BIFURCATIONS OF PLANAR SLIDING HOMOCLINICS

JAN AWREJCEWICZ, MICHAL FEČKAN, AND PAWEL OLEJNIK

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We study bifurcations from sliding homoclinic solutions to bounded solutions on $\mathbb{R}$ for certain discontinuous planar systems under periodic perturbations. Sufficient conditions are derived for such perturbation problems.

1. Introduction

We start from the planar discontinuous system

$$\begin{aligned} \dot{z} &= f_+(z) + \varepsilon g(z,t,\varepsilon) \quad \text{for } y > 1, \\ \dot{z} &= f_-(z) + \varepsilon g(z,t,\varepsilon) \quad \text{for } y < 1, \end{aligned} \tag{1.1}$$

where $z = (x,y) \in \mathbb{R}^2$, $f_\pm$, g are C^3-smooth, and g is 1-periodic in t. Here we set

$$q_\pm(z,t,\varepsilon) = f_\pm(z) + \varepsilon g(z,t,\varepsilon). \tag{1.2}$$

We suppose the following conditions:

(i) $f_-(0) = 0$, and $Df_-(0)$ has no eigenvalues on the imaginary axis,

(ii) there are two solutions $\gamma_-(s)$, $\gamma_+(s)$ of $\dot{z} = f_-(z)$, $y \le 1$ defined on $\mathbb{R}_- = (-\infty,0]$, $\mathbb{R}_+ = [0,+\infty)$, respectively, such that $\lim_{s\to\pm\infty}\gamma_\pm(s) = 0$ and $\gamma_\pm(s) = (x_\pm(s), y_\pm(s))$ with $y_\pm(0) = 1$, $x_-(0) < x_+(0)$. Moreover, $f_\pm(z) = (f_{\pm 1}(z), f_{\pm 2}(z))$ with $f_{\pm 1}(x,1) > 0$, $f_{+2}(x,1) < 0$ for $x_-(0) \le x \le x_+(0)$. Furthermore, $f_{-2}(x,1) > 0$ for $x_-(0) \le x < x_+(0)$, $f_{-2}(x_+(0),1) = 0$, and $\partial_x f_{-2}(x_+(0),1) < 0$.

Assumptions (i) and (ii) mean that (1.1) for $\varepsilon = 0$ has a sliding homoclinic solution γ, created by $\gamma_\pm$, to a hyperbolic equilibrium 0. We are interested in the bifurcation of γ to bounded solutions on $\mathbb{R}$ of (1.1) under the perturbation $\varepsilon g(z,t,\varepsilon)$.

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The plan of the paper is as follows. In Section 2, we study (1.1) by using functional methods based on [4] along with the implicit function theorem [5]. In Section 3, we generalize results of Section 2 to systems with multiple discontinuous levels. Final Section 4 is devoted to a concrete system of piece-vice linear systems with periodic perturbations.

Sliding periodic solutions of discontinuous differential equations are investigated in [1–3] with both analytical and numerical methods. Qualitative properties of discontinuous systems are studied in [6]. Bifurcations for planar discontinuous ordinary differential systems with small periodic perturbations from homoclinic solutions transversally intersecting levels of discontinuity are studied in [7] to generalize the well-known Melnikov method for a smooth case [4] to a discontinuous one. We note that bifurcations from sliding homoclinic solutions, studied in this paper, are different to [4, 7].

2. Bifurcation result

In this section, we find conditions under which γ persists in (1.1) for $\varepsilon \neq 0$ small. For this purpose, we consider (1.1) as a system in $\mathbb{R}^3$ defined by

$$\begin{gathered} \dot{z} = f_+(z) + \varepsilon g(z,t,\varepsilon) \quad \text{for } y > 1, \\ \dot{z} = f_-(z) + \varepsilon g(z,t,\varepsilon) \quad \text{for } y < 1, \\ \dot{t} = 1, \end{gathered} \tag{2.1}$$

while on $y = 1$ (cf. [1, 6]), we consider the system

$$\begin{aligned} \dot{x} = {} & \frac{q_{+2}(x,1,t,\varepsilon)}{q_{+2}(x,1,t,\varepsilon) - q_{-2}(x,1,t,\varepsilon)} q_{+1}(x,1,t,\varepsilon) \\ & + \frac{q_{-2}(x,1,t,\varepsilon)}{q_{-2}(x,1,t,\varepsilon) - q_{+2}(x,1,t,\varepsilon)} q_{-1}(x,1,t,\varepsilon), \end{aligned} \tag{2.2}$$

where $q_\pm = (q_{\pm 1}, q_{\pm 2})$. We first study the system

$$\begin{gathered} \dot{z} = f_-(z) + \varepsilon g(z,t,\varepsilon) \quad \text{for } y \leq 1, \\ \dot{t} = 1, \qquad y(0) = 1, \qquad t(0) = \alpha, \qquad s \leq 0. \end{gathered} \tag{2.3}$$

Lemma 2.1. *For any ε small, there is a unique bounded solution $z_-(s,\varepsilon,\alpha)$ of (2.3) on $\mathbb{R}_-$, which is near to $\gamma_-(s)$.*

Proof. We consider the Banach space

$$X = \{v = (x(s), y(s)) \in C_b(\mathbb{R}_-, \mathbb{R}^2) \mid y(0) = 0\} \tag{2.4}$$

with the usual sup-norm $\|\cdot\|$. We put $z = \gamma_- + v$ into (2.3) to get

$$\begin{gathered} \dot{v} = Df_-(\gamma_-(s))v + \{f_-(\gamma_-(s)+v) - f_-(\gamma_-(s)) - Df_-(\gamma_-(s))v\} + \varepsilon g(\gamma_-(s)+v, s+\alpha, \varepsilon), \\ v_2(0) = 0, \end{gathered} \tag{2.5}$$

where $v = (v_1, v_2)$. Next, the system

$$\dot{v} = Df_-(\gamma_-(s))v \tag{2.6}$$

has an exponential dichotomy on $\mathbb{R}_-$ (cf. [4]), that is, there are positive constants K, a and a projection $P : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$\begin{aligned} ||V_-(s)PV_-(\theta)^{-1}|| &\le Ke^{-a(s-\theta)} \quad \text{for } \theta \le s \le 0, \\ ||V_-(s)(\mathbb{I}-P)V_-(\theta)^{-1}|| &\le Ke^{a(s-\theta)} \quad \text{for } s \le \theta \le 0, \end{aligned} \tag{2.7}$$

where $V_-(s)$, $V_-(0) = \mathbb{I}$ is the fundamental matrix solution of (2.6). Moreover, since $\dot{\gamma}_-(s)$ solves (2.6), and it is bounded on $\mathbb{R}_-$, and $\dot{\gamma}_-(0)$ is transversal to the x-axis, we can suppose (cf. [4]) that $\operatorname{Im}(\mathbb{I}-P) = \mathbb{R}\dot{\gamma}_-(0)$ and $\operatorname{Im}P$ is the x-axis. Then, (2.5) can be rewritten as a fixed point problem

$$v(s) = \int_{-\infty}^{s} V_-(s)PV_-(\theta)^{-1}h(\theta)d\theta - \int_{s}^{0} V_-(s)(\mathbb{I}-P)V_-(\theta)^{-1}h(\theta)d\theta \tag{2.8}$$

on the Banach space X, where

$$h(\theta) = f_-(\gamma_-(\theta)+v(\theta)) - f_-(\gamma_-(\theta)) - Df_-(\gamma_-(\theta))v(\theta) + \varepsilon g(\gamma_-(\theta)+v(\theta),\theta+\alpha,\varepsilon). \tag{2.9}$$

Since

$$f_-(\gamma_-(\theta)+v) - f_-(\gamma_-(\theta)) - Df_-(\gamma_-(\theta))v = O(|v|^2), \tag{2.10}$$

for ε small, we can solve (2.8) by using the implicit function theorem to obtain a unique small solution $v(s,\alpha,\varepsilon)$ of (2.8), and so

$$z(s,\alpha,\varepsilon) = \gamma_-(s) + v(s,\alpha,\varepsilon) \tag{2.11}$$

solves (2.3). The proof is finished. □

We put

$$\varphi_-(\alpha,\varepsilon) = x(0,\alpha,\varepsilon), \tag{2.12}$$

where

$$z(s,\alpha,\varepsilon) = \big(x(s,\alpha,\varepsilon), y(s,\alpha,\varepsilon)\big). \tag{2.13}$$

Clearly, $\varphi_-(\alpha,0) = x_-(0)$. Next, we consider (2.2) with the initial condition

$$x(0) = \varphi_-(\alpha,\varepsilon). \tag{2.14}$$

If $h(x,s,\varepsilon)$ is the right-hand side of (2.2), then conditions (i) and (ii) imply that $h(x,s,\varepsilon) > 0$ for any $x_-(0) \le x \le x_+(0)$ and ε small. Then assumption (ii) gives the solvability of the equation

$$q_{-2}(x(s_+(\alpha,\varepsilon)),1,s_+(\alpha,\varepsilon)+\alpha,\varepsilon) = 0 \tag{2.15}$$

for the function $s_+(\alpha,\varepsilon) > 0$, where $x(s)$ solves (2.2) and (2.14). So, $s_+(\alpha,\varepsilon)$ is the time when the sliding motion of (2.2) is ending. We put

$$\varphi_+(\alpha,\varepsilon) = x\big(s_+(\alpha,\varepsilon)\big). \tag{2.16}$$

Finally, we consider the initial value problem

$$\begin{gathered} \dot z = f_-(z) + \varepsilon g(z,t,\varepsilon) \quad \text{for } y \le 1, \\ \dot t = 1, \qquad s \ge s_+(\alpha,\varepsilon), \\ z\big(s_+(\alpha,\varepsilon)\big) = \big(\varphi_+(\alpha,\varepsilon),1\big), \qquad t\big(s_+(\alpha,\varepsilon)\big) = s_+(\alpha,\varepsilon) + \alpha. \end{gathered} \tag{2.17}$$

That is the initial value problem

$$\begin{gathered} \dot z = f_-(z) + \varepsilon g(z,s+\alpha,\varepsilon) \quad \text{for } y \le 1, \\ z\big(s_+(\alpha,\varepsilon)\big) = \big(\varphi_+(\alpha,\varepsilon),1\big), \quad s \ge s_+(\alpha,\varepsilon). \end{gathered} \tag{2.18}$$

We note that $\gamma_+(0) = (\varphi_+(\alpha,0),1)$ and we look for a solution z of (2.18) near to $\gamma_+(s - s_+(\alpha,\varepsilon)) = \omega_+(s)$. By taking

$$z(s) = \omega_+(s) + \varepsilon w(s) \tag{2.19}$$

in (2.18), we obtain

$$\begin{gathered} \dot w = Df_-\big(\omega_+(s)\big)w + \frac{1}{\varepsilon}\big\{f_-\big(\omega_+(s)+\varepsilon w\big) - f_-\big(\omega_+(s)\big) - Df_-\big(\omega_+(s)\big)\varepsilon w\big\} \\ + g\big(\omega_+(s)+\varepsilon w, s+\alpha,\varepsilon\big), \quad s \ge s_+(\alpha,\varepsilon), \\ w\big(s_+(\alpha,\varepsilon)\big) = \big(\psi_+(\alpha,\varepsilon),0\big), \end{gathered} \tag{2.20}$$

where

$$\psi_+(\alpha,\varepsilon) = \big(\varphi_+(\alpha,\varepsilon) - \varphi_+(\alpha,0)\big)/\varepsilon. \tag{2.21}$$

By shifting the time $s \leftrightarrow s + s_+(\alpha,\varepsilon)$, $s \ge 0$ in (2.20), we obtain

$$\begin{gathered} \dot w = Df_-\big(\gamma_+(s)\big)w + \frac{1}{\varepsilon}\big\{f_-\big(\gamma_+(s)+\varepsilon w\big) - f_-\big(\gamma_+(s)\big) - Df_-\big(\gamma_+(s)\big)\varepsilon w\big\} \\ + g\big(\gamma_+(s)+\varepsilon w, s_+(\alpha,\varepsilon)+s+\alpha,\varepsilon\big), \quad s \ge 0, \\ w(0) = \big(\psi_+(\alpha,\varepsilon),0\big). \end{gathered} \tag{2.22}$$

We set

$$\eta(\alpha,\varepsilon) = \big(\psi_+(\alpha,\varepsilon),0\big). \tag{2.23}$$

Now we study the problem

$$\begin{gathered} \dot w = Df_-\big(\gamma_+(s)\big)w + h(s), \\ w(0) = u, \end{gathered} \tag{2.24}$$

for $h \in C_b(\mathbb{R}_+, \mathbb{R}^2)$ and $u = (u_1, u_2) \in \mathbb{R}^2$. The system

$$\dot{w} = Df_-(\gamma_+(s))w \tag{2.25}$$

has an exponential dichotomy on $\mathbb{R}_+$ (cf. [4]), that is, there are positive constants M, b and a projection $Q : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$\begin{aligned} \|V_+(s)QV_+(\theta)^{-1}\| &\leq Me^{-b(s-\theta)} \quad \text{for } 0 \leq \theta \leq s, \\ \|V_+(s)(\mathbb{I} - Q)V_+(\theta)^{-1}\| &\leq Me^{b(s-\theta)} \quad \text{for } 0 \leq s \leq \theta, \end{aligned} \tag{2.26}$$

where $V_+(s)$, $V_+(0) = \mathbb{I}$ is the fundamental matrix solution of (2.25). Moreover, since $\dot{\gamma}_+(s)$ solves (2.25) and it is bounded on $\mathbb{R}_+$, we can suppose (cf. [4]) that $\operatorname{Im} Q = \mathbb{R}\dot{\gamma}_+(0)$ and $\operatorname{Im}(\mathbb{I} - Q)$ is orthogonal to the line $\mathbb{R}\dot{\gamma}_+(0)$. On the other hand, condition (ii) implies that

$$\dot{y}_+(0) = f_{-2}(x_+(0), y_+(0)) = f_{-2}(x_+(0), 1) = 0. \tag{2.27}$$

So,

$$\dot{\gamma}_+(0) = (\dot{x}_+(0), \dot{y}_+(0)) = (\dot{x}_+(0), 0). \tag{2.28}$$

Consequently, Q is the orthogonal projection onto the x-axis. Let $\Gamma = \dot{\gamma}_+(0)^\perp$ be a nonzero orthogonal vector onto $\dot{\gamma}_+(0)$. Now, for simplicity, we can take $\Gamma = (0,1)$. So, $\operatorname{Im}(\mathbb{I} - Q) = \mathbb{R}\Gamma$. We note that

$$\mu(t) = V_+^*(s)^{-1}\Gamma \tag{2.29}$$

is a basis of a space of bounded solutions on $\mathbb{R}_+$ of the adjoint system (cf. [4])

$$\dot{w} = -Df_-^*(\gamma_+(s))w. \tag{2.30}$$

We need the following result.

Lemma 2.2. *Problem (2.24) has a bounded solution w on $\mathbb{R}_+$ if and only if*

$$\int_0^{+\infty} (h(s), \mu(s))\,ds = -(\Gamma, u) = -u_2, \tag{2.31}$$

where $(\cdot,\cdot)$ is the usual scalar product on $\mathbb{R}^2$. Moreover, if condition (2.31) holds, then problem (2.24) has a unique bounded solution $w = w(u,h)$ on $\mathbb{R}_+$. Furthermore, there is a constant $c > 0$ such that

$$\|w(u,h)\| \leq c(\|h\| + |u|), \tag{2.32}$$

where $\|\cdot\|$ is the sup-norm on $Y = C_b(\mathbb{R}_+, \mathbb{R}^2)$ and $|\cdot|$ corresponds to $(\cdot,\cdot)$.

Proof. A general form of a bounded solution of equation

$$\dot{w} = Df_-(\gamma_+(s))w + h(s) \tag{2.33}$$

on $\mathbb{R}_+$ is given by

$$w(s) = c\dot{y}_+(s) + \int_0^s V_+(s)QV_+(\theta)^{-1}h(\theta)d\theta - \int_s^{+\infty} V_+(s)(\mathbb{I} - Q)V_+(\theta)^{-1}h(\theta)d\theta. \quad (2.34)$$

Then using the initial condition $w(0) = u$, we get the equation

$$u = c\dot{y}_+(0) - \int_0^{+\infty} (\mathbb{I} - Q)V_+(\theta)^{-1}h(\theta)d\theta, \quad (2.35)$$

which implies

$$u_2 = (u,\Gamma) = -\int_0^{+\infty} (V_+(s)^{-1}h(s),\Gamma)ds = -\int_0^{+\infty} (h(s), V_+^*(s)^{-1}\Gamma)ds = -\int_0^{+\infty} (h(s),\mu(s))ds. \quad (2.36)$$

So, (2.31) is proved. On the other hand, if (2.31) holds, then (2.35) gives

$$u_1 = c\dot{x}_+(0). \quad (2.37)$$

Consequently, the unique bounded solution of (2.24) on $\mathbb{R}_+$ is given by

$$w(s) = \frac{u_1}{\dot{x}_+(0)}\dot{y}_+(s) + \int_0^s V_+(s)QV_+(\theta)^{-1}h(\theta)d\theta - \int_s^{+\infty} V_+(s)(\mathbb{I} - Q)V_+(\theta)^{-1}h(\theta)d\theta. \quad (2.38)$$

Then, (2.32) follows directly from (2.38). The proof is finished. □

Let $S: Y \to Y$ be a projection defined by

$$Sh = h(s) - \int_0^{+\infty} \left[\left(h(\theta), \frac{\mu(\theta)}{\|\mu\|_2^2}\right)d\theta\right]\mu(s), \quad (2.39)$$

where $\|u\|_2^2 = \int_0^{+\infty} \mu(\theta)^2 d\theta$. Then, (2.22) is splitted as follows

$$\begin{gathered}\dot{w} = Df_-(\gamma_+(s))w + S\Big[\frac{1}{\varepsilon}\{f_-(\gamma_+ + \varepsilon w) - f_-(\gamma_+) - Df_-(\gamma_+)\varepsilon w\} \\ + g(\gamma_+ + \varepsilon w, s_+(\alpha,\varepsilon) + s + \alpha, \varepsilon)\Big], \\ w(0) = (\psi_+(\alpha,\varepsilon), 0),\end{gathered} \quad (2.40)$$

$$\begin{gathered}\int_0^{+\infty} \Big(\frac{1}{\varepsilon}\{f_-(\gamma_+(s) + \varepsilon w(s)) - f_-(\gamma_+(s)) - Df_-(\gamma_+(s))\varepsilon w(s)\} \\ + g(\gamma_+(s) + \varepsilon w(s), s_+(\alpha,\varepsilon) + s + \alpha, \varepsilon), \mu(s)\Big)ds = 0.\end{gathered} \quad (2.41)$$

By using Lemma 2.2 together with the implicit function theorem, we can solve (2.40) to obtain its solution

$$w = w(\alpha,\varepsilon,s). \quad (2.42)$$

Then, by plugging it into (2.41), we arrive at a bifurcation equation

$$B(\alpha,\varepsilon)=\int_0^{+\infty}\bigg(\frac{1}{\varepsilon}\{f_-(\gamma_+(s)+\varepsilon w(\alpha,\varepsilon,s))-f_-(\gamma_+(s))-Df_-(\gamma_+(s))\varepsilon w(\alpha,\varepsilon,s)\}\\ +g(\gamma_+(s)+\varepsilon w(\alpha,\varepsilon,s),s_+(\alpha,\varepsilon)+s+\alpha,\varepsilon),\mu(s)\bigg)ds=0. \tag{2.43}$$

We have

$$\bar{M}(\alpha)=B(\alpha,0)=\int_0^{+\infty}\big(g(\gamma_+(s),s_+(\alpha,0)+s+\alpha,0),\mu(s)\big)ds=0. \tag{2.44}$$

Any simple root α_0 of $\bar{M}(\alpha)$; that is, $\bar{M}(\alpha_0)=0$ and $\bar{M}'(\alpha_0)\neq 0$, gives the solvability of $B(\alpha,\varepsilon)=0$ with respect to $\alpha=\alpha(\varepsilon)$ for any ε small with $\alpha(0)=\alpha_0$.

On the other hand, from the definition of function $s_+(\alpha,\varepsilon)$ in (2.15), we see that $\partial_\alpha s_+(\alpha,0)=0$. So, simple roots of $\bar{M}(\alpha)$ are in one-to-one correspondence with simple roots of the function

$$M(\beta)=\int_0^{+\infty}\big(g(\gamma_+(s),\beta+s,0),\mu(s)\big)ds. \tag{2.45}$$

Summarizing we arrive at the following result.

Theorem 2.3. *If there is a simple root β_0 of $M(\beta)$, that is, it holds that $M(\beta_0)=0$ and $M'(\beta_0)\neq 0$, then homoclinic solution γ bifurcates to a bounded solution on $\mathbb{R}$ of (1.1) with $\varepsilon\neq 0$ small.*

3. Generalization to multiple discontinuous systems

The above approach to (1.1) can be generalized to cases when homoclinic orbit $\gamma(s)$ transversally crosses another curve of discontinuity. For simplicity, we suppose that such a discontinuity in (1.1) occurs at the level $y=1/2$, that is, in this section, we deal with the system

$$\begin{aligned}\dot z&=f_+(z)+\varepsilon g(z,t,\varepsilon)\quad \text{for } y>1,\\ \dot z&=f_-(z)+\varepsilon g(z,t,\varepsilon)\quad \text{for } \frac{1}{2}<y<1,\\ \dot z&=F(z)+\varepsilon g(z,t,\varepsilon)\quad \text{for } y<\frac{1}{2},\end{aligned} \tag{3.1}$$

where $z=(x,y)\in\mathbb{R}^2$, $f_\pm$, F, g are C^3-smooth and g is 1-periodic in t. We suppose the following conditions:

(a) $F(0)=0$ and $DF(0)$ has no eigenvalues on the imaginary axis,

(b) there are two solutions η_-,η_+ of $\dot z=f_-(z)$, $1/2\le y\le 1$ defined on $[a_-,0]$, $[0,a_+]$, $a_-<0<a_+$, respectively, such that $\eta_\pm(s)=(\tilde{x}_\pm(s),\tilde{y}_\pm(s))$ with $\tilde{y}_\pm(0)=1$, $\tilde{y}_\pm(a_\pm)=1/2$, $\tilde{x}_-(0)<\tilde{x}_+(0)$, $\tilde{x}_-(a_-)<\tilde{x}_+(a_+)$. Moreover, $f_\pm(z)=(f_{\pm1}(z),f_{\pm2}(z))$ with $f_{\pm1}(x,1)>0$, $f_{+2}(x,1)<0$ for $\tilde{x}_-(0)\le x\le\tilde{x}_+(0)$. Furthermore, $f_{-2}(x,1)>0$ for $\tilde{x}_-(0)\le x<\tilde{x}_+(0)$, $f_{-2}(\tilde{x}_+(0),1)=0$, and $\partial_x f_{-2}(\tilde{x}_+(0),1)<0$. Finally, we suppose that $f_{-2}(\eta_-(a_-))>0$ and $f_{-2}(\eta_+(a_+))<0$,

(c) there are two solutions $\widetilde{\gamma}_-(s)$, $\widetilde{\gamma}_+(s)$ of $\dot{z} = F(z)$, $y \leq 1/2$ defined on $\mathbb{R}_- = (-\infty, 0]$, $\mathbb{R}_+ = [0, +\infty)$, respectively, such that $\lim_{s\to\pm\infty} \widetilde{\gamma}_\pm(s) = 0$ and $\widetilde{\gamma}_\pm(0) = \eta_\pm(a_\pm)$. Moreover, $F(z) = (F_1(z), F_2(z))$ with $F_2(\widetilde{\gamma}_-(0)) > 0$ and $F_2(\widetilde{\gamma}_+(0)) < 0$.

Again, assumptions (a), (b), and (c) imply that (3.1) for $\varepsilon = 0$ has a sliding homoclinic solution $\widetilde{\gamma}$, created by $\eta_\pm$ and $\widetilde{\gamma}_\pm$, to a hyperbolic equilibrium 0. We study in this section bifurcation of $\widetilde{\gamma}$ in system (3.1) for $\varepsilon \neq 0$ small. We can directly follow a method of Section 2. We first solve the equation

$$q_{-2}(\widetilde{\varphi}_+(\alpha,\varepsilon), 1, \alpha, \varepsilon) = 0. \tag{3.2}$$

Since

$$\begin{aligned} q_{-2}(\widetilde{x}_+(0), 1, \alpha, 0) &= f_{-2}(\widetilde{x}_+(0), 1) = 0, \\ \partial_x q_{-2}(\widetilde{x}_+(0), 1, \alpha, 0) &= \partial_x f_{-2}(\widetilde{x}_+(0), 1) \neq 0, \end{aligned} \tag{3.3}$$

we can solve (3.2) with $\widetilde{\varphi}_+(\alpha, 0) = \widetilde{x}_+(0)$. Next, we consider the initial value problem

$$\begin{gathered} \dot{z} = f_-(z) + \varepsilon g(z,t,\varepsilon) \quad \text{for } \frac{1}{2} \leq y \leq 1, \\ \dot{t} = 1, \qquad s \geq 0, \\ z(0) = (\widetilde{\varphi}_+(\alpha,\varepsilon), 1), \qquad t(0) = \alpha, \end{gathered} \tag{3.4}$$

which has a unique solution

$$\widetilde{z}(s,\alpha,\varepsilon) = (\widetilde{x}(s,\alpha,\varepsilon), \widetilde{y}(s,\alpha,\varepsilon)). \tag{3.5}$$

Then condition (b) implies that there is the smallest time $\widetilde{s}_+(\alpha,\varepsilon)$ such that

$$\widetilde{y}(\widetilde{s}_+(\alpha,\varepsilon), \alpha, \varepsilon) = \frac{1}{2}. \tag{3.6}$$

So, $\widetilde{s}_+(\alpha,\varepsilon)$ is the first hitting time for the level $y = 1/2$ of the solution of (3.4). We set

$$\xi(\alpha,\varepsilon) = \widetilde{x}(\widetilde{s}_+(\alpha,\varepsilon), \alpha, \varepsilon). \tag{3.7}$$

Consequently, in order to study the bifurcation of $\widetilde{\gamma}$, we need to show that the point $(\xi(\alpha,\varepsilon), 1/2)$ lies on the stable manifold of a unique small 1-periodic solution of (3.1). So we consider the initial value problem

$$\begin{gathered} \dot{z} = F(z) + \varepsilon g(z,t,\varepsilon), \\ \dot{t} = 1, \\ z(\widetilde{s}_+(\alpha,\varepsilon)) = \left(\xi(\alpha,\varepsilon), \frac{1}{2}\right), \qquad t(\widetilde{s}_+(\alpha,\varepsilon)) = \widetilde{s}_+(\alpha,\varepsilon) + \alpha, \end{gathered} \tag{3.8}$$

that is the initial value problem

$$\begin{gathered} \dot{z} = F(z) + \varepsilon g(z, \widetilde{s}_+(\alpha,\varepsilon) + s + \alpha, \varepsilon), \\ z(0) = \left(\xi(\alpha,\varepsilon), \frac{1}{2}\right), \qquad s \geq 0. \end{gathered} \tag{3.9}$$

We note $\widetilde{\gamma}_+(0) = (\xi(\alpha,0),1/2)$. By taking

$$z(s) = \widetilde{\gamma}_+(s) + \varepsilon w(s) \tag{3.10}$$

in (3.9), we get

$$\begin{gathered}
\dot{w} = DF(\widetilde{\gamma}_+(s))w + \frac{1}{\varepsilon}\{F(\widetilde{\gamma}_+(s)+\varepsilon w) - F(\widetilde{\gamma}_+(s)) - DF(\widetilde{\gamma}_+(s))\varepsilon w\} \\
+ g(\widetilde{\gamma}_+(s)+\varepsilon w, \widetilde{s}_+(\alpha,\varepsilon)+s+\alpha,\varepsilon), \quad s \geq 0, \\
w(0) = (\widetilde{\psi}_+(\alpha,\varepsilon),0) = \Psi_+(\alpha,\varepsilon),
\end{gathered} \tag{3.11}$$

where

$$\widetilde{\psi}_+(\alpha,\varepsilon) = (\xi(\alpha,\varepsilon) - \xi(\alpha,0))/\varepsilon. \tag{3.12}$$

Now we can repeat the above arguments of (2.22) to solve (3.11). So we again take

$$\Gamma = \dot{\widetilde{\gamma}}_+(0)^\perp = \left(\dot{\widetilde{\gamma}}_{+2}(0), -\dot{\widetilde{\gamma}}_{+1}(0)\right). \tag{3.13}$$

The statement of Lemma 2.2 changes as follows.

Lemma 3.1. *Problem*

$$\begin{gathered}
\dot{w} = DF(\widetilde{\gamma}_+(s))w + h, \\
w(0) = u
\end{gathered} \tag{3.14}$$

has a bounded solution w on $\mathbb{R}_+$ for a $h \in C_b(\mathbb{R}_+,\mathbb{R}^2)$ if and only if

$$\int_0^{+\infty} (h(s),\widetilde{\mu}(s))\,ds + \left(\dot{\widetilde{\gamma}}_+(0)^\perp, u\right) = 0. \tag{3.15}$$

Moreover, if condition (3.15) holds, then problem (3.14) has a unique bounded solution $w = \widetilde{w}(u,h)$ on $\mathbb{R}_+$. Furthermore, there is a constant $\widetilde{c} > 0$ such that

$$||\widetilde{w}(u,h)|| \leq \widetilde{c}(\|h\| + |u|). \tag{3.16}$$

Here, $\widetilde{\mu}$ is a bounded solution on $\mathbb{R}_+$ of the adjoint linear equation

$$\dot{w} = -DF(\widetilde{\gamma}_+(s))^* w \tag{3.17}$$

with $w(0) = \Gamma$.

Condition (3.15) yields that instead of projection S from Section 2, we take a mapping $\widetilde{S} : \mathbb{R}^2 \times Y \to Y$ defined by

$$\widetilde{S}(u)h = h - \int_0^{+\infty} \left[\left(h(\theta), \frac{\widetilde{\mu}(\theta)}{\|\widetilde{\mu}\|_2^2}\right) d\theta\right]\widetilde{\mu} - \left(\dot{\widetilde{\gamma}}_+(0)^\perp, u\right)\frac{\widetilde{\mu}}{\|\widetilde{\mu}\|_2^2}. \tag{3.18}$$

Then we have

$$\int_0^{+\infty} (\widetilde{S}(u)h(s),\widetilde{\mu}(s))\,ds + \left(\dot{\widetilde{\gamma}}_+(0)^\perp,u\right) = 0. \tag{3.19}$$

So we split (3.11) as follows:

$$\begin{aligned}\dot{w} = DF(\widetilde{\gamma}_+(s))w + \widetilde{S}(\Psi_+(\alpha,\varepsilon))\Big[\frac{1}{\varepsilon}\{F(\widetilde{\gamma}_+(s)+\varepsilon w) - F(\widetilde{\gamma}_+(s)) - DF(\widetilde{\gamma}_+(s))\varepsilon w\}\\ + g(\widetilde{\gamma}_+(s)+\varepsilon w,\widetilde{s}_+(\alpha,\varepsilon)+s+\alpha,\varepsilon)\Big],\\ w(0) = \Psi_+(\alpha,\varepsilon),\end{aligned} \tag{3.20}$$

$$\begin{aligned}\int_0^{+\infty}\Big(\frac{1}{\varepsilon}\{F(\widetilde{\gamma}_+(s)+\varepsilon w) - F(\widetilde{\gamma}_+(s)) - DF(\widetilde{\gamma}_+(s))\varepsilon w\}\\ + g(\widetilde{\gamma}_+(s)+\varepsilon w,\widetilde{s}_+(\alpha,\varepsilon)+s+\alpha,\varepsilon),\widetilde{\mu}(s)\Big)ds\\ + \left(\dot{\widetilde{\gamma}}_+(0)^\perp,\Psi_+(\alpha,\varepsilon)\right) = 0.\end{aligned} \tag{3.21}$$

By using Lemma 3.1, we can solve (3.20) to obtain its solution

$$w = \widetilde{w}(\alpha,\varepsilon,s). \tag{3.22}$$

Then by inserting it into (3.21), we arrive at a bifurcation equation

$$\begin{aligned}\widetilde{B}(\alpha,\varepsilon) = \int_0^{+\infty}\Big(\frac{1}{\varepsilon}\{F(\widetilde{\gamma}_+(s)+\varepsilon\widetilde{w}(\alpha,\varepsilon,s)) - F(\widetilde{\gamma}_+(s)) - DF(\widetilde{\gamma}_+(s))\varepsilon\widetilde{w}(\alpha,\varepsilon,s)\}\\ + g(\widetilde{\gamma}_+(s)+\varepsilon\widetilde{w}(\alpha,\varepsilon,s),\widetilde{s}_+(\alpha,\varepsilon)+s+\alpha,\varepsilon),\widetilde{\mu}(s)\Big)ds\\ + \left(\dot{\widetilde{\gamma}}_+(0)^\perp,\Psi_+(\alpha,\varepsilon)\right) = 0.\end{aligned} \tag{3.23}$$

We have

$$\widetilde{M}(\alpha) = \widetilde{B}(\alpha,0) = \int_0^{+\infty}(g(\gamma_+(s),a_+ + s+\alpha,0),\widetilde{\mu}(s)) + \dot{\widetilde{\gamma}}_{+2}(0)\widetilde{\psi}_+(\alpha,0) = 0, \tag{3.24}$$

where we use that $\widetilde{s}_+(\alpha,0) = a_+$ and

$$\frac{1}{\varepsilon}\{F(\widetilde{\gamma}_+(s)+\varepsilon\widetilde{w}(\alpha,\varepsilon,s)) - F(\widetilde{\gamma}_+(s)) - DF(\widetilde{\gamma}_+(s))\varepsilon\widetilde{w}(\alpha,\varepsilon,s)\} = O(\varepsilon). \tag{3.25}$$

Any simple root α_0 of $\widetilde{M}(\alpha)$ gives the solvability of $\widetilde{B}(\alpha,\varepsilon) = 0$ with respect to $\alpha = \widetilde{\alpha}(\varepsilon)$ for any ε small with $\widetilde{\alpha}(0) = \alpha_0$.

Furthermore, from (3.6), we get

$$f_{-2}(\eta_+(a_+))\partial_\varepsilon\widetilde{s}_+(\alpha,0) + \partial_\varepsilon\widetilde{y}(a_+,\alpha,0) = 0, \tag{3.26}$$

while (3.7) and (3.12) give

$$\widetilde{\psi}_+(\alpha,0) = \partial_\varepsilon\xi(\alpha,0) = f_{-1}(\eta_+(a_+))\partial_\varepsilon\widetilde{s}_+(\alpha,0) + \partial_\varepsilon\widetilde{x}(a_+,\alpha,0), \tag{3.27}$$

which altogether imply

$$\widetilde{\psi}_+(\alpha,0) = -f_{-1}(\eta_+(a_+))\frac{\partial_\varepsilon\widetilde{y}(a_+,\alpha,0)}{f_{-2}(\eta_+(a_+))} + \partial_\varepsilon\widetilde{x}(a_+,\alpha,0). \tag{3.28}$$

Next, we derive from (3.4) for

$$w(s) = \partial_\varepsilon\widetilde{z}(a_+,\alpha,0) = (\partial_\varepsilon\widetilde{x}(a_+,\alpha,0),\partial_\varepsilon\widetilde{y}(a_+,\alpha,0)) \tag{3.29}$$

the linear variational initial value problem

$$\begin{gathered}\dot{w} = Df_-(\eta_+(s))w + g(\eta_+(s),s+\alpha,0),\\ w(0) = (\partial_\varepsilon\widetilde{\varphi}_+(\alpha,0),0).\end{gathered} \tag{3.30}$$

But (3.2) implies

$$\partial_x f_{-2}(\eta_+(0))\partial_\varepsilon\widetilde{\varphi}_+(\alpha,0) + g_2(\eta_+(0),\alpha,0) = 0. \tag{3.31}$$

So instead of (3.30), we consider the linear initial value problem

$$\begin{gathered}\dot{w} = Df_-(\eta_+(s))w + g(\eta_+(s),s+\alpha,0),\\ w(0) = \left(-\frac{g_2(\eta_+(0),\alpha,0)}{\partial_x f_{-2}(\eta_+(0))},0\right).\end{gathered} \tag{3.32}$$

Summarizing we arrive at the following result.

THEOREM 3.2. *Let function $\widetilde{M}$ be given by (3.24) along with formulas (3.28), (3.29), and (3.32). If there is a simple root of $\widetilde{M}$, then homoclinic solution $\widetilde{\gamma}$ bifurcates to a bounded solution on $\mathbb{R}$ of (3.1) with $\varepsilon \neq 0$ small.*

4. Example

We present in this section an illustrative example. Let a_+ be the unique (positive) solution of the equation

$$e^{a_+}(1-a_+) = \frac{1}{2}. \tag{4.1}$$

We note that $a_+ \sim 0.768039$. Then we set

$$a = e^{a_+}(2-a_+) \sim 2.65554. \tag{4.2}$$

In this section, we consider system (3.1) with

$$\begin{gathered} f_+(z) = \begin{cases}\dot{x} = y,\\ \dot{y} = x - 3y,\end{cases} \qquad f_-(z) = \begin{cases}\dot{x} = y,\\ \dot{y} = 2y - x,\end{cases}\\ F(z) = \begin{cases}\dot{x} = -2ay,\\ \dot{y} = -\dfrac{1}{2a}x,\end{cases} \qquad g(x,t,\varepsilon) = \cos t\begin{pmatrix}0\\1\end{pmatrix}.\end{gathered} \tag{4.3}$$

It is not difficult to see that now we have

$$\eta_+(s) = \begin{cases} e^s(2-s), \\ e^s(1-s), \end{cases} \qquad \widetilde{\gamma}_+(s) = \begin{cases} e^{-s}a, \\ e^{-s}/2, \end{cases}$$
$$\widetilde{\gamma}_-(s) = \begin{cases} -e^s a, \\ e^s/2, \end{cases} \qquad \eta_-(s) = \begin{cases} -ae^{s-a_-} + \left(\frac{1}{2}+a\right)e^{s-a_-}(s-a_-), \\ \frac{1}{2}e^{s-a_-} + \left(\frac{1}{2}+a\right)e^{s-a_-}(s-a_-), \end{cases} \tag{4.4}$$

where $a_- \sim -0.122043$ is the unique (negative) solution of the equation

$$e^{a_-} + \left(\frac{1}{2}+a\right)a_- = \frac{1}{2}. \tag{4.5}$$

We note that $(a,1/2) = \eta_+(a_+)$ and system (3.32) has now the form

$$\dot{w} = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} w + \cos(s+\alpha)\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad w(0) = (\cos\alpha, 0). \tag{4.6}$$

After some computations, function (3.24) has now the form

$$\widetilde{M}(\alpha) = \frac{a}{2}\left(\cos(a_+ + \alpha) - \sin(a_+ + \alpha)\right), + \frac{1}{4(1-a)}w_2(a_+) - \frac{1}{2}w_1(a_+), \tag{4.7}$$

where $w(s) = (w_1(s), w_2(s))$ solves (4.6), that is, we have

$$w_1(a_+) = \left(\cos\alpha + \frac{1}{2}\sin\alpha\right)e^{a_+} - \frac{1}{2}(\cos\alpha+\sin\alpha)e^{a_+}a_+ - \frac{1}{2}\sin(a_+ + \alpha),$$
$$w_2(a_+) = \frac{\cos\alpha}{2}e^{a_+} - \frac{1}{2}(\cos\alpha+\sin\alpha)e^{a_+}a_+ - \frac{1}{2}\cos(a_+ + \alpha). \tag{4.8}$$

Then, (4.7) takes the form

$$\widetilde{M}(\alpha) = -0.441052\cos\alpha - 1.7501\sin\alpha. \tag{4.9}$$

Function (4.9) has two different simple roots over the period 2π. By applying Theorem 3.2, we get the existence of two bounded solutions of (3.1) with (4.3) near to $\widetilde{\gamma}$, which is homoclinic to a small hyperbolic 2π-periodic solution of (3.1) with (4.3).

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Jan Awrejcewicz: Department of Automatics and Biomechanics, Technical University of Lodz, 1/15 Stefanowski Street, Lodz 90-924, Poland
E-mail address: awrejcew@p.lodz.pl

Michal Fečkan: Department of Mathematical Analysis and Numerical Mathematics, Comenius University, Mlynská dolina, 842 48 Bratislava, Slovakia
E-mail address: michal.feckan@fmph.uniba.sk

Pawel Olejnik: Department of Automatics and Biomechanics, Technical University of Lodz, 1/15 Stefanowski Street, Lodz 90-924, Poland
E-mail address: olejnik@p.lodz.pl