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BIFURCATIONS OF THE PERIODIC SOLUTIONS IN SYMMETRIC SYSTEMS

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Summary. Bifurcation phenomena in systems of ordinary differential equations which are invariant with respect to involutive diffeomorphisms, are studied. The "symmetry-breaking" bifurcation is investigated in detail.

1. PRELIMINARIES

This work contains a generalization of the author's results from [3] and also a generalization of some results from [4], [5].

1.1. Let $g \in \text{Diff}(\mathbb{R}^n)$ be such that

(1) $$g \circ g = \text{id}\,,$$

i.e. g is an involutory mapping of $\mathbb{R}^n$ on to itself.

We shall consider a 1-parameter system of ordinary differential equations

(2) $$\dot{x} = v(x, \mu)\,,$$

where $x \in \mathbb{R}^n$, $\mu \in \mathbb{R}^1$. Sometimes we shall write $v_\mu(x) = v(x, \mu)$.

We suppose that

a) the vector field $v(x, \mu)$ is of class C^∞ in both variables x and μ;

b)

(3) $$v_\mu(g(x)) = (g_*)_x\, v_\mu(x)$$

for all $x \in \mathbb{R}^n$ and all $\mu \in \mathbb{R}^1$, that means the vector field $v(x, \mu)$ *is invariant under the diffeomorphism* g for every $\mu \in \mathbb{R}^1$;

c) for every $\mu \in \mathbb{R}^1$, the flow T^t_μ, $t \in \mathbb{R}$, of the system (2) exists;

d) the set

$$\Delta = \text{Fix}(g) = \{x \in \mathbb{R}^n, g(x) = x\}$$

is a smooth connected submanifold of $\mathbb{R}^n$.

Remarks. 1. In the relation (3), $(g_*)_x$ denotes the Jacobi matrix of the mapping g at the point $\boldsymbol{x}$. Sometimes we shall write $(g_*)_x = (\mathbf{d}g)_x$.

2. The diffeomorphism g is called a *symmetry* of the system (2) and such a system we shall call a *symmetric system*.

3. The vector field $\boldsymbol{v}(\boldsymbol{x}, \mu)$ is invariant under the diffeomorphism g; hence if $\boldsymbol{x}(t)$ is a solution of (2), then $g(\boldsymbol{x}(t))$ is also a solution of (2), see [1], and every trajectory γ of (2) has a corresponding trajectory $g(\gamma)$.

This last remark results also from the following well-known lemma, see [2], p. 141:

Lemma 1. *Let T_μ^t be the flow of the vector field $\boldsymbol{v}(\boldsymbol{x}, \mu)$ which is invariant under the diffeomorphism g for all $\mu \in \mathbb{R}$. Then*

$$g \circ T_\mu^t = T_\mu^t \circ g \tag{4}$$

for all $t \in \mathbb{R}$ and $\mu \in \mathbb{R}$.

Lemma 2. *The dimension of the submanifold Δ is equal to the multiplicity of the eigenvalue 1 of the matrix $(\mathbf{d}g)_x$, $\boldsymbol{x} \in \Delta$. The tangent space $T_x\Delta$, $\boldsymbol{x} \in \Delta$, can be naturally identified with the eigenspace of the matrix $(\mathbf{d}g)_x$ belonging to the eigenvalue 1.*

Proof. In virtue of the relation (1), for all $\boldsymbol{x} \in \mathbb{R}^n$ we have ($\mathbf{E}$ denotes the unit matrix)

$$\mathbf{E} = (\mathbf{d}(g \circ g))_x = (\mathbf{d}g)_{g(x)} (\mathbf{d}g)_x$$

and also

$$\mathbf{E} = (\mathbf{d}(g \circ g))_{g(x)} = (\mathbf{d}g)_x (\mathbf{d}g)_{g(x)} .$$

Hence

$$(\mathbf{d}g)_x^{-1} = (\mathbf{d}g)_{g(x)} . \tag{5}$$

For $\tilde{\boldsymbol{x}} \in \Delta$ the relation (5) yields

$$(\mathbf{d}g)_{\tilde{x}} (\mathbf{d}g)_{\tilde{x}} = \mathbf{E} . \tag{6}$$

So, the matrix $(\mathbf{d}g)_{\tilde{x}}$, $\tilde{\boldsymbol{x}} \in \Delta$ has only two eigenvalues 1 and -1 with the multiplicity k and r, respectively, $k + r = n$.

Now we determine $T_{\tilde{x}}\Delta$, $\tilde{\boldsymbol{x}} \in \Delta$. Let $\boldsymbol{c}$: $\mathbb{R} \to \Delta$ be differentiable with $\boldsymbol{c}(0) = \tilde{\boldsymbol{x}}$. Then $\boldsymbol{c}$ is a curve on Δ based at $\tilde{\boldsymbol{x}}$ and

$$\frac{\mathrm{d}\boldsymbol{c}}{\mathrm{d}t}(0) = \boldsymbol{t}_{\tilde{x}} \in T_{\tilde{x}}\Delta . \tag{7}$$

In view of the fact that $\boldsymbol{c}(t) \in \Delta$ for all $t \in \mathbb{R}$ we have

$$g(\boldsymbol{c}(t)) = \boldsymbol{c}(t) \tag{8}$$

for all $t \in \mathbb{R}$. Differentiating both sides of (8) with respect to t, we obtain for $t = 0$

(9) $$(\boldsymbol{d}g)_{\tilde{x}}\, \boldsymbol{t}_{\tilde{x}} = \boldsymbol{t}_{\tilde{x}}\,.$$

Hence every tangent vector $\boldsymbol{t}_{\tilde{x}} \in T_{\tilde{x}}$ must lie in the eigenspace of the matrix $(\boldsymbol{d}g)_{\tilde{x}}$ belonging to the eigenvalue 1.

Thus we have proved that

(10) $$\dim \Delta = \dim T_{\tilde{x}}\Delta \leqq k\,.$$

Let us define the mapping $\boldsymbol{F}\colon \mathbb{R}^n \to \mathbb{R}^n$ by the relation

(11) $$\boldsymbol{F}(\boldsymbol{x}) = g(\boldsymbol{x}) - \boldsymbol{x}\,.$$

It is easy to see that $\Delta = \boldsymbol{F}^{-1}(\boldsymbol{0})$. From (11) we obtain

$$(\boldsymbol{d}\boldsymbol{F})_x = (\boldsymbol{d}g)_x - \mathbf{E}\,.$$

Hence for every $\tilde{\boldsymbol{x}} \in \Delta$

(12) $$\operatorname{rank}(\boldsymbol{d}\boldsymbol{F})_{\tilde{x}} = r\,,$$

because the matrix $(\boldsymbol{d}\boldsymbol{F})_{\tilde{x}}$ has a k-multiple *zero* eigenvalue and an r-multiple eigenvalue -2.

It results from the relation (11) that the points of the set Δ are just the solutions of the following equations

$$\begin{aligned} F_1(x_1, \ldots, x_n) &= 0\,,\\ F_2(x_1, \ldots, x_n) &= 0\,,\\ &\ldots\ldots\ldots\\ F_n(x_1, \ldots, x_n) &= 0\,, \end{aligned}$$

where the functions $F_j(\boldsymbol{x}) = F_j(x_1, \ldots, x_n)$, $j = 1, 2, \ldots, n$, are the coordinate functions of the mapping $\boldsymbol{F}$. Then the matrix $(\boldsymbol{d}\boldsymbol{F})_{\tilde{x}}$ can be written in the from

(13) $$\begin{bmatrix} \dfrac{\partial F_1}{\partial x_1}(\tilde{\boldsymbol{x}}), & \ldots, & \dfrac{\partial F_1}{\partial x_n}(\tilde{\boldsymbol{x}}) \\ \ldots & \ldots & \ldots \\ \dfrac{\partial F_n}{\partial x_1}(\tilde{\boldsymbol{x}}), & \ldots, & \dfrac{\partial F_n}{\partial x_n}(\tilde{\boldsymbol{x}}) \end{bmatrix}.$$

For $\tilde{\boldsymbol{x}} \in \Delta$ the rank $(\boldsymbol{d}\boldsymbol{F})_{\tilde{x}} = r$ and we can suppose that the first r rows in (13) are linearly independent vectors (if it is not the case we must rearrange the equations). Further, there exists such a neighbourhood U of $\tilde{\boldsymbol{x}}$ in Δ that for every $\boldsymbol{x} \in U \subset \Delta$ the first r rows in (13) are linearly independent vectors.

Hence the functions $F_1, F_2, \ldots, F_r$ are independent at each point of $U \subset \Delta = \boldsymbol{F}^{-1}(\boldsymbol{0})$. Thus we have proved that codim $\Delta = r$, i.e.

(14) $$\dim \Delta = n - r = k\,.$$

From (14) it follows that $\dim T_{\tilde{x}}\Delta = k$ and Lemma 2 is proved.

Corollary 1. *For every* $\tilde{x} \in \Delta$, $v_\mu(\tilde{x}) \in T_{\tilde{x}}\Delta$.

Proof. The relation (3) has the form (for $\tilde{x} \in \Delta$):

$$v_\mu(\tilde{x}) = (dg)_{\tilde{x}}\, v_\mu(\tilde{x}) .$$

This means the vector $v_\mu(\tilde{x})$ is an eigenvector of the matrix $(dg)_{\tilde{x}}$ belonging to the eigenvalue 1, so $v_\mu(\tilde{x}) \in T_{\tilde{x}}\Delta$.

2. EXAMPLES

In this section several examples will be given in order to motivate and illustrate the subsequent text.

2.1. Example 1. A two-box model of the reaction-diffusion system with Brusselator kinetics is well-known in the chemical literature. The system is described by the following set of four differential equations:

$$\begin{aligned}
\dot{x}_1 &= A - (B + 1)\, x_1 + x_1^2 y_1 + D_1(x_2 - x_1) \\
\dot{y}_1 &= Bx_1 - x_1^2 y_1 \qquad\qquad\; + D_2(y_2 - y_1) \\
\dot{x}_2 &= A - (B + 1)\, x_2 + x_2^2 y_2 + D_1(x_1 - x_2) \\
\dot{y}_2 &= Bx_2 - x_2^2 y_2 \qquad\qquad\; + D_2(y_1 - y_2) ,
\end{aligned} \tag{15}$$

where A, B, D_1, D_2 are adjusted parameters. The state of the system is determined by the quadruple $x = (x_1, y_1, x_2, y_2) \in \mathbb{R}^4$.

Let us consider a mapping g: $\mathbb{R}^4 \to \mathbb{R}^4$ defined by the relation

$$g(x_1, y_1, x_2, y_2) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{bmatrix},$$

i.e. in a short form

$$g(x_1, y_1, x_2, y_2) = (x_2, y_2, x_1, y_1) .$$

It is easy to see that the following statements are true:

(i) $g \circ g = \text{id}$.

(ii) g is a linear *diffeomorphism of* $\mathbb{R}^4$.

(iii) $\text{Fix}\,(g) = \Delta = \{(x_1, y_1, x_2, y_2) \in \mathbb{R}^4, x_1 = x_2, y_1 = y_2\}$, that is, Δ is the diagonal in $\mathbb{R}^4$.

(iv) The matrix $\boldsymbol{A}$ defining the mapping g has two *double* eigenvalues 1 and -1. The eigenvectors corresponding to them are $e_1 = (1, 0, 1, 0)$, $e_2 = (0, 1, 0, 1)$ and $e_3 = (1, 0, -1, 0)$, $e_4 = (0, 1, 0, -1)$, respectively.

We see that the vectors e_1 and e_2 lie in $T_x\Delta$.

The vector field $\boldsymbol{v}$ on the right hand side of the system (15) is invariant under the diffeomorphism g. Since the mapping g is linear, $(g_*)_x = g$ for all $\boldsymbol{x} \in \mathbb{R}^4$. In this case the relation (3) has the form $\boldsymbol{v}(g(\boldsymbol{x})) = g \cdot \boldsymbol{v}(\boldsymbol{x})$ and its verification is easy. Further, for $\boldsymbol{x} \in \Delta$ we immediately see that $\boldsymbol{v}(\boldsymbol{x}) \in T_x\Delta$ when putting $x_2 = x_1$ and $y_2 = y_1$ in the system (15).

2.2. Example 2. In [4] the following system of ordinary differential equations

$$(16) \qquad \begin{aligned} \dot{x} &= \boldsymbol{u}(\boldsymbol{x}, \boldsymbol{y}), \quad \boldsymbol{x} \in \mathbb{R}^k, \quad k \geqq 2 \\ \dot{y} &= \boldsymbol{v}(\boldsymbol{x}, \boldsymbol{y}), \quad \boldsymbol{y} \in \mathbb{R}^m, \end{aligned}$$

with the symmetry

$$(17) \qquad \begin{aligned} \boldsymbol{u}(-\boldsymbol{x}, \boldsymbol{y}) &= -\boldsymbol{u}(\boldsymbol{x}, \boldsymbol{y}) \\ \boldsymbol{v}(-\boldsymbol{x}, \boldsymbol{y}) &= \boldsymbol{v}(\boldsymbol{x}, \boldsymbol{y}) \end{aligned}$$

has been considered.

The symmetry relations (17) can be expressed in the form of the relation (3) with help of the following diffeomorphism: Let us put $\boldsymbol{z} = (\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^k \times \mathbb{R}^m = \mathbb{R}^{k+m}$. Then $\boldsymbol{w}(\boldsymbol{z}) = \boldsymbol{w}(\boldsymbol{x}, \boldsymbol{y}) = [\boldsymbol{u}(\boldsymbol{x}, \boldsymbol{y}), \boldsymbol{v}(\boldsymbol{x}, \boldsymbol{y})]$ is a vector field on $\mathbb{R}^{k+m}$. The desired diffeomorphism is given by

$$(18) \qquad g(\boldsymbol{z}) = g(\boldsymbol{x}, \boldsymbol{y}) = \begin{bmatrix} -\mathbf{E}_k, & 0 \\ 0, & \mathbf{E}_m \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix} = (-\boldsymbol{x}, \boldsymbol{y}),$$

where the $\mathbf{E}_k$ and $\mathbf{E}_m$ are the unit matrices of the order k and m, respectively.

In this case the diffeomorphism g is also a linear mapping, hence $(g_*)_z = g$ for all $\boldsymbol{z} \in \mathbb{R}^{k+m}$ and the relation (3) has the form

$$\boldsymbol{w}(\boldsymbol{g}(\boldsymbol{z})) = \boldsymbol{g} \cdot \boldsymbol{w}(\boldsymbol{z}),$$

$$\boldsymbol{w}(-\boldsymbol{x}, \boldsymbol{y}) = \begin{bmatrix} -\mathbf{E}_k & 0 \\ 0 & \mathbf{E}_m \end{bmatrix} \begin{bmatrix} \boldsymbol{u}(\boldsymbol{z}) \\ \boldsymbol{v}(\boldsymbol{z}) \end{bmatrix},$$

$$(19) \qquad [\boldsymbol{u}(-\boldsymbol{x}, \boldsymbol{y}), \boldsymbol{v}(-\boldsymbol{x}, \boldsymbol{y})] = [-\boldsymbol{u}(\boldsymbol{x}, \boldsymbol{y}), \boldsymbol{v}(\boldsymbol{x}, \boldsymbol{y})].$$

By comparing the first and second coordinates in (19) we obtain the relations in (17).

Let us summarize the properties of the system (16).

(i) $g \circ g = \mathrm{id}$;

(ii) $\mathrm{Fix}\,(g) = \Delta = \{(\boldsymbol{0}, \boldsymbol{y}) \in \mathbb{R}^k \times \mathbb{R}^m\} = \{\boldsymbol{0}\} \times \mathbb{R}^m$;

(iii) $\boldsymbol{w}(\boldsymbol{0}, \boldsymbol{y}) = [\boldsymbol{u}(\boldsymbol{0}, \boldsymbol{y}), \boldsymbol{v}(\boldsymbol{0}, \boldsymbol{y})] = [\boldsymbol{0}, \boldsymbol{v}(\boldsymbol{0}, \boldsymbol{y})] \in T_{(0,y)}\Delta$.

2.3. Example 3. We shall show here that Example 2 includes the famous *Lorenz equations* (for $k = 2$, $m = 1$):

$$(20) \qquad \begin{aligned} \dot{x} &= \sigma(y - x) \\ \dot{y} &= -y + rx - xz \end{aligned}$$

$$\dot{z} = -bz + xy\,,$$

σ, r, b are positive parameters. In this case Fix $(g) \equiv \{z\text{-axis}\}$. We have

$$\boldsymbol{v}(x, y, z) = \begin{bmatrix} \sigma(y - x) \\ -y + rx - xz \\ -bz + xy \end{bmatrix}$$

and further

$$\boldsymbol{v}(g(x, y, z)) = \boldsymbol{v}(-x, -y, z) =$$

$$= \begin{bmatrix} -\sigma(y - x) \\ y - rx + xz \\ -bz + xy \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma(y - x) \\ -y + rx - xz \\ -bz + xy \end{bmatrix} = g\,.\,\boldsymbol{v}(x, y, z)\,.$$

2.4. Example 4. Let us consider, see [5], the system of nonautonomous ordinary differential equations with an ω-periodic right hand side

(21) $$\dot{x} = \boldsymbol{v}(t, \boldsymbol{x})\,, \quad \boldsymbol{v}(t + \omega, \boldsymbol{x}) = \boldsymbol{v}(t, \boldsymbol{x})\,,$$

where $\boldsymbol{x} \in \mathbb{R}^n$ and $t \in \mathbb{R}$.

We can transform the system (21) into an autonomous system by incorporating the time variable into the phase space. Set $\boldsymbol{z} = (t, \boldsymbol{x}) \in \mathbb{R} \times \mathbb{R}^n$ and $\boldsymbol{w}(\boldsymbol{z}) = [1, \boldsymbol{v}(t, \boldsymbol{x})]$, where 1 denotes the constant scalar function with value one. Then $\boldsymbol{w}$ is a vector field on the extended phase space $\mathbb{R} \times \mathbb{R}^n$. In the periodic case the extended phase space is in fact $\boldsymbol{S}^1 \times \mathbb{R}^n$ due to the natural identification of the points $(t + \omega, \boldsymbol{x})$ and $(t, \boldsymbol{x})$ from the extended phase space $\mathbb{R} \times \mathbb{R}^n$.

Let us define the mapping $g\colon \boldsymbol{S}^1 \times \mathbb{R}^n \to \boldsymbol{S}^1 \times \mathbb{R}^n$ by the relation

(22) $$g(t, \boldsymbol{x}) = \begin{bmatrix} 1 & 0 \\ 0 & -\boldsymbol{E}_n \end{bmatrix} \begin{bmatrix} t \\ \boldsymbol{x} \end{bmatrix} + \begin{bmatrix} \frac{\omega}{2} \\ 0 \end{bmatrix} = \left(t + \frac{\omega}{2}, -\boldsymbol{x}\right).$$

It is easy to see that $g \in \operatorname{Diff}(\boldsymbol{S}^1 \times \mathbb{R}^n)$ and

(i) $g \circ g = \mathrm{id}$

for $g(g(t, \boldsymbol{x})) = g\left(t + \frac{\omega}{2}, -\boldsymbol{x}\right) = (t + \omega, \boldsymbol{x}) \equiv (t, \boldsymbol{x})$;

(ii) Fix $(g) = \emptyset$

(iii) $$(g_*)_z = \begin{bmatrix} 1 & 0 \\ 0 & -\boldsymbol{E}_n \end{bmatrix} \quad \text{for all} \quad z \in \mathbb{R} \times R^n\,.$$

Suppose that the vector field $\boldsymbol{w}(\boldsymbol{z})$ is invariant under the diffeomorphism g. What does it mean for the primary vector field $\boldsymbol{v}$? The invariance relation (3) has in this case the form

$$w(g(t, x)) = \left[1, v\left(t + \frac{\omega}{2}, -x\right)\right] = (g_*)_z\, w(z) = \begin{bmatrix} 1 & 0 \\ 0 & -\mathbf{E}_n \end{bmatrix} \begin{bmatrix} 1 \\ v(t, x) \end{bmatrix} = [1, -v(t, x)].$$

Thus the vector field w is invariant under g, if and only if

$$v(t, x) = -v\left(t + \frac{\omega}{2}, -x\right). \tag{23}$$

An example of a nonautonomous system of ordinary differential equations with the symmetry (22) is the *driven damped pendulum*, see [5].

3. THE PERIOD DOUBLING BIFURCATION OF (HS)

Let us return to the system (2) for which the assumptions a)–d) are fulfilled.

3.1. Definition 1. The periodic solution $x_\mu(t)$ of (2) will be called a *g-invariant solution* iff its trajectory γ_μ is an invariant set of the mapping g, i.e. $g(\gamma_\mu) = \gamma_\mu$.

The g-invariant solution $x_\mu(t)$ *for which* $\gamma_\mu \subset \Delta$ will be called a *homogeneous solution* – *(HS)*.

A g-invariant solution $x_\mu(t)$ for which $\gamma_\mu \cap \Delta = \emptyset$ will be called a *Δ-symmetric solution*.

The following lemma yields a useful characterization of the Δ-symmetric solution.

Lemma 3. *Let $x_\mu(t)$ be a periodic solution of* (2) *and γ_μ its trajectory. Let both the points x and $g(x) \neq x$ lie on γ_μ. Then the point $g(y) \neq y$ lies on γ_μ for every $y \in \gamma_\mu$ and hence $g(\gamma_\mu) = \gamma_\mu$. The phase shift of the points $y \in \gamma_\mu$ and $g(y) \in \gamma_\mu$ is one half of the period of the solution $x_\mu(t)$.*

Proof. (From now on the subscript μ will usually be omitted.) Let ω be the smallest period of the solution $x(t)$. Under our assumption the points x and $g(x) \neq x$ lie on γ, hence $T^\omega(x) = x$ and $T^\omega(g(x)) = g(x)$. Then there exists a number $s \in (0, \omega)$ such that $T^s(x) = g(x)$. From (4) and with help of $g \circ g = \text{id}$ we obtain

$$x = g^2(x) = g(g(x)) = g(T^s(x)) = T^s(g\,x)) = T^s(T^s(x)) = T^{2s}(x).$$

Hence

$$2s = \omega\,, \quad s = \frac{\omega}{2} \quad \text{and} \quad g(x) = T^{\omega/2}(x).$$

Let y be an arbitrary point of γ. A number $r \in (0, \omega)$ can be found such that $y = T^r(x)$. Then

$$T^{\omega/2}(y) = T^{(\omega/2+r)}(x) = T^r(T^{\omega/2}(x)) = T^r(g(x)) = g(T^r(x)) = g(y),$$

QED.

3.2. Let $\gamma_{\mu_0} \subset \Delta$ be the trajectory of a (HS) of the system (2) for $\mu = \mu_0$. A *Poincaré map* will be used for the description of the bifurcation phenomena. Let $x_{\mu_0} \in \gamma_{\mu_0}$.

We consider a section Σ through the point x_{μ_0} transversal to the trajectory γ_{μ_0}. The section Σ may be chosen in such a way (see [6]) that

$$g(\Sigma) = \Sigma . \tag{24}$$

By P_{μ_0} let us denote the Poincaré map associated with the trajectory γ_{μ_0} and the section Σ. We suppose that none of the multipliers of this trajectory equals one. In this case there exists a one-parameter family P_μ of Poincaré maps associated with to closed trajectories γ_μ, $\mu \in O(\mu_0)$ and $O(\mu_0)$ is an appropriate neighbourhood of μ_0.

Lemma 4. *For every* $\mu \in O(\mu_0)$ *we have*

$$g \circ P_\mu = P_\mu \circ g \tag{25}$$

whenever $P_\mu \circ g$ *is defined.*

Proof. The Poincaré map P_μ can be expressed with help of the flow T^t_μ, see [7]. If ω_μ is the period of the corresponding (HS), then

$$P_\mu(x) = T_\mu^{[\omega_\mu + \delta_\mu(x)]}(x) \tag{26}$$

where $\delta_\mu: \Sigma \to \mathbb{R}$, $\delta_\mu(x_\mu) = 0$, $x_\mu \in \Sigma \cap \gamma_\mu$.

Let us denote

$$\omega_\mu(x) = \omega_\mu + \delta_\mu(x) . \tag{27}$$

For $x \in \Sigma$ we have

$$g(P_\mu(x)) = g(T^{\omega_\mu(x)}(x)) = T^{\omega_\mu(g(x))}(g(x)) = P_\mu(g(x)) .$$

The validity of the relation $\omega_\mu(g(x)) = \omega_\mu(x)$ results from the following consideration: The trajectory γ starting at the point $x \in \Sigma$ intersects Σ for the first time at the same moment as the trajectory $g(\gamma)$ starting at the point $g(x) \in \Sigma$ intersects Σ.

3.3. Theorem 1. Case A: $\dim \Delta = 2$. *Then after a generic period doubling bifurcation of a* (HS), *the resulting double period solution is* Δ-*symmetric.*

Case B: $\dim \Delta \geqq 3$. *Then after a generic period doubling bifurcation of a* (HS), *the resulting double period solution is either a* (HS) *or a* Δ-*symmetric solution.*

Proof. Let Γ_μ be trajectory of the double period solution bifurcated from the (HS) in question. It is well-known that after a period doubling bifurcation two fixed points of P^2_μ arise; let us denote them by $x_1(\mu)$ and $x_2(\mu)$. Then

$$P_\mu(x_1(\mu)) = x_2(\mu) \quad \text{and} \quad P_\mu(x_2(\mu)) = x_1(\mu) .$$

The relation (25) yields (the letter μ is omitted)

$$P(g(x_1)) = g(P(x_1)) = g(x_2) ,$$
$$P(g(x_2)) = g(P(x_2)) = g(x_1) ,$$

hence

$$g(x_1) = P(g(x_2)) = P(P(g(x_1))) = P^2(g(x_1)) ,$$

analogously

$$g(x_2) = P^2(g(x_2)) .$$

So we have the quadruple $x_1, x_2, g(x_1), g(x_2)$ of the fixed points of the square Poincaré map P^2. Two possibilities arise: Either

(i) $x_1 = g(x_1)$ and $x_2 = g(x_2)$, i.e. $x_1, x_2 \in \Delta$,

or

(ii) $x_1 = g(x_2)$ and $x_2 = g(x_1)$.

If $\dim \Delta = 2$, the case (i) is not possible, because $\Gamma_\mu \subset \Delta$ which is impossible – a period doubling bifurcation cannot arise in the two-dimensional Δ. Thus the equality $g(x_1) = x_2$ holds and the points x_1 and $x_2 = g(x_1) \neq x_1$ lie on Γ_μ, hence Γ_μ is Δ-symmetric.

If $\dim \Delta \geqq 3$ both cases (i) and (ii) can arise. In the case (i) we obtain after the bifurcation a (HS) and in the case (ii) we obtain a Δ-symmetric solution, QED.

Remark. In the nongeneric case, the points $x_1, x_2, g(x_1), g(x_2)$ can be mutually different and after this nongeneric bifurcation *two* double periodic *nonsymmetric* solutions can arise.

4. THE PERIOD DOUBLING BIFURCATION OF A Δ-SYMMETRIC SOLUTION

4.1. Let γ_μ be the trajectory of a Δ-symmetric solution of the equation (2) with a period ω_μ. Let us denote the cros-section which transversally intersects the trajectory γ_μ at a point x_μ^0 by Σ_0 and let $P_\mu(x)$ be the corresponding Poincaré map. Under our assumption, the point $g(x_\mu^0) \neq x_\mu^0$ must lie on γ_μ. Then $\Sigma_1 = g(\Sigma_0)$ is the cross-section of the trajectory γ_μ at the point $g(x_\mu^0)$. Let us denote by $\tilde{P}_\mu(x)$ the corresponding Poincaré map. It is known that the maps P_μ and $\tilde{P}_\mu$ are locally conjugate, see[7]. In our special case the following lemma is valid.

Lemma 5. *For the maps P_μ and $\tilde{P}_\mu$ defined above we have*

$$\tilde{P}_\mu = g \circ P_\mu \circ g^{-1} = g \circ P_\mu \circ g , \tag{28}$$

whenever $P_\mu \circ g$ is defined.

Proof. We express the maps P and $\tilde{P}$ by the flow T^t: for $x \in \Sigma_0$ we put $P(x) = T^{\omega(x)}(x)$ and for $y \in \Sigma_1$ we put $\tilde{P}(y) = T^{\tilde{\omega}(y)}(y)$. By an argument fully analogous to the one used before (cf. Theorem 1), we obtain the equality

$$\tilde{\omega}(g(x)) = \omega(x) . \tag{29}$$

Then for an arbitrary $x \in \Sigma_0$ we have $g(x) = y \in \Sigma_1$ and

$$\tilde{P}(g(x)) = T^{\tilde{\omega}(g(x))}(g(x)) = g(T^{\tilde{\omega}(g(x))}(x) = g(T^{\omega(x)}(x)) = g(P(x)),$$

hence the relation (28) holds.

4.2. Let us define the maps

$$P_1^0 \colon \Sigma_0 \to \Sigma_1 \quad \text{and} \quad P_0^1 \colon \Sigma_1 \to \Sigma_0$$

by the following relations: for $x \in \Sigma_0$,

$$P_1^0(x) = T^{\beta(x)}(x) \in \Sigma_1,$$

where $\beta(x)$ is the time of the first intersection of the trajectory starting at $x \in \Sigma_0$ with the cross-section Σ_1. Analogously for $y \in \Sigma_1$,

$$P_0^1(y) = T^{\tilde{\beta}(y)}(y) \in \Sigma_0.$$

We note that for $y = g(x)$ the equation

$$\beta(x) = \tilde{\beta}(g(x)). \tag{30}$$

holds.

Remark. It is easy to see that

$$P = P_0^1 \circ P_1^0 \colon \Sigma_0 \to \Sigma_0 \tag{31}$$

is the corresponding Poincaré map.

Lemma 6. *For the maps P_1^0 and P_0^1 defined above we have*

$$P_0^1 \circ g = g \circ P_1^0, \tag{32}$$

whenever $g \circ P_1^0$ is defined.

Proof. We have

$$P_0^1(g(x)) = T^{\tilde{\beta}(g(x))}(g(x)) = g(T^{\beta(x)}(x)) = g(P_1^0(x)), \qquad \text{QED.}$$

Definition 2. Let us put

$$H = g \circ P_1^0 \colon \Sigma_0 \to \Sigma_0. \tag{33}$$

Theorem 2. *The Poincaré map P associated with a Δ-symmetric trajectory γ_μ is the square of the map H, i.e.*

$$P = H \circ H = H^2. \tag{34}$$

Proof. With help of Lemma 6 and the relation (31) we obtain

$$H \circ H = g \circ P_1^0 \circ g \circ P_1^0 = P_0^1 \circ g \circ g \circ P_1^0 = P_0^1 \circ P_1^0 = P, \qquad \text{QED.}$$

Remark. We see from Theorem 2 that the generic bifurcations of a Δ-symmetric solutions correspond to the generic bifurcations of the fixed points of the map H.

4.3. Theorem 3. *The Δ-symmetric solution cannot bifurcate by the period doubling bifurcation in the generic case.*

We give three different proofs of this theorem.

Proof I. Let us suppose that for $\mu = \mu_0$ the "double" trajectory Γ_μ arose from the Δ-symmetric trajectory γ_{μ_0} by the period doubling bifurcation. Hence the two fixed points $x_1(\mu)$ and $x_2(\mu)$ of the mapping P_μ^2 lie on the trajectory Γ_μ and $P_\mu(x_1) = x_2$, $P_\mu(x_2) = x_1$. The points $y_1 = g(x_1)$ and $y_2 = g(x_2)$, however, are also fixed points of the mapping $\tilde{P}_\mu^2$ for

$$\tilde{P}(y_1) = (g \circ P \circ g)(y_1) = g(P(x_1)) = g(x_2) = y_2$$

and

$$\tilde{P}(y_2) = (g \circ P \circ g)(y_2) = g(P(x_2)) = g(x_1) = y_1 .$$

Hence the trajectory Γ_μ is Δ-symmetric, because both the points x_1 and $g(x_1) \neq x_1$ lie on Γ_μ.

Let Ω_μ be the period of the double period solution corresponding to the trajectory Γ_μ. The points x_1, x_2, y_1, y_2 lie on the trajectory Γ_μ in the order x_1, y_1, x_2, y_2, x_1 or in the order x_1, y_2, x_2, y_1, x_1. According to Lemma 3 the phase shift between x_1 and y_1 and also between the points x_2 and y_2 is $\frac{1}{2}\Omega_\mu$. Hence the segments of Γ_μ between the points x_2, y_1 and also x_1, y_2 have no "moving" time. This is in contradiction with our assumption about the existence of a period doubling bifurcation.

Proof II. As in Proof I let x_1 and x_2 be a couple of fixed points of P^2, i.e.

$$P(x_1) = x_2 \quad \text{and} \quad P(x_2) = x_1 , \tag{35}$$

hence

$$P^2(x_i) = x_i , \quad i = 1, 2 . \tag{36}$$

With help of Theorem 2 the relations yield

$$H^4(x_i) = x_i , \quad i = 1, 2 .$$

Let us put

$$y_i = H(x_i) , \quad i = 1, 2, \quad y_i \neq x_i . \tag{37}$$

Then (35) and (37) imply

$$H(y_1) = H^2(x_1) = x_2 \quad \text{and} \quad H(y_2) = H^2(x_2) = x_1 .$$

Further,

$$H^2(y_1) = H(x_2) = y_2 \quad \text{and} \quad H^2(y_2) = H(x_1) = y_1 ,$$

hence

$$H^4(y_i) = y_i , \quad i = 1, 2.$$

The mapping H^4 has four fixed points x_1, x_2, y_1, y_2. As is easy to see, the square of the Poincaré map $P^2 = H^4$ has the same four fixed points. This contradicts the genericity assumption.

Proof III. (see [4].) Let $x_0(\mu)$ be a fixed point of the map H_μ, which means that $x_0(\mu)$ is a fixed point of the Poincaré map P_μ as well. Theorem 2 yields

$$(dP)_{x_0} = (dH)_{x_0} \cdot (dH)_{x_0} = (dH)^2_{x_0} . \tag{38}$$

Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of the matrix $(dP)_{x_0}$ and $\tilde{\lambda}_1, \ldots, \tilde{\lambda}_n$ the eigenvalues of the matrix $(dH)_{x_0}$. From (38) we obtain

$$\lambda_i = \tilde{\lambda}_i^2 , \quad i = 1, 2, \ldots, n . \tag{39}$$

If an eigenvalue λ leaves the unit circle at the point -1, then the two eigenvalues $\tilde{\lambda}_{1,2}$ must leave the unit circle at the points $+i$ and $-i$. But this phenomenon is nongeneric.

4.4. In this section we give the list of generic bifurcations of Δ-symmetric solutions in one-parameter families (2).

As we have mentioned in the remark after Theorem 2, this list must be made with respect to the mapping H.

1. A single eigenvalue of the matrix $(dH)_x$ leaves the unit circle at $+1$. It means a single eigenvalue of the matrix $(dP)_x$ leaves the unit circle at $+1$. Thus in this case the usual saddle-node bifurcation occurs.

2. A single eigenvalue of the matrix $(dH)_x$ leaves the unit circle at -1. It means a single eigenvalue of the matrix $(dP)_x$ leaves the unit circle at $+1$. But, in contradistinction to the previous case, two fixed points of the map H^2 arise. Thus after this bifurcation there exist one unstable fixed point x_0 and two fixed points x_1, x_2 of the mapping H^2. The point x_0 is also a fixed of the corresponding Poincaré map P, as $P(x_0) = H^2(x_0) = x_0$. The points x_1 and x_2 are also fixed points of P, as $P(x_i) = H^2(x_i) = x_i$, $i = 1, 2$. Thus there are three closed trajectories in the phase space. The unstable trajectory γ_0 corresponds to the point x_0 and the two stable trajectories γ_1 and γ_2 correspond to the points x_1 and x_2, respectively.

Theorem 4. *None of the trajectories γ_1 and γ_2 is Δ-symmetric and $g(\gamma_1) = \gamma_2$.*

Proof. If x_1, x_2 are fixed points of the Poincaré map P, then the points $y_1 = g(x_1)$, $y_2 = g(x_2)$ are fixed points of the Poincaré map $\tilde{P}$, (see relation (28)) since

$$\tilde{P}(g(x_i)) = g(P(x_i)) = g(x_i) , \quad i = 1, 2.$$

The trajectory γ_1 starting at the point x_1 cannot intersects Σ_1 at the point $g(x_1)$. We prove this by contradiction. Let the trajectory γ_1 intersect Σ_1 at the point $g(x_1)$. It means that

(40) $$P_1^0(x_1) = g(x_1).$$

Then (40) implies

$$x_1 = g(g(x_1)) = g(P_1^0(x_1)) = H(x_1),$$

i.e. x_1 is a fixed point of the mapping H, which is a contradiction, for only the point x_0 is a fixed point of the mapping H.

Thus the trajectory γ_1 starting at x_1 intersects Σ_1 at $g(x_2)$. Analogously, the trajectory γ_2 starting at x_2 intersects Σ_1 at $g(x_1)$. Hence $g(x_1) \neq x_1$ does not lie on the trajectory γ_1, consequently γ_1 cannot be Δ-symmetric. Analogously, the trajectory γ_2 cannot be Δ-symmetric, either. From the proof it is easy to see that $g(\gamma_1) = \gamma_2$ holds, QED.

The bifurcation just described is called the *symmetry-breaking* bifurcation, because the loss of symmetry occurs on the branch of the stable solution.

3. A pair of complex conjugate eigenvalues of the matrix $(dH)_x$ crosses the unit circle. Assuming that the eigenvalues satisfy a non-resonance condition $\tilde{\lambda}^n \neq 1, n = 1, 2, 3, 4$, we conclude there is an *invariant torus* created or annihilated in the phase space.

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Souhrn

BIFURKACE V SYSTÉMECH S INVOLUTIVNÍ SYMETRIÍ

ALOIS KLÍČ

V práci jsou zkoumány bifurkační jevy v soustavách obyčejných diferenciálních rovnic, jež jsou invariantní vzhledem k involutivnímu difeomorfismu. Podrobně je zkoumána bifurkace „symmetry-breaking".

Резюме

БИФУРКАЦИИ В СИСТЕМАХ С ИНВОЛЮТИВНОЙ СИММЕТРИЕЙ

ALOIS KLÍČ

В статье изучаются бифуркационные явления в системах обыкновенных дифференциальных уравнений, инвариантных относительно инволютивного диффеоморфизма. Подробно изучается ,,нарушающая симметрию" бифуркация.

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