic function;

if $m - s + 1 > u + 1$ then one should read $x_1 = 0$.

Before proving (6.1), we note the following

c) $\max(s, u) < m$ by condition *(a)* above;

d) the vector $\left.\dfrac{\partial}{\partial x_1}\right|_{\bar{x}}$ is not in $T_{\bar{x}}(W^s) + T_{\bar{x}}(W^u)$;

e) the vectors $\left.\dfrac{\partial}{\partial x_{m-s+1}}\right|_{\bar{x}}, \ldots, \left.\dfrac{\partial}{\partial x_{u+1}}\right|_{\bar{x}}$ are in $T_{\bar{x}}(W^s) \cap T_{\bar{x}}(W^u)$.

Proof of (6.1). — Since $\dim(T_{\bar{x}}(W^s) + T_{\bar{x}}(W^u)) = m - 1$, we can choose a $(u+1)$-manifold $\widetilde{W}^u$ containing W^u which is at $\bar{x}$ transversal to W^s. From this it clearly follows that one can choose coordinates $\bar{x}_1, \ldots, \bar{x}_m$ such that

$$W^s = \{\bar{x}_1 = \ldots = \bar{x}_{m-s} = 0\};$$
$$\widetilde{W}^u = \{\bar{x}_{u+2} = \ldots = \bar{x}_m = 0\};$$

W^u is tangent at $\bar{x}$ to $\{\bar{x}_1 = \bar{x}_{u+2} = \bar{x}_{u+3} = \ldots = \bar{x}_m = 0\}$.

Then, for some function f

$$W^u = \{\bar{x}_{u+2} = \ldots = \bar{x}_m = 0,\ \bar{x}_1 = \bar{f}(\bar{x}_2, \ldots, \bar{x}_{u+1})\},$$
$$\bar{f}(0) = 0 \quad \text{and} \quad (d\bar{f})_0 = 0.$$

Now we replace $\bar{x}_1$ by the new coordinate

$$x_1 = \bar{x}_1 - \bar{f}(\bar{x}_2, \ldots, \bar{x}_{u+1}) + \bar{f}(0, \ldots, 0, \bar{x}_{m-s+1}, \ldots, \bar{x}_{u+1});$$

then we have

$$W^s = \{x_1 = \bar{x}_2 = \ldots = \bar{x}_{m-s} = 0\},$$
$$W^u = \{x_1 = \bar{f}(0, \ldots, 0, \bar{x}_{m-s+1}, \ldots, \bar{x}_{u+1}),\ \bar{x}_{u+2} = \ldots = \bar{x}_m = 0\},$$

the first derivative of $(\bar{x}_{m-s+1}, \ldots, \bar{x}_{u+1}) \mapsto \bar{f}(0, \ldots, 0, \bar{x}_{m-s+1}, \ldots, \bar{x}_{u+1})$ is zero and the second derivative is just D and hence has maximal rank. Now we can apply Morse's Lemma [3] and obtain a coordinate change of the form

$$x_j = \bar{x}_j \quad \text{for } 2 \leq j \leq m - s \quad \text{or} \quad u + 2 \leq j \leq m,$$
$$x_j = x_j(\bar{x}_{m-s+1}, \ldots, \bar{x}_{u+1}) \quad \text{for } m - s + 1 \leq j \leq u + 1,$$

for which we have

$$W^s = \{x_1 = \ldots = x_{m-s} = 0\},$$
$$W^u = \{x_1 = f(x_{m-s+1}, \ldots, x_{u+1}),\ x_{u+2} = \ldots = x_m = 0\},$$

with f homogeneous, quadratic and non-degenerate.

Remark **(6.2)**. — In suitable coordinates the generic unfolding of the quasi-transversal intersection puts the manifolds W^s_μ and W^u_μ in the form

$$W^s_\mu = \{x_1 = 0 = \ldots = x_{m-s} = 0\},$$
$$W^u_\mu = \{x_{u+2} = \ldots = x_m = 0,\ x_1 = f(x_{m-s+1}, \ldots, x_{u+1}) \pm (\mu - \bar{\mu})\}.$$

III. — NECESSARY CONDITIONS FOR STABILITY OF ARCS

In this chapter we shall obtain necessary conditions for the various kinds of stabilities of arcs we have defined. Let $I = [0, 1]$ and let $\mathscr{P}$ be the space of C^∞ arcs $\varphi : I \to \mathrm{Diff}^\infty(M)$ with the C^∞ topology. Elements of $\mathscr{P}$ will be denoted by φ, or by $\{\varphi_\mu\}$ when we wish to make the dependence on the parameter $\mu \in I$ explicit. Recall that $\mathscr{A} \subset \mathscr{P}$ is the subset of those arcs $\varphi \in \mathscr{P}$ such that

(1) $\varphi_0 \in MS$,

(2) $b = b(\varphi) = \inf\{\mu \in I : \varphi_\mu \notin MS\} < 1$,

(3) the limit set of φ_b consists of finitely many orbits.

A diffeomorphism f is called elementary if either

(a) there is exactly one quasi-hyperbolic periodic orbit, the other periodic orbits are hyperbolic, and all stable, strong stable, unstable, and strong unstable manifolds meet transversally; or

(b) all periodic orbits are hyperbolic, there is one quasi-transversal orbit of intersections of stable and unstable manifolds, and all other stable and unstable manifold intersections are transverse.

Let $\mathscr{B} \subset \mathscr{A}$ be the set of arcs $\varphi \in \mathscr{A}$ such that φ_b is elementary and the quasi-hyperbolic periodic orbit of φ_b, if it exists, unfolds generically.

We proceed to discuss necessary conditions for stability of arcs in $\mathscr{B}$.

1. The modulus condition (quasi-transversal intersection)

In this section we shall show that left stability of an arc $\{\varphi_\mu\}$ in $\mathscr{B}$ forces all stable and unstable manifolds of periodic points of φ_b to meet transversally. Before this, we consider the effect of a quasi-transversal orbit on topological conjugacy. The next theorem shows that generally such an orbit yields at least a one-parameter family of distinct topological types. We will refer to this phenomenon by saying that moduli occur. It should be noted that this occurs even in a locally isolated codimension one submanifold of the boundary of MS on the 2-sphere (hence, on any manifold of dimension larger than one).

For example, consider an MS diffeomorphism φ_0 on S^2 as in the next figure.

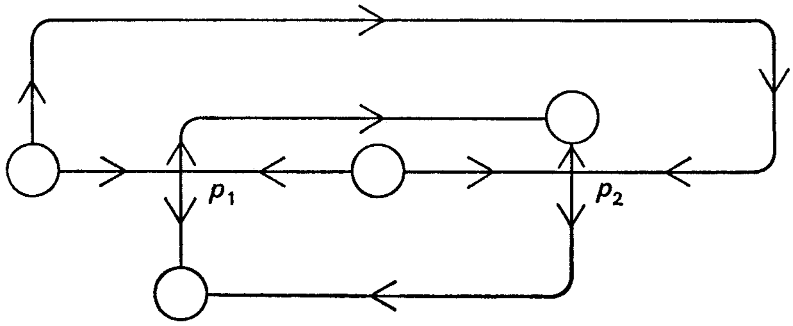

FIG. 2

The circles represent sources and sinks and there are two saddle fixed points p_1 and p_2. We choose a curve of C^2 diffeomorphisms $\{\varphi_\mu\}$, $0 \leq \mu \leq 1$, starting at φ_0 so that p_1 and p_2 remain fixed for each φ_μ, and $W^u(p_1, \varphi_{1/2})$ has a single orbit $o(x)$ of quasi-transversal intersections with $W^s(p_2, \varphi_{1/2})$ as in the next figure.

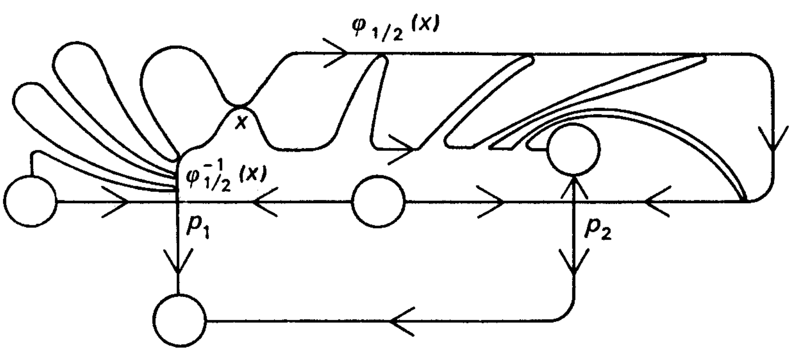

FIG. 3

This can be done so that φ_μ is in MS for $\mu \neq \frac{1}{2}$ and any perturbation $\{\varphi'_\mu\}$ of $\{\varphi_\mu\}$ has a unique bifurcation $b(\varphi')$ near $\frac{1}{2}$.

Let f be a diffeomorphism of M with a hyperbolic fixed point p. Let α^* be the largest modulus of the eigenvalues of $df(p)$ which are inside the unit circle, and let β^* be the smallest modulus of the eigenvalues of $df(p)$ which are outside the unit circle. If there is an eigenvalue α of $df(p)$ such that

(1) $|\alpha| = \alpha^*$,

(2) α has multiplicity one,

(3) any eigenvalue λ of $df(p)$ different from α or the complex conjugate $\bar{\alpha}$ satisfies $|\lambda| \neq \alpha^*$,

then we say that the weakest contracting eigenvalue α of f at p is defined. Similarly, if there is an eigenvalue β of multiplicity one of $df(p)$ such that $|\beta| = \beta^*$ and any eigenvalue $\lambda \neq \bar{\beta}, \beta$ satisfies $|\lambda| \neq \beta^*$, we say the weakest expanding eigenvalue β of f at p is defined.

Let p and q be hyperbolic fixed points of f such that the weakest contracting eigenvalue α at p and the weakest expanding eigenvalue β at q are defined. Let H_p be a C^1 invariant manifold containing $W^u(p)$ and tangent at p to the sum of the eigenspace of α and $T_pW^u(p)$, and let H_q be a C^1 invariant manifold containing $W^s(q)$ and tangent at q to the sum of the eigenspace of β and $T_qW^s(q)$. The existence of H_p and H_q is proved in [12]. They are not unique. However, the tangent bundle of H_p along $W^u(p)$ is unique, and so is that of H_q along $W^s(p)$.

Let r be a quasi-transversal intersection of $W^u(p)$ and $W^s(q)$. We say that r is a regular quasi-transversal intersection if $W^u(p)$ is transverse to H_q at r and $W^s(q)$ is transverse to H_p at r. This definition is independent of the choice of the manifolds H_p and Q_q because it depends only on TH_p, TH_q along $W^u(p)$, $W^s(q)$, respectively.

Note that part of the definition of a regular quasi-transversal intersection of $W^u(p)$ and $W^s(q)$ is that the weakest contracting and expanding eigenvalues α at p and β at q be defined. Note that for arcs φ in a residual subset of $\mathscr{P}$ all quasi-transversal orbits of φ_μ are regular.

Theorem (**1.1**). — *Let f, f' be C^2 diffeomorphisms. Let p and q be hyperbolic fixed points having an orbit $\{f^i(r)\}$ of regular quasi-transversal intersections of $W^u(p)$ and $W^s(q)$. Let α be the weakest contracting eigenvalue of f at p and β be the weakest expanding eigenvalue of f at q. Make analogous assumptions on f' concerning fixed points p', q', eigenvalues α', β', etc.*

If there is a conjugacy from f to f' defined on a neighborhood of the closure of $\{f^i(r)\}$ mapping p to p', q to q', and r to r', then

$$\frac{\log|\alpha|}{\log|\beta|} = \frac{\log|\alpha'|}{\log|\beta'|}.$$

Remark 1. — If p and q are periodic points of period $\tau(p)$ and $\tau(q)$ instead of fixed points, and n is the least common multiple of $\tau(p)$ and $\tau(q)$, the theorem can be applied to f^n. Doing this one obtains that if α is the weakest contracting eigenvalue of $f^{\tau(p)}$ at p and β is the weakest expanding eigenvalue of $f^{\tau(q)}$ at q and α', β' are the corresponding eigenvalues for f', then the existence of a conjugacy between f and f' implies

$$\frac{\log|\alpha|}{\log|\beta|} = \frac{\log|\alpha'|}{\log|\beta'|}.$$

2. Theorem (1.1) shows that quasi-transversal orbits lead to at least one dimensional invariants of topological conjugacy. It is interesting to ask what additional invariants in the presence of such orbits are sufficient to imply the existence of a topological conjugacy in various contexts. For some results in this direction, see [18], [28].

Proof of Theorem (1.1).

A) *The case* $m = 2$; $m = \dim(M)$

In this case the dimensions of the stable and unstable manifolds are both one.

Replacing f, f' by f^2, f'^2, if necessary, we assume $\alpha, \beta, \alpha', \beta' > 0$. So we are in the following situation:

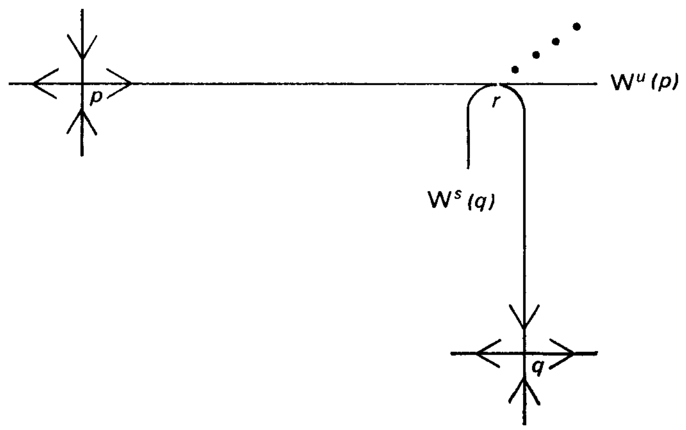

FIG. 4

We consider a sequence of points r_i converging to r but so that $r_i \notin (W^u(p) \cup W^s(q))$ for all i. By choosing a subsequence if necessary, we can arrange that there are sequences of integers $n_i \to \infty$, $m_i \to \infty$ with the property that $f^{-n_i}(r_i)$, $f^{m_i}(r_i)$, has a limit in $W^s(p) - p$, $W^u(q) - q$, respectively. Let $\rho(r_i, W^u(p))$, $\rho(r_i, W^s(q))$ denote the distance from r_i to $W^u(p)$, respectively $W^s(q)$, with respect to some Riemannian metric.

Because f is C^2, it is C^1 linearizable on $W^s(p)$ and $W^u(q)$ [8]. From this, we conclude that $\rho(r_i, W^u(p)) \sim \alpha^{n_i}$ and $\rho(r_i, W^s(q)) \sim \beta^{-m_i}$, where $\sim$ denotes equality up to a positive multiplicative factor, depending on i but uniformly bounded and bounded away from zero. It is clear from the picture, or rather from the normal forms in Chapter II, that a sequence r_i can be chosen so that $\rho(r_i, W^u(p)) \sim \rho(r_i, W^s(q))$. In that case we have

$$\frac{\log \alpha}{\log \beta} = -\lim_{i \to \infty} \frac{m_i}{n_i}.$$

Now we assume that there is a local conjugacy h, defined on a neighborhood of the closure of the orbit of r, as in the theorem. Let $h(r_i) = r'_i$. From the topology of the intersection of W^u and W^s and the position of the r'_i's (see figure) it follows that $\rho(r'_i, W^u(p')) \leq \rho(r'_i, W^s(q'))$; *i.e.*

$$\frac{\rho(r'_i, W^u(p'))}{\rho(r'_i, W^s(q'))} \leq 1.$$

Since $(f')^{-n_i}(r_i')$ and $(f')^{m_i}(r_i')$ must have a limit in $W^s(p') - \{p'\}$ resp. $W^u(q') - \{q'\}$, we conclude that $\rho(r_i', W^u(p')) \sim (\alpha')^{n_i}$ and $\rho(r_i', W^s(q')) \sim (\beta')^{-m_i}$. This, together with

$$\frac{\rho(r_i', W^u(p'))}{\rho(r_i', W^s(q'))} \leq 1$$

implies that $\dfrac{\log \alpha'}{\log \beta'} \leq -\lim_{i \to \infty} \dfrac{m_i}{n_i} = \dfrac{\log \alpha}{\log \beta}$. Reversing the argument, one finds $\dfrac{\log \alpha'}{\log \beta'} \geq \dfrac{\log \alpha}{\log \beta}$ and hence $\dfrac{\log \alpha'}{\log \beta'} = \dfrac{\log \alpha}{\log \beta}$.

Observe that if $W^u(p)$ is transverse to $W^s(q)$ and $W^u(p')$ is transverse to $W^s(q')$, then we cannot conclude for any sequence $r_i \to r$ that $\dfrac{\rho(r_i', W^u(p'))}{\rho(r_i', W^s(q'))}$ is bounded. So we needed some of the properties of the " topology of the intersection ". In the next higher dimensional case we have to analyze this in detail (1).

B) *The case* $\dim(W^u(p)) = \dim(W^u(q)) = m - 1$

As in case A we assume $\alpha, \beta, \alpha', \beta' > 0$ and we consider sequences $r_i \to r$ such that $f^{-n_i}(r_i)$ converges to a point in $W^s(p) - \{p\}$, and $f^{m_i}(r_i)$ converges to a point in $W^u(q) - \{q\}$ for some sequences $n_i, m_i \to \infty$. We also assume that one of the following three possibilities takes place (this can be obtained by taking a subsequence):

$$\frac{\rho(r_i, W^u(p))}{\rho(r_i, W^s(q))} \begin{cases} \to 0\,; \\ \text{is bounded and bounded away from zero (or } \sim 1); \\ \to \infty. \end{cases}$$

In these cases we find that

$$\frac{\log \alpha}{\log \beta} \begin{cases} \leq \liminf\limits_{i \to \infty} \left(-\dfrac{m_i}{n_i}\right); \\ = \lim\limits_{i \to \infty} \left(-\dfrac{m_i}{n_i}\right); \\ \geq \limsup\limits_{i \to \infty} \left(-\dfrac{m_i}{n_i}\right). \end{cases}$$

We assume that there is a local conjugacy h and denote $h(r_i)$ by r_i'. We assume also that $\dfrac{\rho(r_i', W^u(p'))}{\rho(r_i', W^s(q'))}$ either goes to zero, is ~ 1, or goes to ∞; in each of these cases we get a relation between $\dfrac{\log \alpha'}{\log \beta'}$ and " lim " $\left(-\dfrac{m_i}{n_i}\right)$ as above.

(1) After the arguments for sections B, C, and D were written, a cleaner treatment was discovered by S. Van Strien. This treatment uses arguments presented in the proof of Theorem (2.1) *in* [18a].

From this we claim that, if $\dfrac{\log\alpha}{\log\beta} < \dfrac{\log\alpha'}{\log\beta'}$, then, for each sequence $r_i \to r$ as above,

$$\frac{\rho(r_i, W^u(p))}{\rho(r_i, W^s(q))} \to 0 \quad \text{and/or} \quad \frac{\rho(r'_i, W^u(p'))}{\rho(r'_i, W^s(q'))} \to \infty.$$

Indeed suppose that $\dfrac{\log\alpha}{\log\beta} < \dfrac{\log\alpha'}{\log\beta'}$ and $r_i \to r$ as above. If $\dfrac{\rho(r'_i, W^u(p'))}{\rho(r'_i, W^s(q'))}$ does not go to ∞, then for some constant $C > 0$, we have

$$(\alpha')^{n_i}(\beta')^{m_i} \leq C.$$

This gives $n_i \log\alpha' + m_i \log\beta' \leq \log C$ or

$$\frac{\log\alpha'}{\log\beta'} \leq \frac{\log C}{n_i \log\beta'} - \frac{m_i}{n_i}.$$

If $\delta > 1$ is so that $\dfrac{\log\alpha + \log\delta}{\log\beta} = \dfrac{\log\alpha'}{\log\beta'}$, then

$$\frac{\log\alpha\delta}{\log\beta} \leq \frac{\log C}{n_i \log\beta} - \frac{m_i}{n_i},$$

or
$$n_i \log\alpha\delta + m_i \log\beta \leq \log C.\frac{\log\beta}{\log\beta'}.$$

This means that $(\alpha\delta)^{n_i}\beta^{m_i}$ is bounded, so $\alpha^{n_i}\beta^{m_i} \to 0$ as $i \to \infty$ which implies that

$$\frac{\rho(r_i, W^u(p))}{\rho(r_i, W^s(q))} \to 0.$$

This proves the claim. Similarly, for $\dfrac{\log\alpha}{\log\beta} > \dfrac{\log\alpha'}{\log\beta'}$ we have

$$\frac{\rho(r_i, W^u(p))}{\rho(r_i, W^s(q))} \to \infty \quad \text{and/or} \quad \frac{\rho(r'_i, W^u(p'))}{\rho(r'_i, W^s(q'))} \to 0.$$

We want to prove that this cannot happen for every sequence in case of a quasi-transversal intersection of $W^u(p)$ and $W^s(q)$ (and $W^u(p')$ and $W^s(q')$).

For this we take subsets A^s, A^u, A^0 of a neighborhood of r such that:

— $A^s \cup A^u \cup A^0$ is a neighborhood of r;

— if $r_i \to r$ is a sequence such that $\dfrac{\rho(r_i, W^u(p))}{\rho(r_i, W^s(q))} \to 0$, then $r_i \in A^u$ for i big;

— if $r_i \to r$ is a sequence such that $\dfrac{\rho(r_i, W^u(p))}{\rho(r_i, W^s(q))} \to \infty$, then $r_i \in A^s$ for i big;

— if $r_i \to r$ is a sequence such that $r_i \in A^0$ (and $r_i \notin W^u(p) \cup W^s(q)$) then

$$\frac{\rho(r_i, W^u(p))}{\rho(r_i, W^s(q))} \sim 1.$$

A'^u, A'^s and A'^0 are similar sets in a neighborhood of r'. From the above arguments it follows that if $\frac{\log \alpha}{\log \beta} < \frac{\log \alpha'}{\log \beta'}$, then there is a neighborhood U of r such that

$$h(U \cap (A^0 \cup A^s)) \subset A'^s$$

(and if $\frac{\log \alpha}{\log \beta} > \frac{\log \alpha'}{\log \beta'}$,

$$h(U \cap (A^0 \cup A^u)) \subset A'^u).$$

In order to derive a contradiction from this (namely from the assumption that $\left.\frac{\log \alpha}{\log \alpha} < \frac{\log \alpha'}{\log \beta'}\right)$ in the case of quasi-transversal intersection, we proceed to an explicit construction of A^u, A^s and A^0. From Chapter II we know that there are coordinates $x_1, \ldots, x_m$ in a neighborhood of r such that locally

$$W^s(q) = \{x_m = 0\}$$

and

$$W^u(p) = \{x_m = \xi(x_1, \ldots, x_{m-1})\}$$

where ξ is a homogeneous quadratic function. In these coordinates we take

$$A^s = \left\{|x_m| \leq \frac{1}{8}|\xi(x_1, \ldots, x_{m-1})|\right\}$$

$$A^u = \left\{|x_m - \xi(x_1, \ldots, x_{m-1})| \leq \frac{1}{8}|\xi(x_1, \ldots, x_{m-1})|\right\}$$

$$A^0 = \text{closure of the complement of } A^s \cup A^u.$$

Near r', $W^u(p')$ and $W^s(q')$ have the same form, so there is a diffeomorphism Ψ from a neighborhood of r to a neighborhood of r', mapping $W^u(p)$ to $W^u(p')$ and $W^s(q)$ to $W^s(q')$. We define $A'^s = \Psi(A^s)$, $A'^u = \Psi(A^u)$ and $A'^0 = \Psi(A^0)$.

It now follows that $\widetilde{h} = \Psi^{-1} \circ h$ is a local homeomorphism from a neighborhood of r to itself, inducing homeomorphisms in $W^u(p)$ and $W^s(q)$ and mapping $(A^0 \cup A^s) \cap U$ into A^s. Let $U^* \subset U$ be a subset of the form $U^* = \{(x_1, \ldots, x_m) \mid 0 < \Sigma x_i^2 < a^*\}$ and $U^{**} \supset \widetilde{h}(U^*)$ a set of the form $U^{**} = \{(x_1, \ldots, x_m) \mid 0 < \Sigma x_i^2 < a^{**}\}$. From the fact that the subsets $W^u(p)$, $W^s(q)$, A^u, A^s and A^0, restricted to U^* or U^{**} are all cone-formed, and the fact that $\widetilde{h}$ is a local homeomorphism inducing local homeomorphisms in $W^u(p)$ and $W^s(q)$ we conclude that the maps

$$U^* \cap W^s(q) \to U^{**} \cap W^s(q),$$
$$U^* - (W^u(p) \cap U^*) \to U^{**} - (W^u(p) \cap U^{**}),$$

induced by $\widetilde{h}$, induce isomorphisms in the homology. From the definitions it is clear that the inclusions

$$U^* \cap W^s(q) \to U^* \cap A^s, \qquad U^{**} \cap W^s(q) \to U^{**} \cap A^s,$$
$$U^* \cap (A^s \cup A^0) \to U^* - (W^u(q) \cap U^*),$$
$$U^{**} \cap (A^s \cup A^0) \to U^{**} - (W^u(q) \cap U^{**}),$$

are all homology equivalences. Now consider the morphism of long exact sequences:

$$\begin{array}{ccccc}
\to H_i(U^* \cap A^s) & \to & H_i(U^* \cap (A^s \cup A^0)) & \to & H_i(U^* \cap (A^s \cup A^0), U^* \cap A^s) \\
\downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle h_2} & & \downarrow{\scriptstyle h_3} \\
\to H_i(U^{**} \cap A^s) & \to & H_i(U^{**} \cap (A^s \cup A^0)) & \to & H_i(U^{**} \cap (A^s \cup A^0), U^{**} \cap A)
\end{array}$$

h_1, h_2 and h_3 are all induced by $\widetilde{h}$, so h_1, h_2 are isomorphisms. By the Five Lemma, h_3 is then also an isomorphism. But since $\widetilde{h}(U^* \cap (A^s \cup A^0)) \subset U^{**} \cap A^s$, h_3 is the zero morphism. Hence $H_i(U^* \cap (A^s \cup A^0), U^* \cap A^s) = 0$ for all i. This is in contradiction with the following result, which follows from standard arguments:

if $$\xi(x_1, \ldots, x_{m-1}) = x_1^2 + \ldots + x_k^2 - x_{k+1}^2 - \ldots - x_{m-1}^2$$

then

$$H_i(U^* \cap (A^s \cup A^0), U^* \cap A^s) = \begin{cases} \mathbf{Z} \text{ if } i = k \text{ or } i = m-k-1 \text{ provided } k \neq m-k-1; \\ \mathbf{Z}\oplus\mathbf{Z} \text{ if } i = k = m-k-1; \\ 0 \text{ otherwise.} \end{cases}$$

C) *The general case*: α, β, α', β' *real*

We assume that f has fixed points p, q and invariant manifolds H_p and H_q as in the theorem and $\alpha, \beta, \alpha', \beta' > 0$. Then, $\dim H_p = \dim W^u(p) + 1$, and $\dim H_q = \dim W^s(q) + 1$. Let $H_{p'}$, $H_{q'}$ denote the analogous manifolds for f'. Of course, if there is a conjugacy h between f and f', it does not follow that $h(H_p) = H_{p'}$ or $h(H_q) = H_{q'}$. If we knew that $h(H_p) = H_{p'}$ and $h(H_q) = H_{q'}$, we could apply the argument of case B for sequences $r_i \to r$ in $H_p \cap H_q$. For in $H_p \cap H_q$, the manifolds $W^u(p) \cap H_p \cap H_q$ and $W^s(q) \cap H_p \cap H_q$ have codimension one, and intersect "nearly" quasi-transversely. We say nearly because, since H_p and H_q are only C^1 manifolds, a quasi-transverse intersection of $W^u(p) \cap H_p \cap H_q$ and $W^s(q) \cap H_p \cap H_q$ is not defined. However, we can choose C^2 submanifolds $\widetilde{H}_p$ and $\widetilde{H}_q$ (not necessarily invariant) which are C^1 close to H_p and H_q, respectively, such that

$$H_p \cap \widetilde{H}_p \supset W^u(p), \quad H_q \cap \widetilde{H}_q \supset W^s(q),$$
$$W^u(p) \cap \widetilde{H}_p \cap \widetilde{H}_q = W^u(p) \cap H_p \cap H_q,$$
$$W^s(q) \cap \widetilde{H}_p \cap \widetilde{H}_q = W^s(q) \cap H_p \cap H_q,$$

and $W^s(q) \cap \widetilde{H}_p \cap \widetilde{H}_q$ intersects $W^u(p) \cap \widetilde{H}_p \cap \widetilde{H}_q$ quasi-transversely. Hence, the metric properties of the intersection of $W^u(p) \cap H_p \cap H_q$ and $W^s(q) \cap H_p \cap H_q$ in $H_p \cap H_q$ are those of a quasi-transversal intersection.

Since h does not map H_p to $H_{p'}$ or H_q to $H_{q'}$, we must modify our arguments. We shall show that there is a map σ from $h(H_p \cap H_q)$ to $H_{p'} \cap H_{q'}$ such that for any

$x \in h(H_p \cap H_q) - (W^u(p') \cup W^s(q'))$, we have $\rho(x, W^u(p')) \sim \rho(\sigma(x), W^u(p'))$ and $\rho(x, W^s(q')) \sim \rho(\sigma(x), W^s(q'))$.

Let N be a small disk neighborhood of p where we can define a C^2 coordinate chart such that the components of $W^s(p) \cap N$ and $W^u(p) \cap N$ containing p are coordinate planes. In the sequel, we will restrict ourselves to such a neighborhood N and to the components of $W^s(p)$ and $W^{ss}(p)$ containing p. Let $\pi_s : N \to W^s(p)$ be the natural projection and assume that the negative orbit of r lies in N.

Since $W^s(q)$ is transverse to H_p at r, we can characterize $W^{ss}(p)$ as the subset of $W^s(p)$ where the critical points of $\pi_s \mid W^s(q)$ accumulate. This characterization is based in the following fact:

If $v \in W^s(p)$ then $v \notin W^{ss}(p)$ if and only if there are a neighborhood V of v and a neighborhood U of r such that for each $v' \in V \cap W^s(p)$ there is an arbitrarily small neighborhood V' of v' in V such that, for $n \in \mathbf{N}$ suficiently big,

$$\{f^{-n}(W^s(q) \cap U) \cap V'\} \subset \{f^{-n}(W^s(q) \cap U) \cap V\}$$

is a homotopy equivalence. In this formula we should omit from $f^{-n}(W^s(q) \cap U)$ those points whose orbits from $W^s(q) \cap U$ to $f^{-n}(W^s(q) \subset U)$ leave N. From this dynamical characterization it follows that $h(W^{ss}(p)) = W^{ss}(p')$ and similarly $h(W^{uu}(q)) = W^{uu}(q')$.

Thus $h(H_p) \cap W^s(p')$ is an invariant C^0 curve of f' which meets $W^{ss}(p')$ only at p'. Since f' is C^1 linearizable on $W^s(p')$ [8], we see that $h(H_p) \cap W^s(p')$ has to lie in a small sector about $H_{p'} \cap W^s(p')$ which has width zero at $H_{p'} \cap W^u(p')$ as in the following figure.

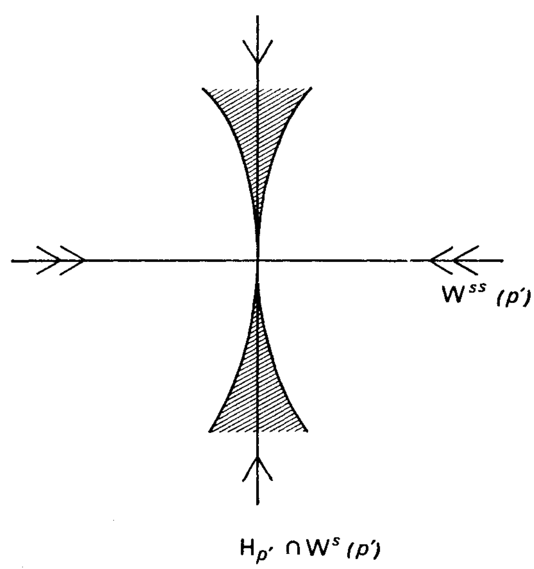

FIG. 5

From this it follows that for any smooth projection of a neighbourhood of $H_{p'}$ to $H_{p'}$, the distance of points in $h(H_{p'})$ to $W^u(p')$ is not changed essentially, *i.e.* the quotient of the new and old distances is uniformly bounded away from zero and infinity. It is not hard to choose such a projection σ_1 on $H_{p'}$ which does not essentially change (at least near r') the distances to $W^s(q')$.

Using the same type of arguments one can define a smooth local projection σ_2 on $H_{q'}$ in such a way that

— σ_2 maps $H_{p'}$ into itself
— σ_2 does not essentially change distances from $W^u(p')$.

As before, it follows that σ_2 does not essentially change the distances from points in $h(H_q)$ to $W^s(q')$. So $\sigma_2 \circ \sigma_1 \mid h(H_p \cap H_q) : h(H_p \cap H_q) \to H_{p'} \cap H_{q'}$ has the required properties.

To see that the whole reasoning of case B applies, we first choose metrics on H_p, H_q, $H_{p'}$, $H_{q'}$ induced by nearby C^2 manifolds $\widetilde{H}_p$, $\widetilde{H}_q$, $\widetilde{H}_{p'}$, $\widetilde{H}_{q'}$, respectively. Defining $\widetilde{A}^s$, $\widetilde{A}^0$, $\widetilde{A}^u \subset \widetilde{H}_p \cap \widetilde{H}_q$ as before and projecting them into H_p and H_q to obtain A^s, A^0, A^u, we then have that the maps $W^u(p) \cap H_p \cap H_q \to A^u$, $W^s(q) \cap H_p \cap H_q \to A^s$, etc., are homotopy equivalences. Now the argument can be completed by observing that

(1) σ roughly preserves distances;
(2) σ induces, locally at r', homotopy equivalences from $W^s(q') \cap h(H_p \cap H_q)$ to $W^s(q') \cap (H_{p'} \cap H_{q'})$ and from $h(H_p \cap H_q) - W^u(p')$ to $(H_{p'} \cap H_{q'}) - W^u(p')$;
(3) if $r'_i \to r'$, $r'_i \in h(H_p \cap H_q)$ are such that $f^{-n_i}(r'_i)$, respectively $f^{m_i}(r'_i)$, has a limit in $W^s(p') - p'$, respectively $W^u(q') - q'$, then $\rho(r'_i, W^u(p')) \sim (\alpha')^{n_i}$ and $\rho(r'_i, W^s(q')) \sim (\beta')^{-m_i}$.

These facts follow easily from the constructions.

D) *The general case*: α, α' *are not real or* β, β' *are not real*

Let Q be the $df(p)$-invariant subspace of $T_pW^s(p)$ complementary to the eigenspace of α, and let $\widetilde{Q}$ be the $df(q)$-invariant subspace of $T_qW^u(q)$ complementary to the eigenspace of β. Let $W^{ss}(p)$ be the invariant manifold in $W^s(p)$ tangent at p to Q, and let $W^{uu}(q)$ be the invariant manifold in $W^u(q)$ tangent at q to $\widetilde{Q}$. If H_p and H_q are the invariant manifolds in the statement of Theorem (1.1), then $W^{ss}(p)$ is transverse to H_p at p and $W^{uu}(q)$ is transverse to H_q at q. The fact that $W^u(p)$ is transverse to H_q at r enables us to get a dynamical characterization of $W^{uu}(q)$. Similarly, we will obtain a dynamical characterization of $W^{ss}(p)$. Once these are obtained, it will follow that a conjugacy h from f to f' as in the statement of Theorem (1.1) will have the property that $h(W^{ss}(p)) = W^{ss}(p')$ and $h(W^{uu}(q)) = W^{uu}(q')$. From this, it will follow as in case C that $h(H_q \cap W^s(p))$ is in a small sector about $H_{p'} \cap W^s(p')$ which meets $W^{ss}(p')$ only at p'. Similarly, $h(H_q \cap W^u(q))$ is in a small sector about $H_{q'} \cap W^u(q')$ which meets $W^{uu}(q')$ only at q'. The arguments then proceed as in case C. In the present situation either or both of $W^u(p)$ and $W^s(q)$ will have codimension 2 in H_p and H_q, respectively. This changes the arguments slightly, but, since these changes are straightforward, they will be left to the reader. Thus to complete the proof of Theorem (1.1), we must show how $W^{uu}(q)$ is characterized dynamically.

Let U be a small disk neighborhood of q containing the forward orbit of r. We can take U as a coordinate chart with a C^2 coordinate system having the components of $W^u(q)$ and $W^s(q)$ containing q as coordinate planes. Let $\pi_u : U \to W^u(q)$ be the natural projection. For an integer $n > 0$ let f_U^n denote $f^n \mid \bigcap_{0 \leq j \leq n} f^{-j}(U)$. Define $W_{loc}^{uu}(q)$ to be the set of points $y \in W^u(q) \cap U$ such that

(*) for each sequence $0 < n_1 < n_2 < \ldots$, the critical points of $\pi_u \mid f_U^{n_i}(W^u(p))$ accumulate on y as $i \to \infty$; in case $\dim(W^u(p)) + \dim(W^s(q)) = \dim(M) - 1)$, $f_U^{n_i}(W^u(p))$ accumulates on y.

We assert that $W^{uu}(q) = \bigcup_{i \geq 0} f^i(W_{loc}^{uu}(q))$. This implies a dynamical characterization of $W^{uu}(q)$.

To prove this assertion, first suppose that $f \mid U$ is linear and, near r, the set of critical points (or fold points) L of $\pi_u \mid W^u(p)$ is an affine subspace. Here we think of U as an open subset of Euclidean space via linearizing coordinates. Note that $\dim(L) = \dim(W^u(q)) - 1$. If the eigenvalue β is real, the assertion follows as in the previous case when the eigenvalues α, β were taken to be real. So assume β is not real. Assume also that H_q, $W^u(q)$ and $W^{uu}(q)$ are linear subspaces of U near q. Choose an affine subspace $L_1 \subset L$ which is complementary to H_q at r. Then $f^n(L_1)$ converges to $\tilde{Q}$ as $n \to \infty$. This implies that (*) holds for any $y \in W^{uu}(q)$ near q. Now let y be in $W^u(q) - W^{uu}(q)$ and near q. Since $f \mid H_q \cap W^u(q)$ near q is a rotation, there is a sequence of integers $n_1 < n_2 < \ldots$ such that $f_U^{n_i}(L \cap H_q)$ does not accumulate on πy as $\tau \to \infty$ where $\pi : W^u(q) \to W^u(q) \cap H_q$ is the natural projection. This implies that $f_U^{n_i}(L)$ does not accumulate on y as $\tau \to \infty$. This proves the assertion if $f \mid U$ is linear and L is affine. The extension to the case where f is not linear or L is not affine near r is straightforward. We leave the details to the reader. Theorem (1.1) is proved.

Lemma (**1.2**). — *Suppose f is an elementary diffeomorphism and x is a quasi-transversal intersection of* $W^u(p, f)$ *and* $W^s(q, f)$ *with p and q periodic points of f. Then f is not topologically conjugate to any Kupka-Smale diffeomorphism. Moreover, if f' is also elementary and h is a topological conjugacy between f and f', then $h(x)$ is a quasi-transversal intersection of* $hW^u(p, f)$ *and* $hW^s(q, f)$.

Proof. — This follows from the fact that a transversal intersection of two submanifolds W^s and W^u of a manifold M is topologically different from any quasi-transversal intersection of W^s and W^u. In a point of quasi-transversal intersection, either

$$\dim W^s + \dim W^u \neq \dim M$$

in which case the intersection is not a manifold of the right dimension or,

$$\dim W^s + \dim W^u = \dim M$$

in which case the intersection number is zero. See Chapter II, section 6.

Note that not all non-transversal intersections are topologically different from transversal ones.

Theorem **(1.3).** — *Suppose f, p, q, α, β, and r are as in the hypotheses of Theorem* (1.1). *Then there is a residual set $\mathscr{P}_1 \subset \mathscr{P}$ such that if $\varphi \in \mathscr{P}_1$, then no φ_μ, $0 \leq \mu \leq 1$, is topologically conjugate to f.*

Proof. — There is a residual set $\mathscr{P}_0 \subset \mathscr{P}$ such that if $\varphi \in \mathscr{P}_0$, then

(1) each φ_μ is elementary or Kupka-Smale;
(2) if some φ_μ has a quasi-transversal orbit, then the hypotheses of Theorem (1.1) with $\varphi_\mu = f'$ are satisfied;
(3) there are at most countably many μ's for which φ_μ has a quasi-transversal orbit.

Now, since each φ_μ has at most one quasi-transversal orbit, it follows that the quotients of the logarithms of the moduli of quasi-transversal orbits occurring in $\{\varphi_\mu\}$ form a set which is at most countable. With standard arguments, one can show that there is a residual subset $\mathscr{P}_1$ of $\mathscr{P}_0$ such that if $\varphi \in \mathscr{P}_1$, then all the quotients occurring for φ are different from $\dfrac{\log|\alpha|}{\log|\beta|}$. Now Theorem (1.3) follows from Theorem (1.1) and Lemma (1.2).

Corollary **(1.4).** — *If $\varphi \in \mathscr{B}$ is left stable, then all stable and unstable manifolds of periodic points of φ_b intersect transversally.*

Proof. — Since $\varphi \in \mathscr{B}$, we have that φ_b is elementary. Thus, if $\varphi_b = f$ has a quasi-transversal orbit, then we may assume the hypotheses of Theorem (1.1) and hence Theorem (1.3) are satisfied. By Theorem (1.3), φ is not left stable.

2. Necessary conditions for mild stability and stability

Suppose $\varphi \in \mathscr{B}$ and $\mathcal{O}(p)$ is a saddle-node for φ_b. We say $\mathcal{O}(p)$ is s-critical for φ_b if there is a periodic orbit $\mathcal{O}(q)$ such that $W^u(\mathcal{O}(q))$ has a non-transversal intersection with the strong stable foliation $\mathscr{F}^{ss}$ of $W^s(\mathcal{O}(p))$. Similarly, we say $\mathcal{O}(p)$ is u-critical if it is s-critical for φ_b^{-1}. If $\mathcal{O}(p)$ is either s-critical or u-critical but not both, we say that $\mathcal{O}(p)$ is semi-critical. If $\mathcal{O}(p)$ is both s-critical and u-critical, we say it is bicritical.

Proposition **(2.1).** — *If φ is mildly stable, then it is left stable and the quasi-hyperbolic periodic orbit is not a Hopf orbit or bi-critical saddle-node.*

Proof. — The first statement is obvious and the second statement was proved in Section 5 of Chapter II. Now suppose $\mathscr{O}(p)$ is a bicritical saddle-node for φ_b. We assume p is a fixed point of φ_b. If q_1 and q_2 are periodic points of φ_b such that $W^u(q_1)$ has a non-transverse intersection with the strong stable foliation of $W^s(p)$ and $W^s(q_2)$ has a non-transverse intersection with the strong unstable foliation of $W^u(p)$, then, perturbing if necessary, we may assume that these intersections are quasi-transverse. For some μ near b, it then follows that $W^u(q_{1\mu}, \varphi_\mu)$ has a quasi-transverse intersection with $W^s(q_{2\mu}, \varphi_\mu)$. Here, of course, $q_{1\mu}$ and $q_{2\mu}$ denote the unique hyperbolic periodic points of φ_μ near q_1 and q_2, respectively. Using Theorem (1.3) we can perturb to φ' in $\mathscr{P}$ so that no φ'_ν is conjugate to φ_μ. This shows that φ is not mildly stable.

Proposition **(2.2)**. — *If φ is mildly stable, then φ_b has no cycle of length bigger than one.*

Proof. — Suppose φ_b has a cycle of length bigger than one. We show φ is not mildly stable. Since $\varphi \in \mathscr{B}$, φ_b has a saddle-node orbit $\mathscr{O}(p_0)$ which is contained in every cycle. Indeed, the transversality of the stable and unstable manifolds implies that if φ_b had any cycle not containing a saddle nodes, then φ_b would have transversal homoclinic points. This would give φ_b infinitely many periodic points [34]. Let $\mathscr{O}(p_0), \ldots, \mathscr{O}(p_r)$ be the distinct orbits in the cycles of φ_b. Replacing $\{\varphi_\mu\}$ by some power $\{\varphi_\mu^n\}$, we assume that all the p_i's are fixed points. Let $W^{ss}(p_0)$, resp. $W^{uu}(p_0)$, denote the strong stable, resp. unstable, manifold of p_0. Since $L(\varphi_b)$ has finitely many orbits it follows that $W^u(p_i) \cap W^{ss}(p_0) = \varnothing$ and $W^s(p_i) \cap W^{uu}(p_0) = \varnothing$ for $0 < i \leq r$. Otherwise, φ_b would again have transversal homoclinic points.

By transversality, we have

$$\dim W^u(p_i) = \dim W^u(p_0) \quad \text{or} \quad \dim W^u(p_i) = \dim W^u(p_0) - 1$$

for each $0 \leq i \leq r$. Replacing $\{\varphi_\mu\}$ by $\{\varphi_\mu^{-1}\}$, if necessary, it is enough to consider the case in which there is a $1 \leq j \leq r$ such that $\dim W^u(p_j) = \dim W^u(p_0)$. This implies $\dim W^s(p_j) \cap W^u(p_0) = 0$ and hence p_0 is u-critical.

Observe that, having $\dim W^u(p_j) = \dim W^u(p_0)$ for some $1 \leq j \leq r$, we may indeed assume that for all $0 \leq i \leq r$

(a) $\dim W^u(p_i) = \dim W^u(p_0)$,
(b) $W^u(p_i)$ is transverse to the strong stable foliation $\mathscr{F}^{ss}$ of $W^s(p_0)$,
(c) $cl\, W^u(p_i) \cap W^{ss}(p_0) = \{p_0\}$.

For suppose *(a)* or *(b)* failed for some $1 \leq i \leq r$. Then p_0 is s-critical. Since $\dim W^u(p_j) = \dim W^u(p_0)$, we already know that p_0 is u-critical. Thus p_0 is bicritical and Proposition (2.1) has already ruled this out. Let us now suppose that *(b)* fails for $i = 0$; that is, $W^u(p_0)$ has a nontransverse intersection with $\mathscr{F}^{ss}$. We already know that $W^u(p_i)$ is transverse to $\mathscr{F}^{ss}$ for $1 \leq i \leq r$. Perturbing φ, if necessary, we may assume that $W^u(p_0)$ has a quasi-transversal intersection with some leaf F in $\mathscr{F}^{ss}$. Choose disks $D_1 \subset W^u(p_0)$ and $D_2 \subset F$ so that D_1 and D_2 have a quasi-transversal intersection. Let ξ

be an invariant center manifold in $W^s(p_0)$. Since $W^u(p_1, b)$ meets $W^s(p_0)$ transversally. and is transverse to $\mathscr{F}^{ss}$, it follows that each component of $W^u(p_1, b) \cap W^s(p_0)$ is an embedded curve whose closure near p_0 can be given as the graph of a smooth function from ξ to $W^{ss}(p_0)$ as in the next figure.

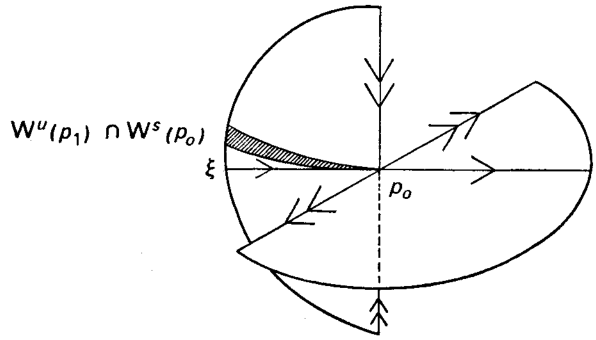

FIG. 6

As μ increases, pieces of $W^u(p_1, \mu)$ will sweep across the local part of $W^s(p_0)$ near p_0, so for each $\mu > b$ near b, $W^u(p_1, \mu)$ contains a disk $D_{1,\mu}C^2$ near D_1.

On the other hand, $W^s(p_1, b)$ accumulates backward on $W^{ss}(p_0)$. For certain μ's near b and greater than b, $W^s(p_1, \mu)$ will contain a disk $D_{2,\mu}C^2$ near D_2. Considering the continuous movements of $D_{1,\mu}$ and $D_{2,\mu}$ as μ varies, one sees that for certain μ's near b and greater than b, $D_{1,\mu}$ has a quasi-transverse intersection with $D_{2,\mu}$. Thus, if *(b)* fails for $i = 0$, there are μ's near b for which $W^u(p_{1,\mu})$ has a quasi-transverse intersection with $W^s(p_{1,\mu})$. In view of Theorem (1.3), we conclude that φ is not mildly stable. Finally if *(c)* failed for some i, then $cl\, W^u(p_i)$ would meet a fundamental domain for $W^{ss}(p_0)$. Since all intersections of $W^u(p_i)$ and $W^s(p_k)$ are transverse, this implies $W^u(p_i) \cap W^{ss}(p_0) \neq \emptyset$ as in the Morse-Smale case ([24]; Lemma (1.5)). As we have already mentioned, this gives transverse homoclinic points—an impossiblity.

Thus, we assume *(a)*, *(b)*, and *(c)* hold for all $0 \leq i \leq r$. We proceed to derive a contradiction from this, and, then, Proposition (2.2) will be proved.

From *(a)*, *(b)*, and *(c)* it follows that each component of $W^u(p_i) \cap W^s(p_0)$, $0 \leq i \leq r$, is an embedded curve whose closure near p_0 is the graph of a smooth function from the center manifold ξ to $W^{ss}(p_0)$.

Let p_{i_0} be a fixed point such that $W^u(p_{i_0}) \cap W^s(p_0) \neq \emptyset$, $W^u(p_0) \cap W^s(p_{i_0}) \neq \emptyset$, and for each $1 \leq i \leq r$ with $i \neq i_0$, $W^u(p_{i_0}) \cap W^s(p_i) = \emptyset$. Then, $W^u(p_{i_0}) \cap W^s(p_0)$ consists of finitely many curves which are permuted by φ_b. Also, the boundary of each such curve is $\{p_0, p_{i_0}\}$. By the boundary of a curve γ, we mean $cl\,\gamma - \gamma$, of course. Since $W^u(p_{i_0})$ is transverse to $W^s(p_0)$, we conclude that the closure of $W^s(p_0)$ near p_{i_0} is a finite union of closed half-spaces bounded by $W^s(p_{i_0})$. Also, since $W^u(p_0)$ has a transverse intersection with $W^s(p_{i_0})$ and $\dim W^u(p_0) = \dim W^u(p_{i_0})$, $W^u(p_0)$ accumulates on $W^u(p_{i_0})$ in the C^1 sense. Thus, we see that $W^u(p_0) \cap W^s(p_0)$ has a component γ which is an embedded curve whose boundary is $\{p_0, x\}$ where $x \in W^s(p_{i_0}) - \{p_{i_0}\} \cap W^u(p_0)$. This is depicted in the next figure.

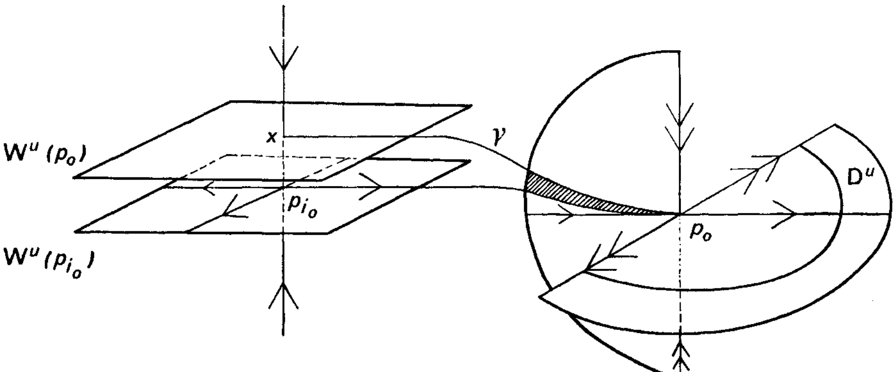

FIG. 7

We will show that the existence of γ leads to a contradiction. Let D be a small half-disk neighborhood about p_0 in $W^u(p_0)$ so that $\varphi_b(\partial D - W^{uu}(p_0)) \cap (\partial D - W^{uu}(p_0)) = \emptyset$. Let $D^u = cl(\varphi_b D - D)$ be a fundamental domain for $W^u(p_0)$. Choose $N > 0$ so that $n \geq N$ implies that $\varphi_b^{-n} x \in D$. It follows that, for $n \geq N$, $\varphi_b^{-n}\gamma$ is an arc whose intersection with $D^u - W^{uu}(p_0)$ has at least one component joining the two components of $\partial(D^u - W^{uu}(p_0))$. We may suppose that ∂D^u is transversal to each curve $\varphi_b^r \gamma$, $r \in \mathbf{Z}$. Hence, the set E of points $y \in D^u - W^{uu}(p_0)$, for which there is a sequence $y_i \in \varphi_b^{-n_i}(\gamma)$, $n_i \to \infty$, such that $y_i \to y$, is uncountable. But $E \subset cl\, W^s(p_0) \cap D^u$.

We assert that $E \cap W^s(p_0) = \emptyset$. For the moment, assume this assertion holds. Then $E \subset \bigcup_{1 \leq i \leq r} W^s(p_i)$. Since $\dim W^s(p_i) + \dim W^u(p_0) = \dim M$, this would put E in the countable set $D^u \cap \bigcup_{1 \leq i \leq r} W^s(p_i))$ which is a contradiction.

Thus, we must prove that $E \cap W^s(p_0) = \emptyset$. Suppose $E \cap W^s(p_0) \neq \emptyset$ and let $y \in E \cap W^s(p_0)$. Choose $y_i \in \varphi_b^{-n_i}(\gamma)$, $n_i \to \infty$, so that $y_i \to y$ as $i \to \infty$. We claim that y_i goes to infinity in $W^s(p_0)$; *i.e.* given any closed disc $F \subset W^s(p_0)$ transverse to D^u there is an $i_0 > 0$ so that $y_i \notin F$ for $i \geq i_0$. This follows from the facts that $\varphi_b^m \gamma \cap \varphi_b^r \gamma = \emptyset$ for $m \neq r$ and $F \cap D^u$ has only finitely many components. Now let D^s and D^{ss} be fundamental domains for $W^s(p_0)$ and $W^{ss}(p_0)$, respectively, and let D^{uu} be a fundamental domain for $W^{uu}(p_0)$. There is an $N > 0$ so that $\varphi_b^N(y) \in D^s$. Since $\varphi_b^N(y_i)$ converges to the point $\varphi_b^N(y)$ in D^s and $\varphi_b^N(y_i)$ goes to infinity in $W^s(p_0)$, there is an $m_i > 0$ such that $\varphi_b^{N+m_i}(y_i)$ accumulates on D^{uu}. Thus, $cl\, W^s(p_0) \cap D^{uu} \neq \emptyset$. But this implies that φ_b has transversal homoclinic points which is impossible. Thus $E \cap W^s(p_0) = \emptyset$ and Proposition (2.2) is proved.

Proposition **(2.3)**. — *If φ is mildly stable, then φ_b cannot have a non-critical* 1*-cycle.*

Proof. — As before, if φ_b has such a cycle, it must have a saddle-node $o(p)$ which is in the cycle. Since the cycle is non-critical, $W^u(o(p))$ is transverse to the strong stable foliation of $W^s(\mathcal{O}(p))$ and $W^s(\mathcal{O}(p))$ is transverse to the strong unstable foliation of $W^u(\mathcal{O}(p))$. Let n_1 be the period of p. Using invariant manifold theory one sees that $W^u(\mathcal{O}(p)) \cap W^s(\mathcal{O}(p))$ is a finite union of C^∞ circles $S_{1b}, \ldots, S_{kb}$ which are permuted by $\varphi_b^{n_1}$. Taking n_1 larger, we may assume $\varphi_b^{n_1}(S_{ib}) = S_{ib}$ for $1 \leq i \leq k$. Replacing φ by a suitable C^∞ approximation, we may assume for μ near b that there is a C^∞ circle $S_{1\mu}$ near S_{1b} which is invariant by $\varphi_\mu^{n_1}$, and the rotation number of $\varphi_\mu^{n_1} \mid S_{1\mu}$ is not constant as a function of μ near b. From the recent work of M. Herman [9] we may find a μ near b such that $\varphi_\mu^{n_1} \mid S_{1\mu}$ is C^∞ conjugate to a geometric rotation through an angle α with $\alpha/2\pi$ irrational. By a small modification of φ to φ', we may make $\varphi_\mu'^{n_1} \mid S_{1\mu}$ C^∞ conjugate to a rotation through an angle α' with $\alpha'/2\pi$ rational. This means that φ'_μ has uncountably many periodic points of the same period. But, since each φ_μ is elementary, it has only finitely many periodic points of a given period. Thus φ is not mildly stable.

Remark. — The previous propositions leave open the possibility that a mildly stable arc might have a critical saddle-node in a 1-cycle at φ_b. We will prove in section 4 that this cannot occur if the stable or unstable manifold of the saddle-node is one dimensional. In particular, it cannot occur if $\dim M = 2$. In general, for $\dim M > 2$, we feel that this cannot occur, but we have no complete proof at this time. The next proposition shows that the situation regarding stability is better.

Proposition **(2.4)**. — *If φ is stable, then φ_b has no cycles and no semi-critical saddle-nodes.*

Proof. — We have already taken care of cycles of length bigger than one and non-critical 1-cycles in Propositions (2.2) and (2.3). Suppose φ_b has a semi-critical 1-cycle containing the saddle-node orbit $\mathcal{O}(p)$. We assume p is fixed by φ_b and $W^u(p)$ has a non-transverse intersection with the strong stable foliation $\mathcal{F}^{ss}$ of $W^s(p)$. The other cases are handled similarly. Perturbing, if necessary, we may assume that $\mathcal{F}^{ss}$ is a C^2 foliation, that all intersections of $W^u(p)$ with leaves of $\mathcal{F}^{ss}$ are transverse or quasi-transverse, and that all the eigenvalues of $d\varphi_b$ on $T_pW^{ss}(p)$ have multiplicity one with distinct norms. Let α be the weakest contracting eigenvalue of $d\varphi_b$ on $T_pW^{ss}(p)$, and let C_p be a C^1 invariant manifold in $W^{ss}(p)$ tangent at p to the eigenspace of α. Let H_p be a C^1 invariant manifold containing $W^u(p)$ and C_p as in the proof of Theorem (1.1). Let $x \in W^u(p) \cap W^s(p)$ be a quasi-transversal intersection of $W^u(p)$ and $\mathcal{F}^{ss}_x$ at x. If γ is a curve in $H_p \cap W^s(p)$ transverse to $W^u(p)$ at x, then γ is transverse to $\mathcal{F}^{ss}_x$ in $W^s(p)$ at x and, hence, projects diffeomorphically near x along the leaves of $\mathcal{F}^{ss}$ into an invariant center manifold ξ for φ_b. Let D be a fundamental domain for $W^{ss}(p)$. Thus D is a compact set in $W^{ss}(p)$ such that $\varphi_b^2 D \cap D = \emptyset$ and $W^{ss}(p) - \{p\} \subset \bigcup_{n \in \mathbb{Z}} \varphi_b^n(D)$. Let U be a small compact neighborhood of D. If $y \in \gamma$ is near x, then $\rho(y, x) \sim |\alpha|^{n(y)}$ where $n(y)$ is the least positive integer n such that $\varphi_b^{-n}(y) \in U$. Here, as in the proof of Theorem (1.1), we use $\rho(y, x) \sim |\alpha|^n$ to mean that $\dfrac{\rho(y, x)}{|\alpha|^n}$ is bounded and bounded away from zero independent of n.

Now let φ' be a perturbation of φ such that $b(\varphi') = b(\varphi)$, $|\alpha'| < |\alpha|$, and $\mathscr{F}^{ss}$ is still a C^2 foliation. Using Chapter II, we may assume that on the center manifold ξ through p, φ_b is the time-one map of a C^∞ vector field X_b which vanishes only at p. Also, there is a corresponding vector field X'_b for φ'_b. Suppose there is a continuous conjugacy $\{h_\mu\}$ between φ and φ'. By Theorems (3.2) and (3.4) of Chapter II, we know that

(1) h_b is C^∞ along $\xi - \{p\}$;

(2) h_b maps X_b to X'_b; and

(3) h_b maps the strong stable foliation $\mathscr{F}^{ss}$ to the corresponding strong stable foliation $\mathscr{F}'^{ss}$.

Since $\mathscr{F}'^{ss}$ is C^2 near $\mathscr{F}^{ss}$, all intersections of $W^u(p')$ and leaves of $\mathscr{F}'^{ss}$ will be transversal or quasi-transversal. As in the proof of Lemma (1.2), this implies that $h_b x$ is a quasi-transversal intersection of $W^u(p')$ and $\mathscr{F}'^{ss}_{h_b x}$. Applying the reasoning in section D of the proof of Theorem (1.1) to $W^{ss}(p)$, we see that, since H_p is transverse to $\mathscr{F}^{ss}_x$, the invariant manifold in $W^{ss}(p)$ tangent to the sum of the eigenspaces complementary to the eigenspace of α has a dynamical characterization. It is the set of points where the backward orbit of $\mathscr{F}^{ss}_x$ accumulates. Thus, $h_b(H_p \cap W^{ss}(p))$ is in a sector about $H'_p \cap W^{ss}(p')$ in $W^{ss}(p')$. This implies that if y is near x in γ and $\rho(x,y) \sim |\alpha|^{n(y)}$, then $\rho(h_b y, h_b x) \sim |\alpha'|^{n(y)}$. Hence,

$$(4) \qquad \lim_{y \to x} \frac{\rho(h_b y, h_b x)}{\rho(y, x)} = 0.$$

If $\pi : W^s(p) \to \xi$ is the projection along the leaves of $\mathscr{F}^{ss}_b$, then (4) implies that h_b has derivative zero at $\pi(x)$. But this contradicts the fact that $h_b(X_b(\pi x)) = X'_b(h_b \pi(x))$.

The proof of the fact that stability of φ implies that φ_b has no saddle-node orbit $\mathscr{O}(p)$ which is semi-critical via some other periodic orbit $\mathscr{O}(q)$ is similar. If say $W^u(\mathscr{O}(q))$ has a non-transverse intersection with $\mathscr{F}^{ss}_b$, we repeat the preceding argument replacing $W^{ss}(\mathscr{O}(p))$ by $W^s(\mathscr{O}(q))$. The remaining case of $W^s(\mathscr{O}(q))$ having a non-transverse intersection with $\mathscr{F}^{uu}_b$ follows replacing $\{\varphi_\mu\}$ by $\{\varphi_\mu^{-1}\}$.

Remark. — We observe that stable arcs in $\mathscr{A}$ must lie in $\mathscr{B}$. This follows from a somewhat more general observation, namely that any stable arc $\{\varphi_\mu\}$, not necessarily starting in MS, with a bifurcation for $\mu = b$ such that the limit set of φ_b has finitely many orbits has the following properties:

— all stable, strong stable, unstable, and strong unstable manifolds of φ_b intersect transversally;

— φ_b has exactly one non-hyperbolic periodic orbit, which is either a flip or a non-critical saddle-node without a cycle; this non-hyperbolic periodic orbit unfolds generically; such a φ_b can have no cycles because it has finitely many orbits in its limit set;

— there is an $\varepsilon > 0$ such that if $\mu \in (b - \varepsilon, b + \varepsilon) - \{b\}$, then φ_μ is MS.

This statement follows from the previous arguments in the following way. If all the periodic orbits of φ_b are hyperbolic then, since the limit set has only finitely many orbits and φ_b is not structurally stable, there must be a non-transversal intersection of a stable and an unstable manifold. By the results of section one of this chapter the arc $\{\varphi_\mu\}$ would then be unstable. Also, the non-hyperbolic periodic points have to unfold generically otherwise we would not even have local stability. It is also clear that no more than one orbit of φ_b can be non-hyperbolic. If φ_b has a Hopf point or a non-critical 1-cycle, then there are nearby arcs with a smooth invariant circle with irrational rotation; as observed in the present section that also contradicts stability of the arc. Also in this section we saw that a saddle-node with criticallity and or a cycle of length bigger than one is impossible if the arc $\{\varphi_\mu\}$ is stable. If φ_b has a non-transverse intersection of the stable, strong stable, unstable, or strong unstable manifold of the non-hyperbolic periodic point with any of the other stable or unstable manifolds, then there is arbitrarily near $\{\varphi_\mu\}$ an arc $\{\varphi'_\mu\}$ and a $\bar{\mu} \in \mathbf{R}$ near b such that $\varphi_{\bar{\mu}}$ has a non-transversal intersection of a stable and an unstable manifold of hyperbolic periodic orbits. Again by the results of section 1 this implies $\{\varphi'_\mu\}$, and hence, $\{\varphi_\mu\}$ is unstable. Finally, we indicate why $\varphi_\mu \in \mathrm{MS}$ for μ near b and $\mu \neq b$. Since φ_b has no cycles and the limit set of φ_b has only finitely many orbits, the Birkhoff center of φ_b must be finite. Then the arguments in [15] show that the limit set of φ_b equals the Birkhoff center, and, hence, must also be finite. Because φ_b is elementary, it then follows that $\varphi_\mu \in \mathrm{MS}$ for μ near b and $\mu \neq b$.

With arguments similar to those in the preceding paragraph, one can show that a mildly stable arc in $\mathscr{A}$ must lie in $\mathscr{B}$. There are, however, left stable arcs in $\mathscr{A}$ which are not in $\mathscr{B}$: φ_b might, for example, have two Hopf points or flips. All such arcs are left conjugate to left stable arcs in $\mathscr{B}$.

3. Endomorphisms of the circle

In this section we first extend the notion of rotation number, defined by H. Poincaré for diffeomorphisms of the circle, to endomorphisms of degree one. Instead of a number we get in general a closed interval of the real line, which we call rotation set. These rotation sets are then used to analyze a class of 1-parameter families of endomorphisms with non-degenerate folds (see *(b)* below). We show that each such family must go through homoclinic trajectories with folds (see Theorem (3.7)). This result has a direct application to the bifurcation of diffeomorphisms exhibiting a saddle-node with one cycle described in the previous section. If the saddle-node has a one dimensional stable or unstable manifold, they must go through a non-transversal homoclinic orbit. An interesting question is if such families of endomorphisms, for which the rotation sets vary, have bifurcation sets of positive measure. For the diffeomorphism case, see [10].

A) *Rotation sets for endomorphisms of the circle*

We identify the circle S^1 with $\mathbf{R}/\mathbf{Z}$. By End(S) we denote the set of continuous maps $\Phi : S^1 \to S^1$ of degree 1. On End(S^1) we use the usual C^0 topology. For each

$\Phi \in \mathrm{End}(S^1)$ there is a $\overline{\Phi} : \mathbf{R} \to \mathbf{R}$ such that $\pi\overline{\Phi} = \Phi\pi$ where $\pi : \mathbf{R} \to S^1 \to \mathbf{R}/\mathbf{Z}$ is the canonical projection. $\overline{\Phi}$ is called a lift of Φ and it is unique up to the addition of an integer. Each such $\overline{\Phi}$ satisfies $\overline{\Phi}(x+1) = \overline{\Phi}(x) + 1$.

For $\Phi \in \mathrm{End}(S^1)$ with lift $\overline{\Phi}$ and $x \in \mathbf{R}$ we define the rotation number $\rho(\overline{\Phi}, x)$ to be $\limsup_{n\to\infty} \frac{1}{n}(\overline{\Phi}^n(x) - x)$ and the rotation set $\rho(\overline{\Phi})$ to be closure $\{\rho(\overline{\Phi}, x) \mid x \in \mathbf{R}\}$. Note that if we take a different lift, say $\overline{\overline{\Phi}}$, of Φ then $\rho(\overline{\Phi}, x)$ and $\rho(\overline{\overline{\Phi}}, x)$, and hence also $\rho(\overline{\Phi})$ and $\rho(\overline{\overline{\Phi}})$, are equal up to translation by some integer. If $x' \in \mathbf{R}$ with $\pi(x') = \pi(x)$ then $\rho(\overline{\Phi}, x') = \rho(\overline{\Phi}, x)$. Hence, up to translation by integers, $\rho(\overline{\Phi}, x)$ and $\rho(\overline{\Phi})$ are invariants of Φ, $\pi(x)$; if no confusion seems possible we may denote them by $\rho(\Phi, \pi(x))$, $\rho(\Phi)$. We note that $\rho(\Phi, p)$ and $\rho(\Phi)$, $p \in S^1$, are topological invariants: if $h : S^1 \to S^1$ is a homeomorphism, then

$$\rho(\Phi, p) = \rho(h\Phi h^{-1}, h(p)) \quad \text{and} \quad \rho(\Phi) = \rho(h\Phi h^{-1}).$$

Finally, if Φ is an orientation preserving homeomorphism, then $\rho(\Phi)$ is the usual rotation number [29].

Lemma (3.1). — *If* $\Phi \in \mathrm{End}(S^1)$ *and* $\overline{\Phi}$ *is a lift of* Φ *and if there is no periodic point of* Φ *with rotation number* $\frac{p}{q}$, $p \in \mathbf{Z}$, $q \in \mathbf{N}$, i.e. *if there is no* $x \in \mathbf{R}$ *with* $\Phi^q(\pi(x)) = \pi(x)$ *and* $\rho(\overline{\Phi}, x) = \frac{p}{q}$, *then* $\rho(\overline{\Phi})$ *is contained in* $\left\{x \in \mathbf{R} \mid x < \frac{p}{q}\right\}$ *or in* $\left\{x \in \mathbf{R} \mid x > \frac{p}{q}\right\}$.

Proof. — If, for some $x \in \mathbf{R}$, $\overline{\Phi}^q(x) - x = p$ then $\pi(x)$ is a periodic point with rotation number $\frac{p}{q}$; so this does not happen. Hence either $\overline{\Phi}^q(x) - x < p$ for all $x \in \mathbf{R}$ or $\overline{\Phi}^q(x) - x > p$ for all $x \in \mathbf{R}$. Since $\overline{\Phi}^q(x) - x$ is periodic in x (with period 1) there is some $\varepsilon > 0$ such that $\overline{\Phi}^q(x) - x < p - \varepsilon$ for all $x \in \mathbf{R}$ or $\overline{\Phi}^q(x) - x > p + \varepsilon$ for all $x \in \mathbf{R}$. But then $\rho(\overline{\Phi}) \subset \left\{x \in \mathbf{R} \mid x \leq \frac{p-\varepsilon}{q}\right\}$, respectively $\rho(\Phi) \subset \left\{x \in \mathbf{R} \mid x \geq \frac{p+\varepsilon}{q}\right\}$.

Corollary (3.2). — *If* $\Phi \in \mathrm{End}(S^1)$, $\alpha, \beta \in \rho(\Phi)$ *and* $\alpha \leq \frac{p}{q} \leq \beta$ *for some rational number* $\frac{p}{q}$, *then* Φ *has a periodic point with rotation number* $\frac{p}{q}$ *and hence* $\frac{p}{q} \in \rho(\Phi)$. *Since* $\rho(\Phi)$ *is closed* (by definition), $\rho(\Phi)$ *must be either a single point in* $\mathbf{R}$ *or a closed interval.*

Now we want to show that $\rho(\Phi)$ depends continuously on Φ. For this we introduce the following notation: if $\Phi \in \mathrm{End}(S^1)$, $\overline{\Phi}$ a lifting of Φ then $\rho_1(\overline{\Phi}), \rho_2(\overline{\Phi}) \in \mathbf{R}$ are such that $\rho(\overline{\Phi}) = [\rho_1(\overline{\Phi}), \rho_2(\overline{\Phi})]$.

Proposition (3.3). — *Let* $U \subset \mathrm{End}(S^1)$ *be some open set such that there is a continuous mapping* $\Phi \to \overline{\Phi}$ *which assigns to each* Φ *a lifting* $\overline{\Phi}$ (in order that $\overline{\Phi}$ depends continuously on Φ, U should not be too big). *Then the functions* $\Phi \mapsto \rho_1(\bar{\Phi})$ *and* $\Phi \mapsto \rho_2(\Phi)$ *on* U *are continuous.*

Proof. — We observe that for any rational number $\frac{p}{q}$, we have that $\frac{p}{q} < \rho_1(\bar{\Phi})$ is equivalent with $\bar{\Phi}^q(x) - x > p$ for all $x \in [0, 1]$ which is an " open condition ", *i.e.* the set of $\Phi \in \mathrm{U}$ with $\rho_1(\bar{\Phi}) > \frac{p}{q}$ is open. Analogously, the set of those $\Phi \in \mathrm{U}$ with $\rho_2(\bar{\Phi}) < \frac{p}{q}$ is open.

Finally, $\frac{p}{q} \in (\rho_1(\bar{\Phi}), \rho_2(\bar{\Phi}))$ if and only if for some big $\mathrm{N} \in \mathbf{N}$, there are $x, y \in [0, 1]$ with $\bar{\Phi}^{\mathrm{N}.q}(x) - x > \mathrm{N}.p + 1$ and $\bar{\Phi}^{\mathrm{N}.q}(y) - y < \mathrm{N}.p - 1$. Also this condition is open, hence $\rho_1(\bar{\Phi})$ and $\bar{\rho}_2(\bar{\Phi})$ depend continuously on Φ.

Proposition (3.4). — *Let* $\Phi \in \mathrm{End}(\mathrm{S}^1)$ *with lifting* $\bar{\Phi}$ *and let* $\frac{p}{q} \in (\rho_1(\bar{\Phi}), \rho_2(\bar{\Phi}))$. *Then there is a periodic point* $s \in \mathrm{S}^1$ *of* Φ *with rotation number* $\frac{p}{q}$ *such that* $\mathrm{W}^u_\Phi(s) = \mathrm{S}^1$, *where* $\mathrm{W}^u_\Phi(s) = \bigcap_{\mathrm{U}} [\bigcup_{i \in \mathbf{N}} \Phi^i(\mathrm{U})]$, *the intersection being taken over neighborhoods* U *of* s.

Proof. — We may assume that $\frac{p}{q} = 0$ (if not we replace Φ by Φ^q and choose an appropriate lifting). Let X denote the projection of the fixed point set of $\bar{\Phi}$ in S^1; since X is closed and non-empty, $\mathrm{S}^1 - \mathrm{X}$ consists of open intervals. Let U be such an interval and $\bar{\mathrm{U}} = (\bar{s}_1, \bar{s}_2)$ a lifting of U. Then $\bar{\Phi}(x) - x$, for $x \in \bar{\mathrm{U}}$, is always positive or always negative. In the first case, $\mathrm{U} \subset \mathrm{W}^u_\Phi(\pi(\bar{s}_1))$, in the second case, $\mathrm{U} \subset \mathrm{W}^u_\Phi(\pi(\bar{s}_2))$. If $\bigcup_{n \geq 0} \Phi^n(\mathrm{U}) = \mathrm{S}^1$ we may take $s = \pi(\bar{s}_1)$ or $s = \pi(s_2)$ and are done. If on the other hand for each component U of $\mathrm{S}^1 - \mathrm{X}$, $\bigcup_{n \geq 0} \Phi^n(\mathrm{U}) \neq \mathrm{S}^1$, then $\Phi^{-n}([0, 1]) \subset [-1, 2]$ for all $n \geq 0$, so we would clearly have $\rho(\bar{\Phi}) = \{0\}$. This would be against the assumptions.

Remark (3.5). — Let $\mathscr{B} \subset \mathrm{End}(\mathrm{S}^1)$ denote the subset of those $\Phi \in \mathrm{End}(\mathrm{S}^1)$ which are C^1, whose first derivative is of bounded variation and whose critical set consists of a finite and non-zero number of generic folds, *i.e.* points where Φ is locally C^0 conjugate with the map $y = x^2$. By a theorem of Block and Franke [2], every Φ in $\mathscr{B}$ has a periodic point. This is equivalent with saying that if $\Phi \in \mathscr{B}$, $\rho(\Phi)$ cannot consist of only one irrational real number. This together with the continuity of ρ_1 and ρ_2 implies that if $\mathbf{R} \ni \sigma \mapsto \Phi_\sigma$ is a continuous arc in $\mathrm{End}(\mathrm{S}^1)$ with image in $\mathscr{B}$ then $\rho(\Phi_\sigma)$ is constant (as function of σ) or, for some $\bar{\sigma}$, $\rho(\Phi_{\bar{\sigma}})$ has a non-empty interior. This observation will be useful in the next section.

Finally, we pose the following conjecture which extends Herman's result for arcs of diffeomorphisms [10]. Let $\{\varphi_\mu\}$ be a C^1 arc of C^∞ endomorphisms of S^1 with rotation sets $\{[\rho_1(\mu), \rho_2(\mu)]\}$.

Conjecture **(3.6)**. — *If the rotation set of* φ_μ *varies with* μ, *then the set of* μ's *for which either* $\rho_1(\mu)$ *or* $\rho_2(\mu)$ *is irrational has positive (Lebesgue) measure.*

B) *Some one-parameter families of endomorphisms of* S^1

In this sub-section we consider continuous (or smooth) families Φ_σ, $\sigma \in \mathbf{R}$, of endomorphisms of S^1 such that

— for each $\sigma \in \mathbf{R}$, $\Phi_\sigma \in \mathscr{B}$, see Remark (3.5);
— the rotation set $\rho(\Phi_\sigma)$ is non-constant, as a function of σ;
— for each rational number $\frac{p}{q}$, there is a locally finite set $\Sigma_{\frac{p}{q}} \subset \mathrm{R}$ such that for $\sigma \notin \Sigma_{\frac{p}{q}}$, all periodic points of Φ_σ with rotation number $\frac{p}{q}$ are hyperbolic.

We shall prove for these families:

Theorem **(3.7)**. — *For* Φ_σ *as above there are an interval* $(\sigma_1, \sigma_2) \subset \mathbf{R}$, *a hyperbolic periodic point* s_σ, $\sigma_1 < \sigma < \sigma_2$ *of* Φ_σ *such that* $W^u_{\Phi_\sigma}(s_\sigma)$ *is* S^1 *for all* $\sigma \in (\sigma_1, \sigma_2)$ *and a point* t_σ, $\sigma_1 < \sigma < \sigma_2$ *in the critical image of* Φ^n_σ (for some n) *such that*:

— *both* s_σ *and* t_σ *depend continuously on* σ;
— *the curves* $\sigma \mapsto (s_\sigma, \sigma)$ *and* $\sigma \mapsto (t_\sigma, \sigma)$ *in* $S^1 \times (\sigma_1, \sigma_2)$ *cross one another*;
— *if* $\overline{\Phi}_\sigma$ *is a lifting of* Φ_σ, *depending continuously on* σ, *and if* $\overline{s}_\sigma$ *is a lifting of* s_σ, *then* t_σ *is the projection of an end point of the interval* $\overline{\Phi}^n_\sigma(\overline{s}_\sigma, \overline{s}_\sigma + 1)$.

Remark **(3.8)**. — For $\Phi \in \mathrm{End}(S^1)$, with lifting $\overline{\Phi}$ one can construct a 1-parameter family of endomorphisms Φ_σ by putting $\Phi_\sigma(\pi(x)) = \pi(\overline{\Phi}(x+\sigma))$. We note that in the set of C^r endomorphisms φ of S^1, $r \geq 2$, which are not diffeomorphisms, there is a residual subset for which the corresponding one parameter families of endomorphisms satisfy the assumptions in Theorem (3.7).

Proof. — Since $\rho(\Phi_\sigma)$ is non-constant, there is some σ_3 such that $\rho(\Phi_{\sigma_3})$ has interior points, see (3.5), and such that $\rho(\Phi_\sigma)$ is not locally constant on a neighborhood of σ_3. Choose a rational number $\frac{p}{q}$ in the interior of the rotation set. Since in any neighborhood of σ_3 there are infinitely many points where the function $\rho(\Phi_\sigma)$ is not locally constant, we can choose σ_4 such that Φ_{σ_4} has a hyperbolic periodic point s with rotation number $\frac{p}{q}$ such that $W^u_{\Phi_{\sigma_4}}(s) = S^1$ (see (3.4)), and such that $\rho(\Phi_\sigma)$ is not locally constant at σ_4. Now if follows that there is some neighborhood U of σ_4 in $\mathbf{R}$ and a continuous function $U \ni \sigma \mapsto s_\sigma \in S^1$ such that, for $\sigma \in U$, s_σ is a hyperbolic periodic point and such that $W^u_{\Phi_\sigma}(s_\sigma) = S^1$. Next, since $\rho(\Phi_\sigma)$ is not locally constant at σ_4 there is a $\overline{\sigma}$ in U for which $\rho(\varphi_{\sigma_4}) \neq \rho(\varphi_{\overline{\sigma}})$. Then one can find two rational numbers $\frac{p_1}{q_1}, \frac{p_2}{q_1}$ with $p_1 < p_2$ such that the closed interval $\left[\frac{p_1}{q_1}, \frac{p_2}{q_1}\right]$ is in $(\rho(\varphi_{\sigma_4}) - \rho(\varphi_{\bar\sigma})) \cup (\rho(\varphi_{\bar\sigma}) - \rho(\varphi_{\sigma_4}))$. We suppose $\left[\frac{p_1}{q_1}, \frac{p_2}{q_1}\right]$ is in $\rho(\varphi_{\sigma_4}) - \rho(\varphi_{\bar\sigma})$ since the other case is similar. Then, with $\bar\Phi_\sigma$, $\bar s_\sigma$ liftings of Φ_σ, s_σ,

$$\bar\Phi^{3q_1}_{\sigma_4}(\bar s_{\sigma_4}, \bar s_{\sigma_4} + 1) \supset [\bar s_{\sigma_4} + 1 + 3p_1, \bar s_{\sigma_4} + 3p_2]$$

and
$$\bar\Phi^{3q_1}_{\bar\sigma}(\bar s_{\bar\sigma}, \bar s_{\bar\sigma} + 1) \subset \text{complement of } [\bar s_{\bar\sigma} + 1 + 3p_1, \bar s_{\bar\sigma} + 3p_2];$$

see also Lemma (3.1). This means that one of the endpoints of the interval $\Phi^{3q_1}_\sigma(\bar s_\sigma, \bar s_\sigma + 1)$, σ between σ_4 and $\bar\sigma$, has to cross over the interval $[\bar s_\sigma + 1 + 3p_1, \bar s_\sigma + 3p_2]$. Since this last interval has length at least 2, one of the endpoints of $\Phi^{3q_1}_\sigma(\bar s_\sigma, \bar s_\sigma + 1)$ crosses, after projection on S^1, over s_σ when σ goes from σ_4 to $\bar\sigma$.

Finally we have to show that if, for σ between σ_4 and $\bar\sigma$, one endpoint of $\bar\Phi^{3q_1}(\bar s_\sigma, \bar s_\sigma + 1)$ lies in $[\bar s_\sigma + 1 + 3p_1, \bar s_\sigma + 3p_2]$ then this endpoint is in the critical image of $\bar\Phi^{3q_1}_\sigma$. To show this, it is enough to show that neither $\bar\Phi^{3q_1}_\sigma(\bar s_\sigma)$ nor $\bar\Phi^{3q_1}_\sigma(\bar s_\sigma + 1)$ can be in $[\bar s_\sigma + 1 + 3p_1, s_\sigma + 3p_2]$. But this follows from the fact that the rotation number of s_σ is $\frac{p}{q}$ and that $\frac{p}{q}$ does not lie between $\frac{p_1}{q_1}$ and $\frac{p_2}{q_1}$ (because otherwise $\frac{p}{q} \notin \rho(\Phi_\sigma)$, see (3.2)).

Now it is clear that we can choose (σ_1, σ_2) to be an interval between σ_4 and $\bar\sigma$ such that if σ goes from σ_1 to σ_2, one endpoint of $\bar\Phi^{3q_1}_\sigma(\bar s_\sigma, \bar s_{\sigma+1})$ crosses over the interval $[\bar s_\sigma + 1 + 3p_1, \bar s_\sigma + 3p_2]$. We choose then $n = 3q_1$ and we choose t_σ to be the projection of that endpoint.

C) *Application to bifurcation theory*

The analysis of families of endomorphisms in the last section leads to results also when analyzing certain families of diffeomorphisms. An explicit formulation of what we need in this direction is:

Remark **(3.9)**. — Let $\varphi_\sigma : S^1 \to S^1$ be a one-parameter family ($\sigma \in \mathbf{R}$) of endomorphisms as in Theorem (3.7). Let $\Phi_{\sigma,\mu} : S^1 \times [0, 1] \to S^1 \times [0, 1]$ be a 2-parameter family of C^2 maps depending continuously on (σ, μ) such that:

— $\Phi_{\sigma,0}(x, t) = (\varphi_\sigma(x), T_\sigma(x))$ for some function T_σ;

— for μ positive, $\Phi_{\sigma,\mu}$ is a diffeomorphism into.

Then, for $\bar\mu$ sufficiently small, there is a $\sigma_{\bar\mu} \in (\sigma_1, \sigma_2)$ such that $\Phi_{\sigma_{\bar\mu},\bar\mu}$ has a non-transversal homoclinic point.

Even more holds: for $\bar\mu$ sufficiently small, and any curve $(\sigma, f(\sigma))$ with $0 < f(\sigma) < \bar\mu$, $\sigma \in (\sigma_1, \sigma_2)$, there is a σ_f such that $\Phi_{\sigma, f(\sigma_f)}$ has a non-transversal homoclinic point.

The proof of this statement follows from the following continuity considerations. The map $\Phi_{\sigma,\mu}$ has a hyperbolic periodic point $s_{\sigma,\mu} = (x_{\sigma,\mu}, t_{\sigma,\mu})$ such that $x_{\sigma,0} = s_\sigma$

as in Theorem (3.7). We observe that $W^s(s_{\sigma,\mu})$ depends continuously on (σ, μ) and $W^s(s_{\sigma,0}) = \{x_{\sigma,0}\} \times [0, 1]$ (note that $\varphi_\sigma : S^1 \to S^1$ was expanding at s_σ). One can choose a local unstable manifold $W^u_{loc}(s_{\sigma,\mu})$ as an embedding (or its image) which depends continuously on (σ, μ). For some k, $\varphi^k_{\sigma,0}(W^u_{loc}(x_{\sigma,0}))$ has a point T_σ in its critical image which moves over $s_{\sigma,0}$ by Theorem (3.7). From this and the above continuity we see that, for each small enough $\mu > 0$, there is a value of σ such that $\Phi^k_{\sigma,0}(W^u_{loc}(s_{\sigma,\mu}))$ is not transversal with respect to $W^s(s_{\sigma,\mu})$.

4. The saddle-node with 1-cycle in dimension 2

Let $\{\varphi_\mu\} : M \to M$ be an arc of diffeomorphisms of a compact 2-manifold M so that for $\mu < b$, φ_μ is Morse-Smale and such that for $\mu = b$, φ_b has a saddle-node at p (we assume p to be a fixed point of φ_b; the case where p is periodic can be handled analogously). Further we assume that φ_b has a 1-cycle containing p; *i.e.* there are non-trivial intersections of $W^u(p)$ with $W^s(p)$ and there is no periodic point q such that both $W^u(p) \cap W^s(q)$ and $W^u(q) \cap W^s(p)$ are non-empty.

Theorem **(4.1)**. — *Under the above hypotheses* $\{\varphi_\mu\}$ *is not mildly stable.*

Remark **(4.2)**. — It will be evident from the proof of Theorem (4.1) that the same result holds if $\dim M > 2$ and φ_b has a saddle-node in a 1-cycle whose stable or unstable manifold is one dimensional; *i.e.* a normally repelling or attracting saddle-node. An open dense set of these arcs will create homoclinic tangencies. Thus, in dimension two they will contain diffeomorphisms with infinitely many sinks or sources [22].

Proof of Theorem (4.1). — An arc $\{\varphi_\mu\}$ is already not mildly-stable if in any neighborhood of φ_μ there is a non mildly-stable arc. Hence we may, without loss of generality, impose generic conditions on $\{\varphi_\mu\}$. In particular, we assume that p unfolds generically. Also, we may assume that the eigenvalues of $(d\varphi_b)_p$ are 1 and α with $0 < \alpha < 1$; in case $\alpha > 1$ we take φ_μ^{-1}, in case $-1 < \alpha < 0$ we take φ_μ^2 and in case $\alpha < -1$ we take φ_μ^{-2} instead of φ.

From the fact that, for $\mu < b$, φ_μ is Morse-Smale and the fact that $\{\varphi_\mu\}$ unfolds generically at p, we may assume that, near p, φ_μ has two fixed points for $\mu < b$, one fixed point for $\mu = b$ and no fixed points for $\mu > b$; we also assume that

$$W^u(p, \varphi_b) \cap \partial(W^s(p, \varphi_b)) = \{p\}.$$

We first observe that $W^u(p, \varphi_b) \subset W^s(p, \varphi_b)$; this follows from the following two facts:

— $W^u(p, \varphi_b) - \{p\} \cap W^s(p, \varphi_b)$ is open in $W^u(p, \varphi_b) - \{p\}$ and non-empty;

— for any filtration $\{M_i\}$ of M for φ_b (see [20]), with $p \in M_{i_0} - M_{i_0-1}$, the set of points in $W^u(p, \varphi_b) - \{p\}$, which go eventually into M_{i_0-1} is open and equals $(W^u(p, \varphi_b) - \{p\}) - W^s(p, \varphi_b)$.

In the case that $W^u(p, \varphi_b)$ closes to a smooth circle, p has a non-critical 1-cycle. Then the instability here was proved in Proposition (2.3). Thus, we may suppose $W^u(p, \varphi_b)$ does not close to a smooth circle. This means that the projection of $W^u(p, \varphi_b)$ on a center manifold in $W^s(p, \varphi_b)$ along the strong stable foliation does not have maximal rank everywhere.

We shall approximate $\{\varphi_\mu\}$ by a family $\{\tilde{\varphi}_\mu\}$ such that non-transversal intersections of stable and unstable manifolds occur for certain μ's arbitrarily near the first bifurcation of $\{\tilde{\varphi}_\mu\}$. From this, we infer that $\{\tilde{\varphi}_\mu\}$, and, hence also $\{\varphi_\mu\}$, is not mildly stable. This will prove Theorem (4.1). For convenience of notation, let us assume that $\{\varphi_\mu\}$ is defined for μ near 0 and that $\mu = 0$ is the bifurcation point instead of $\mu = b$. For each diffeomorphism $\tilde{\varphi}_\mu$ we require that there is a smooth vector field X_μ, defined for μ close to 0 on a neighborhood of p, such that the time 1 map $X_{\mu,1}$ of X_μ, wherever defined, equals $\tilde{\varphi}_\mu$. Also we require that there are smooth coordinates y, z (which may depend on μ) such that X_μ locally has the form

$$X_\mu = Y_\mu(y)\frac{\partial}{\partial y} + Z_\mu(y)\cdot z \cdot \frac{\partial}{\partial z}$$

where Y and Z are smooth functions of (y, μ), $Z_0(0) < 0$, $\dfrac{\partial^2 Y_0(0)}{\partial y^2} > 0$, and $\left\{ Y_\mu(y)\dfrac{\partial}{\partial y} \right\}$ is a saddle-node arc. The fact that such $\tilde{\varphi}_\mu$ exists arbitrarily close to φ_μ follows from [38], [39].

Choose, for $\mu \geq 0$ a fundamental domain $D_{2,\mu}$ for $\tilde{\varphi}_\mu$ in the positive y-axis, smoothly depending on μ. For some big m, $\tilde{\varphi}_0^m(D_{2,0})$ will again be in our coordinate

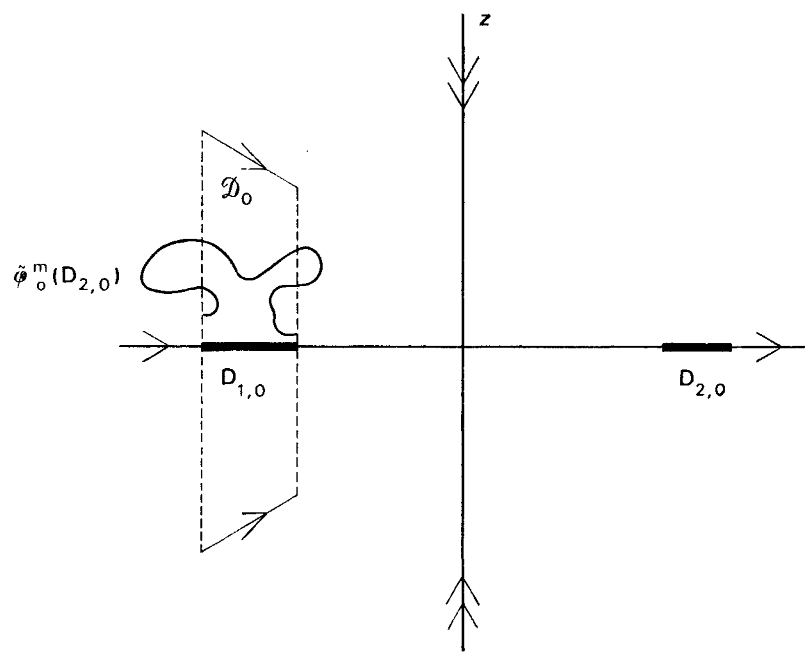

FIG. 8

neighborhood the same holds for $\widetilde{\varphi}_\mu^m(D_{2,\mu})$ if μ is near 0. Define $D_{1,\mu}$ to be the fundamental domain in the negative y-axis whose boundary is the projection, along the z-direction, of the boundary of $\widetilde{\varphi}_\mu^m(D_{2,\mu})$. We take a " rectangle " $\mathscr{D}_\mu$ so that its boundary consists of two pieces of X_μ-integral curves and two pieces of straight lines parallel to the z-axis, such that its projection on the y-axis is $D_{1,\mu}$, and such that for each $q \in \widetilde{\varphi}_\mu^m(D_{2,\mu})$, there is a positive m' such that $\widetilde{\varphi}_\mu^{m'}(q)$ is in the forward orbit of $\mathscr{D}_\mu$. See figure 8.

Let $A_\mu : \mathscr{D}_\mu \to M$, $\mu \geq 0$ be defined as follows:

$$A_0(\mathscr{D}_0) = D_{2,0} \quad \text{and} \quad A_0(y, z) = (a_0(y), 0)$$

where $a_0 : D_{1,0} \to D_{2,0}$ satisfies $(a_0)_* X_0 = X_0$ on $\{z = 0\}$. For $\mu > 0$ there is a positive T_μ such that the time T_μ map of X_μ, $(X_\mu)_{T_\mu}$, satisfies $(X_\mu)_{T_\mu} D_{1,\mu} = D_{2,\mu}$; we define A_μ to be $(X_\mu)_{T_\mu}$. For $\sigma \in [0, 1]$ we define $A_{\sigma,\mu} = (X_\mu)_\sigma \circ A_\mu$.

Next we define $B_{\sigma,\mu} = \widetilde{\varphi}_\mu^m \circ A_{\sigma,\mu}$. For μ close enough to 0, the image of $B_{\sigma,\mu}$ will be in our coordinate neighborhood in the part $\{y < 0\}$, and also there is some positive m' such that the image of $\widetilde{\varphi}_\mu^{m'} \circ B_{\sigma,\mu}$ is in the forward orbit of $\mathscr{D}_\mu$.

Consider now the quotient map

$$\{(y, z) \mid y < 0\} \to \{(y, z) \mid y < 0\}/(y, z) \sim \widetilde{\varphi}_\mu(y, z).$$

Under this quotient, $\mathscr{D}_\mu$ becomes an annulus $\widetilde{\mathscr{D}}_\mu$, $B_{\sigma,\mu}$ becomes a 2-parameter family of mappings $\widetilde{B}_{\sigma,\mu}$ of $\widetilde{\mathscr{D}}_\mu$ into itself and the projection $\pi(y, z) = (y, 0)$ goes over into $\widetilde{\pi}$, the annulus projection. We want to show that Remark (3.9) is applicable to $\widetilde{B}_{\sigma,\mu}$.

In order to prove this we need new coordinate functions $s_\mu : \widetilde{\mathscr{D}}_\mu \to \mathbf{R}/\mathbf{Z}$ and $w_\mu : \mathscr{D}_\mu \to \mathbf{R}$ such that $\widetilde{\pi}(s_\mu, w_\mu) = (s_\mu, 0)$ and such that X_μ has the form $\dfrac{\partial}{\partial s_\mu}$. We shall write s, w instead of s_0, w_0. For some circle endomorphism $\hat{\varphi} : R/Z \to R/Z$ and some map $W : \mathbf{R}/\mathbf{Z} \to \mathbf{R}$ we have $\widetilde{B}_{\sigma,0}(s, w) = (\hat{\varphi}(s + \sigma), W(s))$. The circle endomorphism $\hat{\varphi}$ is determined by $\widetilde{\varphi}_0$; $\widetilde{\varphi}_0$ was obtained from φ_0 by a small, but otherwise arbitrary, perturbation. Hence we may, and do, assume that the family $\{\hat{\varphi}_\sigma\}$ defined by $\hat{\varphi}_\sigma(s) = \hat{\varphi}(s + \sigma)$ satisfies the assumptions of Theorem (3.7); see also Remark (3.8). Now it is clear that $\widetilde{B}_{\sigma,\mu}$ satisfies the assumptions in Remark (3.9).

Now we observe that if $T_\mu + \sigma \in \mathbf{N}$, $\widetilde{B}_{\sigma,\mu}$ is an iterate of $\widetilde{\varphi}_\mu$ (up to the identifications). Hence if such a $\widetilde{B}_{\sigma,\mu}$ has a non-generic tangency of a stable and an unstable manifold then the same holds for $\widetilde{\varphi}_\mu$. Also, $\widetilde{B}_{\sigma,\mu}$ satisfies the conditions of $\Phi_{\sigma,\mu}$ in Remark (3.9). Let (σ_1, σ_2) and $\bar{\mu}$ be as in that remark. Now as $\mu \to 0$, we have $T_\mu \to \infty$. For each $\sigma \in (\sigma_1, \sigma_2)$ choose an $f(\sigma)$ such that $T_{f(\sigma)} + \sigma \in \mathbf{N}$, and $f(\sigma) < \bar{\mu}$. We may arrange for $\sigma \mapsto f(\sigma)$ to be continuous on (σ_1, σ_2). Then as in Remark (3.9), there is a $\sigma_f \in (\sigma_1, \sigma_2)$ so that $\widetilde{B}_{\sigma_f, f(\sigma_f)}$, and hence $\widetilde{\varphi}_{\sigma_f}$, has a non-transversal homoclinic point.

5. On the rigidity of the unfolding of the saddle-node

In Chapter II we have seen that a conjugacy between two C^∞ saddle-node bifurcations must satisfy very restrictive conditions. In particular, at the bifurcation parameter it must be C^∞ along central curves away from the fixed point, and it must preserve adapted saddle-node vector fields along these curves. In the present section, we shall present two more applications of the restrictive nature of such conjugacies. The first concerns certain arcs of diffeomorphisms of the circle, and the second concerns arcs between Anosov diffeomorphisms and so-called DA diffeomorphisms. These furnish more examples where mild conjugacies cannot be strengthened to conjugacies.

A) *One-parameter families of diffeomorphisms of* S^1

We consider one-parameter families $\varphi_\mu : S^1 \to S^1$ of C^∞-diffeomorphisms of S^1 with rotation number $\rho(\mu)$ increasing such that whenever $\rho(\bar\mu)$ is rational, $\varphi_{\bar\mu}$ has two hyperbolic periodic points or one periodic point of saddle-node type which unfolds generically and such that whenever $\rho(\bar\mu)$ is irrational, ρ is not locally constant in $\bar\mu$. Let $\varphi'_\mu : S^1 \to S^1$ be another such family of diffeomorphisms with rotation number $\rho'(\mu)$. If we assume that $\mathrm{Image}(\rho) = \mathrm{Image}(\rho')$, then there is a homeomorphism $h : \mathbf{R} \to \mathbf{R}$ such that $\rho'(h(\mu)) = \rho(\mu)$, see [5], [10]. For each μ, φ_μ and $\varphi'_{h(\mu)}$ are now conjugate: for $\rho(\mu)$ irrational this is Denjoy's theorem [6], for $\rho(\mu)$ rational it follows from the above description. So it is clear that the above two arcs are " mildly conjugate ".

A question, which was raised independently by R. Thom and S. Smale, is: are two arcs as above in general conjugate? We show here that the answer is in general negative.

Let $\{\varphi_\mu\}$, $\{\varphi'_\mu\}$ be a pair of arcs of diffeomorphisms of S^1 as above. Let $\bar\mu$ be a boundary point of $\rho^{-1}\left(\dfrac{p}{q}\right)$ for some rational $\dfrac{p}{q} \in \mathrm{Image}(\rho)$. If there is a conjugacy (h, H_μ) between φ_μ and φ'_μ, *i.e.* $h : \mathbf{R} \to \mathbf{R}$ a homeomorphism and H_μ a conjugacy between φ_μ and $\varphi'_{h(\mu)}$ depending continuously on μ, then $h(\bar\mu)$ is a boundary point of $(\rho')^{-1}\left(\dfrac{p}{q}\right)$. Now $\varphi_{\bar\mu}$ has a unique fixed point which is of saddle-node type. By Chapter II, Theorem (3.2), there is a unique smooth vector field X near the saddle-node orbit such that its time one map X_1 equals $(\varphi_{\bar\mu})^q$ (and such that $(\varphi_{\bar\mu})^q_* X = X$). Also for $\varphi'_{h(\mu)}$ there is such a vector field X'. Again by Chapter II, Theorem (3.2), $H_{\bar\mu}$ has to map X to X'. If we now extend X, and X', to all of S^1 so that $\varphi_{\bar\mu *} X = X$, $\varphi'_{h(\bar\mu)*} X' = X'$, we obtain in general bivalued vector fields. This means that, on the complement of the periodic orbit, $H_{\bar\mu}$ has to respect two different vector fields. This is in general impossible.

B) *Arcs between Anosov and DA diffeomorphisms*

In [34], S. Smale showed that certain Anosov diffeomorphisms may be modified to give Axiom A diffeomorphisms with attractors having intricate topological properties.

Since the latter diffeomorphisms were obtained by modifying Anosov diffeomorphisms, he called them DA (for derived from Anosov) diffeomorphisms. Although Smale's construction was given for Anosov diffeomorphisms of the two torus, it works just as well with codimension one Anosov diffeomorphisms of the n-torus. A brief description of the construction is as follows.

Let L be an Anosov diffeomorphism of T^n and suppose that $\dim W^s(x) = 1$ for each $x \in T^n$. Since L is conjugate to a linear toral automorphism [16], it has a fixed point, say p. Assume that the contracting eigenvalue of p is positive. Locally, near p, one has the usual picture of a hyperbolic saddle fixed point as in Figure 9a.

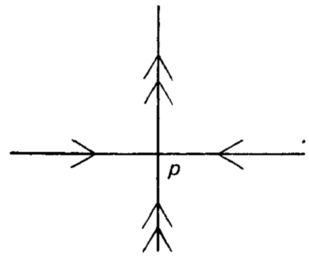

FIG. 9a

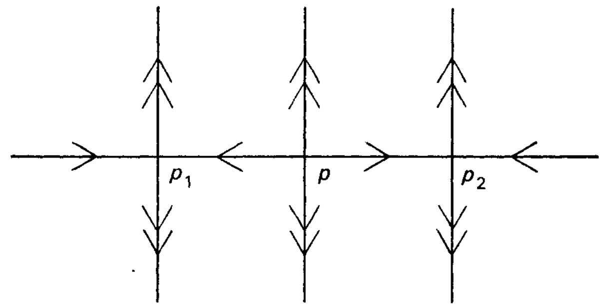

FIG. 9b

FIG. 9

Smale proposed to modify the diffeomorphism L in a disk neighborhood N of p to obtain a diffeomorphism g with two new saddle fixed points p_1 and p_2 on $W^s(p, L)$ and such that p is a fixed source of g (see Fig. 9b). This can be done so that g agrees with L off N, g satisfies Axiom A, and the non-wandering set of g consists of p and an $(n-1)$-dimensional hyperbolic attractor containing p_1 and p_2. In fact, as Williams pointed out in [42], one can choose g so that the foliation $\mathscr{F}^s = \{W^s(x, L) \mid x \in T^n\}$ is g-invariant. Of course, the unstable L-foliation is no longer g-invariant. Somewhat later in a private communication with us, Williams observed that a DA diffeomorphism g could be constructed from an arc in which a saddle-node occurs. The local picture is in Figure 9c.

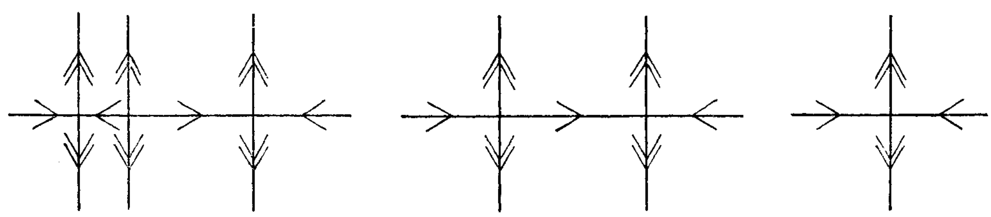

FIG. 9c

If one chooses such an arc carefully, then one can actually make the arc mildly stable with an isolated bifurcation point. This will be proved elsewhere. These mildly stable arcs are such that at the bifurcation point one has a saddle-node p whose stable manifold $W^s(p)$ is one dimensional and such that there is a hyperbolic periodic point q such that $W^u(q) \cap W^s(p) \neq \emptyset$. The next proposition shows that such an arc is never stable.

Proposition. — *Suppose* $\{\varphi_\mu\}$, $0 \leq \mu \leq 1$, *is an arc of diffeomorphisms of* T^n *so that* φ_0 *is Anosov with* $\dim W^s(x) = 1$ *for all* x. *Let* $b = b(\{\varphi_\mu\})$ *be the first bifurcation point of* $\{\varphi_\mu\}$ *and assume* $0 < b < 1$. *Suppose that* φ_b *has a saddle-node periodic point with* $\dim W^s(p) = 1$ *and* φ_b *has a hyperbolic periodic point* q *not in the orbit of* p *such that* $W^u(q) \cap W^s(p) \neq \emptyset$. *Then* $\{\varphi_\mu\}$ *is not stable.*

Proof. — We may assume, by perturbing $\{\varphi_\mu\}$ if necessary, that $W^u(q)$ is transverse to $W^s(p)$. Since q can be continued to a hyperbolic periodic point q_μ for $\mu < b$, and φ_μ is Anosov for $\mu < b$, it follows that $\dim W^u(q_\mu) = \dim M - 1$. As $W^u(q) \cap W^s(p) \neq \emptyset$, this intersection is zero dimensional, and, therefore p is s-critical. Now the method of proof of Proposition (2.4) may be applied to show that $\{\varphi_\mu\}$ is not stable.

C) *One-parameter families of vector fields*

As was pointed out in [41], the rigidity in the conjugacy of a saddle-node arc has consequences for the stability of one-parameter families of vector fields. Consider such a C^∞ one-parameter family $\{X_\mu\}$ on a 2-manifold M such that for $\mu = b$, X_b has a saddle-node closed orbit γ, *i.e.* a closed orbit which is attracting at one side and repelling at the other side and whose Poincaré map has first but not second order contact with the identity. We assume furthermore that γ unfolds generically; by this we mean that if S is a local cross section of $X_{b'}$ intersecting γ, then the Poincaré map $P_\mu : S \to S$, μ near b, is a saddle-node arc, see section 3 of Chapter III. Let $\{X'_\mu\}$ be another one parameter family of vector fields on M (near $\{X_\mu\}$) so that for $\mu = b'$, $X'_{b'}$ has a saddle-node closed orbit γ' which unfolds generically. Let S′ to be a local section of $X'_{b'}$ intersecting γ'. We assume for simplicity that neither $\{X_\mu\}$ nor $\{X'_\mu\}$ has any other saddle-node closed orbit. We say that these two families are topologically equivalent if there exist a homeomorphism $h : \mathbf{R} \to \mathbf{R}$ and a homeomorphism $H_\mu : M \to M$, depending continuously on μ, such that H_μ maps integral curves of X_μ to integral curves of $X'_{h(\mu)}$.

If such an equivalence (h, H_μ) exists, then $h(b) = b'$. Modifying H_μ along the orbits of X, we may assume that $H_\mu(S) = S'$ for μ near b. So $H_\mu \mid S$ has to conjugate the saddle-node arc $\{P_\mu\}$ with $\{P'_\mu\}$. By Chapter III, section 3 this implies that $H_b \mid S$ has to map X to X′, where X, X′ is the unique smooth vector field on S, S′ such that its time 1 map X_1, X'_1 equals P_b, $P'_{b'}$. This means that $H_b \mid S$ is determined as soon as it is determined in two orbits of P_b in S, one on each side of $\gamma \cap S$.

Since H_b has to map separatrices, *i.e* stable or unstable manifolds of saddle points, to separatrices, the map $H_b \mid S$ is essentially fixed for each intersection of S with a separatrix; such an intersection is an orbit of P_b. So in general $\{X_\mu\}$ and $\{X'_\mu\}$ will not be topologically equivalent if at least two separatrices approach γ from the same side. This means that an arc $\{X_\mu\}$ which has a saddle-node closed orbit which is approached from the same side by at least two separatrices is not stable; it has a modulus of stability in the same way as arcs of diffeomorphisms with a tangency of stable and unstable manifolds. In [7] Guckenheimer incorrectly states that some of these arcs are stable.

IV — GLOBAL STABILITY

1. Introduction

In this chapter we complete the proofs of our main results concerning stability of one-parameter families (arcs) of diffeomorphisms, stated in the introduction to the paper. Necessary conditions for stability, mild and left stability at the first bifurcation point were provided in the previous chapter. We now show that these conditions are also sufficient. As a consequence we obtain a characterization for the stability of arcs containing several bifurcation values, under the basic assumption that the limit sets have finitely many orbits.

The main idea here consists of a suitable construction of tubular families or foliations, which will be used to define topological conjugacies. We begin by describing local tubular families for hyperbolic periodic orbits. This concept will then be extended to Hopf orbits, saddle-nodes and flips. The foliations are constructed for a family $\{\varphi_\mu\}$ in $\mathscr{B}$, as defined in the introduction, especially near its first bifurcation value b. In the case of left and mild stability, this interval is of the form $[\mu_1, b]$, for some $\mu_1 < b$. Otherwise, it is of the form $[\mu_1, \mu_2]$ with $b \in (\mu_1, \mu_2)$. For each μ in such intervals, we build up tubular families or foliations requiring them to be φ_μ-invariant. We shall usually assume that the periodic orbits of φ_μ are fixed points. In fact, if a periodic orbit of φ_μ has period k, we can consider φ_μ^k to define the foliation near one of its elements, and use φ_μ-iterates of the leaves to obtain the foliation near the others.

2. Local Tubular Families

We first recall, in a parametrized version, the notion of tubular family for a hyperbolic orbit [24], [25].

Let $\bar{x} \in M$ be a hyperbolic fixed point for $\varphi_{\bar{\mu}}$, $\bar{\mu} \in I$. Let $[\mu_1, \mu_2]$ be a neighborhood of $\bar{\mu}$ in I and U be a cell neighborhood of $\bar{x}$ in M. If these neighborhoods are small enough, there exists a continuous mapping $[\mu_1, \mu_2] \ni \mu \mapsto x_\mu \in U$, where x_μ is the unique (hyperbolic) fixed point for φ_μ in U and $\bar{x} = x_{\bar{\mu}}$. We denote by $W^u(x_\mu)$ and $W^s(x_\mu)$ the unstable and stable manifolds of φ_μ at x_μ. Given the family $\{\varphi_\mu\}$, $\mu \in I$, we define $\Phi : M \times I \to M$ by $\Phi(x, \mu) = (\varphi_\mu(x), \mu)$.

Definition (2.1). — *An unstable tubular family for* $\{\varphi_\mu\}$ *or* Φ *at* $(\bar{x}, \bar{\mu})$ *is a continuous foliation* $\mathscr{F}^u$ *of* $U \times [\mu_1, \mu_2]$, *such that for* $\mu \in [\mu_1, \mu_2]$

a) *the leaves are* C^k *discs varying continuously in the* C^k *topology,* $1 \leq k < \infty$, *and* $\mathscr{F}^u(x_\mu, \mu) = W^u(x_\mu) \times \{\mu\} \cap U \times \{\mu\}$,

b) *each leaf* $\mathcal{F}^u(y, \mu)$ *is contained in* $U \times \{\mu\}$ *for* $y \in U$,

c) *the foliation is* Φ*-invariant in the following sense*

$$\Phi(\mathcal{F}^u(y, \mu)) \supset \mathcal{F}^u(\varphi_\mu(y), \mu)$$

for $y, \varphi_\mu(y) \in U$.

Remark. — For our purposes it is enough to take the leaves of the foliation to be C^2. In fact, in their global setting, we can and will construct the foliation to be C^k $(k \geq 2)$ when restricted to any unstable manifold of a hyperbolic periodic orbit that intersects $W^s(x_\mu) \times \{\mu\}$. See Definition (3.1) of the next section, where even more differentiability is involved. The same remark applies to Definition (2.2) below.

Let us see how a similar foliation can be defined at a Hopf fixed point p of φ_b. We assume, for $\mu < b$ and μ near b, that φ_μ has a hyperbolic fixed point p_μ near p and $\dim W^s(p_\mu) = \dim W^s(p)$. As $\mu \to b$, $\mu < b$, the stable and unstable manifolds of p_μ in U converge to those of p in the C^k topology. So, we can use Definition (2.1) in this case, with the difference that we consider μ in some interval $[\mu_1, b]$.

Let now p be a saddle-node or a flip for φ_b. As a natural extension of Definition (2.1), we present below the concept of strong unstable foliation for $\{\varphi_\mu\}$ or Φ at (p, b).

First we need some basic facts about center manifolds as stated in Chapter II. Let U be a small cell neighborhood of p in M and $[\mu_1, \mu_2] \subset I$ a small interval with $b \in (\mu_1, \mu_2)$. Let W^c be a C^k center manifold for Φ at (p, b), $1 \leq k < \infty$. For each $\mu \in [\mu_1, \mu_2]$, $W^c_\mu = W^c \cap U \times \{\mu\}$ has dimension one and W^c_b is C^∞. We also consider the center stable manifold W^{cs} for Φ at (p, b), which is C^k and has dimension $s + 2$. Here s and u are the number of eigenvalues (with multiplicity) of $d\varphi_b(p)$ with norm less and bigger than one, respectively. Both W^c and W^{cs} are invariant by Φ, and W^c_μ and $W^{cs}_\mu = W^{cs} \cap U \times \{\mu\}$ are invariant by φ_μ. Frequently, here and in the sequel, we will identify a subset $V \times \{\mu\}$ of $M \times \{\mu\}$ with its projection V into M. We recall that W^{cs} is foliated by the strong stable foliation $\mathcal{F}^{ss}$, with leaves s-dimensional C^k discs transverse to W^c. Particular leaves are the strong stable manifolds through the fixed or periodic orbits of φ_μ near p. For each $\mu \in [\mu_1, \mu_2]$, W^{cs}_μ is a union of leaves of $\mathcal{F}^{ss}$. The foliation $\mathcal{F}^{ss}$ is invariant by Φ in the following sense: if $z, \Phi(z) \in U \times [\mu_1, \mu_2]$ and S is the leaf through z, then $\Phi(S)$ is contained in the leaf through $\Phi(z)$. Similarly, we can define the center unstable manifold W^{cu} for Φ at (p, b) and W^{cu} is foliated by the strong unstable foliation $\mathcal{F}^{uu}$. Particular leaves are the strong unstable manifolds of the fixed or periodic orbits of φ_μ near p. Our strong unstable tubular family (or foliation) is an extension of $\mathcal{F}^{uu}$ to a full neighborhood of (p, b) in $M \times I$. For later purposes we need this extended version of $\mathcal{F}^{uu}$, but no such version of $\mathcal{F}^{ss}$.

Suppose (p, b) is a saddle-node. As it can be seen from Chapter II, there are two possibilities for its unfolding. In the first one, for each $\mu < b$ and near b, there are two hyperbolic fixed points $p_{1,\mu}$ and $p_{2,\mu}$ of φ_μ near p, which collapse into p as $\mu \to b$ and then disappear for $\mu > b$. We also may assume that $\dim W^s(p_{2,\mu}) = s + 1$ and

$\dim W^s(p_{1,\mu}) = s$. The other possibility is similar, only the orbits $p_{1,\mu}$ and $p_{2,\mu}$ appear for $\mu > b$. Throughout this chapter we assume the first case.

If (p, b) is a flip, we also assume the following of four similar possibilities for its unfolding. For each $\mu < b$ and near b, there is a hyperbolic fixed point p_μ near p. For $\mu > b$, there is a hyperbolic fixed point $p_{1,\mu}$ and a hyperbolic period two point $p_{2,\mu}$ near p. We assume that $\dim W^s(p_{1,\mu}) = s$ and $\dim W^s(p_{2,\mu}) = s + 1$.

Definition (**2.2**). — *A strong unstable tubular family* $\mathscr{F}^{uu}$ *for* $\{\varphi_\mu\}$ *at the saddle-node or flip* (p, b) *is a continuous foliation of* $U \times [\mu_1, \mu_2]$ *such that*

a) *the leaves are* C^k *discs varying continuously in the* C^k *topology and*

$$\mathscr{F}^{uu}(p, b) = W^{uu}(p, b) \cap U \times \{b\},$$

b) *for each* μ, $U \times \{\mu\}$ *is a union of leaves transverse to* W^{cs}_μ,

c) *the foliation is* Φ*-invariant: if* $\mathscr{F}^{uu}(y, \mu)$ *is the leaf through* $(y, \mu) \in U \times \{\mu\}$, *then* $\Phi(\mathscr{F}^{uu}(y, \mu))$ *is the leaf through* $(\varphi_\mu(y), \mu) \in U \times \{\mu\}$.

These local unstable and strong unstable foliations have already been used in a similar context by several authors; see [26] for references. We provide a construction of them in Proposition (2.3) below, to give a clearer view of some of the main techniques of this chapter.

Proposition (**2.3**). — *There exists a strong unstable foliation for* $\{\varphi_\mu\}$ *at* (p, b), *where* p *is a saddle-node or a flip. Similarly, there exists an unstable foliation at a hyperbolic or a Hopf fixed point.*

Proof. — Let us first consider (p, b) to be a saddle-node or a flip. In $U \times [\mu_1, \mu_2]$ let W^{cs} be the center-stable manifold of Φ at (p, b). As before, Φ is defined by $\Phi(x, \mu) = (\varphi_\mu(x), \mu)$ and $U \ni p$, $[\mu_1, \mu_2] \ni b$ are small neighborhoods. In W^{cs}, we take a closed fundamental domain A. This is a set with the following property. $\varphi_b^2(A) \cap A = \varnothing$ and there is a neighborhood U' of (p, b) in W^{cs} such that if $x \in W^{cs}$ and its negative Φ-orbit leaves U', then the Φ-orbit of x has an element in A, which is unique except if it belongs to ∂A. Let us indicate the construction of such a fundamental domain when (p, b) is a saddle-node. Taking a C^2 coordinate system in the center-stable manifold of φ_b at p, we can write

$$\varphi_b(x,y) = (x + x^2 + x\langle c,y\rangle + o(|x|^3, \|x^2y\|, \|y\|^2), Ay + o(\|xy\|, \|y\|^2)),$$

where $c \in \mathbf{R}^2$ and $\|A\| < 1$. Here $y = 0$ represents the one-dimensional center-manifold and $x = 0$ the s-dimensional strong stable manifold $W^{ss}(p)$. The stable manifold $W^s(p)$ of φ_b at p is the half-plane $x \leq 0$. We take in $W^s(p)$ a hemisphere H of small radius centered at p and slightly extended transversally across $W^{ss}(p)$. It is easy to check from the above expression that $\varphi_b(H) \cap H = \varnothing$. For $\mu_1 \leq \mu \leq b$, $H_\mu = H \times \{\mu\}$ also crosses $W^{ss}(p_1, \mu)$ and $W^{ss}(p_2, \mu)$ transversally in $U \times \{\mu\}$ and

$\varphi_\mu(H_\mu) \cap H_\mu = \emptyset$. Also, for $b < \mu < \mu_2$, $\varphi_\mu(H_\mu) \cap H_\mu = \emptyset$, where $H_\mu = H \times \{\mu\}$. We let A_μ be a closed region bounded by H_μ, $\varphi_\mu(H_\mu)$, and an annulus whose boundary lies in $H_\mu \cap \varphi_\mu(H_\mu)$. Then take $A = \bigcup_{\mu_1 \leq \mu \leq \mu_2} A_\mu$.

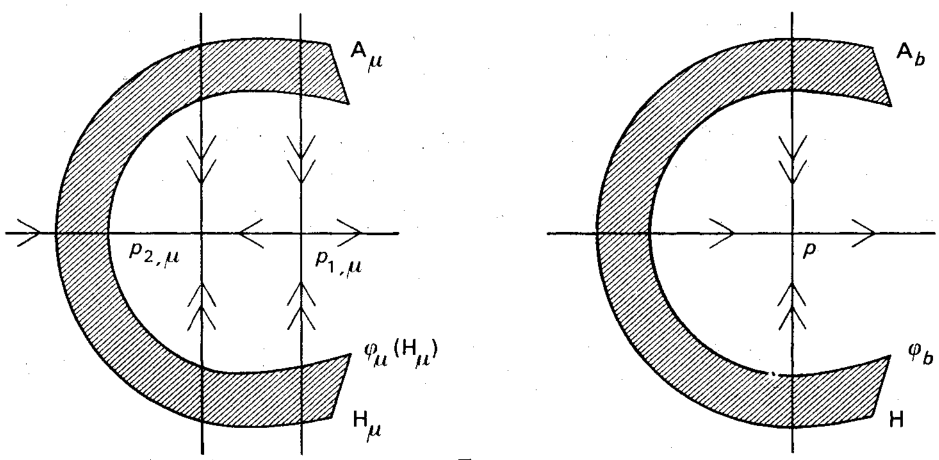

FIG. 10

In the case of a flip, A is diffeomorphic to $A_b \times [\mu_1, \mu_2]$, where A_b is an $s+1$-dimensional annulus. Similarly, if (p, b) is a Hopf point, A is diffeomorphic to $A_b \times [\mu_1, b]$, where A_b is an $(s+2)$-dimensional annulus. If $(\bar{x}, \bar{\mu})$ is a hyperbolic fixed point, A is diffeomorphic to $A_{\bar{\mu}} \times [\mu_1, \mu_2]$, where $A_{\bar{\mu}}$ is an s-dimensional annulus and $\bar{\mu} \in (\mu_1, \mu_2)$. Let us proceed with the construction of the strong unstable foliation for a saddle-node or a flip. Over a neighborhood U_1 of the exterior boundary of A we raise a fibration of class C^k, the fibers being u-dimensional discs transverse to W^{cs} and each of them contained in some $U \times \{\mu\}$. Over the neighborhood $\Phi(U_1)$ of the interior boundary of A, we have a similar fibration, the fibers being the Φ-images of those in U_1. Restricting to smaller neighborhoods of ∂A, we can extend this fibration to a full neighborhood of A. This is done as follows. Over a neighborhood of A, we raise another C^k fibration transverse to W^{cs} and the fibers contained in the sections $U \times \{\mu\}$, but not necessarily φ_μ-invariant. We now define a new fibration, which agrees with the first one in a neighborhood of ∂A and with the second one off a slightly bigger neighborhood of ∂A. Let π_1 and π_2 be the projections, into a neighborhood V of A, defined by the two fibrations. We define a C^k real function $\rho : V \to [0, 1]$, such that ρ is 1 near ∂A and 0 off a small neighborhood of ∂A. The required fibration is then given by the projection $\pi = \rho\pi_1 + (1 - \rho)\pi_2$. Its fibers form the leaves of our foliation. We now simply define it over $W^{cs} - W^c$ through iterates Φ^n or φ_μ^n, all $n \geq 0$: if $(y, \mu) = \Phi^n(x, \mu)$ for $(x, \mu) \in A$ and the fiber through (x, μ) is $\mathscr{F}^{uu}(x, \mu)$, then the fiber through (y, μ) is $\Phi^n(\mathscr{F}^{uu}(x, \mu)) \cap U \times [\mu_1, \mu_2]$. By the generalized λ-lemma [24], [30], $\mathscr{F}^{uu}$ extends over W^c satisfying all the conditions of Definition (2.2). Notice that, on the center unstable manifold of Φ at (p, b), we get the usual strong unstable foliation. The construction at a hyperbolic fixed point is easier and can be done in a similar way. The same applies to a Hopf point (p, b), in which case we take the parameter μ in some interval $[\mu_1, b]$. This finishes the proof of the proposition.

We point out that these unstable and strong unstable foliations are not unique. There is a degree of freedom in their construction, as shown in Proposition (2.3). This is what enables us to globalize them in a compatible way with the tubular families of other periodic orbits. Such globalizations will be performed in the next section.

3. Global Tubular Families

In this section we construct compatible systems of tubular families or foliations for a family $\{\varphi_\mu\}$, μ near its first bifurcation point.

As before, we denote by $b \in I$ the first bifurcation point, so that φ_μ is Morse-Smale for $\mu < b$. From the previous chapter, $\{\varphi_\mu\}$ can be stable in one of our three senses only if one of the periodic orbits of φ_b is an elementary bifurcation (saddle-node, flip or Hopf). Moreover, the (strong) stable and unstable manifolds of all the periodic orbits of φ_b must have transversal intersections.

We recall that, since φ_μ is Morse-Smale for $\mu < b$, its periodic orbits can be partially ordered through the relation $p_{i,\mu} \geq p_{j,\mu}$ if $W^u(p_{i,\mu}) \cap W^s(p_{j,\mu}) \neq \emptyset$. See [24], [25]. We fix a total ordering for the periodic orbits compatible with this relation. If φ_b has a Hopf orbit or a flip, the same ordering applies as well to its periodic orbits: there can not be any cycle, since otherwise b would not be the first bifurcation point [18]. If φ_b has a saddle-node, we may have a cycle containing this orbit. If there is only a 1-cycle, the saddle-node will be counted as $p_{j,\mu}$ and $p_{j+1,\mu}$ for some positive integer j. However, if there is a cycle of larger length, then the saddle-node will be counted as $p_{j,\mu}$ $P_{j+j',\mu}$ where j and j' are positive integers and $j' > 1$. In all cases, this ordering of the periodic orbits will be used to build up global systems of foliations for $\{\varphi_\mu\}$, μ in a small interval in I. This interval is of the form $[\mu_1, b]$ when φ_b has a Hopf point or a saddle-node which is critical or has a cycle, and otherwise $[\mu_1, \mu_2]$ with $b \in (\mu_1, \mu_2)$.

Let $p_{1,\mu} > p_{2,\mu} > \ldots > p_{\ell,\mu}$ be the periodic orbits of φ_μ for some $\mu \in [\mu_1, \mu_2]$, all of them hyperbolic except at most one which is an elementary bifurcation. For convenience, we write p_i instead of $p_{i,\mu}$. Let $U_1, U_2, \ldots, U_\ell$ be neighborhoods of $p_1, p_2, \ldots, p_\ell$. We will consider local unstable or strong unstable foliations in $U_1 \times [\mu_1, \mu_2]$, $U_2 \times [\mu_1, \mu_2]$, $\ldots$, $U_\ell \times [\mu_1, \mu_2]$ as in Definitions (2.1) and (2.2). If the periodic point p_i is a saddle-node with no cycles or a flip, let $\mathscr{F}^u(p_i)$ denote its strong unstable foliation in $U_i \times [\mu_1, \mu_2]$. Otherwise, let $\mathscr{F}^u(p_i)$ denote the unstable foliation in $U_i \times [\mu_1, \mu_2]$. Since $\mathscr{F}^u(p_i)$ is a φ_μ-invariant foliation in $U_i \times [\mu_1, \mu_2]$, it naturally induces one in $\bigcup \varphi_\mu^n(U_i) \times \{\mu\}$ for $\mu \in [\mu_1, \mu_2]$ and $n \in N$ by simply taking iterates φ_μ^n of the leaves. In the sequel, we will be using this extended (or globalized) foliation, which will be still denoted by $\mathscr{F}^u(p_i)$.

Definition (3.1). — *The system of foliations* $\mathscr{F}^u(p_1), \mathscr{F}^u(p_2), \ldots, \mathscr{F}^u(p_l)$ *is compatible when*

a) *if a leaf* F *of* $\mathscr{F}^u(p_i)$ *intersects a leaf* S *of* $\mathscr{F}^u(p_j)$, $i \leq j$, *then* $F \supset S$,

b) *for all* $i \leq j$, *the restriction of* $\mathscr{F}^u(p_j)$ *to each leaf of* $\mathscr{F}^u(p_i)$ *is a foliation of class* C^k, $2 \leq k$.

Remark. — Note that the restriction of these foliations to unstable manifolds of hyperbolic periodic orbits are of class C^k. However, the unstable manifold of a non-critical saddle-node with no cycles or a flip is not a leaf of the strong unstable foliation. Thus, we do not demand differentiability of the foliations when restricted to these unstable manifolds.

Let us show the existence of such systems. First we consider the case where φ_b has a Hopf periodic orbit. This is the same as the parametrized version of the hyperbolic case. Although somewhat simpler than the saddle-node and the flip, it gives a pretty good idea of how to proceed in those cases as well.

Proposition (3.2). — *If* φ_b *has a Hopf periodic orbit, then there exists a compatible system of unstable foliations for* φ_μ, $\mu \in [\mu_1, b]$ *for some* μ_1 *near* b.

Proof. — Let $p_{1,\mu} > p_{2,\mu} > \ldots > p_{l,\mu}$ be an ordering of the periodic orbits of φ_μ for $\mu_1 \leq \mu \leq b$. We simply write p_i instead of $p_{i,b}$. By induction, we may assume that a compatible system of foliations $\mathscr{F}^u(p_1), \ldots, \mathscr{F}^u(p_n)$ has been constructed in neighborhoods $U_1 \times [\mu_1, b], \ldots, U_n \times [\mu_1, b]$, $p_i \in U_i$ for $1 \leq i \leq n$. Let us build $\mathscr{F}^u(p_{n+1})$. We will adapt the proof of Proposition (2.3) to guarantee the compatibility condition. Near p_{n+1}, we consider a closed annulus A_b as a fundamental domain for $W^s(p_{n+1})$ and denote by $\partial_{ex} A_b$ its exterior boundary. Then, the annulus A_μ with boundaries $\partial_{ex} A_b \times \{\mu\}$ and $\varphi_\mu(\partial_{ex} A_b \times \{\mu\})$ is also a fundamental domain for $W^s(p_{n+1}, \mu)$, $\mu \in [\mu_1, b]$ (we take μ_1 closer to b if necessary. If $W^u(p_n)$ intersects $W^s(p_{n+1})$ it must do so transversally. Thus, we can take a C^k fibration of $W^u(p_n)$ near the exterior boundary of A_b, whose fibers are discs transverse to $W^s(p_{n+1})$ with the same dimension as $W^u(p_{n+1})$. The image by φ_b of this fibration induces a similar one near the interior boundary of A_b. On the other hand, it is easy to get a second C^k fibration of $W^u(p_n)$ in a neighborhood of all of its intersection with A_b, if we do not require it to be φ_b-invariant. However, as in Proposition (2.3), this second fibration can be deformed to agree with the first one near the boundaries of A_b and, thus, it becomes φ_b-invariant. We now want to fiber in a similar way the leaves of $\mathscr{F}^u(p_n)$ near A_μ, $\mu_1 \leq \mu \leq b$ with μ_1 close to b. First, we observe that, in $U_n \times [\mu_1, b]$, the leaves of $\mathscr{F}^u(p_n)$ are C^k imbeddings of the disc $W^u(p_n) \cap U_n$, continuously parametrized by $(W^s(p_n) \cap U_n) \times [\mu_1, b]$. Thus, by restricting this parameter space, we have that the leaves of $\mathscr{F}^u(p_n)$ near A_μ are C^k close to $W^u(p_n)$. Using this parametrization and the fact that the foliation $\mathscr{F}^u(p_n)$ is φ_μ-invariant, we can fiber as above the leaves near $W^u(p_n) \cap A_b$. We get a φ_μ-invariant fibration with C^k fibers transverse to $W^s(p_{n+1}, \mu)$, $\mu_1 \leq \mu \leq b$, and varying continuously in the C^k topology. By construction, each fiber is contained in some leaf of

$\mathscr{F}^u(p_n)$. Also, the fibration is C^k when restricted to each leaf of $\mathscr{F}^u(p_n)$. Now we take $\mathscr{F}^u(p_{n-1})$. There are two cases to consider. If $W^u(p_{n-1})$ intersects $W^s(p_{n+1})$ but not $W^s(p_n)$, then $W^u(p_{n-1}) \cap A_b$ is compact and disjoint from $W^u(p_n) \cap A_b$. So, we can proceed as before, fibering the leaves of $\mathscr{F}^u(p_{n-1})$ near $W^u(p_{n-1}) \cap A$. Let us now suppose that $W^u(p_{n-1})$ intersects $W^s(p_n)$. Since $\mathscr{F}^u(p_{n-1})$ and $\mathscr{F}^u(p_n)$ are compatible by the induction hypothesis, the fibration of $\mathscr{F}^u(p_n)$ in a neighborhood V of $W^u(p_n) \cap A_b$ is also a fibration of $\mathscr{F}^u(p_{n-1})$ in V. On the other hand, $B = W^u(p_{n-1}) \cap (A_b - V)$ is compact. So, $\mathscr{F}^u(p_{n-1})$ can be fibered as above in a neighborhood of B. As in Proposition (2.3), we can average these two fibrations near ∂B to get a desired one in some neighborhood of $W^u(p_n) \cup W^u(p_{n-1})$ intersected with A_b. Its fibers form the leaves of our foliation. We repeat the argument to all $\mathscr{F}^u(p_i)$, $1 \leq i \leq n-2$. Once we have the φ_μ-invariant and compatible foliation $\mathscr{F}^u(p_{n+1})$ near $A_b \times [\mu_1, b]$, some μ_1 close to b, we just consider its positive iterates by φ_μ (or Φ). By the generalized λ-lemma [24], [30], it extends to the foliation $W^u(p_{n+1,\mu})$, $\mu \in [\mu_1, b]$, of the center unstable manifold of Φ at (p_{n+1}, b). This finishes the construction of the foliation $\mathscr{F}^u(p_{n+1})$. The proof of the theorem is complete.

Let us now consider the case where φ_b has a saddle-node. Let

$$p_{1,\mu} > p_{2,\mu} > \ldots > p_{\ell,\mu}$$

be an ordering for the periodic points of φ_μ, $\mu_1 \leq \mu < b$. We first consider the case where the saddle node is non-critical and φ_b has no cycles. We assume that $p_{j,\mu}$ and $p_{j+1,\mu}$ coalesce at $\mu = b$, giving rise to the saddle-node $p_j = p_{j+1}$.

We shall construct a compatible system of foliations for $\{\varphi_\mu\}$, with μ in some interval $[\mu_1, \mu_2]$ and $b \in (\mu_1, \mu_2)$. The foliation at the saddle-node $p_j = p_{j+1}$ will be a strong unstable foliation $\mathscr{F}^{uu}(p_j)$. Actually, the construction we just performed in Proposition (3.2) can be adapted to the present case as well as to the flip bifurcation. However, to prove the stability of the family, we also construct a one-dimensional center foliation in a neighborhood of the saddle-node $p_j = p_{j+1}$ in the center stable manifold of Φ at (p_j, b). As usual, $\Phi : M \times I \to M$ is defined by $\Phi(x, \mu) = (\varphi_\mu(x), \mu)$. This center foliation $\mathscr{F}^c$ should be φ_μ-invariant and compatible with the unstable foliations $\mathscr{F}^u(p_i)$, $i < j$, in the sense of Definition (3.1).

Proposition **(3.3)**. — *If φ_b has a non critical saddle-node $p_j = p_{j+1}$ and no cycles, then*

(a) *there is a compatible center foliation $\mathscr{F}^c$ defined in a neighborhood of the center stable manifold W^{cs} of Φ at $(p_{j,b})$,*

(b) *there is a compatible system of unstable foliations $\mathscr{F}^u(p_i)$ for Φ or φ_μ, $\mu \in [\mu_1, \mu_2]$ for some $\mu_1 < b$ and $\mu_2 > b$. The foliation we consider at $p_j = p_{j+1}$ is a strong unstable one, $\mathscr{F}^{uu}$,*

(c) *the union of leaves of $\mathscr{F}^{uu}$ through a leaf of $\mathscr{F}^c$ forms a C^1 submanifold.*

Proof. — Since $p_1 > p_2 > \ldots > p_{j-1}$ are all hyperbolic, Proposition (3.2) provides a compatible system of unstable foliations $\mathscr{F}^u(p_1), \mathscr{F}^u(p_2), \ldots, \mathscr{F}^u(p_{j-1})$. The leaves of these foliations are taken to be C^k, $k \geq 2$. Let us construct a center foliation $\mathscr{F}^c$

at the saddle-node; its leaves will be C^{k-1}. In the center stable manifold of Φ at (p_j, b), we consider a fundamental domain as in Proposition (2.3). We define the center foliation $\mathscr{F}^c$ as the integral curves of a vector field X satisfying the following properties. X is tangent to the leaves of $\mathscr{F}^u(p_i)$ and it is C^{k-1} along these leaves for all $i<j$ such that $W^u(p_i)$ intersects $W^s(p_j)$. The vector field X is also transverse to the strong stable foliation of Φ at p_j and $d\Phi$-invariant. To get such a vector field, we proceed by induction taking $i=j-1, j-2, \ldots, 1$. If $W^u(p_{j-1})$ intersects W^{cs}, we take a C^{k-1} vector field X tangent to $W^u(p_{j-1})$ near its intersection with the exterior boundary of the fundamental domain A. Near the interior boundary, we just consider $d\Phi(X)$. It is easy to extend this vector field X to a full neighborhood of $W^u(p_{j-1}) \cap A$ in $W^u(p_{j-1})$. We now want to define X along the nearby leaves of $\mathscr{F}^u(p_{j-1})$. As we noticed in the proof of Proposition (3.2), these leaves are C^k imbeddings of a disc in $W^u(p_{j-1})$, continuously parametrized by their intersection with $W^s(p_{j-1})$. So we can project X into the leaves of $\mathscr{F}^u(p_{j-1})$ near the exterior boundary of A, consider its image by $d\Phi$ and extend it across A as above. Due to the fact that the saddle-node is noncritical, X is transverse to the strong stable foliation of Φ in the center stable manifold. By induction, let us suppose X defined along the leaves of $\mathscr{F}^u(p_{i+1}), \ldots, \mathscr{F}^u(p_{j-1})$ near the fundamental domain A. If $W^u(p_i) \cap A = \varnothing$, we proceed to the next foliation $\mathscr{F}^u(p_{i-1})$. If $W^u(p_i) \cap A \neq \varnothing$, but $W^u(p_i) \cap W^s(p_k) = \varnothing$ for $i<k<j$, we can proceed as before since $W^u(p_i) \cap A$ is compact and disjoint from $W^u(p_k) \cap A$. Finally, let $W^u(p_i) \cap W^s(p_n) \neq \varnothing$ for some $i<n<j$. The vector field X is already defined in a neighborhood V of $W^u(p_k)$ in A for $i<k<j$. Since $\mathscr{F}^u(p_i)$ is compatible with $\mathscr{F}^u(p_k)$, X is tangent to the leaves of $\mathscr{F}^u(p_i)$ in V. But $W^u(p_i) \cap (A-V)$ is compact, so we are again reduced to the previous case. Thus we can construct X as desired in a neighborhood of A in the center stable manifold. Now we take the integral curves of X and their positive iterates by Φ. By the λ-lemma [24], [30], this partial foliation extends to center manifolds and together they form a center foliation as asserted in (a). Parts (b) and (c) follow as in Proposition (3.2) for the hyperbolic periodic points and also as in Proposition (2.3) for the saddle-node. In this last case, we raise the strong unstable foliation from the center foliation we have just constructed. The proof of Proposition (3.3) is complete.

Remark. — The center foliation $\mathscr{F}^c$ is not a " classical " foliation in two ways

(1) as usual in the present work, we only required that the leaves of $\mathscr{F}^c$, which are C^{k-1} with $k \geq 2$, should vary continuously in the C^{k-1} topology. We did not demand the projection along the leaves into a transverse section (like a leaf of the strong stable foliation) to be differentiable,

(2) $\mathscr{F}^c$ has singularities, as shown in the picture

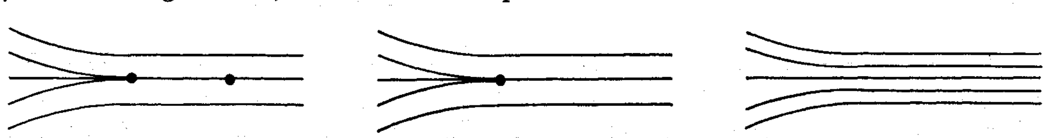

FIG. 11

Now let us consider the situation of a saddle-node $p_j = p_{j+j'}$ for φ_b which is critical or lies in a cycle and $j' \geq 1$. In the case the saddle-node is bicritical or it is part of a cycle, we wish to prove the stability of the family $\{\varphi_\mu\}$ for $\mu \leq b$ (left stability). When the saddle-node is critical but not bicritical and there is no cycle, we wish to prove that $\{\varphi_\mu\}$ is midly stable for $\mu < \mu_1$, some $\mu_1 > b$ and near b. Again, since φ_μ is Morse-Smale for $b < \mu \leq \mu_1$, some $\mu_1 > b$, $\{\varphi_\mu\}$ is stable in this range of the parameter. This follows from [25] or the next section. Thus, it is certainly enough to show that $\{\varphi_\mu\}$ is left stable at b. In conclusion, we can treat these three cases in the same way. As before, we need to construct a compatible system of unstable foliations for $\{\varphi_\mu\}$, $\mu \leq b$.

Proposition (3.4). — *If φ_b has a saddle-node $p_j = p_{j+j'}$, $j' \geq 1$, which is critical or, has a cycle, then there exists a compatible system of unstable foliations for $\mu \leq b$. The leaves of $\mathscr{F}^u(p_{j+j',\mu})$ are one dimension lower than those of $p_{j,\mu}$.*

Proof. — Similar to that of Proposition (3.2), using the unfolding of the saddle-node as in Proposition (2.3). In this case the leaves of $\mathscr{F}^u(p_j, \mu)$ cover a neighborhood of $W^{ss}(p_j, \mu) - \{p_{j,\mu}\}$ near $p_{j,\mu}$ where $W^{ss}(p_{j,\mu})$ is the strong stable manifold of $p_{j,\mu}$.

The leaves of $\mathscr{F}^u(p_{j,\mu})$ intersected with the center stable manifold W^{cs}_μ are represented by horizontal lines in the pictures below. The intersections of the leaves of $\mathscr{F}^u(p_{j+j',\mu})$ with W^{cs}_μ are represented by points.

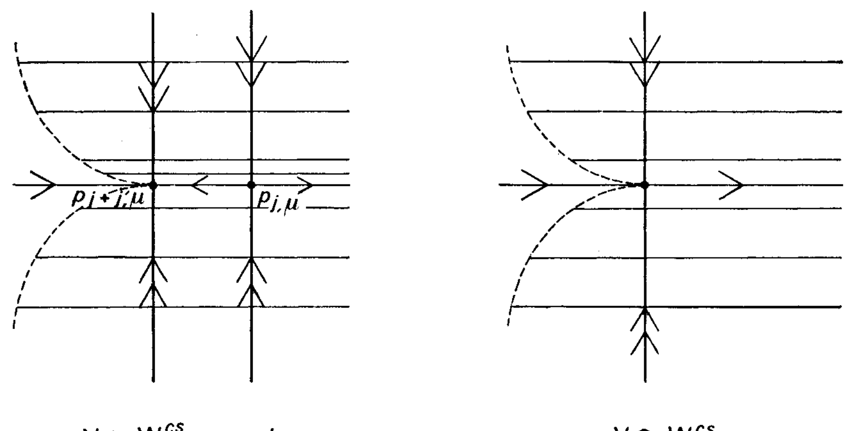

FIG. 12

Notice that the foliations $\mathscr{F}^u(p_{j,\mu})$ and $\mathscr{F}^u(p_{j+j',\mu})$ can be constructed to be compatible in a neighborhood V of $W^{ss}(p_j) - p_j$ in $M \times [\mu_1, b]$ for some $\mu_1 < b$.

Proposition (3.5). — *If φ_b has a flip point $p_{j'}$, then*

(a) *there is a compatible center foliation $\mathscr{F}^c$ defined in a neighborhood of the center stable manifold of Φ at (p_j, b);*

(b) *there is a compatible system of unstable foliations* $\mathscr{F}^u(p_i)$ *for* φ_μ, $\mu \in [\mu_1, \mu_2]$ *for some* $\mu_1 < b$ *and* $\mu_2 > b$; *the foliation we consider at the flip* p_j *is a strong unstable one,* $\mathscr{F}^{uu}$;
(c) *the leaves of* $\mathscr{F}^{uu}$ *through a leaf of* $\mathscr{F}^c$ *form a* C^1 *submanifold.*

Proof. — The proof is entirely similar to that of Proposition (3.3).

We finish this section by summing up in the next theorem the results we have obtained on compatible systems of unstable (strong unstable) foliations.

Theorem **(3.6)**. — *Let* b *be the bifurcation point of the family* $\{\varphi_\mu\}$ *in* $\mathscr{B}$. *Let* $\Phi : M \times I \to M$ *be defined by* $\Phi(x, \mu) = (\varphi_\mu(x), \mu)$. *Then,*

(1) *if* φ_b *has either a noncritical saddle-node and no cycles or a flip, there exists a compatible system of unstable foliations for* Φ *or* $\{\varphi_\mu\}$, $\mu < \mu_1$ *for some* μ_1 *near* b *and* $\mu_1 > b$. *The unstable foliation at the saddle-node or the flip is a strong unstable one,*
(2) *if* φ_b *has a saddle-node or a Hopf periodic orbit, there exists a compatible system of unstable foliations for* Φ *or* $\{\varphi_\mu\}$, $\mu \leq b$.

4. Stability

We culminate this chapter by showing the stability of the families of diffeomorphisms $\{\varphi_\mu\}$ in $\mathscr{B}$. Such families were introduced in Chapter III, studied there and in the previous sections of the present chapter. A family $\{\varphi_\mu\}$ in $\mathscr{B}$ is stable, mildly or left stable according to the structure of the diffeomorphism φ_b, where b is the first bifurcation point of $\{\varphi_\mu\}$. More specifically, the stability depends on the structure of the non-hyperbolic orbit of φ_b. We stress again that the transversality of the stable and unstable manifolds of the periodic orbits is necessary even for left stability, as was proved in Chapter III.

Our results on stability will follow rather naturally and in a unified way from the existence of compatible systems of (strong) unstable tubular families or foliations. Fitting together these foliations provides a kind of global coordinate system from which the topological conjugacies can be constructed. We define the conjugacies inductively on the stable manifolds of the periodic orbits, these orbits being totally ordered as in section 3.

We will make use of the following two known results. The first one states that $\mathscr{B}$ is open in the set $\mathscr{P}$ of C^∞ one-parameter families of C^∞ diffeomorphisms of M with the C^∞ topology. It corresponds to Theorem (2.5) of [19] and Theorem (3.1) of [20]. The second result is an easy extension (a parametrized version) of the Isotopy Extension Theorem [23].

Theorem **(4.1)**. — *Let* $\{\varphi_\mu\} \in \mathscr{B}$ *have the first bifurcation point* b. *There exists a neighborhood* U *of* $\{\varphi_\mu\}$ *such that if* $\{\widetilde{\varphi}_\mu\} \in U$ *has first bifurcation point* $\widetilde{b}$, *then* $\{\widetilde{\varphi}_\mu\} \in \mathscr{B}$ *and* φ_b *and* $\widetilde{\varphi}_{\widetilde{b}}$ *have the same elementary bifurcation. Moreover, there is an order preserving one to one correspondence between the periodic orbits of* φ_b *and* $\widetilde{\varphi}_{\widetilde{b}}$.

Let N be a C^r compact manifold, $r \geq 1$, and A an open subset of $\mathbf{R}^s$. Let M be a C^∞ manifold with $\dim M > \dim N$. We indicate by $C^k_A(N \times A, M \times A)$ the set of C^k mappings $f : N \times A \to M \times A$ such that $\pi = \pi' f$, endowed with the C^k topology, $1 \leq k \leq r$. Here, π and π' denote the natural projections $\pi : N \times A \to A$, $\pi' : M \times A \to A$. Let $\mathrm{Diff}^k_A(M \times A)$ be the set of C^k diffeomorphisms φ of $M \times A$ such that $\pi' = \pi' \varphi$, again with the C^k topology.

Theorem (**4.2**). — *Let* $i \in C^k_A(N \times A, M \times A)$ *be an imbedding and* A′ *a compact subset of* A. *Given neighborhoods* U *of* $i(N \times A)$ *in* $M \times A$ *and* V *of the identity in* $\mathrm{Diff}^k_A(M \times A)$, *there exists a neighborhood* W *of* i *in* $C^k_A(N \times A, M \times A)$ *such that for each* $j \in W$ *there exists* $\varphi \in V$ *satisfying* $\varphi i = j$ *restricted to* $N \times A'$ *and* $\varphi(x) = x$ *for all* $x \notin U$.

Now we prove the main theorem of the present chapter.

Theorem (**4.3**). — *Let* $\{\varphi_\mu\} \in \mathscr{B}$ *with first bifurcation point* b. *Then*

1) *if* φ_b *has a flip or a noncritical saddle-node with no cycles,* $\{\varphi_\mu\}$ *is stable,*
2) *if* φ_b *has a saddle-node which is not bicritical and has no cycles,* $\{\varphi_\mu\}$ *is mildly stable,*
3) *if* φ_b *has a Hopf periodic orbit or a saddle-node which is bicritical or has a cycle then* $\{\varphi_\mu\}$ *is left stable.*

Proof. — First, we observe that the statement in part (2) can be proved as in part (3). In fact, in case (2) we have that $\{\varphi_\mu\}$ is Morse-Smale for $b < \mu \leq \mu_1$, some $\mu_1 > b$, and so it is stable in this range of the parameter. This last fact is an easy consequence of the proof that Morse-Smale diffeomorphisms are stable. By the same reason, in all cases it is enough to consider the stability of $\{\varphi_\mu\}$ in $\mathscr{B}$ for μ near the first bifurcation point.

Let $p_{1,\mu} > p_{2,\mu} > \ldots > p_{\ell,\mu}$ be a total ordering of the periodic orbits of φ_μ for $\mu < b$. As in section 3, this ordering can be naturally extended for $\mu = b$ and even for $\mu > b$ when φ_b is as in parts (1) and (2) of the statement. By Theorem (3.6), we can take a compatible system of (strong) unstable foliations $\mathscr{F}^u(p_1)$, $\mathscr{F}^u(p_2)$, $\ldots$, $\mathscr{F}^u(p_\ell)$ defined in $M \times [\mu_1, \mu_2]$ or $M \times [\mu_1, b]$, for some $\mu_1 < b$ and $\mu_2 > b$ according to case (1) or (3). In case (1), the foliation is a strong unstable one at the flip or the saddle-node. Recall that this strong unstable foliation is constructed from a center foliation $\mathscr{F}^c$ in the center stable manifold of Φ at the flip or the saddle-node, where $\Phi : M \times I \to M \times I$ is defined by $\Phi(x, \mu) = (\varphi_\mu(x), \mu)$. By Theorem (4.1), for a nearby family $\{\widetilde{\varphi}_\mu\}$ with first bifurcation point b, we can consider a corresponding compatible system of (strong) unstable foliations and a compatible center foliation in case (1). Also, given a reparametrization (a homeomorphism) $\rho : I \to I$ near the identity and $\rho(b) = \widetilde{b}$, there is an order preserving continuous correspondence between the periodic orbits $p_{i,\mu}$ and $\widetilde{p}_{i,\rho(\mu)}$ of φ_μ and $\widetilde{\varphi}_{\rho(\mu)}$, for $\mu \in [\mu_1, \mu_2]$ or $\mu \in [\mu_1, b]$. This defines the conjugacy on the periodic orbits. In the construction of a global conjugacy we can take this reparametrization quite arbitrarily, except in the case of a noncritical saddle-node with no cycles. In this situation we must choose ρ as in Chapter II, so that we have a (continuous) conjugacy between Φ and $\widetilde{\Phi}$ restricted to center manifolds at the saddle-nodes.

Our global conjugacy H will be constructed on $M \times [\mu_1, \mu_2]$ or $M \times [\mu_1, b]$ inductively on the stable manifolds of the periodic orbits and, in case (1), at the center stable manifold of Φ at the saddle-node or the flip. Since these stable manifolds cover all of $M \times \{\mu\}$ for each $\mu \in [\mu_1, \mu_2]$ or $\mu \in [\mu_1, b]$, H will be defined on all of $M \times [\mu_1, \mu_2]$ or $M \times [\mu_1, b]$ and maps onto $M \times [\rho(\mu_1), \rho(\mu_2)]$ or $M \times [\rho(\mu_1), \widetilde{b}]$. We fix fundamental domains A_k for $W^s(p_k)$ or $W^{cs}(p_k)$ with exterior boundaries transverse to all $W^u(p_i)$, $i < k \leq \ell$. The conjugacy H will be constructed with the following properties:

a) $H(M \times \{\mu\}) = M \times \{\rho(\mu)\}$, μ in $M \times [\mu_1, \mu_2]$ or in $M \times [\mu_1, b]$,
b) it sends leaves of $\mathscr{F}^u(p_i)$ into leaves of $\mathscr{F}^u(\widetilde{p}_i)$, $1 \leq i \leq \ell$,
c) it is differentiable along each leaf of $\mathscr{F}^u(p_i)$ off the stable manifold of Φ at p_i, $1 \leq i \leq \ell$,
d) if $W^u(p_i) \cap W^s(p_k) \neq \emptyset$, $i < k$, then for each leaf F of $\mathscr{F}^u(p_i)$, in a neighborhood of $F \cap A_k$, H is C^1 near the inclusion map.

As the first step H takes the sources of $\{\varphi_\mu\}$ onto the sources of $\{\widetilde{\varphi}_{\rho(\mu)}\}$. Suppose H has been constructed on the stable manifolds of Φ at $p_1 > p_2 > \ldots > p_n$. Notice that the space of leaves of $\mathscr{F}^u(p_i)$ is parametrized by the intersection of the leaves with $W^s(p_i)$ or $W^{cs}(p_i)$. Thus, in particular, H defines a map from the space of leaves of $\mathscr{F}^u(p_i)$ onto the space of leaves of $\mathscr{F}^u(\widetilde{p}_i)$ for $1 \leq i \leq n$: $F \in \mathscr{F}^u(p_i)$ is associated to $\widetilde{F} \in \mathscr{F}^u(\widetilde{p}_i)$ if $H(F \cap W^s(p_i)) = \widetilde{F} \cap W^s(\widetilde{p}_i)$. Let us now consider $W^s(p_{n+1})$, $W^s(\widetilde{p}_{n+1})$ if p_{n+1} is either hyperbolic, a Hopf orbit or a saddle-node which is critical or has a cycle. Let $A_b = A_{n+1} \cap (M \times \{b\})$ and $\widetilde{A}_{\widetilde{b}} = \widetilde{A}_{n+1} \cap (M \times \{\widetilde{b}\})$. If $W^u(p_n) \cap A_b \neq \emptyset$, we choose H as a diffeomorphism near the inclusion of a neighborhood of $W^u(p_n) \cap \partial_{ex} A_b$ in $W^u(p_n)$ onto a neighborhood of $W^u(\widetilde{p}_n) \cap \partial_{ex}\widetilde{A}_{\widetilde{b}}$ in $W^u(\widetilde{p}_n)$. Such a diffeomorphism exists because $W^u(p_n) \cap A_b$ is close to $W^u(\widetilde{p}_n) \cap \widetilde{A}_{\widetilde{b}}$ for $\{\varphi_\mu\}$ near $\{\widetilde{\varphi}_\mu\}$. Near the interior boundary of A_b, H is defined by $H\varphi_b = \widetilde{\varphi}_{\widetilde{b}} H$. From Theorem (4.2) this partial diffeomorphism can be extended to all of $W^u(p_n) \cap A_b$. To extend H to the leaves of $\mathscr{F}^u(p_n)$ near A_{n+1}, $\mu_1 \leq \mu \leq b$, we proceed in an entirely similar way using Theorem (4.2), since we already know which leaf of $\mathscr{F}^u(\widetilde{p}_n)$ is associated to a given leaf of $\mathscr{F}^u(p_n)$. This completes the construction of the conjugacy on the intersection of the leaves of $\mathscr{F}^u(p_n)$ with the fundamental domain A_{n+1}. Next, we take $\mathscr{F}^u(p_{n-1})$ and suppose that $W^u(p_{n-1}) \cap A_b \neq \emptyset$. There are two cases to consider. If $W^u(p_{n-1}) \cap W^s(p_n) = \emptyset$, then $W^u(p_{n-1}) \cap A_b$ is compact and disjoint from $W^u(p_n) \cap A_b$. Thus, the construction of the conjugacy on the leaves of $\mathscr{F}^u(p_{n-1})$ restricted to A_{n+1} is the same as on the leaves of $\mathscr{F}^u(p_n)$. So, assume that $W^u(p_{n-1}) \cap W^s(p_n) \neq \emptyset$. Since $\mathscr{F}^u(p_{n-1})$ and $\mathscr{F}^u(p_n)$ are compatible, H is already defined on the leaves of $\mathscr{F}^u(p_{n-1})$ in a neighborhood V of $W^u(p_n) \cap A_b$ in A_{n+1}. But $B = W^u(p_{n-1}) \cap (A_b - V)$ is compact. So, we can extend H to B using Theorem (4.2) and also to the leaves of $\mathscr{F}^u(p_{n-1})$ in a neighborhood of B in A_{n+1}. We repeat the argument to all $\mathscr{F}^u(p_i)$, $1 \leq i \leq n-2$. If p_{n+1} is a

Hopf orbit or a critical saddle-node with no cycle, then H is defined on A_{n+1} satisfying $H\varphi_\mu = \widetilde{\varphi}_{\rho(\mu)}H$ or $H\varphi = \widetilde{\varphi}H$, $\mu_1 \leq \mu \leq b$ or $\mu_1 \leq \mu \leq \mu_2$ with $\mu_1 < b$ and $b < \mu_2$. The same equation allows us to define H on all of the stable manifold of φ at p_{n+1}. If the saddle-node occurs in a cycle so that $p_{n+1} = p_{n+1+j'}$ for some $j' \geq 1$, then we have only defined H on a neighborhood V of $W^{ss}(p_{n+1,b}) - \{p_{n+1,b}\}$ in $W^s(p_{n+1})$. We continue as before for p_{n+1+j} with $j < j'$. When we come to $p_{n+1+j'}$, the conjugacy H as already been defined on the part V of $W^s(p_{n+1+j'})$. We continue the process as above obtaining a conjugacy H_1 defined on $(\bigcup_{k \leq n+1+j'} W^u(p_k)) \cap W^s(p_{n+1+j'})$ such that $H_1 = H$ off a neighborhood of $W^s(p_{n+1+j'}) - V$ in $W^s(p_{n+1+j'})$. Thus, H can be defined on all of $W^s(p_{n+1})$.

Now we consider the case where p_{n+1} is a flip or a non-critical saddle-node with no cycles. Here, H will be defined in the center stable manifold of Φ at p_{n+1} as the " product " of two partial conjugacies. One of them, which we call H^c, is defined at the center manifold of Φ at p_{n+1}, as constructed in section 3 of Chapter II. It corresponds to a conjugacy on the space of leaves of the strong stable foliations $\mathscr{F}^{ss}(p_{n+1})$, $\mathscr{F}^{ss}(\widetilde{p}_{n+1})$. The other, which we call H^s, is to be defined on the space of leaves of the center foliations $\mathscr{F}^c(p_{n+1})$, $\mathscr{F}^c(\widetilde{p}_{n+1})$. However, since the center foliations may have several leaves going through the periodic orbits p_{n+1}, $\widetilde{p}_{n+1}$, we cannot express the stable manifolds of such orbits exactly as the product of these two foliations. That is the reason we will make a slight modification of the center foliation $\mathscr{F}^c(\widetilde{p}_{n+1})$ already constructed. The second partial conjugacy H^s will then send leaves of $\mathscr{F}^c(p_{n+1})$ onto leaves of the modified center foliation for $\widetilde{p}_{n+1}$. To do this, we first consider a continuous family of leaves $\{F_\mu\}$, $\mu_1 \leq \mu \leq \mu_2$ with $F_\mu \in \mathscr{F}^{ss}(p_{n+1}, \mu)$, such that F_μ meets all the leaves of the center foliation $\mathscr{F}^c(p_{n+1}, \mu)$. We also choose a family of disks $\{D_\mu\}$, $D_\mu \subset F_\mu$, such that the exterior boundary $\partial_{ex} D_\mu$ of D_μ is transverse to the unstable manifolds $W^u(p_i, \mu)$ for all $1 \leq i \leq n$ and each $\mu \in [\mu_1, \mu_2]$. We now form a fundamental domain $A = \bigcup A_\mu$ for the center stable manifold of p_{n+1} as follows. For each μ, the exterior boundary $\partial_{ex} A_\mu$ is made of the disk D_μ and a " cylinder " formed with the leaves of $\mathscr{F}^c(p_{n+1}, \mu)$ through the points of $\partial_{ex} D_\mu$. Notice that H^c determines a corresponding family $\{\widetilde{F}_{\rho(\mu)}\}$ of leaves of $\mathscr{F}^{ss}(\widetilde{p}_{n+1})$. We then take a family of disks $\{\widetilde{D}_{\rho(\mu)}\}$, $\widetilde{D}_{\rho(\mu)} \subset \widetilde{F}_{\rho(\mu)}$ for each μ, and construct a similar fundamental domain $\widetilde{A} = \bigcup \widetilde{A}_{\rho(\mu)}$ for the center stable manifold of $\widetilde{p}_{n+1}$. We now define $H^s : \bigcup D_\mu \to \bigcup \widetilde{D}_{\rho(\mu)}$ as follows. Of course, we want it to be compatible with the unstable foliations $\mathscr{F}^u(p_i)$ and the conjugacies already defined on the stable manifolds $W^s(p_i)$, $1 \leq i \leq n$. Near the exterior boundaries of D_μ, $\widetilde{D}_{\rho(\mu)}$ we proceed by induction on the indices $i = n, n-1, \ldots, 1$. We start with a diffeomorphism $H^s : V \to \widetilde{V}$ near the inclusion map, where V is a neighborhood of $W^u(p_n) \cap \partial_{ex} D_\mu$ in $W^u(p_n) \cap F_\mu$ and $\widetilde{V}$ is a neighborhood of $W^u(\widetilde{p}_n) \cap \partial_{ex} \widetilde{D}_{\rho(\mu)}$ in $W^u(\widetilde{p}_n) \cap \widetilde{F}_{\rho(\mu)}$. Similarly for the intersections of the leaves of $\mathscr{F}^u(p_n)$ with $\partial_{ex} D_\mu$ which are mapped onto the intersections of the corresponding leaves of $\mathscr{F}^u(\widetilde{p}_n)$ with $\partial_{ex} \widetilde{D}_{\rho(\mu)}$. Notice that the correspondence between the leaves of $\mathscr{F}^u(p_n)$ and those of $\mathscr{F}^u(\widetilde{p}_n)$ is determined by $H: W^s(p_n) \to W^s(\widetilde{p}_n)$ already defined. The extension to the leaves of $\mathscr{F}^u(p_{n-1}), \ldots, \mathscr{F}^u(p_1)$ is done in the same way as in the previous case. In order to define $H^s: \bigcup D_\mu \to \bigcup \widetilde{D}_{\rho(\mu)}$, we consider the annuli $A^*_\mu \subset D_\mu$ and $\widetilde{A}^*_{\rho(\mu)} \subset \widetilde{D}_{\rho(\mu)}$, each $\mu \in [\mu_1, \mu_2]$, obtained as follows. The exterior boundary of A^*_μ is $\partial_{ex} D_\mu$ and its interior boundary is the projection of $\varphi_\mu(\partial_{ex} D_\mu)$ into D_μ via the leaves of $\mathscr{F}^c(p_{n+1}, \mu)$. Similarly for $\widetilde{A}^*_{\rho(\mu)}$. Notice that $H^s: \partial_{ex} A^*_\mu \to \partial_{ex} \widetilde{A}^*_{\rho(\mu)}$ constructed above induces a map $H^s: \partial_{in} A^*_\mu \to \partial_{in} \widetilde{A}^*_{\rho(\mu)}$. In fact, for $x \in \partial_{in} A^*_\mu$ let ℓ_x be the leaf of $\mathscr{F}^c(p_{n+1}, \mu)$ through x and let y be the intersection of $\varphi_\mu^{-1}(\ell_x)$ with $\partial_{ex} \widetilde{A}^*_\mu$. If $\widetilde{\ell}$ is the leaf of $\mathscr{F}^c(\widetilde{p}_{n+1}, \rho(\mu))$ through $H^s(y)$, we set $H^s(x) = z$, where z is the intersection of $\widetilde{\varphi}_{\rho(\mu)}(\widetilde{\ell})$ with $\partial_{in} \widetilde{A}^*_{\rho(\mu)}$. We now extend H^s first to all of A^*_μ using the Isotopy Extension Theorem as before. Finally, we can extend H^s to all of D_μ sending leaves of $\mathscr{F}^u(p_i, \mu)$ to leaves of $\mathscr{F}^u(\widetilde{p}_i, \rho(\mu))$, $1 \leq i \leq n$. Again, this can be done as before since $W^u(p_i, \mu)$ and $W^u(\widetilde{p}_i, \rho(\mu))$ are transverse to D_μ and $\widetilde{D}_{\rho(\mu)}$, respectively. At this point, we would like to define the conjugacy $H: W^{cs}(p_{n+1}) \to W^{cs}(\widetilde{p}_{n+1})$ as the "product" of H^c and H^s using the strong stable and the center foliations. To do this we have to modify the center foliation $\mathscr{F}^c(\widetilde{p}_{n+1})$ in $W^{cs}(\widetilde{p}_{n+1})$. For each $\mu \in [\mu_1, \mu_2]$, it is enough to do so in the region bounded by $\widetilde{D}_{\rho(\mu)} - \widetilde{A}^*_{\rho(\mu)}$, $\widetilde{\varphi}_{\rho(\mu)}(\widetilde{D}_{\rho(\mu)})$ and the cylinder formed with the leaves of $\mathscr{F}^c(\widetilde{p}_{n+1})$ through points of $\partial_{in} \widetilde{A}^*_{\rho(\mu)}$. Let $\eta: \widetilde{D}_{\rho(\mu)} - \widetilde{A}^*_{\rho(\mu)} \to \widetilde{\varphi}_{\rho(\mu)}(\widetilde{D}_{\rho(\mu)})$ be the homeomorphism defined by $\eta(x) = y$, where $(H^s)^{-1}(x)$ and $\varphi_\mu (H^s)^{-1} \widetilde{\varphi}^{-1}_{\rho(\mu)}(y)$ belong to the same leaf of $\mathscr{F}^c(p_{n+1}, \mu)$. Let $\lambda: \widetilde{D}_{\rho(\mu)} - \widetilde{A}^*_{\rho(\mu)} \to \widetilde{\varphi}_{\rho(\mu)}(\widetilde{D}_{\rho(\mu)})$ be defined by $\lambda(x) = z$, where x and z belong to the same leaf of $\mathscr{F}^c(\widetilde{p}_{n+1}, \rho(\mu))$. If $\lambda = \eta$ then no modification of $\mathscr{F}^c(\widetilde{p}_{n+1}, \rho(\mu))$ is needed. So let us change $\mathscr{F}^c(\widetilde{p}_{n+1}, \rho(\mu))$ to get the second map to be equal to the first. Notice that $\eta(x)$ and $\lambda(x)$ belong to the same leaf of $\mathscr{F}^u(\widetilde{p}_i, \rho(\mu))$ for some $1 \leq i \leq n$. Moreover, $\lambda\eta^{-1}$ along such a leaf is C^1 near the identity map. Let $\widetilde{X}$ be the vector field whose integral curves are the leaves of $\mathscr{F}^c(\widetilde{p}_{n+1}, \rho(\mu))$. We modify $\widetilde{X}$ near but off $\widetilde{\varphi}_{\rho(\mu)}(\widetilde{D}_{\rho(\mu)})$ so that the corresponding map $\bar{\lambda}$ satisfies $\bar{\lambda}\eta^{-1} = 1$ on $\widetilde{\varphi}_{\rho(\mu)}(\widetilde{D}_{\rho(\mu)})$. Observe that the modification required for $\widetilde{X}$ along $W^u(\widetilde{p}_n, \rho(\mu))$ is well known and it can be performed in a parametrized way along the leaves of $\mathscr{F}^u(\widetilde{p}_n, \rho(\mu))$. Using the fact that the foliations $\mathscr{F}^u(\widetilde{p}_i, \rho(\mu))$ are compatible, we proceed by downward induction on the indices $1 \leq i \leq n$. It is clear that the new center foliation coincides with the previous one on the boundary $\partial\widetilde{A} = \bigcup \partial\widetilde{A}_{\rho(\mu)}$ of the fundamental domain $\widetilde{A}$. Once it is defined on $\widetilde{A}$, it can be defined on all of $W^{cs}(\widetilde{p}_{n+1})$ simply through iterations by $\widetilde{\Phi}$, where $\widetilde{\Phi}(x, \mu) = (\widetilde{\varphi}_\mu(x), \mu)$ as before. Now we can define H from $W^{cs}(p_{n+1})$ onto $W^{cs}(\widetilde{p}_{n+1})$ using the center foliation, the strong stable foliation and the conjugacies H^s and H^c on their spaces of leaves. H is clearly one to one, onto and also continuous since these foliations and the maps H^s, H^c are continuous. Moreover, H is differentiable along the leaves of $\mathscr{F}^u(p_i)$, $1 \leq i \leq n$, in $W^{cs}(p_{n+1})$. In fact, $\mathscr{F}^c$ was constructed in section 3 to be differentiable along these leaves; the same is true with the modified $\mathscr{F}^c$ and the map H^s constructed above. Being codimension one, the foliation $\mathscr{F}^{ss}$ is C^1 (see section 2, Chapter II). Finally, the differentiability of H^c off the periodic points is in section 3 and 4 of Chapter II. Together these facts imply our statement. Thus, the proof of the induction step is finished. Our map H is defined on $M \times [\mu_1, \mu_2]$ or $M \times [\mu_1, b]$ onto $M \times [\rho(\mu_1), \rho(\mu_2)]$ or $M \times [\rho(\mu_1), \tilde{b}]$, it is clearly one to one and satisfies the conjugacy equation $H\varphi_\mu = \tilde{\varphi}_{\rho(\mu)}H$. Let us prove that H is continuous. We have to show that H is continuous at the stable (center stable) manifolds of the periodic orbits of $\{\varphi_\mu\}$. Here we indicate by W^s both the stable and center stable manifolds; also, let I_1 indicate either $[\mu_1, \mu_2]$ or $[\mu_1, b]$. By construction, H is continuous along these manifolds and, in particular, along the stable manifolds of the sinks. By induction we may assume that $H : W \to \tilde{W}$ is a homeomorphism, where W and $\tilde{W}$ are the union of $W^s(p_{k,\mu})$ and $W^s(p_{k,\rho(\mu)})$ for $i < k \leq \ell$ and $\mu \in I_1$. Let us now show that H is also continuous at $W^s(p_{i,\mu})$. Consider a sequence $(x_n, \mu_n) \to (x, \mu) \in W^s(p_{i,\mu})$, $x_n \in M$ and $\mu_n \in I_1$. Since H restricted to $W^s(p_{i,\mu})$ is continuous, we may assume, via a subsequence, that $(x_n, \mu_n) \in W$ for all n. Let F_n, F be the leaves of $\mathscr{F}^u(p_i)$ containing (x_n, μ_n), (x, μ) and $\tilde{F}_n$, $\tilde{F}$ the leaves of $\mathscr{F}^u(\tilde{p}_i)$ containing $H(x_n, \mu_n)$ and $H(x, \mu)$, respectively. We have that $F_n \to F$ and since H is continuous restricted to the stable manifolds, we have $\tilde{F}_n \to \tilde{F}$. Thus, it is enough to show that the sequence $H(x_n, \mu_n)$ accumulates on $W^s(p_{i,\rho(\mu)})$. In fact, $H(x_n, \mu_n)$ cannot accumulate on W because $H : W \to W$ is a homeomorphism. Also, $H(x_n, \mu_n)$ cannot accumulate on the union $\tilde{Z}$ of $W^s(\tilde{p}_{j,\rho(\mu)})$, $1 \leq j < i$, because $\tilde{F}_n$ and $\tilde{Z}$ are far apart. This proves our assertion and so H is continuous on all of $M \times I_1$. This finishes the proof of the theorem.

We now complete the proof of our second main theorem stated in the introduction, the first part of which was done in section 2 of Chapter III.

Let us denote by $\mathscr{C} \subset \mathscr{P}$ the set of arcs $\{\varphi_\mu\}$ such that the limit set of each φ_μ has finitely many orbits, $\mu \in I = [0, 1]$. We also denote by $\mathscr{S} \subset \mathscr{C}$ the set of arcs $\{\varphi_\mu\}$ such that there are only finitely many bifurcation values for $\{\varphi_\mu\}$ say $b_1, \ldots, b_s$ in $(0, 1)$ and for each $1 \leq i \leq s$, φ_{b_i} has the following properties:

— all stable, strong stable, unstable and strong unstable manifolds intersect transversally

— φ_{b_i} has no cycles and has exactly one non-hyperbolic periodic orbit, which is either a flip or a non-critical saddle-node; this non-hyperbolic orbit unfolds generically.

As we mentioned before, it turns out that for arcs $\{\varphi_\mu\} \in \mathscr{S}$, φ_μ is a Morse-Smale diffeomorphism if μ is not a bifurcation value.

Theorem (4.4). — *The arcs in* $\mathscr{S}$ *are stable.*

Proof. — As in Theorem (4.1), it follows from [19], [20] that $\mathscr{S}$ is an open subset of $\mathscr{P}$. Thus, if $\{\varphi_\mu\} \in \mathscr{S}$ and has bifurcation values $b_1, \ldots, b_s$ in $(0, 1)$, then a nearby arc $\{\varphi'_\mu\}$ also belongs to $\mathscr{S}$ and has nearby bifurcation values $b'_1, \ldots, b'_s$ in $(0, 1)$. Moreover, the non-hyperbolic periodic orbits of φ_{b_i} and $\varphi'_{b'_i}$ are both saddle-nodes or flips. To produce a conjugacy between $\{\varphi_\mu\} \in \mathscr{S}$ and a nearby arc $\{\varphi'_\mu\}$, we first assume that

$\varphi_\mu = \varphi'_\mu$ for $\mu \in I - U$, where U is a small open subinterval of I containing at most one bifurcation value. In this case, by Theorem (3.6) we can construct unstable tubular families or foliations for $\{\varphi_\mu\}$ and $\{\varphi'_\mu\}$ and $\mu \in V$, where V is an open subinterval of I containing the closure $\overline{U}$ of U. Moreover, using the Isotopy Extension Theorem (4.2), we may construct these unstable foliations to be the same for $\{\varphi_\mu\}$ and $\{\varphi'_\mu\}$ if $\mu \in V - W$, where W is an open subinterval of I such that $\overline{U} \subset W$ and $\overline{W} \subset V$. Again using the Isotopy Extension Theorem, the construction of the conjugacy $\{h_\mu\}$ between $\{\varphi_\mu\}$ and $\{\varphi'_\mu\}$ for $\mu \in V$, as performed in Theorem (4.3), can be done so that h_μ is the identity map on M for $\mu \in V - W$. Since $\varphi_\mu = \varphi'_\mu$ for $\mu \in I - U$, we can extend this conjugacy to all of I by defining it to be the identity for $\mu \in I - V$. Finally, let $\{U_i\}$, $1 \leq i \leq n$, be a covering of I by small subintervals, each containing at most one bifurcation value. It is immediate that we can decompose any small perturbation of $\{\varphi_\mu\}$ into perturbations, each with support in one of the subintervals U_i. So the construction of the conjugacy between $\{\varphi_\mu\}$ and a nearby arc is reduced to the previous case. The proof of theorem is complete.

REFERENCES

[1] A. A. Andronov, E. A. Leontovich, I. I. Gordon, A. G. Maier, Theory of bifurcations of dynamic systems on a plane, *Isr. Program Scientific Transl.*, Jerusalem, 1971.

[2] L. Block, J. Franke, Existence of periodic points of maps of S^1, *Invent. Math.*, **22** (1973), 69-73.

[3] Th. Bröcker, L. C. Lander, Differentiable germs and catastrophes, *London Math. Soc. Lecture Notes*, **17**, Cambridge Univ. Press, 1975.

[4] P. Brunovski, On one-parameter families of diffeomorphisms, I and II, *Comment. Mat. Univ. Carolinae*, **11** (3) (1970), 559-582; **12** (4) (1971), 765-784.

[5] P. Brunovski, Generic properties of the rotation number of one-parameter diffeomorphisms of the circle, *Czech. Math. J.*, **24** (1974), 74-90.

[6] A. Denjoy, Les trajectoires à la surface du tore, *C. r. Acad. Sci.*, **223** (1946), 5-8.

[7] J. Guckenheimer, One-parameter families of vector fields on two-manifolds: another nondensity theorem, *Dynamical Systems*, Academic Press, 1973, 111-128.

[8] P. Hartman, On local homeomorphisms of Euclidean spaces, *Bol. Soc. Mat. Mexicana*, **5** (2) (1960), 220-241.

[9] M. Herman, Sur la conjugaison différentiable des difféomorphismes du cercle à des rotations, *Publ. Math. I.H.E.S.*, **49** (1979), 5-234.

[10] M. Herman, Mesure de Lebesgue et nombre de rotation, *Geometry and Topology*, *Springer Lecture Notes in Math.*, **597** (1977), 271-293.

[11] M. W. Hirsch, C. C. Pugh, Stable manifolds and hyperbolic sets, *Proc. Symp. Pure Math. A.M.S.*, **14** (1970), 133-164.

[12] M. W. Hirsch, C. C. Pugh, M. Shub, Invariant manifolds, *Springer Lecture Notes in Math.*, **583** (1977).

[13] I. Kupka, Contributions à la théorie des champs génériques, *Contributions to differential equations*, vol. 2 (1963), 475-484 and vol. 3 (1964), 411-420.

[14] D. C. Lewis, Formal power series transformations, *Duke Math. J.*, **5** (1939), 794-805.

[15] I. P. Malta, Hyperbolic Birkhoff centers, *Trans. A.M.S.*, **262** (1980), 181-193; announced in *An. Acad. Brasil. Ciências*, **51** (1979), (1), 27-29.

[16] A. Manning, There are no new Anosov diffeomorphisms on tori, *Amer. J. Math.*, **96** (1974), 422-429.

[17] J. E. Marsden, M. McCracken, The Hopf bifurcation and its applications, *Appl. Math. Sci.*, **19**, Springer-Verlag, 1976.

[18] W. C. de Melo, Moduli and stability of two-dimensional diffeomorphisms, *Topology*, **19** (1980), 9-21.

[18a] W. de MELO, J. PALIS and S. VAN STRIEN, Characterizing diffeomorphisms with modulus of stability one, *Dynamical Systems and Turbulence*, Warwick 1980, *Springer Lecture Notes in Math.*, **898** (1981), 266-286.
[19] S. NEWHOUSE, J. PALIS, Bifurcations of Morse-Smale dynamical systems, *Dynamical Systems*, Academic Press, 1973, 303-366.
[20] S. NEWHOUSE, J. PALIS, Cycles and bifurcation theory, *Astérisque*, **31** (1976), 43-140.
[21] S. NEWHOUSE, J. PALIS, F. TAKENS, Stable arcs of diffeomorphisms, *Bull. A.M.S.*, **82** (1976), 499-502.
[22] S. NEWHOUSE, The abundance of wild hyperbolic sets and non-smooth stable sets, *Publ. Math. I.H.E.S.*, **50** (1979), 101-152.
[23] R. PALAIS, Local triviality of the restriction map for embeddings, *Comm. Math. Helvet.*, **34** (1960), 305-312.
[24] J. PALIS, On Morse-Smale dynamical systems, *Topology*, **8** (1969), 385-405.
[25] J. PALIS, S. SMALE, Structural stability theorems, *Proc. Symp. Pure Math. A.M.S.*, **14** (1970), 223-232.
[26] J. PALIS, F. TAKENS, Topological equivalence of normally hyperbolic dynamical systems, *Topology*, **16** (1977), 335-345.
[26a] J. PALIS and F. TAKENS, *Stability of parametrized families of gradient vector fields*, Preprint IMPA, to appear.
[27] J. PALIS, A differentiable invariant of topological conjugacies and moduli of stability, *Astérisque*, **51** (1978), 335-346.
[28] J. PALIS, Moduli of stability and bifurcation theory, *Proc. Int. Congres of Math. Helsinki* (1978), 835-839.
[29] H. POINCARÉ, *Œuvres complètes*, t. 1, Gauthier-Villars, 1952, 137-158.
[30] C. C. PUGH, M. SHUB, Linearization of normally hyperbolic diffeomorphisms and flows, *Invent. Math.*, **10** (1970), 187-198.
[31] C. ROBINSON, Global structural stability of a saddle-node bifurcation, *Trans. A.M.S.*, **236** (1978), 155-172.
[32] D. RUELLE, F. TAKENS, On the nature of turbulence, *Comm. Math. Phys.*, **20** (1971), 167-192; A note concerning our paper on the nature of turbulence, *Comm. Math. Phys.*, **23** (1971), 343-344.
[33] S. SMALE, Stable manifolds for differential equations and diffeomorphisms, *Ann. Scuola Norm. Sup. Pisa*, **18** (1963), 97-116.
[34] S. SMALE, Differentiable dynamical systems, *Bull. A.M.S.*, **73** (1967), 747-817.
[35] J. SOTOMAYOR, Generic one parameter families of vector fields, *Publ. Math. I.H.E.S.*, **43** (1974), 4-56.
[36] J. SOTOMAYOR, Generic bifurcations of dynamical systems, *Dynamical Systems*, Academic Press 1973, 549-560.
[37] S. J. VAN STRIEN, Center manifolds are not C^∞, *Math. Z.*, **166** (1979), 143-145.
[38] F. TAKENS, Partially hyperbolic fixed points, *Topology*, **10** (1971), 133-147.
[39] F. TAKENS, Normal forms for certain singularities of vector fields, *Ann. Inst. Fourier*, **23** (1973), (2), 163-195.
[40] F. TAKENS, Forced oscillations and bifurcations, *Applications of Global Analysis* I, *Comm. of the Math. Inst.*, R.U. Utrecht (Holland), 1974.
[41] F. TAKENS, Global phenomena in bifurcations of dynamical systems with simple recurrence, *Jber. d. Dt. Math.-Verein*, **81** (1979), 87-96.
[42] R. WILLIAMS, The D.A. map of Smale and structural stability, *Proc. Symp. Pure Math. A.M.S.*, **14** (1970), 329-334.

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