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Big Pieces of $C^{1,\alpha}$-Graphs for Minimizers of the Mumford-Shah Functional

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Abstract. We consider the generalization of the Mumford-Shah functional defined by

$$J(u, K) = \int_{\Omega \setminus K} |u - g|^2 + \int_{\Omega \setminus K} |\nabla u|^2 + H^{n-1}(K),$$

where Ω is a bounded domain in $\mathbb{R}^n$ ($n \geq 2$), g a bounded measurable function on Ω, K a relatively closed subset of Ω, $H^{n-1}(K)$ denotes the $(n-1)$-dimensional Hausdorff measure of K and $u \in W^{1,2}(\Omega \setminus K)$. We prove here that there exist $\alpha \in (0, 1)$ and $C > 1$ such that if (u, K) is an irreducible minimizer for J and $B(x, r)$ a ball centered on K, contained in Ω, with radius $r \leq 1$, then there is a ball B centered on K, contained in $B(x, r)$, with radius $\geq C^{-1}r$, such that $K \cap B$ is a $C^{1,\alpha}$-hypersurface. Moreover the constants α, C and the $C^{1,\alpha}$-constant for $K \cap B$ depend only on n and $\|g\|_\infty$. In particular the Hausdorff dimension of the set of points in K around which K is not a $C^{1,\alpha}$-hypersurface is strictly less than $n - 1$.

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1. – Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$ ($n \geq 2$) and g be a bounded measurable function on Ω. We consider the following functional defined by

$$J(u, K) = \int_{\Omega \setminus K} |u - g|^2 + \int_{\Omega \setminus K} |\nabla u|^2 + H^{n-1}(K)$$

where the integrals are taken with respect to the Lebesgue measure in $\mathbb{R}^n$, H^{n-1} denotes the $(n-1)$-Hausdorff measure and the competitors (u, K) satisfy the following conditions:

$$\begin{cases} \text{the set } K \text{ is relatively closed in } \Omega, \\ \text{the function } u \text{ lies in } W^{1,2}(\Omega \setminus K). \end{cases} \tag{1.1}$$

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This is a generalization of the Mumford-Shah functional introduced in [13] which corresponds to $n = 2$. The existence of minimizers for J was proved in [11]: using a larger class of competitors, E. De Giorgi, M. Carriero and A. Leaci prove the existence of "generalized" minimizers and then they show that these "generalized" minimizers are equivalent to competitors which satisfy (1.1) (see also [2]).

Here we are interested in the regularity properties of the set K when (u, K) minimizes the functional J. Note that if (u, K) is a minimizer, adding to K any closed set of H^{n-1}-Hausdorff measure 0 and keeping the same function u doesn't change the value of the functional. This gives a new minimizer $(u, \tilde{K})$ for which the set $\tilde{K}$ may look much uglier than the original set K. This is the reason why we shall always restrict ourselves to "irreducible minimizers".

DEFINITION 1.2. *Let (u, K) be a competitor which satisfies* (1.1). *We say that (u, K) is an irreducible minimizer for J if*

$$J(u, K) = \inf\{J(u', K') : K' \text{ is relatively closed in } \Omega \text{ and } u' \in W^{1,2}(\Omega\backslash K')\}$$

and if there is no proper relatively closed set $\tilde{K} \subset K$ such that u has an extension in $W^{1,2}(\Omega\backslash\tilde{K})$.

Note also that, for each minimizer (u, K), there is an irreducible minimizer $(u, \tilde{K})$ with $\tilde{K} \subset K$. From now on, we will restrict ourselves to the case where $\|g\|_\infty \leq 1$ which is not really restrictive by a scaling argument. We will prove the following

THEOREM 1.3. *There exist $\alpha \in (0, 1)$ and $C > 1$ such that if (u, K) is an irreducible minimizer for J and $B(x, r)$ a ball centered on K, contained in Ω, with radius $r \leq 1$, then there is a ball B centered on K, contained in $B(x, r)$, with radius $\geq C^{-1}r$, such that $K \cap B$ is a $C^{1,\alpha}$-hypersurface. Moreover the constants α, C and the $C^{1,\alpha}$-constant for $K \cap B$ depend only on n.*

This is the n-dimensional version of the main result in [7] which gives this regularity property when $n = 2$. To prove this, we will need a uniform control on two quantities. The first one is the function

$$\omega_2(x, t) = t\left\{t^{-n}\int_{B(x,t)\backslash K} |\nabla u|^2\right\}$$

defined for all $x \in K$ and $t \in (0, 1]$ such that $B(x, t) \subset \Omega$. The second one is the number $\beta(x, t)$ which measures the flatness of K inside a ball $B(x, t)$ and is defined for all $x \in K$ and $t > 0$ by

$$\beta(x, t) = \inf_P\left\{\sup_{y\in K\cap\overline{B}(x,t)} t^{-1}\,\mathrm{dist}(y, P)\right\}$$

where the infimum is taken on the set of all affine hyperplanes in $\mathbb{R}^n$. See Lemma 3.1 below. Then we will use a criterion proved in [3] which involves these two quantities to conclude. Let us point out that, using the equivalence between our irreducible minimizers and minimizers in the SBV setting, it can be shown that irreducible minimizers are quasi minimizers as in Definition 2.2 in [3]. More precisely, following the notations of [3], it can be shown that if (u, K) is an irreducible minimizer for J, then $u \in \mathcal{M}_\omega(\Omega)$ with $\omega(t) = Ct$ for some absolute constant C and $K = K(u)$ (see Definition 2.2 and page 42 in [3] for the definitions of $\mathcal{M}_\omega(\Omega)$ and $K(u)$ and Remark 2.3 in [3] for the arguments of a proof, see also [11]). Thus a direct application of Theorem 3.1 in [3] gives the $C^{1,\alpha}$-regularity of the set K in a small neighborhood of H^{n-1}-almost every points in K but without the kind of uniform estimates that we want. This is the reason why we shall first obtain a uniform control on the numbers ω_2 and β defined above. The ideas used here are a generalization in dimension greater than 2 of ideas developed in [7].

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2. – Some known properties of irreducible minimizers

From now on, (u, K) will denote a fixed irreducible minimizer for J. We will first recall some known properties of u and K. First u is bounded, $\|u\|_\infty \le \|g\|_\infty$, and $u \in C^1(\Omega \backslash K)$ (because, when K is fixed, u is a solution of a Neumann problem in $\Omega\backslash K$).

LEMMA 2.1. *There is an absolute constant $C_0 > 1$ such that*

$$(2.2)\quad C_0^{-1} t^{n-1} \le H^{n-1}(K \cap B(x,t)) \le C_0 t^{n-1} \quad \text{for all } x \in K \text{ and } t \in (0,1] \text{ such that } B(x,t) \subset \Omega.$$

The upper estimate follows from a simple truncation argument. The lower estimate was proved in [5] when $n = 2$ and follows from Remark 3.13 in [4] when $n \ge 2$. Set

$$\Delta = \{(x,t) \in K \times (0,1] : B(x,t) \subset \Omega\}.$$

For each $1 \le p \le 2$, we will need a L^p version of the function ω_2 defined above. So, for each $(x,t) \in \Delta$, set

$$\omega_p(x,t) = t\left\{t^{-n} \int_{B(x,t)\backslash K} |\nabla u|^p\right\}^{\frac{2}{p}}$$

LEMMA 2.3. *There is an absolute constant $C > 0$ such that*

$$\omega_2(x,t) \le C \text{ for all } (x,t) \in \Delta. \tag{2.4}$$

For each $1 \le p < 2$, there is a constant $C_p > 0$ such that

$$\iint_{y \in K \cap B(x,\frac{t}{2}), 0<s<\frac{t}{2}} \omega_p(y,s)\, \frac{dH^{n-1}(y)ds}{s} \le C_p t^{n-1} \text{ for all } (x,t) \in \Delta. \tag{2.5}$$

The estimate (2.4) follows from the same truncation argument as for the upper estimate of (2.2). The "Carleson measure estimate" (2.5) follows from (2.4), Hölder, Fubini and the size estimate (2.2). The proof is the same as the proof of Proposition 4.5 in [9], modulo minor modifications due to the dimension $n \ge 2$. We will also need some consequences of the uniform rectifiability property of the set K. It is proven in [10] that the set K is contained in a uniformly rectifiable set (at least locally). See Theorem 5.4 and Theorem 5.48 there. From the properties of uniformly rectifiable sets (see e.g. [8]) we deduce the

LEMMA 2.6 (Weak Geometric Lemma). *For each $\varepsilon > 0$, there is a constant $C(\varepsilon) > 0$ such that*

$$\iint_{y \in B(x,\frac{t}{2}), 0<s<\frac{t}{2}} \chi_{\mathcal{D}_\varepsilon}(y,s)\, \frac{dH^{n-1}(y)ds}{s} \le C(\varepsilon) t^{n-1} \text{ for all } (x,t) \in \Delta \tag{2.7}$$

where $\mathcal{D}_\varepsilon = \{(y,s) \in \Delta : \beta(y,s) \ge \varepsilon\}$ and $\chi_{\mathcal{D}_\varepsilon}$ denotes the characteristic function of $\mathcal{D}_\varepsilon$.

We will only need here the following consequence of the "Carleson measure estimates" (2.5) and (2.7):

COROLLARY 2.8. *For each $\varepsilon > 0$ and $1 \le p < 2$, there is a constant $C(\varepsilon, p) > 1$ such that for each $(x,t) \in \Delta$, there is a pair $(y,s) \in \Delta$ with $B(y,s) \subset B(x,t)$, $s \ge C(\varepsilon,p)^{-1} t$ and*

$$\omega_p(y,s) + \beta(y,s) < \varepsilon. \tag{2.9}$$

PROOF. Let $\varepsilon > 0$, $1 \le p < 2$ be fixed and $(x,t) \in \Delta$. We argue by contradiction and assume that we can not find (y,s) as in the statement. Then all pairs $(y,s) \in \Delta$ with $y \in B(x,\frac{t}{2})$ and $C^{-1}t < s < \frac{t}{2}$, for some $C > 2$ to be fixed later, would be such that $\omega_p(y,s) \ge \frac{\varepsilon}{2}$ or $\beta(y,s) \ge \frac{\varepsilon}{2}$. Then we would have $\omega_p(y,s) + \chi_{\mathcal{D}_{\frac{\varepsilon}{2}}}(y,s) \ge \min(\frac{\varepsilon}{2}, 1)$ and hence

$$\begin{aligned}
\iint_{y \in B(x,\frac{t}{2}), 0<s<\frac{t}{2}} &\left(\omega_p(y,s) + \chi_{\mathcal{D}_{\frac{\varepsilon}{2}}}(y,s)\right) \frac{dH^{n-1}(y)ds}{s} \\
&\ge \ln\left(\frac{C}{2}\right) \min\left(\frac{\varepsilon}{2}, 1\right) H^{n-1}\left(K \cap B\left(x, \frac{t}{2}\right)\right) \\
&\ge C' \ln\left(\frac{C}{2}\right) t^{n-1}
\end{aligned}$$

by (2.2). The constant C' depends only on the constant C_0 in (2.2) and on ε. Then, if C is large enough, this contradicts (2.5) and (2.7). □

3. – Uniform estimate on $\omega_2(x,t)$ and $\beta(x,t)$

As announced before Theorem 1.3 will follow from the same kind of estimate as (2.9) but with ω_p replaced by ω_2 and from Theorem 3.1 in [3]. More precisely we will prove the following

LEMMA 3.1. *For each $\tau > 0$, there is a constant $C > 1$ and a radius $t_0 \in (0,1)$ such that for each $(x,t) \in \Delta$ with $t \le t_0$, there is a pair $(y,s) \in \Delta$ with $B(y,s) \subset B(x,t)$, $s \ge C^{-1}t$, and*

$$\omega_2(y,s) + \beta(y,s) < \tau. \tag{3.2}$$

Before giving the proof of this Lemma, let us explain how Theorem 1.3 follows. We recall Theorem 3.1 of [3].

THEOREM 3.3 ([3]). *There exist $\alpha \in (0,1)$ and some absolute constants $\tau > 0$ and $C > 1$ such that if the pair $(x,t) \in \Delta$ satisfies*

$$\omega_2(x,t) + \beta(x,t) < \tau, \tag{3.4}$$

then $K \cap B(x, C^{-1}t)$ is a $C^{1,\alpha}$-hypersurface.

Clearly Theorem 1.3 follows from Lemma 3.1 and Theorem 3.3. To be more precise Theorem 3.1 in [3] is more general and $\beta(x,t)$ is replaced by a L^2 version of it,

$$\beta_2(x,t) = \inf_P \left\{ t^{1-n} \int_{K\cap\overline{B}(x,t)} \left(t^{-1}\operatorname{dist}(y,P)\right)^2 dH^{n-1}(y) \right\}^{\frac{1}{2}}$$

where the infimum is taken on the set of all affine hyperplanes in $\mathbb{R}^n$. Clearly from (2.2) and the definitions of the numbers β and β_2 we have that $\beta_2(x,t) \le C\beta(x,t)$ with a constant C which depends only on the constant C_0 in (2.2). On the other hand we also have that $\beta(x,t) \le C\beta_2(x,2t)^{2/n+1}$ (see e.g. the relation (1.73) on page 27 in [8]). And hence condition (3.4) here and condition (3.2) in [3] are equivalent.

Thus it remains to prove Lemma 3.1. Taking into account Corollary 2.8 and the fact that $\beta(y,s) \le C\beta(x,t)$ whenever $B(y,s) \subset B(x,t)$ and $s \ge C^{-1}t$, we only need to find for each $(x,t) \in \Delta$ a ball $B(y,s)$ centered on K, contained in $B(x,t)$, of size comparable to t, where $\omega_2(y,s)$ is as small as we want.

Let us first give a few more notations. For $1 \le p \le 2$ and $(x,t) \in \Delta$, we define the analogue of $\omega_p(x,t)$ where balls are replaced by spheres,

$$h_p(x,t) = t^{\frac{p}{2}} t^{1-n} \int_{\partial B(x,t)\setminus K} |\nabla u|^p \, d\mu,$$

where $\mu = H^{n-1}$. From now on we will always set $\mu = H^{n-1}$ when looking at integrals over $(n-1)$-dimensional sets. It will be also useful to introduce some subsets of Δ. For each $\beta \in (0,1)$, we define $\Delta(\beta)$ as the set of pairs $(x,t) \in \Delta$ for which there is a hyperplane P through x such that

$$K \cap \overline{B}(x,t) \subset \{y \in \overline{B}(x,t) : \operatorname{dist}(y,P) \leq \beta t\}.$$

If we want to specify a hyperplane P for which this inclusion holds, we will say that $(x,t) \in \Delta(\beta)$ with hyperplane P. Note that if $\beta(x,t) \leq \beta$ then $(x,t) \in \Delta(2\beta)$ with hyperplane a hyperplane through x parallel to some hyperplane that realizes $\beta(x,t)$. If $(x,t) \in \Delta(\beta)$ with hyperplane P, the set $U = \{y \in \overline{B}(x,t) : \operatorname{dist}(y,P) > \beta t\}$ has two connected components (because $\beta < 1$). Call them $U^+(t)$ and $U^-(t)$ and set $\partial^\pm(t) = \partial U^\pm(t) \cap U = U^\pm(t) \cap \partial B(x,t)$. Because $u \in C^1(\Omega \backslash K)$ the values of u on $\partial^\pm(t)$ are well defined. We will denote by $m^\pm(t)$ its mean value on $\partial^\pm(t)$,

$$m^\pm(t) = \frac{1}{\mu(\partial^\pm(t))} \int_{\partial^\pm(t)} u \, d\mu.$$

Clearly the sets $U^\pm(t)$, $\partial^\pm(t)$ and the quantities $m^\pm(t)$ don't depend only on t but also on β, x and P. To specify these dependances when needed, we will say that $U^\pm(t)$, $\partial^\pm(t)$ and $m^\pm(t)$ are defined with respect to $(x,t) \in \Delta(\beta)$ with hyperplane P.

First we want to get a lower bound on the jump $t^{-\frac{1}{2}}|m^+(t) - m^-(t)|$ when $(x,t) \in \Delta(\beta)$ for β small and with $h_p(x,t)$ small. We will need the following Lemma to fix which p we consider.

LEMMA 3.5. *Let B_1 be the unit ball in $\mathbb{R}^n$ and $p \in (\frac{2(n-1)}{n}, 2)$. There is a constant $C(p,n) > 0$ such that if $h \in C^0(\partial B_1) \cap W^{1,p}(\partial B_1)$ and if v is the harmonic extension of h inside B_1, then*

$$\int_{B_1} |\nabla v|^2 \leq C(p,n) \left(\int_{\partial B_1} |\nabla h|^p \, d\mu \right)^{\frac{2}{p}}.$$

The Lemma follows from well known Sobolev imbeddings (see e.g. [1]). Now we fix once and for all some $p \in (\frac{2(n-1)}{n}, 2)$ so that the previous Lemma holds.

LEMMA 3.6. *There exist some absolute constants $\varepsilon > 0$, $t_0 \in (0,1)$ and $C > 0$ such that if $(x,t) \in \Delta(\beta)$ with $t \leq t_0$ and $\beta + h_p(x,t) \leq \varepsilon$, then*

$$|m^+(t) - m^-(t)| \geq C t^{\frac{1}{2}}. \tag{3.7}$$

PROOF. Let $C_1 > 1$ be a (large) constant to be chosen later. Its value will depend only on n and on the constant C_0 in (2.2). Let $\varepsilon > 0$ be a small constant to be fixed later, $\varepsilon \leq \frac{C_1^{-1}}{100}$ for the time being. Consider $(x,t) \in \Delta(\beta)$

with hyperplane P and assume that $\beta + h_p(x,t) \leq \varepsilon$. Consider a function $\varphi \in C_c^\infty(\mathbb{R}^n)$ such that $0 \leq \varphi \leq 1$,

$$\varphi \equiv 1 \quad \text{on } \{y \in \partial B(x,t) : \operatorname{dist}(y,P) \geq 2C_1^{-1}t\},$$
$$\varphi \equiv 0 \quad \text{on } \{y \in \partial B(x,t) : \operatorname{dist}(y,P) \leq C_1^{-1}t\},$$

and $|\nabla\varphi(y)| \leq Ct^{-1}$ for $y \in \partial B(x,t)$, where C depends only on n and C_1. We define a function h on $\partial B(x,t)$ by

$$h(y) = \varphi(y)u(y) + (1-\varphi(y))m^+(t).$$

We have that $h \in C^0(\partial B(x,t))$ because $\varepsilon \leq \frac{C_1^{-1}}{100}$. We now define a new function v. Inside $B(x,t)$, we take v to be the harmonic extension of h. On $\Omega\backslash(K \cup \overline{B}(x,t))$ we set $v = u$. We also set $\tilde{K} = (K\backslash B(x,t)) \cup J$ where

$$J = \{y \in \partial B(x,t) : \operatorname{dist}(y,P) \leq 3C_1^{-1}t\}.$$

Then $(v,\tilde{K})$ is a competitor which satisfies (1.1). Because $\|v\|_\infty \leq \|u\|_\infty \leq 1$, we have

$$\int_{\Omega\backslash\tilde{K}} |v-g|^2 \leq \int_{\Omega\backslash K} |u-g|^2 + Ct^n,$$

and we get from the minimality of (u,K) that

$$\int_{B(x,t)\backslash K} |\nabla u|^2 + H^{n-1}(K \cap \overline{B}(x,t))$$
$$\leq \int_{B(x,t)} |\nabla v|^2 + H^{n-1}(\tilde{K} \cap \overline{B}(x,t)) + Ct^n.$$

We have $K \cap \partial B(x,t) \subset J$ and hence

$$H^{n-1}(\tilde{K} \cap \overline{B}(x,t)) = H^{n-1}(J) \leq CC_1^{-1}t^{n-1}.$$

On the other hand, by (2.2), we also have

$$H^{n-1}(K \cap \overline{B}(x,t)) \geq C_0^{-1}t^{n-1}.$$

Thus if we fix C_1 large enough and if t is small enough, $t \leq t_0$ for some t_0 depending only n and C_0, we get

$$\int_{B(x,t)} |\nabla v|^2 \geq \frac{C_0^{-1}}{2}t^{n-1}.$$

Lemma 3.5 gives

$$\int_{B(x,t)} |\nabla v|^2 \leq Ct^n t^{-\frac{2(n-1)}{p}} \left(\int_{\partial B(x,t)} |\nabla h|^p \, d\mu\right)^{\frac{2}{p}}$$

and hence we have

$$t^{-\frac{2(n-1)}{p}} t \left(\int_{\partial B(x,t)} |\nabla h|^p \, d\mu \right)^{\frac{2}{p}} \geq C.$$

Thus (3.7) will follow if we show that

$$\int_{\partial B(x,t)} |\nabla h|^p \, d\mu \leq C \int_{\partial B(x,t)\setminus K} |\nabla u|^p \, d\mu + C t^{n-1} t^{-p} |m^+(t) - m^-(t)|^p. \tag{3.8}$$

Indeed if this inequality holds we get that

$$\begin{aligned} 1 &\leq C t^{-\frac{2(n-1)}{p}} t \left(\int_{\partial B(x,t)\setminus K} |\nabla u|^p \, d\mu \right)^{\frac{2}{p}} + C t^{-1} |m^+(t) - m^-(t)|^2 \\ &\leq C h_p(x,t)^{\frac{2}{p}} + C t^{-1} |m^+(t) - m^-(t)|^2 \\ &\leq C \varepsilon^{\frac{2}{p}} + C t^{-1} |m^+(t) - m^-(t)|^2, \end{aligned}$$

and if ε is small enough we get (3.7). Let us prove (3.8). We have

$$\nabla h(y) = (u(y) - m^+(t)) \nabla \varphi(y) + \varphi(y) \nabla u(y),$$

and hence

$$\begin{aligned} &\int_{\partial B(x,t)} |\nabla h|^p \, d\mu \\ &\qquad \leq C \int_{\partial B(x,t)\setminus K} |\nabla u|^p \, d\mu + C t^{-p} \int_{\partial^+(t) \cup \partial^-(t)} |u(y) - m^+(t)|^p \, d\mu(y). \end{aligned}$$

We have $H^{n-1}(\partial^+(t)) \geq C t^{n-1}$ for some absolute constant $C > 0$ because $\varepsilon \leq \frac{C_1^{-1}}{100}$ and we have already fixed C_1, and Poincaré's inequality applied to the function u on $\partial^+(t)$ gives us

$$\begin{aligned} \int_{\partial^+(t)} |u(y) - m^+(t)|^p \, d\mu(y) &\leq C t^{1-n} \int_{\partial^+(t)} \int_{\partial^+(t)} |u(y) - u(z)|^p \, d\mu(z) \, d\mu(y) \\ &\leq C t^p \int_{\partial^+(t)} |\nabla u|^p \, d\mu. \end{aligned}$$

Similarly we have

$$\int_{\partial^-(t)} |u(y) - m^-(t)|^p \, d\mu(y) \leq C t^p \int_{\partial^-(t)} |\nabla u|^p \, d\mu.$$

Hence we get

$$\begin{aligned} &t^{-p} \int_{\partial^+(t) \cup \partial^-(t)} |u(y) - m^+(t)|^p \, d\mu(y) \\ &\qquad \leq C \int_{\partial B(x,t)\setminus K} |\nabla u|^p \, d\mu + C t^{n-1} t^{-p} |m^+(t) - m^-(t)|^p \end{aligned}$$

and finally we get (3.8). And this achieves the proof of Lemma 3.6. □

Next we want to say that if $(x,r) \in \Delta(\beta)$ then $m^+(t)$ (and $m^-(t)$) does not vary too much when t describes $(\frac{r}{2}, r)$. Note that if $\beta < \frac{1}{2}$ and $(x,r) \in \Delta(\beta)$ with hyperplane P then $(x,t) \in \Delta(2\beta)$ with the same hyperplane P for all $t \in (\frac{r}{2}, r)$.

LEMMA 3.9. *Let $\beta < \frac{1}{2}$ and $(x,r) \in \Delta(\beta)$ with hyperplane P. Then*

$$|m^+(t_1) - m^+(t_2)| \le C r^{\frac{1}{2}} \omega_p(x,r)^{\frac{1}{2}} \tag{3.10}$$

and

$$|m^-(t_1) - m^-(t_2)| \le C r^{\frac{1}{2}} \omega_p(x,r)^{\frac{1}{2}}. \tag{3.11}$$

for all $t_1, t_2 \in (\frac{r}{2}, r)$, where $m^\pm(t_i)$ are defined with respect to $(x, t_i) \in \Delta(2\beta)$ with hyperplane P. The constant C in (3.10) *and* (3.11) *depends only on n.*

PROOF. This does not come from the minimality of (u, K) but from the fact that $u \in C^1(\Omega\backslash K)$ and that the segment joining any point in $\partial^+(t_1)$ to any point in $\partial^+(t_2)$ stays in $\Omega\backslash K$. Let $\beta < \frac{1}{2}$ and $(x,r) \in \Delta(\beta)$ with hyperplane P. In the following proof $U^\pm(t)$, $\partial^\pm(t)$ and $m^\pm(t)$ will always be defined with respect to $(x,t) \in \Delta(2\beta)$ with hyperplane P when $t \in (\frac{r}{2}, r)$. Let $\frac{r}{2} < t_1 < t_2 < r$. The set

$$U^+(t_1, t_2) = \bigcup_{t \in [t_1, t_2]} U^+(t)$$

is convex and $u \in C^1(U^+(t_1,t_2))$. Thus for each $y \in \partial^+(t_1)$ and $z \in \partial^+(t_2)$ we have

$$|u(y) - u(z)| \le \int_0^1 |\nabla u(sy + (1-s)z)||y - z| \, ds.$$

Integrating over $y \in \partial^+(t_1)$ and $z \in \partial^+(t_2)$ and using Fubini's Theorem, we get

$$\begin{aligned}
&\int_{\partial^+(t_1)} \int_{\partial^+(t_2)} |u(y) - u(z)| \, d\mu(z) \, d\mu(y) \\
&\quad \le \int_{\partial^+(t_1)} \int_{\partial^+(t_2)} \int_0^{\frac{1}{2}} |\nabla u(sy + (1-s)z)||y - z| \, ds \, d\mu(z) \, d\mu(y) \\
&\quad + \int_{\partial^+(t_2)} \int_{\partial^+(t_1)} \int_{\frac{1}{2}}^1 |\nabla u(sy + (1-s)z)||y - z| \, ds \, d\mu(y) \, d\mu(z).
\end{aligned}$$

We have

$$\begin{aligned}
\int_{\partial^+(t_2)} \int_0^{\frac{1}{2}} |\nabla u(sy + (1-s)z)||y - z| \, ds \, d\mu(z) &\le C \int_{U^+(t_1,t_2)} |\nabla u| \\
&\le C r^n r^{-\frac{1}{2}} \omega_1(x,r)^{\frac{1}{2}}
\end{aligned}$$

and similarly

$$\int_{\partial^+(t_1)} \int_{\frac{1}{2}}^{1} |\nabla u(sy + (1-s)z)||y - z| \, ds \, d\mu(y) \le Cr^n r^{-\frac{1}{2}} \omega_1(x, r)^{\frac{1}{2}}.$$

Hence we get

$$\int_{\partial^+(t_1)} \int_{\partial^+(t_2)} |u(y) - u(z)| \, d\mu(z) \, d\mu(y) \le Cr^{2n-1} r^{-\frac{1}{2}} \omega_1(x, r)^{\frac{1}{2}},$$

and then

$$\begin{aligned} |m^+(t_1) - m^+(t_2)| &\le C t_1^{1-n} t_2^{1-n} \int_{\partial^+(t_1)} \int_{\partial^+(t_2)} |u(y) - u(z)| \, d\mu(z) \, d\mu(y) \\ &\le C r^{\frac{1}{2}} \omega_1(x, r)^{\frac{1}{2}} \le C r^{\frac{1}{2}} \omega_p(x, r)^{\frac{1}{2}} \end{aligned}$$

by Hölder's inequality. This is exactly (3.10). We get (3.11) by the same argument. And this achieves the proof of Lemma 3.9. □

Combining Lemma 3.6 and Lemma 3.9 we get the following

LEMMA 3.12. *There exist $\varepsilon > 0$, $r_0 \in (0, 1)$ and $C > 0$ such that if $(x, r) \in \Delta(\beta)$ with hyperplane P, $r \le r_0$ and $\beta + \omega_p(x, r) \le \varepsilon$, then*

$$|m^+ - m^-| \ge C r^{\frac{1}{2}} \tag{3.13}$$

where

$$m^\pm = \frac{1}{|U^\pm|} \int_{U^\pm} u \qquad \text{with} \qquad U^\pm = \left(\bigcup_{t \in (\frac{r}{2}, r)} U^\pm(t) \right) \setminus \overline{B}\left(x, \frac{r}{2}\right).$$

Here $U^\pm(t)$ is defined with respect to $(x, t) \in \Delta(2\beta)$ with hyperplane P as in Lemma 3.9.

PROOF. First by Fubini and Tchebychev we can find $t_0 \in (\frac{r}{2}, r)$ such that

$$\int_{\partial B(x, t_0) \setminus K} |\nabla u|^p \, d\mu \le C r^{-1} \int_{B(x, r) \setminus K} |\nabla u|^p,$$

i.e. such that $h_p(x, t_0) \le C \omega_p(x, r)^{\frac{p}{2}}$. Since $(x, t_0) \in \Delta(2\beta)$ with hyperplane P, if ε and r_0 are small enough, we get by Lemma 3.6 that

$$|m^+(t_0) - m^-(t_0)| \ge C r^{\frac{1}{2}}.$$

We may assume without loss of generality that $m^+(t_0) > m^-(t_0)$. By Lemma 3.9 we have

$$|m^+(t) - m^+(t_0)| \le Cr^{\frac{1}{2}}\omega_p(x,r)^{\frac{1}{2}} \le C\varepsilon^{\frac{1}{2}}r^{\frac{1}{2}}$$

and

$$|m^-(t) - m^-(t_0)| \le Cr^{\frac{1}{2}}\omega_p(x,r)^{\frac{1}{2}} \le C\varepsilon^{\frac{1}{2}}r^{\frac{1}{2}}$$

for all $t \in (\frac{r}{2}, r)$. Hence if ε is small enough we get

$$m^+(t) - m^-(t) \ge Cr^{\frac{1}{2}} \tag{3.14}$$

for all $t \in (\frac{r}{2}, r)$ and for some absolute constant $C > 0$. On the other hand we have

$$\begin{aligned} m^+ - m^- &= Cr^{-n}\int_{\frac{r}{2}}^{r}\left(\int_{\partial^+(t)} u\,d\mu - \int_{\partial^-(t)} u\,d\mu\right) dt \\ &\ge C'r^{-n}\int_{\frac{r}{2}}^{r} t^{n-1}(m^+(t) - m^-(t))\,dt, \end{aligned}$$

because $H^{n-1}(\partial^+(t)) = H^{n-1}(\partial^-(t)) \ge Ct^{n-1}$ for some absolute $C > 0$ (recall that $\partial^{\pm}(t)$ is defined with respect to $(x,t) \in \Delta(2\beta)$ for all $t \in (\frac{r}{2}, r)$ and we consider only small values of β). Combining this inequality and (3.14) we get

$$m^+ - m^- \ge Cr^{\frac{1}{2}}$$

which is exactly what we want. □

The fact that the jump $r^{-\frac{1}{2}}|m^+ - m^-|$ is large will tell us that K separates well a large portion of $U^+(r)$ from a large portion of $U^-(r)$. Therefore replacing $K \cap B(x,t)$ by $P \cap B(x,t)$ for some t comparable to r won't increase too much the surface of the singularity set.

Lemma 3.15. *There exist $\varepsilon_0 > 0$, $r_0 \in (0,1)$ and $C > 0$ such that for each $\varepsilon \le \varepsilon_0$ and $(x,r) \in \Delta(\beta)$ with hyperplane P, $r \le r_0$ and $\beta + \omega_p(x,r) \le \varepsilon$, there exists $t \in (\frac{r}{20}, r)$ such that*

$$H^{n-1}(P \cap B(x,t)) \le H^{n-1}(K \cap B(x,t)) + C\varepsilon^{\frac{1}{2}}t^{n-1}. \tag{3.16}$$

Proof. We want to construct a function $u^* \in C^1(B(x,r)\backslash K)$ such that its values on a large portion of $U^+(r)$ are far from its values on a large portion of $U^-(r)$. The point is to deduce from (3.13) pointwise estimates on the jump of u^*. Then if we also have that $\int_{B(x,r)\backslash K} |\nabla u^*|$ is small, we can easily conclude. The function u^* will be a mollified version of u. First we can assume that ε and r are small enough so that (3.13) holds, and that $\varepsilon < \frac{1}{100}$. Denote by x_1 and x_2 the intersections of the line through x perpendicular to P with respectively $\partial^+(\frac{r}{2})$ and $\partial^-(\frac{r}{2})$. Here $\partial^+(\frac{r}{2})$ and $\partial^-(\frac{r}{2})$ are defined with respect to $(x, \frac{r}{2}) \in \Delta(2\beta)$ with hyperplane P. Set $B_1 = B(x_1, \frac{r}{20})$ and $B_2 = B(x_2, \frac{r}{20})$. If ε is small enough then $2B_1 = B(x_1, \frac{r}{10}) \subset U^+(r)$ and $2B_2 = B(x_2, \frac{r}{10}) \subset U^-(r)$ (with $U^+(r)$ and $U^-(r)$ defined with respect to $(x, r) \in \Delta(2\beta)$ with hyperplane P). Set $\tilde{u} = \eta * u$ where $\eta \in C_c^\infty(\mathbb{R}^n)$ is such that supp $\eta \subset B(0, \frac{r}{20})$, $\int \eta = 1$ and $0 \leq \eta \leq Cr^{-n}$. We want to replace u by $\tilde{u}$ inside $B_1 \cup B_2$. Let $\varphi \in C_c^\infty(\mathbb{R}^n)$ be such that $0 \leq \varphi \leq 1$,

$$\varphi \equiv 1 \quad \text{on} \quad \left\{ y \in B(x, r) : \ \text{dist}(y, P) \leq \frac{r}{10} \right\},$$
$$\varphi \equiv 0 \quad \text{on} \quad \left\{ y \in B(x, r) : \ \text{dist}(y, P) \geq \frac{r}{5} \right\},$$

and $\|\nabla \varphi\|_\infty \leq Cr^{-1}$. Set

$$u^* = u\varphi + (1 - \varphi)\tilde{u}.$$

Then $u^* \in C^1(B(x, r) \backslash K)$ and $u^* \equiv \tilde{u}$ on $B_1 \cup B_2$. Let us show that

$$|u^*(y_1) - u^*(y_2)| \geq Cr^{\frac{1}{2}} \tag{3.17}$$

for all $y_1 \in B_1$ and $y_2 \in B_2$ and for some absolute constant $C > 0$. Keeping the same notations as in Lemma 3.12, we have for each $y \in B_1$,

$$\begin{aligned} |\tilde{u}(y) - m^+| &\leq \int_{2B_1} \eta(y - w)|u(w) - m^+| \, dw \\ &\leq Cr^{-2n} \int_{2B_1} \int_{U^+} |u(w) - u(z)| \, dz \, dw \\ &\leq Cr^{-n} r \int_{B(x,r)\backslash K} |\nabla u| \end{aligned}$$

by Poincaré's inequality. Thus

$$|\tilde{u}(y) - m^+| \leq Cr^{\frac{1}{2}} \omega_1(x, r)^{\frac{1}{2}} \leq Cr^{\frac{1}{2}} \omega_p(x, r)^{\frac{1}{2}}$$

for all $y \in B_1$. Similarly we have

$$|\tilde{u}(y) - m^-| \leq Cr^{\frac{1}{2}} \omega_1(x, r)^{\frac{1}{2}} \leq Cr^{\frac{1}{2}} \omega_p(x, r)^{\frac{1}{2}}$$

for all $y \in B_2$. Hence combining these inequalities and (3.13), we get (3.17) if ε is small enough. Set $S = (P \cap B(x, \frac{r}{20})) \backslash \Pi_P(K \cap B(x, r))$ where Π_P denotes the orthogonal projection onto P. The inequality (3.16) will follow for some t a little larger than $\frac{r}{20}$ if we show that

$$H^{n-1}(S) \leq C\varepsilon^{\frac{1}{2}} r^{n-1}. \tag{3.18}$$

Indeed if $t^2 = (\frac{r}{20})^2 + (\beta r)^2$, we have

$$H^{n-1}(P \cap B(x,t)) \le H^{n-1}\left(P \cap B\left(x, \frac{r}{20}\right)\right) + C\beta^2 r^{n-1}$$
$$\le H^{n-1}\left(P \cap B\left(x, \frac{r}{20}\right)\right) + C\varepsilon^{\frac{1}{2}} t^{n-1}$$

if $\varepsilon < 1$ is small enough, and

$$K \cap \Pi_P^{-1}\left(B\left(x, \frac{r}{20}\right) \cap P\right) \cap B(x,r) \subset K \cap B(x,t).$$

Hence we get that

$$H^{n-1}(P \cap B(x,t)) \le H^{n-1}(K \cap B(x,t)) + H^{n-1}(S) + C\varepsilon^{\frac{1}{2}} t^{n-1}$$
$$\le H^{n-1}(K \cap B(x,t)) + C\varepsilon^{\frac{1}{2}} t^{n-1}$$

which is exactly what we want. To prove (3.18) set $\partial_1 = B_1 \cap \partial B(x, \frac{r}{2})$ and $\partial_2 = B_2 \cap \partial B(x, \frac{r}{2})$. If $z \in S$ then the segment $[z_1, z_2]$ perpendicular to P and joining ∂_1 to ∂_2 through z does not meet K. Since u^* is C^1 outside of K we have that

$$|u^*(z_1) - u^*(z_2)| \le \int_0^1 |\nabla u^*(sz_1 + (1-s)z_2)||z_1 - z_2|\, ds.$$

Hence by (3.17) we have

$$Cr^{\frac{1}{2}} \le \int_0^1 |\nabla u^*(sz_1 + (1-s)z_2)||z_1 - z_2|\, ds,$$

and integrating over S we get

$$H^{n-1}(S) r^{\frac{1}{2}} \le C \int_{(\Pi_P^{-1}(S) \cap B(x, \frac{r}{2}))\setminus K} |\nabla u^*| \le C \int_{B(x,\frac{r}{2})\setminus K} |\nabla u^*|.$$

If we show that

$$\int_{B(x,\frac{r}{2})\setminus K} |\nabla u^*| \le C \int_{B(x,r)\setminus K} |\nabla u|, \tag{3.19}$$

we will have

$$H^{n-1}(S) \le Cr^{-\frac{1}{2}} \int_{B(x,r)\setminus K} |\nabla u| \le Cr^{n-1}\omega_1(x,r)^{\frac{1}{2}} \le C\varepsilon^{\frac{1}{2}} r^{n-1},$$

which is exactly (3.18). Thus let us prove (3.19). We have

$$\nabla u^* = \varphi \nabla u + (u - \tilde{u})\nabla\varphi + (1-\varphi)\nabla\tilde{u},$$

hence

$$\int_{B(x,\frac{r}{2})\setminus K} |\nabla u^*| \le \int_{B(x,r)\setminus K} |\nabla u| + Cr^{-1}\int_D |u-\tilde{u}| + C\int_D |\nabla\tilde{u}|$$

with $D = \{y \in B(x,\frac{r}{2}) : \operatorname{dist}(y,P) \ge \frac{r}{10}\}$. For $y \in D$ we have

$$\begin{aligned}|u(y) - \tilde{u}(y)| &\le \int_{B(y,\frac{r}{20})} \eta(y-w)|u(y) - u(w)|\,dw\\ &\le Cr^{-n}\int_{D'} |u(y) - u(w)|\,dw\end{aligned}$$

with $D' = \{y \in B(x,r) : \operatorname{dist}(y,P) \ge \frac{r}{20}\}$. Then if ε is small enough we can apply Poincaré's inequality to the C^1 function u on the convex domains $D' \cap U^{\pm}(r)$ and we get

$$\begin{aligned}r^{-1}\int_D |u-\tilde{u}| &\le Cr^{-n}r^{-1}\int_D\int_{D'} |u(y) - u(w)|\,dw\,dy\\ &\le C\int_{B(x,r)\setminus K} |\nabla u|.\end{aligned}$$

It remains to estimate $|\nabla\tilde{u}|$. For $y \in D$ we have

$$|\nabla\tilde{u}(y)| = \left|\int_{B(y,\frac{r}{20})} \eta(y-w)\nabla u(w)\,dw\right| \le Cr^{-n}\int_{B(x,r)\setminus K} |\nabla u|,$$

hence

$$\int_D |\nabla\tilde{u}| \le C\int_{B(x,r)\setminus K} |\nabla u|,$$

and finally we get (3.19). This achieves the proof of Lemma 3.15. □

Next we claim that if (3.16) holds then $t^{1-n}\int_{P(\eta)\setminus K} |\nabla u|^2$ is as small as we want provided η is small enough (and ε even smaller), where

$$P(\eta) = \{y \in \overline{B}(x,t) : \operatorname{dist}(y,P) \le \eta t\}.$$

LEMMA 3.20. *For each $\tau > 0$, there exist $\eta > 0$, $\varepsilon > 0$ and $t_0 \in (0,1)$ such that if $(x,t) \in \Delta(\beta)$ with hyperplane P, $t \le t_0$, $\beta + \omega_p(x,t) \le \varepsilon$ and if (3.16) holds, then*

$$t^{1-n}\int_{P(\eta)\setminus K} |\nabla u|^2 \le \tau. \tag{3.21}$$

PROOF. We want to replace $K \cap B(x,t)$ by the union of a thin torus of size a little larger than ηt near $\partial B(x,t)$ and $P \cap B(x,t)$ and build a new function $\tilde{u} = u(\psi)$, where ψ is a piecewise C^1-diffeomorphism whose image is contained in the domain of u and also not meeting $P(\eta)$. And we may choose ψ such that $D\psi - \mathrm{Id}$ is small. Thus let $\tau > 0$ be fixed. Let $\eta < \frac{1}{100}$ and $\varepsilon > 0$ be two positive constants to be chosen small later and $(x,t) \in \Delta(\beta)$ with hyperplane P be as in the statement. If ε is small enough (depending on η but we will first fix η and then ε) then $K \cap \overline{B}(x,t) \subset P(\eta)$. Set

$$T = \{y \in \mathbb{R}^n : \operatorname{dist}(y, P \cap \partial B(x,t)) < 20\eta t\}.$$

We define a new singularity set $\tilde{K}$ by

$$\tilde{K} = (K \backslash B(x,t)) \cup ((P \cap B(x,t)) \backslash T) \cup \partial(B(x,t) \cap T).$$

We have

$$\begin{aligned} H^{n-1}(\tilde{K}) &\leq H^{n-1}(K) - H^{n-1}(K \cap B(x,t)) + H^{n-1}(P \cap B(x,t)) + C\eta t^{n-1} \\ &\leq H^{n-1}(K) + C(\varepsilon^{\frac{1}{2}} + \eta)t^{n-1} \end{aligned}$$

by (3.16) and the definition of T. Thus if η and ε are small enough, we get

$$H^{n-1}(\tilde{K}) \leq H^{n-1}(K) + \frac{\tau}{2} t^{n-1}. \tag{3.22}$$

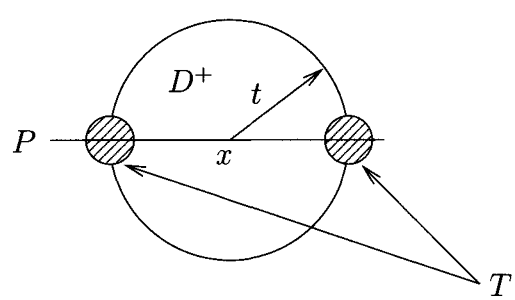

Fig. 1. The Domain D^+.

We now want to construct the function $\tilde{u}$. To simplify the notations we may identify $\mathbb{R}^n$ with $\mathbb{R}^{n-1} \times \mathbb{R}$ and denote by $y = (y', y_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ a generic point in $\mathbb{R}^n$. We may assume that $x = 0$ and $P = \{y = (y', y_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : y_n = 0\}$. Let D^+ denote the connected component of $B(x,t) \backslash (\overline{T} \cup P)$ that lies above P (i.e. $y_n > 0$ for each $y = (y', y_n) \in D^+$). See Figure 1. We will define $\tilde{u}$ on D^+ by

$$\tilde{u}(y) = u(\psi(y))$$

where ψ is a C^1 diffeomorphism from D^+ onto a subset $\psi(D^+)$ of $B(x,t)$ contained in the region

$$V^+ = \{y = (y', y_n) \in B(x,t) : y_n > \eta t\}.$$

This mapping will be continuous up to the boundary ∂D^+ and we will have

$$\psi(y) = y \text{ on a neighborhood of } \partial B(x,t) \cap \partial D^+. \tag{3.23}$$

To define ψ, we first choose a non negative C^∞ function h defined on $\mathbb{R}^+$ such that

$$\begin{cases} h(u) = 1 & \text{for} \quad u \le \frac{1}{4}, \\ h(u) = 0 & \text{for} \quad u \ge \frac{1}{2}, \\ 0 \le h \le 1, \end{cases}$$

and $\|h'\|_\infty \le 4$. We also consider a non negative C^∞ function φ defined on $\mathbb{R}^+$ such that

$$\begin{cases} \varphi(s) = 0 & \text{for} \quad s \le 5\eta t, \\ \varphi(s) = \eta t & \text{for} \quad s \ge 10\eta t, \\ 0 \le \varphi \le \eta t, \end{cases}$$

and $\|\varphi'\|_\infty \le 5^{-1}$. We first define $\psi = (\psi_1, \psi_2)$ on

$$W^+ = \left\{ y = (y', y_n) \in B(x,t) : \ y_n > 0 \text{ and } y_n < \frac{t - |y'|}{2} \right\}$$

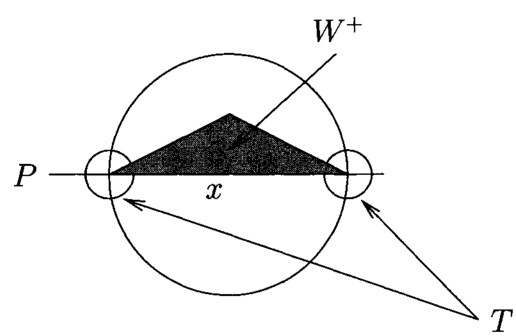

Fig. 2. The Domain W^+.

(see Figure 2) by

$$\psi_1(y', y_n) = y'$$

and

$$\psi_2(y', y_n) = y_n + \varphi(t - |y'|)h\left(\frac{y_n}{t - |y'|}\right).$$

On $D^+ \backslash W^+$ we set $\psi(y) = y$. Clearly (3.23) holds. If $y \in D^+$ with $y_n \le \eta t$, we have $y_n \le \frac{t-|y'|}{4}$ otherwise y would be in T, and $t - |y'| \ge 10\eta t$ for the same reason, so that $\psi_2(y', y_n) > \eta t$ and $\psi(D^+) \subset V^+$ as announced before. The mapping ψ is a continuous and piecewise C^1 function on D^+. Let us estimate its derivatives. By definition of ψ we only need to worry about ψ_2. For $y \in W^+$ set $y' = (y_1, \dots, y_{n-1})$. For all $1 \le i \le n-1$, we have

$$\frac{\partial \psi_2}{\partial y_i} = -\frac{y_i}{|y'|}\varphi'(t-|y'|)h\left(\frac{y_n}{t-|y'|}\right) + \frac{y_n}{(t-|y'|)^2}\frac{y_i}{|y'|}\varphi(t-|y'|)h'\left(\frac{y_n}{t-|y'|}\right).$$

The first term can only exist when $y_n < \frac{t-|y'|}{2}$ because of h and $5\eta t < t-|y'| < 10\eta t$ because of φ'. And so it vanishes on $W^+ \backslash T$. The second term can only exist when $\frac{t-|y'|}{4} < y_n < \frac{t-|y'|}{2}$ because of h' and $t-|y'| > 5\eta t$ because of φ, and is dominated by $2(t-|y'|)^{-1}\eta t$. Thus for all $y \in D^+$ and all $1 \le i \le n-1$, we have

$$\left|\frac{\partial \psi_2}{\partial y_i}(y', y_n)\right| \le 2(t-|y'|)^{-1}\eta t \chi_{(\frac{1}{4},\frac{1}{2})}\left(\frac{y_n}{t-|y'|}\right)\chi_{(5\eta t,1)}(t-|y'|) \le \frac{2}{5}. \tag{3.24}$$

Similarly we have

$$\frac{\partial \psi_2}{\partial y_n}(y', y_n) - 1 = (t-|y'|)^{-1}\varphi(t-|y'|)h'\left(\frac{y_n}{t-|y'|}\right)$$

and the same computations give

$$\left|\frac{\partial \psi_2}{\partial y_n}(y', y_n) - 1\right| \le 4(t-|y'|)^{-1}\eta t \chi_{(\frac{1}{4},\frac{1}{2})}\left(\frac{y_n}{t-|y'|}\right)\chi_{(5\eta t,1)}(t-|y'|) \le \frac{4}{5}. \tag{3.25}$$

In particular, when y' is fixed, ψ_2 is a strictly increasing function of y_n and so ψ is one to one from D^+ onto $\psi(D^+)$ and is in fact a C^1-diffeomorphism on D^+. To estimate $\int_{W^+} |\nabla \tilde{u}|^2$, we cut W^+ into pieces

$$E_k = \{y \in W^+ : \ 2^{k-1}(20\eta t) < \operatorname{dist}(y, P \cap \partial B(x,t)) \le 2^k(20\eta t)\},$$

$k = 1, 2, \dots, k(\eta)$ where $2^{k(\eta)} = C\eta^{-1}$. By (3.24) and (3.25) we have

$$\int_{E_k} |\nabla \tilde{u}|^2 \le \left(1 + \frac{C}{2^k}\right)\int_{\psi(E_k)} |\nabla u|^2$$

for some absolute constant C. We also have that

$$\int_{\psi(E_k)} |\nabla u|^2 \le C 2^k \eta t^{n-1}.$$

This can be viewed using a simple truncation argument. We set $v = u$ outside of $\psi(E_k) \cup K$, $v = 0$ inside $\psi(E_k)$ and $K' = (K \backslash \psi(E_k)) \cup \partial\psi(E_k)$. Then

(v, K') is a competitor which satisfies (1.1), such that $\|v\|_\infty \le \|u\|_\infty$, and we get from the minimality of (u, K) that

$$\int_{\psi(E_k)} |\nabla u|^2 \le H^{n-1}(\partial\psi(E_k)) + C|\psi(E_k)|.$$

We have $H^{n-1}(\partial\psi(E_k)) \le CH^{n-1}(\partial E_k) \le C2^k\eta t^{n-1}$ and $|\psi(E_k)| \le C|E_k| \le C2^k\eta t^n \le C2^k\eta t^{n-1}$ (because $t \le 1$) and we get the required estimate. Then we have

$$\int_{E_k} |\nabla\tilde{u}|^2 \le \int_{\psi(E_k)} |\nabla u|^2 + C\eta t^{n-1}.$$

A summation over k gives

$$\int_{W^+} |\nabla\tilde{u}|^2 \le \int_{\psi(W^+)} |\nabla u|^2 + C\eta t^{n-1}\ln\frac{C}{\eta}$$

and thus

$$\int_{D^+} |\nabla\tilde{u}|^2 \le \int_{V^+} |\nabla u|^2 + C\eta t^{n-1}\ln\frac{C}{\eta}.$$

Let us now define $\tilde{u}$ on the whole $\Omega\backslash\tilde{K}$. Let D^- be the image of D^+ by the orthogonal symmetry σ with respect to P. We define $\tilde{u}$ on D^- in a similar way. More precisely, we set $\tilde{u}(y) = u(\psi^-(y))$ for $y \in D^-$, where $\psi^-(y) = \sigma(\psi(\sigma(y)))$. The restriction of $\tilde{u}$ to D^- has the same sort of properties as its restriction to D^+. In particular, $\tilde{u}$ coincide with u on a neighborhood $\partial B(x,t)\backslash\overline{T}$, and

$$\int_{D^-} |\nabla\tilde{u}|^2 \le \int_{\sigma(V^+)} |\nabla u|^2 + C\eta t^{n-1}\ln\frac{C}{\eta}.$$

And hence, we have

$$\int_{D^+\cup D^-} |\nabla\tilde{u}|^2 \le \int_{B(x,t)\backslash P(\eta)} |\nabla u|^2 + C\eta t^{n-1}\ln\frac{C}{\eta}. \tag{3.26}$$

On the complement of $B(x,t)$, we set $\tilde{u}(y) = u(y)$ and on $T \cap B(x,t)$, we set $\tilde{u}(y) = 0$. The pair $(\tilde{u}, \tilde{K})$ satisfies the conditions (1.1) and is a good candidate for the functionnal J. Because $\|\tilde{u}\|_\infty \le \|u\|_\infty$, we have

$$\int_{\Omega\backslash\tilde{K}} |\tilde{u} - g|^2 \le \int_{\Omega\backslash K} |u - g|^2 + Ct^n. \tag{3.27}$$

And by (3.22), (3.26) and (3.27), we have

$$J(\tilde{u}, \tilde{K}) \le J(u, K) - \int_{P(\eta)\backslash K} |\nabla u|^2 + \frac{\tau}{2}t^{n-1} + C\eta t^{n-1}\ln\frac{C}{\eta} + C't^n.$$

If t and η are small enough (depending on τ), we then deduce (3.21) from the minimality of (u, K). □

Now, using the size estimate (2.2) and (3.21), we will find a ball $B(y, \frac{\eta t}{4})$ centered on K such that

$$(\eta t)^{1-n} \int_{B(y, \frac{\eta t}{4})} |\nabla u|^2 \leq C\tau.$$

We keep the same notations as above and (x, t) is as in the statement of Lemma 3.20. If ε is small enough then $B(y, \frac{\eta t}{4}) \subset P(\eta)$ for all $y \in K \cap B(x, \frac{t}{2})$. We consider a maximal family $\mathcal{A}$ of points in $K \cap B(x, \frac{t}{2})$ such that $|y - z| \geq \frac{\eta t}{2}$ for all y, $z \in \mathcal{A}$, $y \neq z$. Then there exists $y \in \mathcal{A}$ such that the previous inequality holds. Otherwise, because the balls $B(y, \frac{\eta t}{4})$ are disjoints, we would have

$$\int_{P(\eta)\setminus K} |\nabla u|^2 \geq \sum_{y \in \mathcal{A}} \int_{B(y, \frac{\eta t}{4})\setminus K} |\nabla u|^2.$$

Then the size estimate (2.2) and the fact that the balls $B(y, \frac{\eta t}{2})$, $y \in \mathcal{A}$, cover $K \cap B(x, \frac{t}{2})$ would give

$$\begin{aligned}\int_{P(\eta)\setminus K} |\nabla u|^2 &\geq C\tau \sum_{y \in \mathcal{A}} (\eta t)^{n-1} \\ &\geq CC_0^{-1} 2^{n-1} \tau \sum_{y \in \mathcal{A}} H^{n-1}\left(K \cap B\left(y, \frac{\eta t}{2}\right)\right) \\ &\geq CC_0^{-1} 2^{n-1} \tau H^{n-1}\left(K \cap B\left(x, \frac{t}{2}\right)\right) \\ &\geq CC_0^{-2} \tau t^{n-1},\end{aligned}$$

which contradicts (3.21) if C is large enough. Let us resume all these Lemmas in the following

LEMMA 3.28. *For each $\tau > 0$, there exist $\varepsilon > 0$, $r_0 \in (0, 1)$ and $C > 1$ such that if $(x, r) \in \Delta(\beta)$ with $r \leq r_0$ and $\beta + \omega_p(x, r) \leq \varepsilon$, then there are $y \in K \cap B(x, r)$ and $s \geq C^{-1} r$ such that $B(y, s) \subset B(x, r)$ and*

$$\omega_2(y, s) \leq \tau.$$

PROOF. Indeed, if ε and r_0 are small enough and if $(x, r) \in \Delta(\beta)$ with hyperplane P is as in the statement, Lemma 3.15 gives us $t \in (\frac{r}{20}, r)$ such that (3.16) holds. Because $(x, t) \in \Delta(20\beta)$ with hyperplane P and $\omega_p(x, t) \leq C\omega_p(x, r)$, we may apply Lemma 3.20 and the argument following it to get the required conclusion. □

The proof of Lemma 3.1 is now almost finished. Let $\tau > 0$ be given and $(x,t) \in \Delta$ with $t \leq t_0$, for some t_0 to be fixed small later. First by Corollary 2.8 we may find for each $\varepsilon > 0$ a pair $(z,r) \in \Delta$ with $B(z,r) \subset B(x,t)$, $r \geq C_\varepsilon^{-1} t$ and $\beta(z,r) + \omega_p(z,r) \leq \varepsilon$. Then if ε and t_0 are small enough, we may apply Lemma 3.28 to $(z,r) \in \Delta(2\beta(z,r))$. This gives us $y \in K$ and $s \geq C_\tau^{-1} r$ such that $B(y,s) \subset B(z,r)$ and $\omega_2(y,s) \leq \frac{\tau}{2}$. Finally we note that $\beta(y,s) \leq C_\tau \beta(z,r)$ and if ε is small enough, we also have that $\beta(y,s) \leq \frac{\tau}{2}$. And Lemma 3.1 follows.

REMARK 3.29. A consequence of Theorem 1.3 is that the Hausdorff dimension of the set Σ of points in K around which K is not a $C^{1,\alpha}$-hypersurface (see a few lines below for a precise definition) is strictly less than $n-1$. Note that it was only known before that $H^{n-1}(\Sigma) = 0$ (see [3]). More precisely we set for all $x \in K$,

$$\rho(x) = \sup\{r > 0 : \ K \cap B(x,r) \text{ is a } C^{1,\alpha}\text{-hypersurface}\}$$

and we define

$$\Sigma = \{x \in K : \ \rho(x) = 0\}.$$

It follows from Theorem 1.3 and from [7, pages 809-810] that $H^d(\Sigma) = 0$ for some $d < n-1$. Since, once we have Theorem 1.3, the arguments to prove that $H^d(\Sigma \cap B(x,r)) = 0$ for all $x \in K$ and $r \leq 1$ whenever d is close enough to $n-1$ are not really new, we just sketch them here. The key point is the existence for every Ahlfors-regular set of a decomposition into "dyadic cubes" (see [6]). Using this decomposition, Theorem 3.1 and arguing as in [7, pages 809-810], it can be shown that for all $k \in \mathbb{N}^*$, $\Sigma \cap B(x,r)$ is contained in a set E_k. This set E_k is the union of cubes Q of diameter comparable to 2^{-kN} (where N is a fixed constant) and such that $H^{n-1}(Q)$ is comparable to $2^{-k(n-1)N}$. Furthermore, we have

$$H^{n-1}(E_k) \leq C(1-\eta)^k r^{n-1},$$

where η is a fixed (small) constant (we tried here to keep the same notations as in [7, pages 809-810] and the set E_k has nothing to do with the set E_k of the proof of Lemma 3.20). Hence we can cover $\Sigma \cap B(x,r)$ with at most $C(1-\eta)^k 2^{k(n-1)N}$ sets of diameter less than $C2^{-kN}$. Thus

$$H^d_{C2^{-kN}}(\Sigma \cap B(x,r)) \leq C(1-\eta)^k 2^{k(n-1)N} 2^{-kdN}$$

which tends to 0 when $k \to +\infty$ if d is close enough to $n-1$. □

REMARK 3.30. After the present article had been accepted for publication, the author was told that an analogous result was obtained independently by F. Maddalena and S. Solimini ([12]).

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