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BIHARMONIC CURVES IN CONTACT GEOMETRY

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Abstract. We study biharmonic curves in contact geometry whose mean curvature vector field is in the kernel of Laplacian. We give some results for biharmonic curves in Sasakian 3-space. We also give some characterizations for Legendre curves in the same space.

1. **Introduction and Preliminaries.**

Let M be a smooth manifold. A contact form η on M is a 1-form such that $(d\eta)^n \wedge \eta \neq 0$. A manifold M together with a contact form is called a contact manifold [4,10]. The distribution D defined by the Phaffian equation $\eta = 0$ is called the contact structure determined by η. That is,

$$D = \{x \in \chi(M)| \quad \eta : \chi(M) \to C^\infty(M, \mathrm{R}),\ \eta(X) = 0\}$$

(see, for instance, [4,10]). The maximum dimension of integral submanifold of D is $(\dim M - 1)/2$. An integral submanifold of D of maximum dimension is called a Legendre submanifold of (M, η) [3].

The reel vector field ζ (killing vector field) is defined by

$$\eta(\zeta) = 1,\ \ d\eta(\zeta, .) = 0$$

(see [10]).

On a contact manifold (M, η), there exist an endomorphism field ϕ and a Riemannian metric g satisfying

$$\begin{aligned} \phi^2 &= -I + \eta \otimes \zeta, \\ g(\phi X, \phi Y) &= g(X, Y) - \eta(X)\eta(Y), \\ d\eta(X, Y) &= 2g(X, \phi Y), \end{aligned}$$

for all vector fields X and Y on M. The structure tensors (ζ, ϕ, g) is called the associated almost contact structure of η [10].

A contact manifold $(M, \eta, \zeta, \phi, g)$ is said to be a Sasaki manifold if M satisfies

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$$(\nabla_X \phi)Y = g(X, Y)\zeta - \eta(Y)X$$

(see [10]).

Since ζ is a globally defined unit vector field, the contact manifold M admits a Lorentz metric. In fact let us define h by

$$h = g - 2\eta \otimes \eta.$$

Then h is a Lorentz metric which satisfies the following formulae:

$$\begin{aligned} h(X,\zeta) &= -\eta(X), \\ h(\phi X, \phi Y) &= h(X, Y) + \eta(X)\eta(Y), \\ d\eta(X, Y) &= 2h(X, \phi Y). \end{aligned}$$

We call the Lorentz metric h by associated Lorentz metric of M (see [8], [9]).

Let us define by ∇ the Levi-Civita connection of the Lorentz metric h. Then ∇ is related to the Levi-Civita connection ∇^g of g by the following formula:

$$\nabla_X Y = \nabla^g_X Y + 2\left(\eta(X)\phi Y + \eta(Y)\phi X\right).$$

Now let $(M, \eta, \zeta, \phi, g)$ be a Sasaki manifold. Then the associated Lorentz metric h satisfies the following equation (Theorem 3 in [2]):

$$\begin{aligned} (\nabla_X \phi)Y &= h(X,Y)\zeta + \eta(Y)X, \\ \nabla_X \zeta &= -\phi X. \end{aligned}$$

The reel vector field ζ is globally defined timelike killing vector field on the Lorentz manifold (M,h). The resulting manifold $(M, \eta, \zeta, \phi, g)$ is called a Lorentz-Sasaki manifold or Sasakian spacetime (see [2], [6]).

Now let $M^3 = (M, \eta, \zeta, \phi, g)$ be a contact 3-manifold with an associated metric g. A curve $\gamma = \gamma(s) : \mathrm{I} \to M$ parameterized by the arclength parameter is said to be a Legendre curve if γ is tangent to contact distribution D of M. It is obvious that γ is Legendre if and only if $\eta(\gamma') = 0$.

Let γ be a Legendre curve on M^3. Then we can take a Frenet frame $\{V_1, V_2, V_3\}$ so that $V_1 = \gamma'$ and $V_3 = \zeta$.

Now we assume that M is a Sasaki manifold. Then the following equality is defined

$$(\nabla_X \phi)Y = g(X, Y)\zeta - \eta(Y)X, \qquad X, Y \in \chi(M).$$

The Frenet-Serret formulae of γ are given explicitly by

$$\begin{bmatrix} \nabla'_\gamma V_1 \\ \nabla'_\gamma V_2 \\ \nabla'_\gamma V_3 \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}. \tag{1.1}$$

The function κ is called the curvature of γ. Namely, every Legendre curve has constant torsion 1 (see [1]). In particular, a curve γ parametrized by the arclength is said to be a geodecis if $\kappa = 0$. Note that if $\kappa = 0$, then automatically $\tau = 0$.

More generally a curve with constant curvature and zero torsion is called a (Riemannian) circle. Geodesics are regarded as Riemannian circles of zero curvatures.

A circular helix is a curve whose curvature and torsion are constants. Geodesics and Riemannian circles are regarded as degenerate helices. Helices which are neither geodesics nor circles are frequently called proper helices.

Let us denote by Δ the Laplace-Beltrami operator of γ and by H the mean curvature vector field along γ.

The Frenet-Serret formulae of γ imply that the mean curvature vector field H is given by

$$\mathrm{H} = \nabla'_{\gamma}\gamma' = \nabla'_{\gamma}V_1 = \kappa V_2, \tag{1.2}$$

where κ is the curvature of γ.

The Laplace-Beltrami operator of γ is defined by

$$\Delta\mathrm{H} = -\nabla'^{2}_{\gamma}\mathrm{H} \tag{1.3}$$

(see [5], [7]).

Definition 1.1. A unit speed Legendre curve $\gamma = \mathrm{I} \to M^3$ on Sasakian 3-manifold is said to be biharmonic if $\Delta\mathrm{H} = 0$(i.e., $\Delta\mathrm{H} = \Delta^2\gamma = 0$).

Chen and Ishikawa [5] classified biharmonic curves in semi-Euclidean space E^n_v. In this paper we shall give the characterizations of biharmonic curves in contact geometry in terms of curvature.

2. Biharmonic Curves on Sasakian 3-Manifolds with Rimennian Metric.

In this section we give the characterizations for biharmonic curves on Sasakian 3-manifolds. By using the obtained results we give some conditions for these curves to be helix.

Theorem 2.1. *Let γ be a unit speed Legendre curve on Sasakian 3-manifold and let λ be a real constant. Then Legendre curve γ is a circular helix if and only if the following differential equation satisfies*

$$\Delta\mathrm{H} + \lambda\mathrm{H} = 0 \tag{2.1}$$

where $\lambda = -\kappa^2 - 1$.

Proof. By the use of (1.1), (1.2) and (1.3) we get that

$$\Delta\mathrm{H} = 3\kappa\kappa' V_1 + (\kappa^3 + \kappa - \kappa'')V_2 - 2\kappa' V_3, \tag{2.2}$$

$$\lambda\mathrm{H} = \lambda\kappa V_2. \tag{2.3}$$

So, the proof follows from (2.1), (2.2) and (2.3). □

Theorem 2.2. *Let γ be a unit speed Legendre curve on Sasakian 3-manifold. Then γ satisfies the following differential equation*

$$\nabla'^3_\gamma V_1 + \lambda_1 \nabla'^2_\gamma V_1 + \lambda_2 \nabla'_\gamma V_1 + \lambda_3 V_1 = 0, \tag{2.4}$$

where

$$\lambda_1 = -2\tfrac{\kappa'}{\kappa},$$
$$\lambda_2 = \kappa^2 + 1 - \tfrac{\kappa''^2 - 2\kappa(\kappa'^2}{\kappa^3},$$
$$\lambda_3 = \kappa\kappa'.$$

Proof. By the use of Frenet-Serret formulae (1.1), we have

$$\nabla'^2_\gamma V_1 = -\kappa^2 V_1 + \kappa' V_2 + \kappa V_3. \tag{2.5}$$

By differentiating (2.5) with respect to arc parameter we obtain

$$\nabla'^3_\gamma V_1 = -2\kappa\kappa' V_1 - \kappa^2 \nabla'_\gamma V_1 + \kappa'' V_2 + \kappa' \, \nabla'_\gamma V_2 + \kappa' V_3 + \kappa \nabla'_\gamma V_3. \tag{2.6}$$

From the second equation of (1.1) we get

$$V_3 = \nabla'_\gamma V_2 + \kappa V_1, \tag{2.7}$$

and from the first equation of (1.1) we have

$$V_2 = \frac{1}{\kappa} \nabla'_\gamma V_1. \tag{2.8}$$

Substituting (2.8) into (2.7) we obtain

$$V_3 = \left(\frac{1}{\kappa}\right)' \nabla'_\gamma V_1 + \frac{1}{\kappa} \nabla'^2_\gamma V_1 + \kappa V_1. \tag{2.9}$$

Differentiating (2.9) gives

$$\nabla'_\gamma V_3 = \frac{1}{\kappa} \nabla'^3_\gamma V_1 + 2\left(\frac{1}{\kappa}\right)' \nabla'^2_\gamma V_1 + \left[\kappa + \left(\frac{1}{\kappa}\right)''\right] \nabla'_\gamma V_1 + \kappa' V_1. \tag{2.10}$$

Substituting (2.8) into the third equation of (1.1) we have

$$\nabla'_\gamma V_3 = -\frac{1}{\kappa} \nabla'_\gamma V_1. \tag{2.11}$$

Substituting (2.11) into the third equation of (2.10) and doing the regulations we have

$$\nabla'^3_\gamma V_1 + \lambda_1 \nabla'^2_\gamma V_1 + \lambda_2 \nabla'_\gamma V_1 + \lambda_3 V_1 = 0,$$

where

$$\lambda_1 = -2\frac{\kappa'}{\kappa}, \quad \lambda_2 = \kappa^2 + 1 - \frac{\kappa''^2 - 2\kappa(\kappa'^2}{\kappa^3}, \quad \lambda_3 = \kappa\kappa', \tag{2.12}$$

that finishes the proof. □

Corollary 1. *Let γ be a unit speed Legendre curve on Sasakian 3-manifold. If the Legendre curve γ is geodesic, then the curve γ is a circular helix.*

Proof. Let γ be a geodesic curve. Then $\nabla'_\gamma V_1 = 0$ which gives that κ is constant i.e., γ is a circular helix. □

Corollary 2. *Let γ be a unit speed Legendre curve on Sasakian 3-manifold. If the Legendre curve γ is a circular helix, then the differential equation characterizing the curve γ is*

$$\nabla'^3_\gamma V_1 + (\kappa^2 + 1)\nabla'_\gamma V_1 = 0. \tag{2.13}$$

Proof. The proof follows from (2.4) immediately. □

Theorem 2.3. *Let γ be a unit speed Legendre curve on Sasakian 3-manifold. Then γ satisfies the following differential equation*

$$\nabla'^3_\gamma V_2 + \lambda_1 \nabla'^2_\gamma V_2 + \lambda_2 \nabla'_\gamma V_2 + \lambda_3 V_2 = 0, \tag{2.14}$$

where

$$\lambda_1 = -\frac{\kappa''}{\kappa'}, \quad \lambda_2 = -(1 + \kappa^2), \quad \lambda_3 = 3\kappa\kappa' - \frac{\kappa''^2)}{\kappa'}.$$

Proof. The proof is obtained immediately by considering the similar way used in the proof of Theorem 2.1. □

Corollary 3. *Let γ be a unit speed Legendre curve on Sasakian 3-manifold. If the Legendre curve γ is a circular helix, then the differential equation characterizing the curve γ is*

$$\nabla'^3_\gamma V_2 + (\kappa^2 + 1)\nabla'_\gamma V_2 = 0.$$

Proof. The proof follows from (2.4) immediately. □

Theorem 2.4. *Let γ be a unit speed Legendre curve on Sasakian 3-manifold. Then γ satisfies the following differential equation*

$$\nabla'^3_\gamma V_3 + \lambda_1 \nabla'^2_\gamma V_3 + \lambda_2 \nabla'_\gamma V_3 + \lambda_3 V_3 = 0, \tag{2.15}$$

where

$$\lambda_1 = -\frac{\kappa'}{\kappa} = \lambda_3, \quad \lambda_2 = 1 + \kappa^2.$$

Proof. From the third equation of equation (1.1) we have

$$\nabla'^2_\gamma V_3 = -\nabla'_\gamma V_2. \tag{2.16}$$

Substituting the second equation of equation (1.1) into (2.16) we get

$$\nabla'^2_\gamma V_3 = \kappa V_1 - V_3. \tag{2.17}$$

Differentiating (2.17) gives

$$\nabla'^3_\gamma V_3 = \kappa' V_1 + \kappa \nabla'_\gamma V_1 - \nabla'_\gamma V_3. \tag{2.18}$$

From the third equation of Equation (1.1) we have

$$\nabla'_\gamma V_3 = -V_2. \tag{2.19}$$

From the first equation of Equation (1.1) and (2.19) we get

$$\nabla'_\gamma V_1 = -\kappa \nabla'_\gamma V_3. \tag{2.20}$$

Similarly, from the second equation of Equation (1.1) we have

$$V_1 = \frac{1}{\kappa} V_3 - \frac{1}{\kappa} \nabla'_\gamma V_2. \tag{2.21}$$

From (2.16) and (2.21) we obtain

$$V_1 = \frac{1}{\kappa} V_3 - \frac{1}{\kappa} \nabla'^2_\gamma V_3. \tag{2.22}$$

Then, writing (2.20) and (2.22) in the (2.18) we have (2.15). □

Corollary 4. *Let γ be a unit speed Legendre curve on Sasakian 3-manifold. If the Legendre curve γ is a circular helix, then the differential equation characterizing the curve γ is*

$$\nabla'^3_\gamma V_3 + (1+\kappa^2) \nabla'_\gamma V_3 = 0. \tag{2.23}$$

Proof. The proof is clear from Theorem 2.4. □

3. Biharmonic Curves on Sasakian 3-Manifolds with Lorentzian Metric.

Let γ be a Legendre curve on Sasakian 3-manifold M. Then according to the Lorentzian metric the Frenet-Serret formulae of γ are given explicitly by

$$\begin{bmatrix} \nabla'_\gamma V_1 \\ \nabla'_\gamma V_2 \\ \nabla'_\gamma V_3 \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \varepsilon \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}, \tag{3.1}$$

where $\varepsilon = \mp 1$ is the torsion of γ [3]. The Laplacian operator Δ and the mean curvature H of γ are defined, respectively, by

$$\Delta = -\varepsilon \nabla'^2_\gamma, \tag{3.2}$$

and

$$\mathrm{H} = \nabla'_\gamma V_1. \tag{3.3}$$

Theorem 3.1. *Let γ be a unit speed Legendre curve on Sasakian 3-manifold and let λ be a real constant. Then Legendre curve γ is a circular helix if and only if the following differential equation satisfies*

$$\Delta\mathrm{H} = \lambda\mathrm{H},$$

then κ is a constant and

$$\lambda = \varepsilon + \kappa^2. \tag{3.4}$$

Proof. By (3.1), (3.2) and (3.3), we get (3.4). The converse statement of Theorem 3.1. is also fine. □

Corollary 5. *Let γ be a unit speed Legendre curve on Sasakian 3-manifold (M, g). Then γ is a Legendre circular helix whose Killing vector field is timelike (respectively spacelike) if and only if $\Delta\mathrm{H} = \lambda\mathrm{H}$ where $\lambda = 1 - \kappa^2$(respectively $\lambda = 1 + \kappa^2$).*

Proof. The proof is easily seen by Theorem 3.1. □

Corollary 6. *Let γ be a unit speed Legendre curve on Sasakian 3-manifold (M, g). Then γ is a Legendre biharmonic circular helix, whose Killing vector field is timelike (respectively spacelike) if and only if $\kappa = \pm 1$ (respectively, $\kappa = \pm i$) .*

Proof. The proof is easily seen by Theorem 5. □

Theorem 3.2. *Let γ be a unit speed curve with Lorentzian metric on Sasakian 3-manifold (M, g). Then γ satisfies the following differential equation*

$$\nabla'^3_\gamma V_1 + \lambda_1 \nabla'^2_\gamma V_1 + \lambda_2 \nabla'_\gamma V_1 + \lambda_3 V_1 = 0, \tag{3.5}$$

where

$$\lambda_1 = -2\frac{\kappa'}{\kappa}, \quad \lambda_2 = \kappa^2 + \varepsilon - \frac{\kappa''^2 - 2\kappa(\kappa'^2}{\kappa^3}, \quad \lambda_3 = \kappa\kappa'.$$

Proof. We get (3.5) by using (3.1). □

Corollary 7. *Let γ be a unit speed Legendre curve on Sasakian 3-manifold with Lorentzian metric. If the Legendre curve γ is geodesic then the equation characterizing the curve γ is*

$$\kappa\kappa' = 0. \tag{3.6}$$

Proof. Let γ be a geodesic curve. Then the proof is clear from Corollary 1. □

Corollary 8. *Let γ be a unit speed Legendre curve on Sasakian 3-manifold. If the Legendre curve γ is a circular helix then the differential equation characterizing the curve γ is*

$$\nabla'^3_\gamma V_1 + (\kappa^2 + \varepsilon)\nabla'_\gamma V_1 = 0. \tag{3.7}$$

Proof. By (3.5), we get (3.7). □

Theorem 3.3. *Let γ be a unit speed curve with Lorentzian metric on Sasakian 3-manifold (M, g). Then γ satisfies the following differential equation*

$$\nabla'^3_\gamma V_2 + \lambda_1 \nabla'^2_\gamma V_2 + \lambda_2 \nabla'_\gamma V_2 + \lambda_3 V_2 = 0, \tag{3.8}$$

where

$$\lambda_1 = -\frac{\kappa''}{\kappa'}, \quad \lambda_2 = \kappa^2 + \varepsilon, \quad \lambda_3 = 3\kappa\kappa' - \frac{\kappa''(1+\kappa^2)}{\kappa'}.$$

Proof. We get (3.8) by using (3.1). □

Corollary 9. *Let γ be a unit speed Legendre curve on Sasakian 3-manifold. If the Legendre curve γ is a circular helix, then the differential equation characterizing the curve γ is*

$$\nabla'^3_\gamma V_2 + (\kappa^2 + \varepsilon)\nabla'_\gamma V_2 = 0. \tag{3.9}$$

Proof. By (3.8), we get (3.9). □

Theorem 3.4. *Let γ be a unit speed curve with Lorentzian metric on Sasakian 3-manifold (M, g). Then γ satisfies the following differential equation*

$$\nabla'^3_\gamma V_3 + \lambda_1 \nabla'^2_\gamma V_3 + \lambda_2 \nabla'_\gamma V_3 + \lambda_3 V_3 = 0, \tag{3.10}$$

where

$$\lambda_1 = -\frac{\kappa'}{\kappa}, \quad \lambda_2 = \kappa^2 + \varepsilon, \quad \lambda_3 = -\frac{\varepsilon\kappa'}{\kappa}.$$

Proof. We get (3.10) by using (3.1). □

Corollary 10. *Let γ be a unit speed Legendre curve on Sasakian 3-manifold. If the Legendre curve γ is a circular helix, then the differential equation characterizing the curve γ is*

$$\nabla'^3_\gamma V_3 + (\kappa^2 + \varepsilon)\nabla'_\gamma V_3 = 0. \tag{3.11}$$

Proof. By (3.10), we get (3.11). □

Özet: Bu çalışmada, contact geometride ortalama eğrilik vektör alanı, Laplasyan operatörünün çekirdeğinde olan biharmonik eğrileri inceleriz. Sasakian 3-uzayında biharmonic eğriler için bazı sonuçlar veririz. Bununla birlikte, aynı uzayda Legendre eğriler için bazı karakterizasyonları veririz.

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