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Biharmonic curves in 3-dimensional Sasakian space forms

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Abstract We show that every proper biharmonic curve in a 3-dimensional Sasakian space form of constant holomorphic sectional curvature H is a helix (both of whose geodesic curvature and geodesic torsion are constants). In particular, if $H \neq 1$, then it is a slant helix, that is, a helix which makes constant angle α with the Reeb vector field with the property $\kappa^2 + \tau^2 = 1 + (H-1)\sin^2\alpha$. Moreover, we construct parametric equations of proper biharmonic herices in Bianchi–Cartan–Vranceanu model spaces of a Sasakian space form.

Keywords Harmonic and biharmonic curves · Sasakian space forms · Bianchi–Cartan–Vranceanu model spaces

Mathematics Subject Classification (2000) 53C25 · 53C43 · 54D10

1 Introduction

Let M be a Riemannian 3-manifold with Levi-Civita connection ∇ and Riemannian curvature R. A curve γ parametrized by the arc-length in M is said to be *biharmonic* if it satisfies $\nabla^3_T T + R(\kappa N, T)T = 0$. Here T, N and κ denote the tangent vector field,

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principal normal vector field and geodesic curvature, respectively. Obviously geodesics are biharmonic. Non-geodesic biharmonic curves are called *proper biharmonic curves*.

Chen and Ishikawa [13] showed nonexistence of proper biharmonic curves in Euclidean 3-space $\mathbb{E}^3$. (Dimitrić [14] obtained the same result independently.) Moreover they classified all proper biharmonic curves in Minkowski 3-space $\mathbb{E}^3_1$. See also [17].

In Caddeo et al. [7] studied biharmonic curves in the unit 3-sphere. More precisely, they showed that proper biharmonic curves in S^3 are circles of geodesic curvature 1 or helices which are geodesics in the Clifford minimal torus.

On the other hand, there are few results on biharmonic curves in arbitrary Riemannian manifolds.

The Heisenberg group $\mathbb{H}_3$ is a Cartesian 3-space $\mathbb{R}^3(x, y, z)$ furnished with group structure

$$(x, y, z) \cdot (x', y', z') = \left(x + x', y + y', z + z' + \frac{1}{2}(xy' - x'y)\right).$$

Let us denote by η the canonical contact form of the Heisenberg group;

$$\eta = \mathrm{d}z + \frac{1}{2}(y\mathrm{d}x - x\mathrm{d}y).$$

Then $\mathrm{d}x^2 + \mathrm{d}y^2 + \eta \otimes \eta$ is a naturally reductive left-invariant Riemannian metric on $\mathbb{H}_3$.

In recent paper Caddeo et al. [10], studied biharmonic curves in $\mathbb{H}_3$. Caddeo–Oniciuc–Piu actually showed that biharmonic curves in $\mathbb{H}_3$ are helices, that is curves with constant geodesic curvature κ and geodesic torsion τ. They gave an explicit formula for proper biharmonic helices in $\mathbb{H}_3$.

Moreover they proved the nonexistence of proper biharmonic Legendre curves in $\mathbb{H}_3$. This nonexistence result has been obtained by the second named author independently [18]. In [18], it is shown that there are proper biharmonic Legendre curves in Sasakian space form of constant φ-holomorphic sectional curvature greater than 1.

As is well known the unit 3-sphere S^3 admits a structure of Sasakian space form compatible to the metric of constant curvature 1. The Heisenberg group is homothetic to the Sasakian space form $\mathbb{R}^3(-3)$ of constant φ-holomorphic sectional curvature -3. It is well known that complete and simply connected 3-dimensional Sasakian space forms are realized as the following unimodular Lie groups together with left invariant Sasakian structures – special unitary group SU(2), the Heisenberg group $\mathbb{H}_3$ or the universal covering group of special linear group $\mathrm{SL}_2\mathbb{R}$.

These facts motivate us to generalize results in [10] to general Sasakian space forms. In this paper we study biharmonic curves in unimodular homogeneous contact Riemannian 3-manifolds. We shall deduce the biharmionic equations for curves in general 3-dimensional unimodular Lie groups with left invariant contact Riemannian structures.

In Sect. 3, we shall show that every proper biharmonic curve in a 3-dimensional Sasakian space form of constant holomorphic sectional curvature H is a helix (with constant geodesic curvature and geodesic torsion). In particular, if $H \neq 1$, then it is a *slant helix*, that is, a helix which makes a constant angle α with the Reeb vector field such that $\kappa^2 + \tau^2 = 1 + (H - 1)\sin^2\alpha$.

Furthermore, in Sect. 4 we shall obtain explicitly parametric equations of proper biharmonic helices in Bianchi–Cartan–Vranceanu model spaces of Sasakian space forms.

2 Preliminaries

2.1

Let (N,h) and (M,g) be Riemannian manifolds. Denote by R^N and R the Riemannian curvature tensors of N and M, respectively. We use the sign convention:

$$R^N(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}, \quad X, Y \in \Gamma(TN).$$

For a smooth map $\phi : N \to M$, the Levi-Civita connection ∇ of (N,h) induces a conncetion ∇^ϕ on the pull-back bundle $\phi^*TM = \cup_{p\in N} T_{\phi(p)}M$.

The section $\mathcal{T}(\phi) := \mathrm{tr}\, \nabla^\phi d\phi$ is called the *tension field* of ϕ. A map ϕ is said to be harmonic if its tension field vanishes identically.

Definition 2.1 *A smooth map $\phi : N \to M$ is said to be biharmonic if it is a critical point of the bienergy functional*:

$$\mathcal{E}_2(\phi) = \int_N \frac{1}{2}|\mathcal{T}(\phi)|^2 \mathrm{d}v_h.$$

The Euler–Lagrange equation of the bienergy is given by $\mathcal{T}_2(\phi) = 0$. Here the section $\mathcal{T}_2(\phi)$ is defined by

$$\mathcal{T}_2(\phi) = -\triangle_\phi \mathcal{T}(\phi) + \mathrm{tr}\, R(\mathcal{T}(\phi), \mathrm{d}\phi)\mathrm{d}\phi$$

and called the *bitension field* of ϕ. The operator $\triangle_\phi$ is the *rough Laplacian* acting on $\Gamma(\phi^*TM)$ defined by

$$\triangle_\phi := -\sum_{i=1}^{n} \left(\nabla^\phi_{e_i} \nabla^\phi_{e_i} - \nabla^\phi_{\nabla^N_{e_i} e_i} \right),$$

where $\{e_i\}_{i=1}^n$ is a local orthonormal frame field of N. Obviously, every harmonic map is biharmonic. Non-harmonic biharmonic maps are called *proper biharmonic maps*.

In particular, if the target manifold M is the Euclidean space $\mathbb{E}^m$, the biharmonic equation of a map $\phi : N \to \mathbb{E}^m$ is

$$\triangle_h \triangle_h \phi = 0,$$

where $\triangle_h$ is the Laplace–Beltrami operator of (N,h).

2.2

For later use, here, we recall the biharmonic equation for curves. Let $\gamma(s) : I \to M$ be a curve defined on an open interval I and parametrized by arc-length. Then the bitension field is given by

$$\mathcal{T}_2(\gamma) = \nabla_{\gamma'}\nabla_{\gamma'}\nabla_{\gamma'}\gamma' + R(\nabla_{\gamma'}\gamma', \gamma')\gamma'.$$

Now let us consider biharmonicity of curves in 3-dimensional Riemannian manifolds. Let (T, N, B) be the Frenet frame field along γ. Then the Frenet frame satisfies the following Frenet–Serret equations:

$$\begin{cases} \nabla_T T = \kappa N, \\ \nabla_T N = -\kappa T + \tau B, \\ \nabla_T B = -\tau N, \end{cases} \tag{2.1}$$

where $\kappa = |\mathcal{T}(\gamma)| = |\nabla_T T|$ is the *geodesic curvature* of γ and τ its *geodesic torsion*.

A *helix* is a curve with constant geodesic curvature and geodesic torsion. In particular, curves with constant nonzero geodesic curvature and zero geodesic torsion are called (*Riemannian*) *circles*. Note that geodesics are regarded as helices with zero geodesics curvature and torsion.

By using (2.1), we find that a curve $\gamma : I \to M$ is biharmonic if and only if

$$\begin{cases} \kappa\kappa' = 0, \\ R(\kappa N, T)T + (\kappa'' - \kappa^3 - \kappa\tau^2)N + (2\tau\kappa' + \kappa\tau')B = 0. \end{cases} \tag{2.2}$$

In particular, if M^3 is of constant curvature c, then the Riemannian curvature R is given explicitly by

$$R(X, Y)Z = c\{g(Y, Z)X - g(Z, X)Y\},$$

which gives $R(\nabla_T T, T)T = c\kappa N$. This together with (2.2) implies that there are no proper biharmonic curves in Euclidean 3-space $\mathbb{E}^3$ (cf. [13,14]) or in hyperbolic 3-space H^3 (cf. [8]).

On the other hand, Caddeo–Montaldo–Oniciuc classified proper biharmonic curves in S^3.

Theorem 2.1 ([7]) *Let γ be a proper biharmonic curve in S^3. Then $\kappa \leq 1$ and have two cases*:

- $\kappa = 1$ *and γ is a circle of radius* $1/\sqrt{2}$;
- $0 < \kappa < 1$ *and γ is a helix, which is a geodesic in the Clifford minimal torus* $S^1(1/\sqrt{2}) \times S^1(1/\sqrt{2})$.

In the former case, γ is congruent to

$$\frac{1}{\sqrt{2}}\left(\cos(\sqrt{2}s), \sin(\sqrt{2}s), c_1, c_2\right), \ c_1^2 + c_2^2 = 1.$$

In the latter case, γ is congruent to

$$\frac{1}{\sqrt{2}}\Big(\cos(as), \sin(as), \cos(bs), \sin(bs)\Big).$$

Remark 2.1 The biharmonicity condition $\mathcal{T}_2(\phi) = 0$ makes sense for maps between semi-Riemannian manifolds. Although, the Euclidean 3-space $\mathbb{E}^3$ does not admit a proper biharmonic curve, Minkowski 3-space $\mathbb{E}^3_1$ contains proper biharmonic curves. See Chen-Ishikawa's paper [13]. In [17], it is pointed out that proper biharmonic curves in Minkowski 3-space are helices with property $\kappa^2 - \tau^2 = 0$.

For general informations on biharmonic maps, we refer to [9] and references therein.

3 Biharmonic curves in contact 3-manifolds

3.1

A 3-dimensional smooth manifold M^3 is called a *contact manifold*, if it admits a global differential 1-form η such that $\eta \wedge (\mathrm{d}\eta) \neq 0$ everywhere on M. This form η is called the *contact form* of M. Given a contact form η, we have a unique vector field ξ, which is called the *characteristic vector field*, satisfying $\eta(\xi) = 1$ and $\mathrm{d}\eta(\xi, X) = 0$ for any vector field X.

A Riemannian metric g is an associate metric to a contact structure η if there exists a tensor field φ of type (1.1) satisfying

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(X) = g(X, \xi), \quad \mathrm{d}\eta(X, Y) = g(X, \varphi Y).$$

We refer to (η, g) or (φ, ξ, η, g) as a contact metric structure.

For a 3-dimensional contact metric manifold M^3, one may define naturally an almost complex structure J on $M \times \mathbb{R}$ by

$$J\left(X, f\frac{\mathrm{d}}{\mathrm{d}t}\right) = \left(\varphi X - f\xi, \eta(X)\frac{\mathrm{d}}{\mathrm{d}t}\right),$$

where X is a vector field tangent to M, t the coordinate of $\mathbb{R}$ and f a function on $M \times \mathbb{R}$. If the almost complex structure J is integrable, then the contact metric manifold M is said to be a *normal contact metric manifold* or a *Sasakian manifold*.

A plane section Π_x at a point x of a contact Riemannian 3-manifold is called a *holomorphic plane* if it is invariant under φ_x. The sectional curvature function of holomorphic planes is called the *holomorphic sectional curvature*. In particular, Sasakian 3-manifolds of constant holomorphic sectional curvature are called 3-dimensional *Sasakian space forms*.

Simply connected and complete 3-dimensional Sasakian space forms are classified as follows:

Proposition 3.1 ([6]) *Simply connected and complete 3-dimensional Sasakian space forms $\mathcal{M}^3(H)$ of constant holomorphic sectional curvature H are isomorphic to one of the following unimodular Lie groups with left invariant Sasakian structures*: *the special unitary group $SU(2)$ for $H > -3$, the Heisenberg group for $H = -3$, or the universal covering group $\widetilde{SL}(2, \mathbf{R})$ of the special linear group $SL(2, \mathbf{R})$ for $H < -3$. The Sasakian space form $\mathcal{M}^3(1)$ is the unit 3-sphere S^3 with the canonical Sasakian structure.*

As is well known, the maximum dimension of the isometry group of Riemannian 3-manifold is 6. Moreover Riemannian 3-manifolds with 6-dimensional isometry group are of constant curvature. Since, there are no Riemannian 3-manifolds with 5-dimensional isometry group (see e.g., [20, p. 47, Theorem 3.2]), it is natural to study biharmonic curves in Riemannian 3-manifolds with 4-dimensional isometry group.

The 3-dimensional Sasakian space forms are naturally reductive homogeneous spaces and have four-dimensional isometry group.

On these reasons, we shall study biharmonic curves in homogeneous contact Riemannian 3-manifolds, especially, Sasakian space forms. Hereafter we investigate homogeneous contact Riemannian 3-manifolds which are unimodular (as Lie groups). Our general reference is Perrone's paper [24]. For general theory of contact Riemannian geometry, especially Sasakian geometry, we refer to [5].

3.2

Let M be a 3-dimensional unimodular Lie group with a left-invariant Riemannian metric $g = \langle\cdot,\cdot\rangle$. Then M admits its compatible left-invariant contact Riemannian structure if and only if there exists an orthonormal basis $\{e_1, e_2, e_3\}$ of the Lie algebra $\mathfrak{m}$ such that (cf. [24]):

$$[e_1, e_2] = 2e_3,\ [e_2, e_3] = c_2 e_1,\ [e_3, e_1] = c_3 e_2.$$

The Reeb vector filed ξ is obtained by left translation of e_3. The *contact distribution* $\mathfrak{D}$ is spanned by e_1 and e_2.

By the Koszul formula, one can calculate the Levi-Civita connection ∇ in terms of the basis $\{e_1, e_2, e_3 = \xi\}$ as follows:

$$\begin{aligned} \nabla_{e_1} e_2 &= \frac{1}{2}(c_3 - c_2 + 2)e_3, \quad \nabla_{e_1} e_3 = -\frac{1}{2}(c_3 - c_2 + 2)e_2, \\ \nabla_{e_2} e_1 &= \frac{1}{2}(c_3 - c_2 - 2)e_3, \quad \nabla_{e_2} e_3 = -\frac{1}{2}(c_3 - c_2 - 2)e_1, \\ \nabla_{e_3} e_1 &= \frac{1}{2}(c_3 + c_2 - 2)e_2, \quad \nabla_{e_3} e_2 = -\frac{1}{2}(c_3 + c_2 - 2)e_1, \end{aligned} \tag{3.1}$$

all others are zero.

In particular, M is Sasakian if and only if $c_2 = c_3 = c$, and it is of constant φ-holomorphic sectional curvature $H = -3 + 2c$ (cf. [24]). The Riemannian curvature R is given by

$$\begin{aligned} R(e_1, e_2)e_2 &= \left\{\frac{1}{4}(c_3 - c_2)^2 - 3 + c_3 + c_2\right\}e_1, \\ R(e_1, e_3)e_3 &= \left\{-\frac{1}{4}(c_3 - c_2)^2 - \frac{1}{2}({c_3}^2 - {c_2}^2) + 1 - c_2 + c_3\right\}e_1, \\ R(e_2, e_1)e_1 &= \left\{\frac{1}{4}(c_3 - c_2)^2 - 3 + c_3 + c_2\right\}e_2, \\ R(e_2, e_3)e_3 &= \left\{\frac{1}{4}(c_3 + c_2)^2 - {c_2}^2 + 1 + c_2 - c_3\right\}e_2, \\ R(e_3, e_1)e_1 &= \left\{-\frac{1}{4}(c_3 - c_2)^2 - \frac{1}{2}({c_3}^2 - {c_2}^2) + 1 - c_2 + c_3\right\}e_3, \\ R(e_3, e_2)e_2 &= \left\{\frac{1}{4}(c_3 + c_2)^2 - {c_2}^2 + 1 + c_2 - c_3\right\}e_3. \end{aligned} \tag{3.2}$$

3.3

Now we study biharmonic curves in homogeneous contact Riemannian 3-manifold M. Let $\gamma : I \to M$ be a curve parametrized by arc-length with the Frenet frame (T, N, B). Expand T, N, B as $T = T_1 e_1 + T_2 e_2 + T_3 e_3$, $N = N_1 e_1 + N_2 e_2 + N_3 e_3$, $B = B_1 e_1 + B_2 e_2 + B_3 e_3$ with respect to the basis $\{e_1, e_2, e_3 = \xi\}$. Since (T, N, B) is positively oriented,

$$B_1 = T_2 N_3 - T_3 N_2, \quad B_2 = T_3 N_1 - T_1 N_3, \quad B_3 = T_1 N_2 - T_2 N_1. \tag{3.3}$$

We call the angle function between T and ξ, the *contact angle* of γ. By definition, the contact angle $\alpha(s)$ is computed by the formula $\cos\alpha(s) = g(T,\xi)$. Curves with constant contact angle are called *slant curves*.

By using the curvature formula (3.2), we have

$$
\begin{aligned}
R(\kappa N, T)T &= \kappa R(N_1e_1 + N_2e_2 + N_3e_3, T_1e_1 + T_2e_2 + T_3e_3)(T_1e_1 + T_2e_2 + T_3e_3)\\
&= \kappa\Big[\Big\{{B_1}^2\Big(\frac{1}{4}(c_3+c_2)^2 - {c_2}^2 + 1 + c_2 - c_3\Big)\\
&\quad - {B_2}^2\Big(\frac{1}{4}(c_3-c_2)^2 + \frac{1}{2}({c_3}^2 - {c_2}^2) - 1 + c_2 - c_3\Big)\\
&\quad + {B_3}^2\Big(\frac{1}{4}(c_3-c_2)^2 - 3 + c_2 + c_3\Big)\Big\}N\\
&\quad \Big\{ - B_1N_1\Big(\frac{1}{4}(c_3+c_2)^2 - {c_2}^2 + 1 + c_2 - c_3\Big)\\
&\quad + B_2N_2\Big(\frac{1}{4}(c_3-c_2)^2 + \frac{1}{2}({c_3}^2 - {c_2}^2) - 1 + c_2 - c_3\Big)\\
&\quad - B_3N_3\Big(\frac{1}{4}(c_3-c_2)^2 - 3 + c_2 + c_3\Big)\Big\}B\Big].
\end{aligned} \tag{3.4}
$$

Here we have used the relations (3.3). Hence, from (2.2) and (3.4) the biharmonic equation for γ becomes:

$$
\begin{aligned}
\mathcal{T}_2(\gamma) &= \nabla^3{}_T T + R(\kappa N, T)T\\
&= (-3\kappa\kappa')T + \Big[(\kappa'' - \kappa^3 - \kappa\tau^2) + \kappa\Big\{{B_1}^2\Big(\frac{1}{4}(c_3+c_2)^2 - {c_2}^2 + 1 + c_2 - c_3\Big)\\
&\quad -{B_2}^2\Big(\frac{1}{4}(c_3-c_2)^2 + \frac{1}{2}({c_3}^2 - {c_2}^2) - 1 + c_2 - c_3\Big)\\
&\quad +{B_3}^2\Big(\frac{1}{4}(c_3-c_2)^2 - 3 + c_2 + c_3\Big)\Big\}\Big]N \qquad (3.5)\\
&\quad +\Big[(2\tau\kappa' + \kappa\tau') + \kappa\Big\{ - B_1N_1\Big(\frac{1}{4}(c_3+c_2)^2 - {c_2}^2 + 1 + c_2 - c_3\Big)\\
&\quad +B_2N_2\Big(\frac{1}{4}(c_3-c_2)^2 + \frac{1}{2}({c_3}^2 - {c_2}^2) - 1 + c_2 - c_3\Big)\\
&\quad -B_3N_3\Big(\frac{1}{4}(c_3-c_2)^2 - 3 + c_2 + c_3\Big)\Big\}\Big]B\\
&= 0.
\end{aligned}
$$

Thus we have

Theorem 3.1 *Let $\gamma : I \to M$ be a curve parametrized by arc-length in a 3-manifold unimodular Lie group M with a left-invariant contact Riemannian metric. Then γ is a* *proper biharmonic curve if and only if*

$$\begin{aligned}\kappa &= \text{constant} \neq 0,\\ \kappa^2+\tau^2 &= {B_1}^2\Big(\frac{1}{4}(c_3+c_2)^2-{c_2}^2+1+c_2-c_3\Big)\\ &\quad -{B_2}^2\Big(\frac{1}{4}(c_3-c_2)^2+\frac{1}{2}({c_3}^2-{c_2}^2)-1+c_2-c_3\Big)\\ &\quad +{B_3}^2\Big(\frac{1}{4}(c_3-c_2)^2-3+c_2+c_3\Big),\\ \tau' &= B_1N_1\Big(\frac{1}{4}(c_3+c_2)^2-{c_2}^2+1+c_2-c_3\Big)\\ &\quad -B_2N_2\Big(\frac{1}{4}(c_3-c_2)^2+\frac{1}{2}({c_3}^2-{c_2}^2)-1+c_2-c_3\Big)\\ &\quad +B_3N_3\Big(\frac{1}{4}(c_3-c_2)^2-3+c_2+c_3\Big).\end{aligned}$$

In particular, if M is Sasakian, then γ is a proper biharmonic curve if and only if

$$\begin{cases}\kappa = \text{constant} \neq 0\\ \kappa^2+\tau^2 = 1+2(c-2)\eta(B)^2\\ \tau' = 2(c-2)\eta(B)\eta(N),\end{cases} \tag{3.6}$$

where $c = c_2 = c_3$.

In addition, γ is a proper biharmonic helix in a Sasakian space form M if and only if the geodesic curvature κ and geodesic torsion τ satisfy the following conditions:

$$\begin{cases}\kappa^2+\tau^2 = 1+2(c-2)\eta(B)^2,\\ (c-2)\eta(B)\eta(N) = 0.\end{cases}$$

Since the condition $c = 2$ reflects the case of a space of constant curvature 1, we get the following result.

Proposition 3.2 *Let M be a 3D unimodular Lie group with a left-invariant Sasakian structure except the case of constant curvature 1 and let $\gamma : I \to M$ be a non-geodesic curve parametrized by arc-length whose geodesic curvature κ is constant. If $\eta(B)\eta(N) \neq 0$, then γ is never biharmonic.*

Proof This can be verified using an argument similar to [10, Proposition 3.3]. □

From Theorem 3.1 and Proposition 3.2, we have

Theorem 3.2 *Let M be a 3-dimensional Sasakian space form and let $\gamma : I \to M$ be a non-geodesic curve parametrized by arc-length.*

1. *If M is not of constant curvature 1, then γ is biharmonic if and only if γ is a helix with the property*:

$$\begin{cases}\kappa \neq 0,\\ \eta(B)\eta(N) = 0,\\ \kappa^2+\tau^2 = 1+2(c-2)\eta(B)^2.\end{cases} \tag{3.7}$$

2. *If there exist proper biharmonic helices such that $\eta(B)\eta(N) \neq 0$, then M is of constant curvature 1.*

The following result can be proved in a similar way to that of [10, Proposition 4.1].

Proposition 3.3 *Let M be a 3-dimensional Sasakian space form and let $\gamma : I \to M$ be a non-geodesic curve parametrized by arc length. If $\eta(B) = 0$, then $\tau^2 = 1$ and γ is not biharmonic.*

This implies the following (cf. [10, Remark 5.8]).

Corollary 3.1 *Let M be a 3-dimensional Sasakian space form except the case of constant curvature 1 and let $\gamma : I \to M$ be a non-geodesic biharmonic helix parametrized by arc-length. Then*

$$\begin{cases} \eta(B) = \text{constant} \neq 0, \\ \eta(N) = 0, \\ \kappa^2 + \tau^2 = 1 + 2(c-2)\eta(B)^2. \end{cases}$$

Now, we investigate non-geodesic biharmonic helices in a 3-dimensional Sasakian space form.

Lemma 3.1 (cf. [10]) *Let M be a 3-dimensional Sasakian space form and let $\gamma : I \to M$ be a non-geodesic curve parametrized by arc-length. If $\eta(N) = 0$, then γ is a slant curve. More precisely, the tangent vector filed has the form*:

$$T(s) = \sin\alpha_0 \cos\beta(s) e_1 + \sin\alpha_0 \sin\beta(s) e_2 + \cos\alpha_0 e_3, \tag{3.8}$$

where α_0 is the constant contact angle.

Proof Since M is Sasakian, by using the 1st Frenet-Serret equation, it follows that

$$\kappa N = (T_1' - T_2 T_3(c-2))e_1 + (T_2' + T_1 T_3(c-2))e_2 + T_3' e_3. \tag{3.9}$$

From (3.9), we get easily $N_3 = 0$ if and only if $T_3' = 0$. Hence we obtain the required result. □

Theorem 3.3 *Let M be a 3-dimensional Sasakian space form except the case of constant curvature 1 and let $\gamma : I \to M$ be a non-geodesic biharmonic curve parametrized by arc-length. Then γ is obtained by integrating the following ordinary differential equation*:

$$\frac{d\gamma}{ds}(s) = \sin\alpha_0 \cos(As + a)\, e_1 + \sin\alpha_0 \sin(As + a)\, e_2 + \cos\alpha_0\, e_3,$$

where $A = -(c-1)\cos\alpha_0 \pm \sqrt{(-2c+5)\cos^2\alpha_0 + 2(c-2)}$.
Moreover, if $c < 2$, then $\alpha_0 \in \left(0, \cos^{-1}\sqrt{\frac{2c-4}{2c-5}}\,\right] \bigcup \left[\cos^{-1}\left(-\sqrt{\frac{2c-4}{2c-5}}\right), \pi\right)$.

Proof In the proof of Proposition 3.3, since $\alpha_0 \in \mathbb{R}$, we have obtained

$$\begin{aligned} \nabla_T T &= -\sin\alpha_0(\beta' + (c-2)\cos\alpha_0)(\sin\beta e_1 - \cos\beta e_2) \\ &= \kappa N, \end{aligned}$$

where $\kappa = |-\sin\alpha_0(\beta' + (c-2)\cos\alpha_0)|$. We assume that $-\sin\alpha_0(\beta' + (c-2)\cos\alpha_0 > 0$. Then

$$\kappa = -\sin\alpha_0(\beta' + (c-2)\cos\alpha_0) \tag{3.10}$$

and

$$N = \sin\beta e_1 - \cos\beta e_2.$$

Thus we get

$$B = \cos\beta\cos\alpha_0 e_1 + \sin\beta\cos\alpha_0 e_2 - \sin\alpha_0 e_3. \quad (3.11)$$

But, by using (3.1) ($c_2 = c_3 - c$) we have

$$\nabla_T N = \cos\beta(\beta' + (c-1)\cos\alpha_0)e_1 + \sin\beta(\beta' + (c-1)\cos\alpha_0)e_2 - \sin\alpha_0 e_3. \quad (3.12)$$

So, from (3.11) and (3.12) the geodesic torsion τ of γ is given by

$$\tau = (\cos\alpha_0)(\beta' + (c-2)\cos\alpha_0) + 1. \quad (3.13)$$

If γ is a curve with $\gamma' = T$, then it is a proper biharmonic curve if and only if

$$\begin{cases} \beta' = \text{constant}, (\tau = \text{constant}) \\ \beta' \neq (c-2)\cos\alpha_0, (\kappa \neq 0) \\ \kappa^2 + \tau^2 = 1 + 2(c-2)B_3^2. \end{cases} \quad (3.14)$$

From (3.11), (3.12), (3.13) and (3.14) we obtain

$$(\beta')^2 + 2(c-1)\cos\alpha_0\beta' + (c-2)\big((c+2)\cos^2\alpha_0 - 2\big) = 0,$$

from which we get

$$\beta' = -(c-1)\cos\alpha_0 \pm \sqrt{(-2c+5)\cos^2\alpha_0 + 2(c-2)} = A.$$

□

For the case $c = 0$, the ODE in Theorem 3.3 can be integrated explicitly. See Sect. 4.2 and [10].

Theorem 3.3 implies that non-geodesic biharmonic curves in 3-dimensional Sasakian space forms except the case of constant curvature 1 are *slant helices*.

To characterize biharmonic curves in Sasakian space forms geometrically, here we use the canonical fibering of Sasakian space forms.

Let M be a 3-dimensional Sasakian manifold. Then M is said to be *regular* if its Reeb vector field ξ generates a one-parameter group K of isometries on M, such that the action of K on M is simply transitive. The Killing vector field ξ induces a regular one-dimensional (1D) Riemannian foliation on M. We denote by $\overline{M} := M/\xi$ the orbit space (the space of all leaves) of a regular Sasakian 3-manifold M under the K-action.

The Sasakian structure on M induces a Kähler structure on the orbit space $\overline{M}$. Further the natural projection $\pi : M \to \overline{M}$ is a Riemannian submersion [23]. It is easy to see that M is a Sasakian space form of constant φ-holomorphic sectional curvature H if and only if $\overline{M}$ is a space form of curvature $H + 3$ (cf. [22]).

Hereafter we assume that M is a regular Sasakian space form of dimension 3. Let $\gamma(s)$ be a non-geodesic biharmonic curve as before. Then its tangent vector field and principal normal vector field are given by

$$T(s) = \sin\alpha_0\cos(As + a)\, e_1 + \sin\alpha_0\sin(As + a)\, e_2 + \cos\alpha_0\, e_3,$$

$$N(s) = \sin(As + a)e_1 - \cos(As + a)e_2.$$

Here we note that the contact distribution $\mathfrak{D}$ is spanned by e_1, e_2. Let $\overline{\gamma} = \pi \circ \gamma$ be the projection of γ onto $\overline{M}$. Direct computation shows that the arc-length parameter $\bar{s}$ of $\overline{\gamma}$ is

$$\bar{s} = s\sin\alpha_0.$$

This formula says somewhat more. In fact, this implies that s is the arc-length parameter of $\overline{\gamma}$ if and only if γ is a Legendre curve. In such a case, γ is a horizontal lift of $\overline{\gamma}$.

The Frenet frame $\{\overline{T}(\bar{s}), \overline{N}(\bar{s})\}$ of $\overline{\gamma}$ is given by

$$\overline{T}(\bar{s}) = \frac{1}{\sin\alpha_0}\pi_* T(s), \quad \overline{N}(\bar{s}) = \pm\pi_* N(s).$$

Thus the geodesic curvature $\bar{\kappa}$ of $\overline{\gamma}$ is given by

$$\overline{\kappa}(\bar{s}) = \frac{\pm 1}{\sin^2\alpha_0}\kappa(s).$$

Hence $\overline{\gamma}$ is a Riemannian circle, *i.e.*, a curve of constant (nonzero) geodesic curvature.

Corollary 3.2 *The projected curve of a non-geodesic biharmonic curve in a regular* 3*D Sasakian space form is a Riemannian circle in the orbit space.*

Let $S = \pi^{-1}\{\overline{\gamma}\}$ be the inverse image of the projected curve $\overline{\gamma}$. Then it is known that S is a flat surface with constant mean curvature $\bar{\kappa}/2$. This flat surface is called the *Hopf cylinder* over $\overline{\gamma}$. (See [3,29]).

Corollary 3.2 implies that every proper biharmonic helix lies in a Hopf cylinder over a Riemannian circle (Compare with Remark 5.3 of [7]).

Remark 3.1 In [10], it is pointed out that Hopf cylinders over Riemannian circles in the Heisenberg group $\mathcal{M}^3(-3)$ are not biharmonic (Remark 5.3). More generally there are no proper biharmonic cylinders in Sasakian space form $\mathcal{M}^3(H)$ of constant φ-holomorphic sectional curvature $H \leq 1$ [18].

4 Bianchi–Cartan–Vranceanu spaces

To describe proper biharmonic curves in 3-dimensional Sasakian space form explicitly, it is convenient to use another model of Sasakian space form.

4.1 Bianchi–Cartan–Vranceanu model space

Let c be a real number and set

$$\mathcal{D} = \left\{(x,y,z) \in \mathbb{R}^3(x,y,z) \mid 1 + \frac{c}{2}(x^2+y^2) > 0\right\}.$$

Note that $\mathcal{D}$ is the whole $\mathbb{R}^3(x,y,z)$ for $c \geq 0$. On the region $\mathcal{D}$, we equip the following Riemannian metric:

$$g_c = \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{\{1 + \frac{c}{2}(x^2+y^2)\}^2} + \left(\mathrm{d}z + \frac{y\mathrm{d}x - x\mathrm{d}y}{1 + \frac{c}{2}(x^2+y^2)}\right)^2. \tag{4.1}$$

Take the following orthonormal frame field on $(\mathcal{D}, g_c)$:

$$u_1 = \left\{1 + \frac{c}{2}(x^2+y^2)\right\}\frac{\partial}{\partial x} - y\frac{\partial}{\partial z}, \quad u_2 = \left\{1 + \frac{c}{2}(x^2+y^2)\right\}\frac{\partial}{\partial y} + x\frac{\partial}{\partial z}, \quad u_3 = \frac{\partial}{\partial z}.$$

Then the Levi-Civita connection ∇ of this Riemannian 3-manifold is described as

$$\nabla_{u_1}u_1 = c\,yu_2, \quad \nabla_{u_1}u_2 = -c\,yu_1 + u_3, \quad \nabla_{u_1}u_3 = -u_2,$$

$$\nabla_{u_2} u_1 = -c\,x u_2 - u_3, \quad \nabla_{u_2} u_2 = c\,x u_1, \quad \nabla_{u_2} u_3 = u_1, \tag{4.2}$$

$$\nabla_{u_3} u_1 = -u_2, \quad \nabla_{u_3} u_2 = u_1, \quad \nabla_{u_3} u_3 = 0.$$

$$[u_1, u_2] = -c\,y u_1 + c\,x u_2 + 2u_3, \quad [u_2, u_3] = [u_3, u_1] = 0. \tag{4.3}$$

Define the endomorphism field φ by

$$\varphi u_1 = u_2, \quad \varphi u_2 = -u_1, \quad \varphi u_3 = 0.$$

The dual one-form η of the vector field $\xi = u_3$ is a contact form on $\mathcal{D}$ and satisfies

$$d\eta(X, Y) = g(X, \varphi Y), \quad X, Y \in \mathfrak{X}(\mathcal{D}).$$

Moreover the structure (φ, ξ, η, g) is Sasakian. Moreover, $(\mathcal{D}, g_c)$ is of constant holomorphic sectional curvature $H = -3+2c$. (cf. [3,29]). Hereafter we denote this model $(\mathcal{D}, g_c)$ of a Sasakian space form by $\mathcal{M}^3(H)$. The one-parameter family of Riemannian 3-manifolds $\{\mathcal{M}^3(H)\}_{H\in\mathbb{R}}$ is classically known by L. Bianchi [4], E. Cartan [12] and G. Vranceanu [31] (See also Kobayashi [20]). The model $\mathcal{M}^3(H)$ of Sasakian 3-space form is called the Bianchi–Cartan–Vranceanu model of 3D Sasakian space form.

The Reeb flows are the translations in the z-directions. Hence the orbit space $\overline{\mathcal{M}^2}(H+3) = \mathcal{M}^3(H)/\xi$ is given explicitly by

$$\overline{\mathcal{M}^2} = \left(\left\{(x, y) \in \mathbb{R}^2 \mid 1 + \frac{c}{2}(x^2 + y^2) > 0\right\}, \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{\{1 + \frac{c}{2}(x^2 + y^2)\}^2}\right)$$

The natural projection $\pi : \mathcal{M}^3(H) \to \overline{\mathcal{M}^2}(H+3)$ is

$$\pi(x, y, z) = (x, y).$$

The frame field $\{u_1, u_2, u_3\}$ is not left invariant. Thus we can not choose $e_i = u_i$ and apply our results in Section 2 directly, except the case $c = 0$.

However, since the Reeb vector field ξ is u_3 and it is left invariant, thus we may choose $e_3 = u_3$. (Note that in the case $c = 0$, we can choose $u_1 = e_1, u_2 = e_2$.)

For a curve $\gamma(s) = (x(s), y(s), z(s))$ in $\mathcal{M}^3(H)$ parameterized by arc-length, we can expand the tangent vector field as

$$T = T_1 e_1 + T_2 e_2 + T_3 e_3 = \hat{T}_1 u_1 + \hat{T}_2 u_2 + \hat{T}_3 u_3.$$

In these expansions, we have $T_3 = \hat{T}_3$. Analogously, we consider expansions for $N(s)$ and $B(s)$ and get the relations $N_3 = \hat{N}_3,\ B_3 = \hat{B}_3$.

Then in a similar way as in section 4, we compute

$$R(\kappa N, T)T = \kappa\Big[\big\{\hat{B}_3^2(2c-3) + \hat{B}_2^2 + \hat{B}_1^2\big\}N - \big\{\hat{B}_3\hat{N}_3(2c-3) + \hat{B}_2\hat{N}_2 + \hat{B}_1\hat{N}_1\big\}B\Big]$$

and

$$\begin{aligned} \mathcal{T}_2(\gamma) &= \nabla^3{}_T T + R(\kappa N, T)T \\ &= (-3\kappa\kappa')T + \Big[(\kappa'' - \kappa^3 - \kappa\tau^2) + \kappa\big\{B_3^2(2c-3) + \hat{B}_2^2 + \hat{B}_1^2\big\}\Big]N \\ &\quad + \Big[(2\tau\kappa' + \kappa\tau') - \kappa\big\{B_3 N_3(2c-3) + \hat{B}_2\hat{N}_2 + \hat{B}_1\hat{N}_1\big\}\Big]B \end{aligned} \tag{4.4}$$

with respect to $\{u_1, u_2, u_3\}$.

Lemma 3.1 is adjusted in the following way:

Lemma 4.1 *Let $\mathcal{M}^3(H)$ be a Bianchi–Cartan–Vranceanu model space of the 3D Sasakian space form and $\gamma : I \to M$ a non-geodesic curve parametrized by arc-length. If $N_3 = 0$, then γ is a slant curve. More precisely, the tangent vector filed has the form*:

$$T(s) = \sin\alpha_0 \cos\hat{\beta}(s)u_1 + \sin\alpha_0 \sin\hat{\beta}(s)u_2 + \cos\alpha_0 u_3, \tag{4.5}$$

where α_0 is the constant contact angle.

In the case $c = 0$, we notice that $\hat{\beta}(s) = \beta(s)$, where $\beta(s)$ is defined in the proof of Proposition 3.3.

Let $\gamma(s) = (x(s), y(s), z(s))$ be a curve in $\mathcal{M}^3(H)$. Then the tangent vector field T of γ is

$$T = \left(\frac{\mathrm{d}x}{\mathrm{d}s}, \frac{\mathrm{d}y}{\mathrm{d}s}, \frac{\mathrm{d}z}{\mathrm{d}s}\right) = \frac{\mathrm{d}x}{\mathrm{d}s}\frac{\partial}{\partial x} + \frac{\mathrm{d}y}{\mathrm{d}s}\frac{\partial}{\partial y} + \frac{\mathrm{d}z}{\mathrm{d}s}\frac{\partial}{\partial z}.$$

Using the relations:

$$\frac{\partial}{\partial x} = \frac{1}{1+\frac{c}{2}(x^2+y^2)}(u_1 + yu_3), \quad \frac{\partial}{\partial y} = \frac{1}{1+\frac{c}{2}(x^2+y^2)}(u_2 - xu_3), \quad \frac{\partial}{\partial z} = u_3,$$

we get

$$\frac{\mathrm{d}x}{\mathrm{d}s} = \left\{1 + \frac{c}{2}(x^2+y^2)\right\}\hat{T}_1, \quad \frac{\mathrm{d}y}{\mathrm{d}s} = \left\{1 + \frac{c}{2}(x^2+y^2)\right\}\hat{T}_2,$$

$$\frac{\mathrm{d}z}{\mathrm{d}s} = \hat{T}_3 - \frac{1}{1+\frac{c}{2}(x^2+y^2)}\left(\frac{\mathrm{d}x}{\mathrm{d}s}y - x\frac{\mathrm{d}y}{\mathrm{d}s}\right).$$

Hence we obtain the following ODE:

Corollary 4.1 *Let $\gamma : I \to \mathcal{M}^3(H)$ be a non-geodesic biharmonic helix parametrized by arc-length in the Bianchi-Cartan-Vranceanu model of Sasakian space form. Then the parametric equations of γ are given by*

$$\frac{\mathrm{d}x}{\mathrm{d}s}(s) = \sin\alpha_0 \cos\hat{\beta}(s)\left\{1 + \frac{c}{2}(x(s)^2 + y(s)^2)\right\}, \tag{4.6}$$

$$\frac{\mathrm{d}y}{\mathrm{d}s}(s) = \sin\alpha_0 \sin\hat{\beta}(s)\left\{1 + \frac{c}{2}(x(s)^2 + y(s)^2)\right\}, \tag{4.7}$$

$$\frac{\mathrm{d}z}{\mathrm{d}s}(s) = \cos\alpha_0 + \frac{1}{\{1+\frac{c}{2}(x(s)^2+y(s)^2)\}}\left\{x(s)\frac{\mathrm{d}y}{\mathrm{d}s}(s) - y(s)\frac{\mathrm{d}x}{\mathrm{d}s}(s)\right\}. \tag{4.8}$$

4.2 Explicit formulas of biharmonic curves

In this section, we give the solution of the ODE-system (4.6)–(4.8) explicitly.

We calculate along a slant curve γ with the contact angle $\hat{\alpha} = \alpha_0$. Then we have

$$\kappa = -\sin\alpha_0(\hat{\beta}' + cy\sin\alpha_0\cos\hat{\beta} - cx\sin\alpha_0\sin\hat{\beta} - 2\cos\alpha_0), \tag{4.9}$$

$$\tau = \cos\alpha_0(\hat{\beta}' + cy\sin\alpha_0\cos\hat{\beta} - cx\sin\alpha_0\sin\hat{\beta} - 2\cos\alpha_0) + 1. \tag{4.10}$$

From this, by using Corollary 3.1, we find that γ is a proper biharmonic curve if and only if (cf. [10, Remark 5.8])

$$\begin{cases} \hat{\beta}' + c\sin\alpha_0(y\cos\hat{\beta} - x\sin\hat{\beta}) - 2\cos\alpha_0 = \text{nonzero constant}, \\ \kappa^2 + \tau^2 = 1 + 2(c-2)B_3^2. \end{cases} \tag{4.11}$$

From (4.9), (4.10) and the first equation of (4.11), we obtain

$$c\sin\alpha_0(x\sin\hat{\beta} - y\cos\hat{\beta}) = \hat{\beta}' - \cos\alpha_0 \mp \sqrt{2(c-2) - (2c-5)\cos^2\alpha_0} \tag{4.12}$$

under the assumption $2(c-2) - (2c-5)\cos^2\alpha_0 \geq 0$.

Let us solve the ODE for the following divided cases.

(I) Case-1 $c = 0$: Then (4.12) is reduced to

$$\hat{\beta}' = \cos\alpha_0 \pm \sqrt{5\cos^2\alpha_0 - 4} = \text{constant}.$$

Namely, $\hat{\beta}'$ is a constant, say A, hence $\hat{\beta}(s) = As + b,\ b \in \mathbb{R}$. Thus, from (4.6) and (4.7) we have the following result (cf. [10]):

$$x(s) = \frac{1}{A}\sin\alpha_0\sin(As+b) + x_0, \tag{4.13}$$

$$y(s) = -\frac{1}{A}\sin\alpha_0\cos(As+b) + y_0, \tag{4.14}$$

$$\begin{aligned} z(s) = {} & \{\cos\alpha_0 + \sin^2\alpha_0/(2A)\}s \\ & -\frac{\sin\alpha_0}{2A}\{x_0\cos(As+b) + y_0\sin(As+b)\} + z_0. \end{aligned} \tag{4.15}$$

(II) Case-2 $c \neq 0$ ($c \neq 2$): Then together with (4.12), we see that the equation (4.8) becomes

$$\frac{\mathrm{d}z}{\mathrm{d}s} = \frac{1}{c}\{\hat{\beta}' + (c-1)\cos\alpha_0 \mp \sqrt{2(c-2) - (2c-5)\cos^2\alpha_0}\}.$$

Thus we have

$$z(s) = \frac{1}{c}\hat{\beta}(s) + \frac{1}{c}\{(c-1)\cos\alpha_0 \mp \sqrt{2(c-2) - (2c-5)\cos^2\alpha_0}\}s + z_0 \tag{4.16}$$

where z_0 is a constant. We now compute the x- and y-coordinates. We put $h(s) := 1 + \frac{c}{2}(x(s)^2 + y(s)^2)$. Then (4.6), (4.7) becomes

$$\frac{\mathrm{d}x}{\mathrm{d}s} = \sin\alpha_0\cos\hat{\beta}(s)h(s), \quad \frac{\mathrm{d}y}{\mathrm{d}s} = \sin\alpha_0\sin\hat{\beta}(s)h(s),$$

respectively. We note that the function $h(s)$ satisfies the following ODE:

$$\frac{\mathrm{d}}{\mathrm{d}s}\log|h(s)| = c\sin\alpha_0(\cos\hat{\beta}(s)x(s) + \sin\hat{\beta}(s)y(s)).$$

On the other hand, from the first equation of (4.11), we have

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}\hat{\beta}(s) = \frac{\mathrm{d}\hat{\beta}}{\mathrm{d}s}(s)\frac{\mathrm{d}}{\mathrm{d}s}\log|h(s)|. \tag{4.17}$$

First, if $\mathrm{d}\hat{\beta}/\mathrm{d}s = 0$, then $(x(s), y(s))$ is a line in the orbit space. Hence we have the following parametrization:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \sin\alpha_0 \cos\hat{\beta}_0 \int h(s)\mathrm{d}s \\ \sin\alpha_0 \sin\hat{\beta}_0 \int h(s)\mathrm{d}s \\ \frac{1}{c}\{(c-1)\cos\alpha_0 \mp \sqrt{2(c-2)-(2c-5)\cos^2\alpha_0}\}s \end{pmatrix} + \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}.$$

Next, we assume that $\mathrm{d}\hat{\beta}/\mathrm{d}s \neq 0$. Then from (4.17) we get

$$h(s) = r\frac{\mathrm{d}\hat{\beta}}{\mathrm{d}s}(s), \quad r \in \mathbb{R}\backslash\{0\}.$$

Hence we obtain

$$(x(s), y(s)) = (r\sin\alpha_0 \sin\hat{\beta}(s) + x_0, -r\sin\alpha_0\cos\hat{\beta}(s) + y_0).$$

It only remains to obtain $z(s)$ in this case $d\hat{\beta}/ds \neq 0$.

(a) Subcase-1 $c > 0$:

In this case, then the orbit space is the whole plane $\mathbb{R}^2(x, y)$. The projected curve $\bar{\gamma}(s)$ is a circle $(x-x_0)^2 + (y-y_0)^2 = r^2\sin^2\alpha_0$. Since, M is homogeneous, we may assume that $\bar{\gamma}$ is a circle centered at $(0,0)$. Then the angle function $\hat{\beta}$ is given by

$$\hat{\beta}(s) = \left(\frac{c}{2}r\sin^2\alpha_0 + \frac{1}{r}\right)s.$$

Hence the z-coordinate is given by

$$z(s) = \frac{1}{c}\left\{\frac{c}{2}r\sin^2\alpha_0 + \frac{1}{r} + (c-1)\cos\alpha_0 \mp \sqrt{2(c-2)-(2c-5)\cos^2\alpha_0}\right\}s + z_0.$$

(b) Subcase-2 $c < 0$:

In this case, the orbit space is the disk $x^2 + y^2 < R^2$, where $R = \sqrt{-2/c}$. According to the radius of the circle, there are three possibilities: closed circles, horocycles or open circles.

We consider a circle through the origin $(0,0)$. Namely $(x(s), y(s)) = (r\sin\hat{\beta}(s) + r, -r\cos\hat{\beta}(s))$. Then the above three possibilities correspond to the following conditions:

1. $\bar{\gamma}$ is a closed circle if and only if $r < R/2$,
2. $\bar{\gamma}$ is a horocycle if and only if $r = R/2$,
3. $\bar{\gamma}$ is an open circle if and only if $r > R/2$.

For the circle $(x(s), y(s)) = (r\sin\hat{\beta}(s) + r, -r\cos\hat{\beta}(s))$, $h(s) = 1 + cr^2(1 + \sin\hat{\beta}(s))$. From (4.12), the angle function $\hat{\beta}$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}s}\hat{\beta}(s) = cr\sin\alpha_0(1+\sin\hat{\beta}(s)) + \cos\alpha_0 \pm \sqrt{2(c-2)-(2c-5)\cos^2\alpha_0}.$$

From this, we get

$$\int \frac{\mathrm{d}\hat{\beta}}{\mathcal{A}\sin\hat{\beta} + \mathcal{B}} = s + s_0,$$

where

$$\mathcal{A} = \mathrm{cr}\sin\alpha_0, \quad \mathcal{B} = \mathrm{cr}\sin\alpha_0 + \cos\alpha_0 \pm \sqrt{2(c-2)-(2c-5)\cos^2\alpha_0}.$$

Introducing a new variable $t := \tan(\hat{\beta}/2)$, then when $\mathcal{B} \neq 0$, we have

$$\frac{2}{\lambda - \mu} \log \left| \frac{t - \lambda}{t - \mu} \right| = s + s.$$

where

$$\lambda = \frac{-\mathcal{A} + \sqrt{\mathcal{A}^2 - \mathcal{B}^2}}{\mathcal{B}}, \quad \mu = \frac{-\mathcal{A} - \sqrt{\mathcal{A}^2 - \mathcal{B}^2}}{\mathcal{B}},$$

and hence (4.16) gives

$$\begin{aligned} z(s) &= \frac{2}{c} \tan^{-1} \left\{ \frac{\mu \exp(\frac{(\lambda-\mu)}{2} \mathcal{B}(s + s_0)) - \lambda}{\exp(\frac{(\lambda-\mu)}{2} \mathcal{B}(s + s_0)) - 1} \right\} \\ &\quad + \frac{1}{c} \{(c-1) \cos \alpha_0 \mp \sqrt{2(c-2) - (2c-5) \cos^2 \alpha_0}\} s + z_0. \end{aligned}$$

Next, if $\mathcal{B} = 0$, then

$$\text{cr} \sin \alpha_0 (s + s_0) = \int \frac{d\hat{\beta}}{\sin \hat{\beta}} = \log \left| \tan \frac{\hat{\beta}}{2} \right|.$$

From this, it follows that

$$\hat{\beta}(s) = 2 \tan^{-1} \{\exp(\text{cr} \sin \alpha_0 (s + s_0))\}.$$

Then we get

$$\begin{aligned} z(s) &= \frac{2}{c} \tan^{-1} \{\exp(\text{cr} \sin \alpha_0 (s + s_0))\} \\ &\quad + \frac{1}{c} \{(c-1) \cos \alpha_0 \mp \sqrt{2(c-2) - (2c-5) \cos^2 \alpha_0}\} s + z_0. \end{aligned}$$

Remark 4.1 The explicit parametrizations of biharmonic Legendre helices (*i.e.*, the case $\cos \alpha_0 = 0$) in the Bianchi-Cartan-Vranceanu model of $\mathcal{M}^3(H)$ are independently obtained by R. Caddeo, C. Oniciuc and P. Piu. Their result was presented at the poster session of the conference "Curvature and Geometry, in honor of Lieven Vanhecke", Lecce, 11–14 June 2003.

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