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# Biharmonic hypersurfaces in Riemannian symmetric spaces II

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**ABSTRACT.** We study biharmonic homogeneous hypersurfaces in Riemannian symmetric spaces associated to the exceptional Lie groups $E_6$ and $G_2$ as well as real, complex and quaternion Grassmannian manifolds.

## 1. Introduction

This is a continuation of our previous work [8] (named Part I).

In this paper we study biharmonicity of homogeneous hypersurfaces in the following compact Riemannian symmetric spaces:

- the space $SU(n)/SO(n)$ with $n > 2$ (type AI),
- the space $SU(2n)/Sp(n)$ with $n > 2$ (type AII),
- the real Grassmannian manifolds $\widetilde{Gr}_k(\mathbf{R}^n)$ of oriented $k$-planes in $\mathbf{R}^n$ with $2 < k < n$ (type BDI),
- the space $SO(2n)/U(n)$ with $n > 2$ (type DIII),
- the space $Sp(n)/U(n)$ with $n \geq 2$ (type CI),
- the quaternion Grassmannian manifold $Gr_k(\mathbf{H}^n)$ $(2 \leq k < n)$, (type CII),
- $E_6/SU(6) \cdot SU(2)$ (type EII),
- $E_6/((Spin(10) \times U(1))/\mathbf{Z}_4)$ (type EIII),
- $E_6/F_4$ (type EIV), and
- the space $G_2/SO(4)$ (type G).

In Part I we have studied biharmonic tubes around $Gr_k(\mathbf{C}^{n-1}) \subset Gr_k(\mathbf{C}^n)$, $2 < k \leq n$, biharmonic tubes around $Gr_2(\mathbf{C}^{n+1}) \subset Gr_2(\mathbf{C}^{n+2})$, $(n > 2)$ and biharmonic tubes around $\mathbf{H}P^n \subset Gr_2(\mathbf{C}^{2n+2})$. As a suppliment to Part I, in this Part II, we study biharmonic tubes around $Gr_{k-1}(\mathbf{C}^{n-1})$ in $Gr_k(\mathbf{C}^n)$ for $2 \leq k < n-k$ and $(k,n) \neq (2,2m)$, $m > 2$.

Our results can be summarized as follows:

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THEOREM. *The following compact Riemannian symmetric spaces contain proper biharmonic homogeneous hypersurfaces:*

- *the real Grassmannian manifolds* $\widetilde{\mathrm{Gr}}_k(\mathbf{R}^n)$ *of oriented k-planes,*
- *the complex Grassmannian manifolds* $\mathrm{Gr}_k(\mathbf{C}^n)$,
- *the space* $\mathrm{SO}(2n)/\mathrm{U}(n)$ *with* $n > 2$ (*type* DIII),
- *the space* $\mathrm{Sp}(n)/\mathrm{U}(n)$ *with* $n \geq 2$ (*type* CI),
- *the quaternion Grassmannian manifolds* $\mathrm{Gr}_k(\mathbf{H}^n)$,
- *the space* $\mathrm{E}_6/((\mathrm{Spin}(10) \times \mathrm{U}(1))/\mathbf{Z}_4)$ (*type* EIII) *and*
- *the space* $\mathrm{G}_2/\mathrm{SO}(4)$ (*type* G).

This Part II is organized as follows. First we recall basic facts on biharmonic map theory in Section 2. In particular we recall a criterion of biharmonicity of constant mean curvature hypersurfaces in Einstein manifolds due to Ou [15]. Next in Section 3, we prepare useful formulas of orbits in compact Riemannian symmetric spaces under cohomogeneity one actions due to Verhóczki [18]. In the next ten successive sections, we study biharmonicity of homogeneous hypersurfaces in compact Riemannian symmetric spaces of type AI, AII, AIII, BDI, DIII, CI, CII, EII, EIII and EIV by using the principal curvature formulas prepared in Section 3. In the final section, we study biharmonicity of homogeneous hypersurfaces in the quaternionic symmetric space $\mathrm{G}_2/\mathrm{SO}(4)$.

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## 2. Preliminaries

Here we recall basic ingredients of biharmonic map theory.

**2.1.** Let $(M^m, g)$ and $(N^n, \tilde{g})$ be Riemannian manifolds and $\phi : M \to N$ a smooth map. Then $\phi$ is said to be *harmonic* if it is a critical point of the *energy functional*:

$$E(\phi) = \int \frac{1}{2} |\mathrm{d}\phi|^2 \mathrm{d}v_g.$$

The Euler-Lagrange equation of this variational problem is

$$\tau(\phi) = 0$$

with respect to any compact-supported variations through $\phi$. The vector field $\tau(\phi)$ along $\phi$ is called the *tension field* of $\phi$ and defined by

$$\tau(\phi) = \sum_{i=1}^{m} \{\tilde{\nabla}_{\mathrm{d}\phi(e_i)}\, \mathrm{d}\phi(e_i) - \mathrm{d}\phi(\nabla_{e_i} e_i)\}.$$

Here $\nabla$ and $\tilde{\nabla}$ are the Levi-Civita connections of $M$ and $N$, respectively and $\{e_1, e_2, \dots, e_m\}$ is a local orthonormal frame field of $M$.

More generally, a smooth map $\phi$ is said to be *biharmonic* if it is a critical point of the *bienergy functional*

$$E_2(\phi) = \int \frac{1}{2} |\tau(\phi)|^2 \mathrm{d}v_g.$$

The Euler-Lagrange equation of this variational problem is

$$\Delta_\phi \tau(\phi) + \sum_{i=1}^{m} \tilde{R}(\mathrm{d}\phi(e_i), \tau(\phi))\mathrm{d}\phi(e_i) = 0. \tag{1}$$

Here $\tilde{R}$ is the Riemannian curvature of $N$. The operator $\Delta_\phi$ is the *rough Laplacian* acting on the space $\Gamma(\phi^* TN)$ of all smooth vector fields along $\phi$ defined by

$$\Delta_\phi := -\sum_{i=1}^{m} \{\tilde{\nabla}_{\mathrm{d}\phi(e_i)} \tilde{\nabla}_{\mathrm{d}\phi(e_i)} - \tilde{\nabla}_{\mathrm{d}\phi(\nabla_{e_i} e_i)}\},$$

where $\{e_1, e_2, \dots, e_m\}$ is a local orthonormal frame field on $M$ as before.

**2.2.** In case that $\phi : (M^m, g) \to (N^{m+1}, \tilde{g})$ is an isometric immersion of codimension 1, the mean curvature vector field $\boldsymbol{H}$ and the tension field are related by $\tau(\phi) = m\boldsymbol{H}$. This formula implies that a hypersurface immersion $\phi : M \to N$ is minimal if and only if it is a harmonic map.

Since the harmonicity of isometric immersions is equivalent to minimality of isometric immersions, biharmonic isometric immersions are regarded as generalizations of minimal immersions.

In [15], Ou obtained the following criterion for biharmonicity of hypersurfaces in Einstein manifolds.

Theorem 1 ([15]). *Let $\phi : (M^m, g) \to (N^{m+1}, \tilde{g})$ $(m \geq 2)$ be a hypersurface with shape operator $A$ in an Einstein manifold $N$ with $\widetilde{\mathrm{Ric}} = \lambda \tilde{g}$. Assume that the mean curvature $H = |\boldsymbol{H}|$ of the hypersurface is constant. Then $\phi$ is biharmonic if and only if either $\phi$ is minimal or non-minimal with*

$$|A|^2 = \lambda. \tag{2}$$

*Furthermore, in the latter case, both the ambient space and the hypersurface must have positive scalar curvatures:*

$$\tilde{\rho} = (m+1)\lambda > 0, \qquad \rho = (m-2)\lambda + m^2 H^2 > 0.$$

**2.3.** Hereafter we assume that the ambient space $(N, \tilde{g})$ is an irreducible compact Riemannian symmetric space $G/K$ with compact semi-simple $G$. Let us denote by $B$ the Killing form of the Lie algebra $\mathfrak{g}$ of $G$. Then $B$ is negative definite on $\mathfrak{g}$ since $G$ is semi-simple. Thus $-B$ is an $\mathrm{Ad}(G)$-invariant inner product on $\mathfrak{g}$. Moreover the tangent space $T_oN$ of $N$ at the origin $o = K$ is identified with the orthogonal complement $\mathfrak{p}$ of the Lie algebra $\mathfrak{k}$ of $K$ in $\mathfrak{g}$. The orthogonal decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

is a reductive decomposition of $\mathfrak{g}$, that is, $\mathfrak{p}$ satisfies $\mathrm{ad}(\mathfrak{k})\mathfrak{p} \subset \mathfrak{p}$. Moreover, since $N$ is a symmetric space, we have

$$[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}.$$

The restriction $-B|_{\mathfrak{p}}$ of $-B$ to $\mathfrak{p}$ induces a $G$-invariant Riemannian metric $\tilde{g}$ on $N$. This Riemannian metric is called the *Killing metric* of $N$. The *rank* of a Riemannian symmetric space $N = G/K$ is the maximum dimension of a flat totally geodesic submanifold of $N$. The Ricci tensor $\widetilde{\mathrm{Ric}}$ of $N$ with respect to the Killing metric $\tilde{g}$ computed at the origin is

$$\widetilde{\mathrm{Ric}}_o = -\frac{1}{2} B|_{\mathfrak{p}}.$$

This formula shows that $N$ is an Einstein manifold. Ou's criterion is rephrased as:

Theorem 2. *Let $N = G/K$ be a compact semi-simple Riemannian symmetric space equipped with the Killing metric. Then a hypersurface $\phi : M \to G/K$ with non-zero constant mean curvature is proper biharmonic if and only if its shape operator $A$ has constant square norm*

$$|A|^2 = \frac{1}{2}.$$

## 3. Cohomogeneity one actions

Let $\psi : L \times N \to N$ be an isometric action of a compact connected Lie group $L$ on a Riemannian manifold $N = (N, \tilde{g})$.

An orbit $L(p)$ of a point $p \in N$ is said to be *principal* if for any $q \in N$, there exists an element $g \in L$ such that the isotropy subgroup $L_p$ satisfies $L_p \subset g L_q g^{-1}$. By definition principal orbits are orbits of maximum dimension. The *cohomogeneity* of the action $\psi$ is the codimension of principal orbits.

An orbit $L(q)$ is said to be *singular* if its dimension is less than that of the principal orbits.

A closed totally geodesic submanifold $C$ of $N$ is said to be a *section* if $C$ intersects orthogonally all the orbits of $L$, and in this case the action is called *polar*. An isometric action $\psi$ is said to be *hyperpolar* if it admits sections which are flat totally geodesic submanifolds. Actions of isotropy subgroups on Riemannian symmetric spaces are typical examples of hyperpolar actions.

Now let $N = G/K$ be a Riemannian symmetric space of compact type with *simply connected* $G$ and the metric $\tilde{g}$ which is induced from the inner product $-c^2 B$ on the Lie algebra $\mathfrak{g}$ of $G$. Here $B$ is the Killing form of $\mathfrak{g}$ and $c > 0$ a constant. Denote by $\sigma$ the corresponding involution, *i.e.*, $\sigma$ characterize $K$ as $K = G_\sigma = \{g \in G \mid \sigma(g) = g\}$.

A connected subgroup $L$ of $G$ is called *symmetric* if there exists an involutive automorphism $\tau$ of $G$ such that $L$ coincides with $G_\tau = \{g \in G \mid \tau(g) = g\}$. In this paper we assume that the involution $\tau$ commutes with $\sigma$.

Note that $L$ is a connected compact Lie group since $G$ is simply connected. The natural action of $L$ on $G/K$ is called the *Hermann action*.

Kollross classified hyperpolar isometric actions on compact Riemannian symmetric spaces [14]. It should be remarked that *cohomogeneity one* isometric actions on compact irreducible Riemannian symmetric spaces are always hyperpolar.

We denote by $L(o)$ the orbit of the origin under the induced action of $L$. Then $L(o) = L/(L \cap K)$ is a Riemannian symmetric space and totally geodesic in $G/K$. In our setting, $\tau\sigma$ is an involution on $G$ since $\tau$ and $\sigma$ commute with each other. Let us take a symmetric subgroup $H = G_{\tau\sigma}$. Then the orbit $H(o) = H/(H \cap K)$ is also a totally geodesic submanifold of $G/K$. It is known that the $L$-action is cohomogeneity one if and only if $H(o)$ is a Riemannian symmetric space of rank 1.

Remark 1. For the classification of totally geodesic submanifolds in compact Riemannian symmetric spaces of rank 2, we refer to [5, 9, 10, 11, 12, 13].

Proposition 1 ([18]). *If the action of $L$ is of cohomogeneity one and $L(o)$ is a singular orbit, then the other orbits of $L$ coincide with the tubular hypersurfaces around $L(o)$.*

Thus hereafter we assume that $L$-action is of cohomogeneity one with *singular orbit* $L(o)$.

Take the normal bundle $T^\perp L(o)$ in $N = G/K$. At the origin, we have

$$T_o G/K = T_o L(o) \oplus T_o^\perp L(o), \qquad T_o^\perp L(o) = T_o H(o).$$

Let $u$ be a unit vector in $T_o^\perp L(o)$ and $\gamma(t) = \exp(tu)$ the closed geodesic with initial velocity $u$. Then the image $C$ of $\gamma$ intersects orthogonally all the orbits of $L$. Let $a$ be the maximal eigenvalue of the Jacobi operator $R_u$ on $T_oL(o)$ and $h$ the length of $C$. We put $r = \min\{\pi/(2\sqrt{a}), h/2\}$.

Consider the *principal orbit* $M_r := L(\gamma(r))$ which is realized as a tube of radius $r \in (0, r)$ around the singular orbit $L(o)$. Then the principal curvature of $M_r$ can be computed in the following way (see *e.g.*, [18]):

(1) Take non-negative eigenvalues $a_1, a_2, \ldots, a_s$ of $R_u$ on $T_oL(o)$. Denote by $m_1, m_2, \ldots, m_s$ the multiplicities of these eigenvalues.

(2) Take eigenvalues $b_1 = \chi$, $b_2 = \chi/4$, $b_3 = 0$ of $R_u$ on $T_o^{\perp}L(o)$. Denote by $k_1, k_2, k_3 = 1$ the multiplicities of these eigenvalues. Here $\chi$ is the maximal sectional curvature of the Riemannian symmetric space $H(o)$ of rank 1.

(3) When $H(o)$ is of constant curvature, *i.e.*, $H(o)$ is the $\ell$-sphere $\mathbf{S}^{\ell}$ or real projective $\ell$-space $\mathbf{R}P^{\ell}$, then $k_2 = 0$.

(4) When $H(o) = \mathbf{F}P^{\ell}$ with $\mathbf{F} = \mathbf{C}$ (the field of complex numbers), $\mathbf{H}$ (the skew field of quaternions) or $\mathfrak{O}$ (the Cayley algebra), then $k_1 = \dim_{\mathbf{R}} \mathbf{F} - 1$.

Note that $h = 2\pi/\sqrt{\chi}$ if $H(o)$ is not a real projective space. In case $H(o) = \mathbf{R}P^{\ell}$ $(\ell \geq 2)$, $h = \pi/\sqrt{\chi}$. By using these data, principal curvatures of $M_r$ is computed as follows:

Theorem 3 ([17, 18]). *The constant principal curvatures of $M_r$ are*

$$\mu_i = \sqrt{a_i}\tan(\sqrt{a_i}r), \qquad i = 1, 2, \ldots, s$$

*with multiplicity $m_i$ and*

$$\hat{\mu}_j = -\sqrt{b_j}\cot(\sqrt{b_j}r), \qquad j = 1, 2$$

*with multiplicity $\hat{m}_j = k_j$.*

## 4. Riemannian symmetric space of type AI

**4.1.** Let us consider the Riemannian symmetric space $\mathrm{AI}(n) := \mathrm{SU}(n)/\mathrm{SO}(n)$ with $n > 2$. With respect to the Killing metric, this symmetric space is an $(n+2)(n-1)/2$-dimensional Riemannian symmetric space of rank $n-1$. The maximal sectional curvature is $\kappa = 1/n$. Totally geodesic singular orbits under cohomogeneity one actions are ([2, 14, 18]):

- $\{\mathrm{AI}(n-1) \times \mathrm{U}(1)\}/\mathbf{Z}_{n-1}$ for $n \neq 4$ and
- $\{\mathrm{AI}(3) \times \mathrm{U}(1)\}/\mathbf{Z}_3$ and $\widetilde{\mathrm{Gr}}_2(\mathbf{R}^5)$ for $n = 4$. In this case $\mathrm{AI}(4) = \widetilde{\mathrm{Gr}}_3(\mathbf{R}^6)$ because of the isomorphism $\mathrm{SU}(4) \cong \mathrm{Spin}(6)$.

**4.2.** In this section we consider biharmonic tubes around the singular orbit $\{\mathrm{AI}(n-1) \times \mathrm{U}(1)\}/\mathbf{Z}_{n-1}$ with the symmetric subgroup $L = \mathrm{S}(\mathrm{U}(n-1) \times$

U(1)). Then $H = \mathrm{SO}(n)$ and $H(o) = \mathbf{R}P^{n-1}$. Moreover, as in [18], we have

$$a_1 = \kappa, \qquad a_2 = \kappa/4, \qquad a_3 = 0, \qquad m_1 = 1, \qquad m_2 = n-2$$

and $r = \pi/(2\sqrt{\kappa})$. For a positive $r < \pi/(2\sqrt{\kappa})$, the tube around $\{\mathrm{AI}(n-1) \times \mathrm{U}(1)\}/\mathbf{Z}_{n-1}$ is a homogeneous hypersurface with principal curvatures:

$$\mu_1 = \sqrt{\kappa}\tan(\sqrt{\kappa}r), \qquad \mu_2 = \frac{\sqrt{\kappa}}{2}\tan\left(\frac{\sqrt{\kappa}}{2}r\right), \qquad \mu_3 = 0,$$

$$\hat{\mu}_1 = -\frac{\sqrt{\kappa}}{2}\cot\left(\frac{\sqrt{\kappa}}{2}r\right), \qquad \hat{\mu}_2 = -\frac{\sqrt{\kappa}}{4}\cot\left(\frac{\sqrt{\kappa}}{4}r\right)$$

with multiplicities

$$m_1 = 1, \qquad m_2 = n-2, \qquad m_3 = (n-1)(n-2)/2, \qquad \hat{m}_1 = n-2, \qquad \hat{m}_2 = 0.$$

Theorem 4. *The tube around the singular orbit* $\{\mathrm{AI}(n-1) \times \mathrm{U}(1)\}/\mathbf{Z}_{n-1}$ *of radius*

$$r = \sqrt{n}\tan^{-1}\sqrt{n-2}$$

*is minimal in* $\mathrm{AI}(n) = \mathrm{SU}(n)/\mathrm{SO}(n)$ $(n > 2)$.

*The only biharmonic tubes around* $\{\mathrm{AI}(n-1) \times \mathrm{U}(1)\}/\mathbf{Z}_{n-1}$ *in* $\mathrm{AI}(n)$ $(n > 2)$ *are the minimal ones.*

Proof. The mean curvature $H$ of the tube around the singular orbit $\{\mathrm{AI}(n-1) \times \mathrm{U}(1)\}/\mathbf{Z}_{n-1}$ is computed as

$$\begin{aligned}\frac{1}{2}(n^2+n-4)H &= \mu_1 + (n-2)\mu_2 + (n-2)\hat{\mu}_1 \\ &= \sqrt{\kappa}\left\{\tan(\sqrt{\kappa}r) + \frac{n-2}{2}\tan\left(\frac{\sqrt{\kappa}}{2}r\right) - \frac{n-2}{2}\cot\left(\frac{\sqrt{\kappa}}{2}r\right)\right\}.\end{aligned}$$

Now we put $t = \tan(\sqrt{\kappa}r/2)$. Then we have

$$\frac{1}{2}(n^2+n-4)H = -\frac{\sqrt{\kappa}}{2t(1-t^2)}\{(n-2)t^4 - 2nt^2 + n - 2\}.$$

Since $t^2 < 1$, we obtain that $M_r$ is minimal if and only if

$$r = \frac{2}{\sqrt{\kappa}}\tan^{-1}\left(\sqrt{\frac{n-2\sqrt{n-1}}{n-2}}\right) = \sqrt{n}\tan^{-1}\sqrt{n-2}.$$

Next, the square norm $|A|^2$ is

$$\begin{aligned}|A|^2 &= \mu_1^2 + (n-2)\mu_2^2 + (n-2)\hat{\mu}_1^2 \\ &= \kappa \tan^2(\sqrt{\kappa}r) + \frac{(n-2)\kappa}{4}\tan^2\frac{\sqrt{\kappa}r}{2} + \frac{(n-2)\kappa}{4}\cot^2\frac{\sqrt{\kappa}r}{2} \\ &= \kappa\left\{\frac{4t^2}{(1-t^2)^2} + \frac{(n-2)t^2}{4} + \frac{(n-2)}{4t^2}\right\} \\ &= \frac{\kappa}{4t^2(1-t^2)^2}\{16t^4 + (n-2)t^4(1-t^2)^2 + (n-2)(1-t^2)^2\}.\end{aligned}$$

Now we consider the biharmonicity equation $|A|^2 = 1/2$:

$$16t^4 + (n-2)t^4(1-t^2)^2 + (n-2)(1-t^2)^2 = 2nt^2(1-t^2)^2.$$

Hence we have

$$n = \frac{2(t^8 - 2t^6 - 6t^4 - 2t^2 + 1)}{(t^2-1)^4},$$

which implies

$$\begin{aligned}(t^2-1)^4(3-n) &= t^8 - 8t^6 + 30t^4 - 8t^2 + 1 \\ &= (t^4 - 4t^2 + 1)^2 + 12t^4.\end{aligned}$$

The equation has no real solutions satisfying $0 < t < 1$ for $n \geq 3$. □

## 5. Riemannian symmetric space of type AII

**5.1.** Let us consider the compact Riemannian symmetric space $\mathrm{AII}(n) := \mathrm{SU}(2n)/\mathrm{Sp}(n)$ of type AII ($n \geq 2$) equipped with the Killing metric. This Riemannian symmetric space is $(n-1)(2n+1)$-dimensional and of rank $n-1$. The maximal sectional curvature is $\kappa = 1/(4n)$. Moreover we have $r = \pi/(2\sqrt{\kappa})$. Since $\mathrm{AII}(2) = \mathbf{S}^5(\kappa)$, hereafter we assume that $n > 2$. Totally geodesic singular orbits under cohomogeneity one actions are ([2, 14, 18]):

- $\{\mathrm{AII}(n-1) \times \mathrm{U}(1)\}/\mathbf{Z}_{n-1}$, $n > 3$.
- $\{\mathrm{AII}(2) \times \mathrm{U}(1)\}/\mathbf{Z}_2 = (\mathbf{S}^5(\kappa) \times \mathbf{S}^1)/\mathbf{Z}_2$ and $\mathrm{SU}(3)$, $n = 3$.

**5.2.** In this section we consider biharmonicity of tubes around the singular orbit $\{\mathrm{AII}(n-1) \times \mathrm{U}(1)\}/\mathbf{Z}_{n-1}$ of dimension $(2n^2 - 5n + 3)$ with the corresponding symmetric subgroup $L = \mathrm{S}(\mathrm{U}(2n-2) \times \mathrm{U}(2))$. Then we have $H = \mathrm{Sp}(n)$ and $H(o) = \mathbf{H}P^{n-1}$ of maximal sectional curvature $\kappa$.

The eigenvalues of Jacobi operator are given by ([18]):

$$a_1 = \kappa, \qquad a_2 = \frac{\kappa}{4}, \qquad a_3 = 0$$

with multiplicities $m_1 = 1$ and $m_2 = 4n - 8$. From these, we obtain the principal curvatures and their multiplicities of the tube:

$$\mu_1 = \sqrt{\kappa}\tan(\sqrt{\kappa}r), \qquad m_1 = 1,$$

$$\mu_2 = \frac{\sqrt{\kappa}}{2}\tan\left(\frac{\sqrt{\kappa}}{2}r\right), \qquad m_2 = 4n - 8,$$

$$\mu_3 = 0, \qquad m_3 = 2n^2 - 9n + 10,$$

$$\hat{\mu}_1 = -\sqrt{\kappa}\cot(\sqrt{\kappa}r), \qquad \hat{m}_1 = 3,$$

$$\hat{\mu}_2 = -\frac{\sqrt{\kappa}}{2}\cot\left(\frac{\sqrt{\kappa}}{2}r\right), \qquad \hat{m}_2 = 4n - 8.$$

THEOREM 5. *A tube around the singular orbit* $\{\mathrm{AII}(n-1)\times \mathrm{U}(1)\}/\mathbf{Z}_{n-1}$ *of radius* $r$ *in* $\mathrm{AII}(n) = \mathrm{SU}(2n)/\mathrm{Sp}(n)$ $(n \geq 3)$ *is minimal if and only if*

$$r = 2\sqrt{n}\tan^{-1}\sqrt{4n-5}.$$

*The only biharmonic tubes around* $\{\mathrm{AII}(n-1)\times \mathrm{U}(1)\}/\mathbf{Z}_{n-1}$ *are minimal ones.*

PROOF. The mean curvature $H$ is

$$\begin{aligned}(2n^2 - n - 2)H &= \mu_1 + 4(n-2)\mu_2 + 3\hat{\mu}_1 + 4(n-2)\hat{\mu}_2 \\ &= \sqrt{\kappa}\left\{\tan(\sqrt{\kappa}r) + 2(n-2)\tan\left(\frac{\sqrt{\kappa}}{2}\right)\right. \\ &\qquad \left. - 3\cot(\sqrt{\kappa}r) - 2(n-2)\cot\left(\frac{\sqrt{\kappa}}{2}\right)\right\} \\ &= \sqrt{\kappa}\left\{2(n-2)\left(t - \frac{1}{t}\right) + \frac{-3t^4 + 10t^2 - 3}{2t(1-t^2)}\right\},\end{aligned}$$

where $t = \tan(\sqrt{\kappa}r/2)$.

Thus, it follows from $0 < t < 1$ that $M_r$ is minimal if and only if

$$r = \frac{2}{\sqrt{\kappa}}\tan^{-1}\sqrt{\frac{4n-3-4\sqrt{n-1}}{4n-5}} = 2\sqrt{n}\tan^{-1}\sqrt{4n-5}.$$

The square norm $|A|^2$ is computed as

$$\begin{aligned}
|A|^2 &= \mu_1^2 + 4(n-2)\mu_2^2 + 3\hat{\mu}_1^2 + 4(n-2)\hat{\mu}_2^2 \\
&= \kappa\left\{\tan^2(\sqrt{\kappa}r) + (n-2)\tan^2\left(\frac{\sqrt{\kappa}}{2}r\right)\right. \\
&\qquad \left. + 3\cot^2(\sqrt{\kappa}r) + (n-2)\cot^2\left(\frac{\sqrt{\kappa}}{2}r\right)\right\} \\
&= \kappa\left\{\frac{4t^2}{(1-t^2)^2} + (n-2)t^2 + \frac{3(1-t^2)^2}{4t^2} + \frac{n-2}{t^2}\right\}.
\end{aligned}$$

The biharmonicity equation $|A|^2 = 1/2$ is equivalent to

$$n = \frac{5t^8 - 4t^6 - 18t^4 - 4t^2 + 5}{4(1-t^2)^4}.$$

It follows that

$$\begin{aligned}
4(1-t^2)^4(3-n) &= 7t^8 - 44t^6 + 90t^4 - 44t^2 + 7 \\
&= (\sqrt{7}t^4 - 9t^2 + \sqrt{7})^2 \\
&\quad + t^2\{(18\sqrt{7} - 44)t^4 - 5t^2 + (18\sqrt{7} - 44)\} > 0.
\end{aligned}$$

Therefore, the equation has no real solutions satisfying $0 < t < 1$ for $n \geq 3$. □

## 6. Riemannian symmetric space of type AIII

**6.1.** Let us denote by $\mathrm{Gr}_k(\mathbf{C}^n)$ the Grassmannian manifold of all complex linear $k$-subspaces in complex Euclidean $n$-space $\mathbf{C}^n$. The Grassmannian manifold $\mathrm{Gr}_k(\mathbf{C}^n)$ is represented by $\mathrm{Gr}_k(\mathbf{C}^n) = \mathrm{SU}(n)/\mathrm{S}(\mathrm{U}(k) \times \mathrm{U}(n-k))$ as a homogeneous space. We equip the Grassmannian manifold $\mathrm{Gr}_k(\mathbf{C}^n)$ with the Killing metric $\tilde{g}$ induced from $-B$. Then the resulting homogeneous Riemannian space is a real $2k(n-k)$-dimensional compact Riemannian symmetric space of rank $\min(k, n-k)$. Moreover $\mathrm{Gr}_k(\mathbf{C}^n)$ admits a $\mathrm{SU}(n)$-invariant complex structure $J$ which is compatible to the metric $\tilde{g}$. Hence $(\mathrm{Gr}_k(\mathbf{C}^n), \tilde{g}, J)$ is a Hermitian symmetric space of type AIII. The maximal sectional curvature is $\kappa = 1/n$.

**6.2.** Totally geodesic singular orbits in $\mathrm{Gr}_k(\mathbf{C}^n)$ under cohomogeneity one actions are ([2, 14, 18]):

(1) $\mathrm{Gr}_k(\mathbf{C}^{n-1})$ and $\mathrm{Gr}_{k-1}(\mathbf{C}^{n-1})$ for $2 \leq k < n-k$, $(k, n) \neq (2, 2m)$ for $m > 2$.

(2) $\mathrm{Gr}_{k-1}(\mathbf{C}^{2k-1}) = \mathrm{Gr}_k(\mathbf{C}^{2k-1})$, if $n = 2k$ and $k \geq 3$.
(3) $\mathrm{Gr}_2(\mathbf{C}^{2l-1})$, $\mathbf{C}P^{2l-2}$ and the quaternion projective space $\mathbf{H}P^{l-1}$ if $k = 2$ and $n = 2l$.

In Part I, we have classified:
(1) biharmonic tubes around $\mathrm{Gr}_k(\mathbf{C}^{n-1}) \subset \mathrm{Gr}_k(\mathbf{C}^n)$, $2 < k \leq n$ ([Theorem 4, Part1]),
(2) biharmonic tubes around $\mathrm{Gr}_2(\mathbf{C}^{n+1}) \subset \mathrm{Gr}_2(\mathbf{C}^{n+2})$, $n > 2$ ([Theorem 5, Part1]),
(3) biharmonic tubes around $\mathbf{H}P^n \subset \mathrm{Gr}_2(\mathbf{C}^{2n+2})$, $n > 2$ ([Theorem 6, Part1]).

In this section we study biharmonic tubes around $\mathrm{Gr}_{k-1}(\mathbf{C}^{n-1})$ in $\mathrm{Gr}_k(\mathbf{C}^n)$ for $2 < k < n-k$ and $(k,n) \neq (2,2m)$, $m > 2$. The symmetric subgroup $L$ is $L = S(U(1) \times U(n-1))$. Hence $H = S(U(k-1) \times U(n-k+1))$ and $H(o) = \mathbf{C}P^{n-k}$. The maximal sectional curvature of $H(o)$ is $\chi = \kappa$. Moreover we have $r = \pi/\sqrt{\kappa}$.

The eigenvalue of Jacobi operator are given by ([18]):

$$a_1 = \frac{\kappa}{4}, \qquad a_2 = 0$$

with multiplicity $m_1 = 2k-2$. Hence the principal curvature of the tube are given by

$$\mu_1 = \frac{\sqrt{\kappa}}{2}\tan\left(\frac{\sqrt{\kappa}}{2}r\right), \qquad \mu_2 = 0,$$

$$\hat{\mu}_1 = -\sqrt{\kappa}\cot(\sqrt{\kappa}r), \qquad \hat{\mu}_2 = -\frac{\sqrt{\kappa}}{2}\cot\left(\frac{\sqrt{\kappa}}{2}r\right)$$

with multiplicities

$$m_1 = 2(k-1), \qquad m_2 = 2(k-1)(n-k-1), \qquad \hat{m}_1 = 1, \qquad \hat{m}_2 = 2(n-k-1).$$

Theorem 6. *A tube $M_r$ around $\mathrm{Gr}_{k-1}(\mathbf{C}^{n-1})$ of radius $r$ in $\mathrm{Gr}_k(\mathbf{C}^n)$ $(2 \leq k \leq n-k)$ with $(k,n) \neq (2,2m)$, $m > 2$ is minimal if and only if*

$$r = 2\sqrt{n}\tan^{-1}\left(\sqrt{\frac{2n-2k-1}{2k-1}}\right).$$

*A tube $M_r$ is proper biharmonic if and only if*

$$r = 2\sqrt{n}\tan^{-1}\left(\sqrt{\frac{n+1 \pm \sqrt{(n-2k)^2+4n}}{2k-1}}\right).$$

PROOF. The mean curvature $H$ is computed as

$$(2nk - 2k^2 - 1)H = 2(k-1)\mu_1 + \hat{\mu}_1 + 2(n-k-1)\hat{\mu}_2$$
$$= \sqrt{\kappa}\left\{(k-1)t - \frac{1-t^2}{2t} - \frac{n-k-1}{t}\right\},$$

where $t = \tan(\sqrt{\kappa}r/2)$. The equation $H = 0$ implies that $M_r$ is minimal if and only if

$$r = \frac{2}{\sqrt{\kappa}} \tan^{-1}\left(\sqrt{\frac{2n-2k-1}{2k-1}}\right) = 2\sqrt{n} \tan^{-1}\left(\sqrt{\frac{2n-2k-1}{2k-1}}\right).$$

Next, the square norm $|A|^2$ is computed as

$$|A|^2 = 2(k-1)\mu_1^2 + \hat{\mu}_1^2 + 2(n-k-1)\hat{\mu}_2^2$$
$$= \kappa\left\{\frac{(k-1)t^2}{2} + \frac{(1-t^2)^2}{4t^2} + \frac{n-k-1}{2t^2}\right\}.$$

The biharmonicity equation $|A|^2 = 1/2$ becomes

$$f(t) := (2k-1)t^4 - 2(n+1)t^2 + 2n - 2k - 1 = 0,$$

which gives us

$$t^2 = \frac{n+1 \pm \sqrt{(n-2k)^2 + 4n}}{2k-1}.$$

Moreover, we have

$$f\left(\sqrt{\frac{2n-2k-1}{2k-1}}\right) = \frac{4(2k-2n+1)}{2k-1} < 0.$$

Thus $M_r$ is proper biharmonic if and only if

$$r = 2\sqrt{n} \tan^{-1}\left(\sqrt{\frac{n+1 \pm \sqrt{(n-2k)^2 + 4n}}{2k-1}}\right). \qquad \square$$

## 7. Riemannian symmetric space of type BDI

**7.1.** Let $N = \mathrm{SO}(n)/\mathrm{SO}(k) \times \mathrm{SO}(n-k)$ be the compact Riemannian symmetric space of type BDI. This symmetric space is the real Grassmannian manifold $\widetilde{\mathrm{Gr}}_k(\mathbf{R}^n)$ of *oriented* $k$-planes in $\mathbf{R}^n$ with $\dim N = k(n-k)$ and of rank $\min(k, n-k)$. With respect to the Killing metric, $\widetilde{\mathrm{Gr}}_k(\mathbf{R}^n)$ has maximal sectional curvature $\kappa = 1/(n-2)$. Even if $\mathrm{SO}(n)$ is not simply connected, the procedure of computing the principal curvatures of tubes around singular orbits still works for this space.

Totally geodesic singular orbits in $\widetilde{\mathrm{Gr}}_k(\mathbf{R}^n)$ under cohomogeneity one actions are (see [2]):

- reflective submanifolds:
  - $\widetilde{\mathrm{Gr}}_{k-1}(\mathbf{R}^{n-1})$ and $\widetilde{\mathrm{Gr}}_k(\mathbf{R}^{n-1})$, if $2 \le k < n-k$ and $(k, n) \neq (2, 2m)$ for $m > 2$.
  - $\widetilde{\mathrm{Gr}}_{k-1}(\mathbf{R}^{2k-1}) = \widetilde{\mathrm{Gr}}_k(\mathbf{R}^{2k-1})$, if $n = 2k$ and $k \ge 4$.
  - $\mathbf{S}^{2l-2}$, $\widetilde{\mathrm{Gr}}_2(\mathbf{R}^{2l-1})$ and $\mathbf{C}P^{l-1}$ if $k = 2$ and $n = 2l$.
  - $\widetilde{\mathrm{Gr}}_3(\mathbf{R}^5)$ and $\mathrm{U}(3)/\mathrm{SO}(3)$ if $k = 3$ and $n = 6$. In this case $\widetilde{\mathrm{Gr}}_3(\mathbf{R}^6) = \mathrm{AI}(4)$.
- non-reflective submanifolds:
  - $\mathrm{G}_2/\mathrm{SO}(4) \subset \widetilde{\mathrm{Gr}}_3(\mathbf{R}^7)$.

**7.2.** In this section we consider tubes around the singular orbit $\widetilde{\mathrm{Gr}}_{k-1}(\mathbf{R}^{n-1})$ with $k > 2$ and $n > 4$. This orbit is obtained by the cohomogeneity one action of the symmetric subgroup $L = \mathrm{SO}(n-1)$. Under the action of $L$, we have $H = \mathrm{SO}(k-1) \times \mathrm{SO}(n-k+1)$ and $H(o) = \mathbf{S}^{n-k}(\chi)$ is the $(n-k)$-sphere of curvature $\chi = \kappa/2$. We have $r = \frac{\pi}{2\sqrt{\chi}}$. The eigenvalues of the Jacobi operator are ([18]):

$$a_1 = \frac{\kappa}{2}, \qquad a_2 = 0$$

with multiplicity $m_1 = k-1$.

The principal curvatures of the tube $M_r$ of radius $r$ around $L(o)$ are

$$\mu_1 = \sqrt{\frac{\kappa}{2}} \tan\left(\sqrt{\frac{\kappa}{2}} r\right), \qquad \mu_2 = 0, \qquad \hat{\mu}_1 = -\sqrt{\frac{\kappa}{2}} \cot\left(\sqrt{\frac{\kappa}{2}} r\right)$$

with multiplicities

$$m_1 = k-1, \qquad m_2 = (k-1)(n-k-1), \qquad \hat{m}_1 = n-k-1.$$

Theorem 7. *A tube $M_r$ around $\widetilde{\mathrm{Gr}}_{k-1}(\mathbf{R}^{n-1})$ of radius $r$ in $\widetilde{\mathrm{Gr}}_k(\mathbf{R}^n)$ $(2 \le k < n)$ is minimal if and only if*

$$r = \sqrt{2(n-2)} \tan^{-1}\left(\sqrt{\frac{n-k-1}{k-1}}\right).$$

*In case $n \neq 2k$, $M_r$ is proper biharmonic if and only if*

$$r = \frac{\pi}{4}\sqrt{2(n-2)}.$$

*In case $n = 2k$, the only biharmonic tubes are minimal ones.*

Proof. The mean curvature is computed as

$$\begin{aligned}(nk - k^2 - 1)H &= (k-1)\mu_1 + (n-k-1)\hat{\mu}_1 \\ &= \sqrt{\frac{\kappa}{2}}\left\{(k-1)t - \frac{n-k-1}{t}\right\},\end{aligned}$$

where $t = \tan(\sqrt{\kappa/2}r)$.

Hence, $H = 0$ if and only if

$$r = \sqrt{\frac{2}{\kappa}}\tan^{-1}\left(\sqrt{\frac{n-k-1}{k-1}}\right).$$

Note that in case $k = n-k$, *i.e.*, $n = 2k$, two singular orbits $\widetilde{\mathrm{Gr}}_k(\mathbf{R}^{n-1})$ and $\widetilde{\mathrm{Gr}}_{k-1}(\mathbf{R}^{n-1})$ coincide. The mean curvature is

$$H = \frac{1}{k+1}\sqrt{\frac{\kappa}{2}}\left(t - \frac{1}{t}\right).$$

Hence when $n = 2k$, $M_r$ is minimal if and only if

$$r = \frac{\pi}{2\sqrt{2\kappa}} < \frac{\pi}{\sqrt{2\kappa}}.$$

Next, the square norm $|A|^2$ is computed as

$$|A|^2 = \frac{\kappa}{2}\left\{(k-1)t^2 + \frac{n-k-1}{t^2}\right\}.$$

By solving the biharmonicity equation $|A|^2 = 1/2$, we get

$$t^2 = 1, \qquad \frac{n-k-1}{k-1}.$$

As shown above, if $t^2 = \frac{n-k-1}{k-1}$, then $M_r$ is minimal. Next we notice that $\frac{n-k-1}{k-1} = 1$ when $n = 2k$. Thus $M_r$ is proper biharmonic if and only if $n \neq 2k$ and

$$r = \frac{\pi}{4}\sqrt{2(n-2)}. \qquad \square$$

Remark 2. In case $k = 2$ and $n = 2l$, $\widetilde{\mathrm{Gr}}_2(\mathbf{R}^{2l})$ is the complex quadric $\mathcal{Q}_{2l-2} \subset \mathbf{C}P^{2l-1}$. Biharmonic tubes around $\mathbf{C}P^{l-1}$ are classified in [Theorem 7, Part I].

## 8. Riemannian symmetric space of type DIII

**8.1.** In this section we consider the Riemannian symmetric space $N = G/K$ of type DIII ($n > 2$). The Riemannian symmetric space $\mathrm{DIII}(n) := \mathrm{SO}(2n)/\mathrm{U}(n)$ is $n(n-1)$-dimensional and of rank $\lfloor n/2 \rfloor$. The maximal sectional curvature is $\kappa = 1/\{2(n-1)\}$ with respect to the Killing metric. As in Section 7, the procedure of computing the principal curvatures of tubes around singular orbits still works for this space. Since $\mathrm{DIII}(3) = \mathbf{C}P^3$ and $\mathrm{DIII}(4) = \widetilde{\mathrm{Gr}}_2(\mathbf{R}^8)$ is the complex quadric $\mathcal{Q}_6 \subset \mathbf{C}P^7$, hereafter we restrict our attention to the case $n > 4$.

**8.2.** Totally geodesic singular orbits in $\mathrm{DIII}(n)$ under cohomogeneity one actions are congruent to $\mathrm{DIII}(n-1)$ (see *e.g.* [2, 14]). The corresponding symmetric subgroup is $L = \mathrm{SO}(2n-2) \times \mathrm{SO}(2)$ of maximal sectional curvature $\kappa$. In this case $H = \mathrm{U}(n)$ and $H(o) = \mathbf{C}P^{n-1}$ of maximal sectional curvature $\chi = \kappa$ and $r = \pi/\sqrt{\kappa} = \sqrt{2(n-1)}\pi$.

The eigenvalues of the Jacobi operator on $T_oL(o)$ are

$$a_1 = \frac{\kappa}{4}, \qquad a_2 = 0.$$

The multiplicity of $a_1$ is $m_1 = 2(n-2)$ (see [18]). From these data together with Theorem 3, the principal curvatures of the tube $M_r$ around $\mathrm{DIII}(n-1)$ are given by

$$\mu_1 = \frac{\sqrt{\kappa}}{2}\tan\left(\frac{\sqrt{\kappa}}{2}r\right), \qquad \mu_2 = 0,$$

$$\hat{\mu}_1 = -\sqrt{\kappa}\cot(\sqrt{\kappa}r), \qquad \hat{\mu}_2 = -\frac{\sqrt{\kappa}}{2}\cot\left(\frac{\sqrt{\kappa}}{2}r\right)$$

with multiplicities

$$m_1 = 2(n-2), \qquad m_2 = (n-2)(n-3), \qquad \hat{\mu}_1 = 1, \qquad \hat{\mu}_2 = 2(n-2).$$

Theorem 8. *A tube $M_r$ around $\mathrm{DIII}(n-1)$ of radius $r$ in $\mathrm{DIII}(n)$ is minimal if and only if $r = \sqrt{n-1}\pi/\sqrt{2}$.*

*A tube $M_r$ is proper biharmonic if and only if*

$$r = 2\sqrt{2n-2}\tan^{-1}\left(\frac{\sqrt{2n-2} \pm 1}{\sqrt{2n-3}}\right).$$

Proof. The mean curvature $H$ of $M_r$ is computed as

$$\begin{aligned}(n^2-n-1)H &= 2(n-2)\mu_1+\hat{\mu}_1+2(n-2)\hat{\mu}_2\\ &= (n-2)\sqrt{\kappa}\tan\left(\frac{\sqrt{\kappa}}{2}r\right)-\sqrt{\kappa}\cot(\sqrt{\kappa}r)-(n-2)\cot\left(\frac{\sqrt{\kappa}}{2}r\right)\\ &= \sqrt{\kappa}\left\{(n-2)t-\frac{1-t^2}{2t}-(n-2)\frac{1}{t}\right\}\\ &= \frac{\sqrt{\kappa}}{2t}(2n-3)(t^2-1),\end{aligned}$$

where $t=\tan(\sqrt{\kappa}r/2)$. Thus $M_r$ is minimal if and only if $t=1$. Namely,

$$r=\frac{2}{\sqrt{\kappa}}\tan^{-1}1=\frac{\sqrt{n-1}}{\sqrt{2}}\pi<\sqrt{2(n-1)}\pi.$$

Next, we have

$$\begin{aligned}|A|^2 &= 2(n-2)\mu_1^2+\hat{\mu}_1^2+2(n-2)\hat{\mu}_2^2\\ &= \kappa\left\{\frac{n-2}{2}\tan^2\left(\frac{\sqrt{\kappa}}{2}r\right)+\cot^2(\sqrt{\kappa}r)+\frac{n-2}{2}\cot^2\left(\frac{\sqrt{\kappa}}{2}r\right)\right\}\\ &= \kappa\left\{\frac{(n-2)t^2}{2}+\frac{(1-t^2)^2}{4t^2}+\frac{n-2}{2t^2}\right\}.\end{aligned}$$

Solving the biharmonicity equation $|A|^2=1/2$, we obtain

$$t^2=\frac{2n-1\pm2\sqrt{2n-2}}{2n-3}\quad(\neq 1).$$

Therefore $M_r$ is proper biharmonic if and only if

$$\begin{aligned}r &= 2\sqrt{2n-2}\tan^{-1}\left(\sqrt{\frac{2n-1\pm2\sqrt{2n-2}}{2n-3}}\right)\\ &= 2\sqrt{2n-2}\tan^{-1}\left(\frac{\sqrt{2n-2}\pm1}{\sqrt{2n-3}}\right).\end{aligned}$$

□

## 9. Riemannian symmetric space of type CI

**9.1.** Now we consider $n(n+1)$-dimensional Riemannian symmetric space $N=\mathrm{CI}(n):=\mathrm{Sp}(n)/\mathrm{U}(n)$ of type CI ($n\geq 2$). The Riemannian symmetric space

$\mathrm{CI}(n)$ is of rank $n$. With respect to the Killing metric, the maximal sectional curvature is $\kappa = 1/(n+1)$. Note that $\mathrm{CI}(2) = \widetilde{\mathrm{Gr}}_3(\mathbf{R}^5)$ because of the isomorphism $\mathrm{Sp}(2) \cong \mathrm{Spin}(5)$.

**9.2.** Totally geodesic singular orbits under cohomogeneity one actions are congruent to $\mathrm{CI}(n-1) \times \mathbf{S}^2$ ([2, 14]). The corresponding symmetric subgroup is $L = \mathrm{Sp}(n-1) \times \mathrm{Sp}(1)$. In this case we have $H = \mathrm{U}(n)$ and $H(o) = \mathbf{C}P^{n-1}$ of maximal sectional curvature $\chi = \kappa/2$. Moreover we have $r = \pi/\sqrt{2\kappa}$. In case $n = 2$, $\mathrm{CI}(n-1) \times \mathbf{S}^2 = \widetilde{\mathrm{Gr}}_2(\mathbf{R}^4) \subset \widetilde{\mathrm{Gr}}_3(\mathbf{R}^5)$.

The eigenvalues of the Jacobi operator on $T_oL(o)$ are

$$a_1 = \frac{\kappa}{2}, \qquad a_2 = \frac{\kappa}{8}, \qquad a_3 = 0$$

with multiplicities $m_1 = 2$, $m_2 = 2n-4$ (see [18]). From these data together with Theorem 3, we get

$$\mu_1 = \frac{\sqrt{\kappa}}{\sqrt{2}} \tan\left(\frac{\sqrt{\kappa}}{\sqrt{2}} r\right), \qquad \mu_2 = \frac{\sqrt{\kappa}}{2\sqrt{2}} \tan\left(\frac{\sqrt{\kappa}}{2\sqrt{2}} r\right), \qquad \mu_3 = 0,$$

$$\hat{\mu}_1 = -\frac{\sqrt{\kappa}}{\sqrt{2}} \cot\left(\frac{\sqrt{\kappa}}{\sqrt{2}} r\right), \qquad \hat{\mu}_2 = -\frac{\sqrt{\kappa}}{2\sqrt{2}} \cot\left(\frac{\sqrt{\kappa}}{2\sqrt{2}} r\right)$$

with multiplicities

$$m_1 = 2, \qquad m_2 = 2n-4, \qquad m_3 = n^2 - 3n + 4, \qquad \hat{m}_1 = 1, \qquad \hat{m}_2 = 2n-4.$$

Theorem 9. *A tube $M_r$ around $\mathrm{CI}(n-1) \times \mathbf{S}^2$ of radius $r$ in $\mathrm{CI}(n)$ is minimal if and only if*

$$r = \sqrt{2(n+1)} \tan^{-1} \frac{\sqrt{2n-3}}{\sqrt{2}}.$$

*For $n \geq 3$, the only biharmonic tubes are minimal ones.*

*For $n = 2$, the only biharmonic tube $M_r$ around $\widetilde{\mathrm{Gr}}_2(\mathbf{R}^4)$ is the tube of radius*

$$r = \frac{\sqrt{6}}{4}\pi.$$

Proof. First we look for minimal tubes. We get

$$\begin{aligned}(n^2+n-1)H &= 2\mu_1 + (2n-4)\mu_2 + \hat{\mu}_1 + (2n-4)\hat{\mu}_2 \\ &= 2 \cdot \frac{\sqrt{\kappa}}{\sqrt{2}} \tan\left(\frac{\sqrt{\kappa}}{\sqrt{2}} r\right) + (2n-4) \cdot \frac{\sqrt{\kappa}}{2\sqrt{2}} \tan\left(\frac{\sqrt{\kappa}}{2\sqrt{2}} r\right) \\ &\quad - \frac{\sqrt{\kappa}}{\sqrt{2}} \cot\left(\frac{\sqrt{\kappa}}{\sqrt{2}} r\right) - (2n-4) \cdot \frac{\sqrt{\kappa}}{2\sqrt{2}} \cot\left(\frac{\sqrt{\kappa}}{2\sqrt{2}} r\right)\end{aligned}$$

$$
\begin{aligned}
&= \sqrt{2}\sqrt{\kappa}\cdot\frac{2t}{1-t^2} + (2n-4)\cdot\frac{\sqrt{\kappa}}{2\sqrt{2}}t \\
&\quad -\frac{\sqrt{\kappa}}{\sqrt{2}}\frac{1-t^2}{2t} - (2n-4)\cdot\frac{\sqrt{\kappa}}{2\sqrt{2}}\frac{1}{t} \\
&= \frac{\sqrt{\kappa}}{2\sqrt{2}t(1-t^2)}\{(3-2n)t^4 + 2(2n+1)t^2 + 3 - 2n\},
\end{aligned}
$$

where $t = \tan(\sqrt{\kappa}r/(2\sqrt{2}))$.

Since $0 < t < 1$, $M_r$ is minimal if and only if

$$
r = \frac{2\sqrt{2}}{\sqrt{\kappa}}\tan^{-1}\left(\sqrt{\frac{2n+1-2\sqrt{4n-2}}{2n-3}}\right) = \sqrt{2(n+1)}\tan^{-1}\frac{\sqrt{2n-3}}{\sqrt{2}}.
$$

The square norm $|A|^2$ is computed as

$$
\begin{aligned}
|A|^2 &= 2\mu_1^2 + (2n-4)\mu_2^2 + \hat{\mu}_1^2 + (2n-4)\hat{\mu}_2^2 \\
&= \kappa\left\{\tan^2\left(\frac{\sqrt{\kappa}}{\sqrt{2}}r\right) + \frac{n-2}{4}\tan^2\left(\frac{\sqrt{\kappa}}{2\sqrt{2}}r\right)\right. \\
&\quad \left. + \frac{1}{2}\cot^2\left(\frac{\sqrt{\kappa}}{\sqrt{2}}r\right) + \frac{n-2}{4}\cot^2\left(\frac{\sqrt{\kappa}}{2\sqrt{2}}r\right)\right\} \\
&= \kappa\left\{\frac{4t^2}{(1-t^2)^2} + \frac{(n-2)t^2}{4} + \frac{(1-t^2)^2}{8t^2} + \frac{n-2}{4t^2}\right\}.
\end{aligned}
$$

The biharmonicity equation $|A|^2 = 1/2$ is equivalent to

$$
n = \frac{3t^8 - 38t^4 + 3}{2(1-t^2)^4}.
$$

It follows that

$$
\begin{aligned}
2(1-t^2)^4(3-n) &= 3t^8 - 24t^6 + 74t^4 - 24t^2 + 3 \\
&= (\sqrt{3}t^4 - 8t^2 + \sqrt{3})^2 + (16\sqrt{3} - 24)t^6 + 4t^4 + (16\sqrt{3} - 24)t^2.
\end{aligned}
$$

Hence, in case $n \geq 3$ the equation has no real solutions. On the other hand, in case $n = 2$, we obtain

$$
t^2 = 3 - 2\sqrt{2},\ 5 - 2\sqrt{6},
$$

because $t^2 < 1$. As shown above, if $n = 2$ and $t^2 = 5 - 2\sqrt{6}$, then $H = 0$. Thus, $M_r$ is proper biharmonic if and only if

$$
r = 2\sqrt{6}\tan^{-1}(\sqrt{3 - 2\sqrt{2}}) = \frac{\sqrt{6}}{4}\pi. \qquad \square
$$

In case $n = 2$, the above classification coincides with the one in Theorem 7 with $k = 3$ and $n = 5$.

## 10. Riemannian symmetric space of type CII

**10.1.** The quaternion Grassmannian manifold $\mathrm{Gr}_k(\mathbf{H}^n)$ $(2 \le k \le n-1)$ is the manifold of all quaternion linear $k$-subspaces in quaternion Euclidean $n$-space $\mathbf{H}^n$. The quaternion Grassmannian manifold $\mathrm{Gr}_k(\mathbf{H}^n)$ is represented by $\mathrm{Gr}_k(\mathbf{H}^n) = \mathrm{Sp}(n)/\mathrm{Sp}(k) \times \mathrm{Sp}(n-k)$ as a Riemannian symmetric space of real dimension $4k(n-k)$ and of rank $\min(k, n-k)$. The maximal sectional curvature is $\kappa = 1/(n+1)$ with respect to the metric induced from $-B$.

**10.2.** Totally geodesic singular orbits in $\mathrm{Gr}_k(\mathbf{H}^n)$ under cohomogeneity one actions are ([2, 14]):

(1) $\mathrm{Gr}_k(\mathbf{H}^{n-1})$ and $\mathrm{Gr}_{k-1}(\mathbf{H}^{n-1})$ for $2 \le k < n-k$.

(2) $\mathrm{Gr}_{k-1}(\mathbf{H}^{2k-1}) = \mathrm{Gr}_k(\mathbf{H}^{2k-1})$, if $n = 2k$ and $k \ge 2$.

Consider the singular orbit $\mathrm{Gr}_{k-1}(\mathbf{H}^{n-1})$ with the corresponding symmetric subgroup $L = \mathrm{Sp}(1) \times \mathrm{Sp}(n-1)$. Then we have $H = \mathrm{Sp}(k-1) \times \mathrm{Sp}(n-k+1)$ and $H(o) = \mathbf{H}P^{n-k}$. The maximal sectional curvature of $H(o)$ is $\chi = \kappa/2$. Moreover we have $r = \sqrt{2}\pi/\sqrt{\kappa}$. For $k > 2$, $L(o) = \mathrm{Gr}_{k-1}(\mathbf{H}^{n-1})$ is of maximal sectional curvature $\kappa$ and for $k = 2$, $L(o) = \mathbf{H}P^{n-k}$ is of maximal sectional curvature $\kappa/2$.

The eigenvalues of the Jacobi operator on $T_oL(o)$ are given by [18]:

$$a_1 = \frac{\kappa}{8}, \qquad a_2 = 0$$

and $m_1 = 4k-4$. Thus by using Theorem 3, the principal curvatures of the tube $M_r$ around $\mathrm{Gr}_{k-1}(\mathbf{H}^{k-1})$ are computed as:

$$\mu_1 = \frac{\sqrt{\kappa}}{2\sqrt{2}} \tan\left(\frac{\sqrt{\kappa}}{2\sqrt{2}} r\right), \qquad \mu_2 = 0,$$

$$\hat{\mu}_1 = -\frac{\sqrt{\kappa}}{\sqrt{2}} \cot\left(\frac{\sqrt{\kappa}}{\sqrt{2}} r\right), \qquad \hat{\mu}_2 = -\frac{\sqrt{\kappa}}{2\sqrt{2}} \cot\left(\frac{\sqrt{\kappa}}{2\sqrt{2}} r\right).$$

Theorem 10. *A tube $M_r$ around* $\mathrm{Gr}_{k-1}(\mathbf{H}^{n-1})$ *of radius $r$ in* $\mathrm{Gr}_k(\mathbf{H}^n)$ $(2 \le 2 < n-k)$ *is minimal if and only if*

$$r = 2\sqrt{2(n+1)} \tan^{-1} \sqrt{\frac{4n-4k-1}{4k-1}}.$$

*The only proper biharmonic tube* $M_r$ *is the tube of radius*

$$r = 2\sqrt{2(n+1)}\,\tan^{-1}\left(\sqrt{\frac{2n+5 \pm 2\sqrt{(n-2k)^2+6n+6}}{4k-1}}\right).$$

PROOF. The mean curvature is computed as

$$\begin{aligned}(4nk-4k^2-1)H &= 4(k-1)\mu_1 + 3\hat{\mu}_1 + 4(n-k-1)\hat{\mu}_2 \\ &= \sqrt{\frac{\kappa}{2}}\left\{(2k-2)t - \frac{3(1-t^2)}{2t} - \frac{2(n-k-1)}{t}\right\},\end{aligned}$$

where $t = \tan(\sqrt{\kappa}r/(2\sqrt{2}))$.

Hence, $M_r$ is minimal if and only if

$$r = \frac{2\sqrt{2}}{\sqrt{\kappa}}\tan^{-1}\sqrt{\frac{4n-4k-1}{4k-1}}.$$

Next, we compute the square norm $|A|^2$.

$$\begin{aligned}|A|^2 &= 4(k-1)\mu_1^2 + 3\hat{\mu}_1^2 + 4(n-k-1)\hat{\mu}_2^2 \\ &= \kappa\left\{\frac{(k-1)t^2}{2} + \frac{3(1-t^2)^2}{8t^2} + \frac{n-k-1}{2t^2}\right\}.\end{aligned}$$

The biharmonicity equation $|A|^2 = 1/2$ can be written as

$$g(t) := (4k-1)t^4 - 2(2n+5)t^2 + 4n - 4k - 1 = 0.$$

By solving this equation, we obtain

$$t^2 = \frac{2n+5 \pm 2\sqrt{(n-2k)^2+6n+6}}{4k-1}.$$

Moreover, we have

$$g\left(\sqrt{\frac{4n-4k-1}{4k-1}}\right) = \frac{12(4k-4n+1)}{4k-1} < 0.$$

Therefore, $M_r$ is proper biharmonic if and only if

$$r = 2\sqrt{2(n+1)}\,\tan^{-1}\left(\sqrt{\frac{2n+5 \pm 2\sqrt{(n-2k)^2+6n+6}}{4k-1}}\right). \qquad \square$$

## 11. Symmetric space of type EII

**11.1. Exceptional Lie group** $\mathrm{E}_6$**.** Let us denote by $\mathfrak{J}$ the real linear space of all 3 by 3 Hermitian matrices of octonions. On this linear space the *Jordan product* $\circ$ is defined by

$$X \circ Y := \frac{1}{2}(XY + YX), \qquad X, Y \in \mathfrak{J}.$$

The real algebra $\mathfrak{J}$ equipped with Jordan product is called the *exceptional Jordan algebra.*

The automorphism group $\mathrm{F}_4$ of the Jordan algebra $\mathfrak{J}$ is a simply connected compact simple Lie group of dimension 52. The exceptional Jordan algebra $\mathfrak{J}$ is parametrized as

$$\mathfrak{J} = \left\{ (\Xi, X) := \begin{pmatrix} \xi_1 & x_3 & \overline{x_2} \\ \overline{x_3} & \xi_2 & x_1 \\ x_2 & \overline{x_1} & \xi_3 \end{pmatrix} \middle| \, \Xi = (\xi_1, \xi_2, \xi_3) \in \mathbf{R}^3, X = (x_1, x_2, x_3) \in \mathfrak{O}^3 \right\}.$$

The *trace* $\mathrm{tr}(\Xi, X)$ of $(\Xi, X)$ is defined by $\mathrm{tr}(\Xi, X) = \xi_1 + \xi_2 + \xi_3$.

The inner product $(\cdot\,,\cdot)$ on $\mathfrak{J}$ and trilinear form $\mathrm{tr}(\cdot\,,\cdot\,,\cdot)$ are defined by ([21])

$$(X, Y) = \mathrm{tr}(X \circ Y), \qquad \mathrm{tr}(X, Y, Z) = (X, Y \circ Z).$$

Next, the *Freudental product* $\times$ is defined by

$$X \times Y = \frac{1}{2}(2X \circ Y - \mathrm{tr}(X)Y - \mathrm{tr}(Y)X + (\mathrm{tr}(X)\,\mathrm{tr}(Y) - (X, Y))E), \qquad X, Y \in \mathfrak{J}.$$

By using $\times$, triple product $(X, Y, Z)$ and determinant function det are defined by

$$(X, Y, Z) = (X, Y \times Z), \qquad \det X = \frac{1}{3}(X, X, X).$$

Now we consider the *complexification* $\mathfrak{J}^{\mathbf{C}}$ of $\mathfrak{J}$. The resulting complex algebra

$$\mathfrak{J}^{\mathbf{C}} = \left\{ \begin{pmatrix} \xi_1 & x_3 & \overline{x_2} \\ \overline{x_3} & \xi_2 & x_1 \\ x_2 & \overline{x_1} & \xi_3 \end{pmatrix} \in \mathfrak{J}^{\mathbf{C}} \middle| \, \xi_1, \xi_2, \xi_3 \in \mathbf{C}, x_1, x_2, x_3 \in \mathfrak{O}^{\mathbf{C}} \right\}$$

is called the *exceptional complex Jordan algebra*. On the complexification $\mathfrak{J}^{\mathbf{C}}$ of $\mathfrak{J}$, *Hermitian inner product* $\langle \cdot\,,\cdot \rangle$ is defined by

$$\langle X, Y \rangle = (X^{\dagger}, Y),$$

where † is the complex conjugation of the exceptional complex Jordan algebra $\mathfrak{J}^{\mathbf{C}}$.

The simply connected compact Lie group $E_6$ is given by ([21, § 3.1]):

$$E_6 = \{\alpha \in GL(\mathfrak{J}^{\mathbf{C}}) \mid \det(\alpha(X)) = \det(X), \langle \alpha(X), \alpha(Y) \rangle = \langle X, Y \rangle\}.$$

The Lie group $E_6$ is a 78-dimensional Lie subgroup of $U(27) = U(\mathfrak{J}^{\mathbf{C}})$. For more informations on $E_6$, we refer to [21].

**11.2.** The simply connected compact Riemannian symmetric space of type EII is represented by $N = E_6/SU(6) \cdot SU(2)$. This is a 40-dimensional *quaternionic symmetric space* of rank 4. The maximal sectional curvature is $\kappa = 1/12$ with respect to the Killing metric. Totally geodesic singular orbits under cohomogeneity one actions are congruent to $FI = F_4/Sp(3) \cdot Sp(1)$ with maximal sectional curvature $\kappa$. The symmetric subgroup corresponding to FI is $L = F_4$ ([2, 14, 20]). In this case, $H = Sp(4)/\mathbf{Z}_2$ and $H(o) = \mathbf{H}P^3$ with maximal sectional curvature $\chi = \kappa/2$ and $r = \pi/(\sqrt{2\kappa}) = \sqrt{6}\pi$.

We consider tubes of radius $r < \pi/(\sqrt{2\kappa})$ around $F_4/Sp(3) \cdot Sp(1)$. The principal curvatures of $M_r$ are given by Verhóczki [20, (11), 8 Proposition]:

$$\mu_1 = \frac{\sqrt{\kappa}}{\sqrt{2}} \tan\left(\frac{\sqrt{\kappa}}{\sqrt{2}} r\right), \qquad \mu_2 = \frac{\sqrt{\kappa}}{2\sqrt{2}} \tan\left(\frac{\sqrt{\kappa}}{2\sqrt{2}} r\right), \qquad \mu_3 = 0,$$

$$\hat{\mu}_1 = -\frac{\sqrt{\kappa}}{\sqrt{2}} \cot\left(\frac{\sqrt{\kappa}}{\sqrt{2}} r\right), \qquad \hat{\mu}_2 = -\frac{\sqrt{\kappa}}{2\sqrt{2}} \cot\left(\frac{\sqrt{\kappa}}{2\sqrt{2}} r\right)$$

with multiplicities

$$m_1 = 5, \qquad m_2 = 8, \qquad m_3 = 15, \qquad \hat{m}_1 = 3, \qquad \hat{m}_2 = 8.$$

Theorem 11. *A tube $M_r$ around $F_4/Sp(3) \cdot Sp(1)$ of radius $r$ in $E_6/SU(6) \cdot SU(2)$ is minimal if and only if*

$$r = 4\sqrt{6} \tan^{-1}\left(\frac{4 - \sqrt{5}}{\sqrt{11}}\right) < \sqrt{6}\pi.$$

*The only biharmonic tubes are minimal ones.*

Proof. We put $t = \tan(\sqrt{\kappa} r/\sqrt{8})$ and $\alpha = \sqrt{\kappa/8}$, then we have

$$\begin{aligned} 39H &= 5\mu_1 + 8\mu_2 + 3\hat{\mu}_1 + 8\hat{\mu}_2 \\ &= 5\left\{2\alpha\left(\frac{2t}{1-t^2}\right)\right\} + 8(\alpha t) + 3\left\{-2\alpha\left(\frac{1-t^2}{2t}\right)\right\} + 8\left(-\frac{\alpha}{t}\right) \\ &= \frac{-\alpha(11t^4 - 42t^2 + 11)}{t(1-t^2)}. \end{aligned}$$

Thus $M_r$ is minimal if and only if

$$r = \sqrt{\frac{8}{\kappa}}\tan^{-1}\left(\sqrt{\frac{21-8\sqrt{5}}{11}}\right) = 4\sqrt{6}\tan^{-1}\left(\frac{4-\sqrt{5}}{\sqrt{11}}\right) < \sqrt{6}\pi.$$

The square norm $|A|^2$ is computed as follows:

$$\begin{aligned}|A|^2 &= 5\mu_1^2 + 8\mu_2^2 + 3\hat{\mu}_1^2 + 8\hat{\mu}_2^2 \\ &= \kappa\left\{\frac{10t^2}{(1-t^2)^2} + t^2 + \frac{3(1-t^2)^2}{8t^2} + \frac{1}{t^2}\right\}.\end{aligned}$$

The biharmonicity equation $|A|^2 = 1/2$ becomes

$$11t^8 - 76t^6 + 210t^4 - 76t^2 + 11 = 0.$$

However, the LHS of the equation can be transformed into

$$(\sqrt{11}t^4 - 13t^2 + \sqrt{11})^2 + (26\sqrt{11} - 76)t^6 + 19t^4 + (26\sqrt{11} - 76)t^2,$$

which shows that the biharmonicity equation has no real solutions. □

## 12. Symmetric spaces of type EIII

**12.1.** We consider the complex projective space $P(\mathfrak{J}^{\mathbf{C}}) = \mathbf{C}P^{26}$ over $\mathfrak{J}^{\mathbf{C}}$. According to Atsuyama [1], the simply connected 32-dimensional Riemannian symmetric space of type EIII is realized as $\mathrm{EIII} = (\widetilde{\mathrm{EIII}}\backslash\{0\})/\mathbf{C}^{\times} \subset P(\mathfrak{J}^{\mathbf{C}}) = \mathbf{C}P^{26}$, where

$$\widetilde{\mathrm{EIII}} := \left\{ (\Xi, X) = \begin{pmatrix} \xi_1 & x_3 & \overline{x_2} \\ \overline{x_3} & \xi_2 & x_1 \\ x_2 & \overline{x_1} & \xi_3 \end{pmatrix} \in \mathfrak{J}^{\mathbf{C}} \;\middle|\; \begin{array}{l} \xi_2\xi_3 = |x_1|^2,\ \xi_3\xi_1 = |x_2|^2,\ \xi_1\xi_2 = |x_3|^2, \\ x_2x_3 = \xi_1\overline{x_1},\ x_3x_1 = \xi_2\overline{x_2},\ x_1x_2 = \xi_3\overline{x_3} \end{array} \right\}$$

This symmetric space EIII is represented by $E_6/((\mathrm{Spin}(10)\times \mathrm{U}(1))/\mathbf{Z}_4)$ and of rank 2. The maximal sectional curvature is $\kappa = 1/12$.

**12.2.** The only singular orbits under cohomogeneity one actions are $\mathfrak{O}P^2$ of maximal sectional curvature $\kappa/2$ ([2, 14, 20]). In this case $L = H = \mathrm{F}_4$ and $L(o) = H(o) = \mathfrak{O}P^2$ of maximal sectional curvature $\chi = \kappa/2$. In addition we have $r = \pi/\sqrt{2\kappa} = \sqrt{6}\pi$. The principal curvatures of a tube $M_r$ around $\mathfrak{O}P^2$ with radius $r < \pi/\sqrt{2\kappa}$ are given by Verhóczki [20, 12 Proposition]:

$$\mu_1 = \frac{\sqrt{\kappa}}{\sqrt{2}} \tan\left(\frac{\sqrt{\kappa}}{\sqrt{2}} r\right), \qquad m_1 = 1,$$

$$\mu_2 = \frac{\sqrt{\kappa}}{2\sqrt{2}} \tan\left(\frac{\sqrt{\kappa}}{2\sqrt{2}} r\right), \qquad m_2 = 8,$$

$$\mu_3 = 0, \qquad m_3 = 7,$$

$$\hat{\mu}_1 = -\frac{\sqrt{\kappa}}{\sqrt{2}} \cot\left(\frac{\sqrt{\kappa}}{\sqrt{2}} r\right), \qquad \hat{m}_1 = 7,$$

$$\hat{\mu}_2 = -\frac{\sqrt{\kappa}}{2\sqrt{2}} \cot\left(\frac{\sqrt{\kappa}}{2\sqrt{2}} r\right), \qquad \hat{m}_2 = 8.$$

THEOREM 12. *A tube $M_r$ of radius $r$ around $\mathfrak{O}P^2$ in* $E_6/((\mathrm{Spin}(10) \times \mathrm{U}(1))/\mathbf{Z}_4)$ *is minimal if and only if*

$$r = 4\sqrt{6} \tan^{-1}\left(\sqrt{\frac{3}{5}}\right).$$

*A tube $M_r$ is proper biharmonic if and only if $r = (2\sqrt{6}\pi)/3$ or $r = 2\sqrt{6} \tan^{-1} \sqrt{5}$.*

PROOF. The mean curvature is computed as

$$31H = \sqrt{\frac{\kappa}{8}}\left(\frac{4t}{1-t^2} + 8t - \frac{7(1-t^2)}{t} - \frac{8}{t}\right),$$

where $t = \tan(\sqrt{\kappa}r/\sqrt{8})$.

Thus $H = 0$ if and only if $15t^4 - 34t^2 + 15 = 0$. It follows from $0 < t < 1$ that $M_r$ is minimal if and only if

$$r = \sqrt{\frac{8}{\kappa}} \tan^{-1}\left(\sqrt{\frac{3}{5}}\right) = 4\sqrt{6} \tan^{-1}\left(\sqrt{\frac{3}{5}}\right) < \sqrt{6}\pi.$$

Next, we have

$$|A|^2 = \frac{\kappa}{8}\left(\frac{16t^2}{(1-t^2)^2} + 8t^2 + \frac{7(1-t^2)^2}{t^2} + \frac{8}{t^2}\right).$$

The biharmonicity equation $|A|^2 = 1/2$ can be written as

$$15t^8 - 92t^6 + 170t^4 - 92t^2 + 15 = 0.$$

It follows from $0 < t < 1$ that $M_r$ is proper biharmonic if and only if

$$r = 4\sqrt{6} \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{2\sqrt{6}}{3}\pi,$$

or

$$r = 4\sqrt{6}\tan^{-1}\left(\sqrt{\frac{7-2\sqrt{6}}{5}}\right) = 2\sqrt{6}\tan^{-1}\sqrt{5}. \qquad \square$$

## 13. Symmetric spaces of type EIV

The simply connected compact Riemannian symmetric space of type EIV is realized as

$$\{X \in \mathfrak{J}^{\mathbf{C}} \mid \det X = 1, \langle X, Y \rangle = 3\}.$$

On this manifold, $E_6$ acts transitively and the isotropy subgroup at the identity matrix $E$ is $F_4$ (see [21, § 3.7, § 3.8]). With respect to the Riemannian metric induced from the inner product $-B$ on $\mathfrak{e}_6$, $E_6/F_4$ is a 26-dimensional compact simply connected Riemannian symmetric space of rank 2. The maximal sectional curvature is $\kappa = 1/24$.

We consider tubes of radius $r < \pi/(2\sqrt{\kappa}) = \sqrt{6}\pi$ around SU(6)/Sp(3). The corresponding symmetric subgroup $L$ of SU(6)/Sp(3) is $L = \mathrm{SU}(6)\cdot \mathrm{SU}(2)$. In this case we have $H = \mathrm{Sp}(4)/\mathbf{Z}_2$ and $H(o) = \mathbf{H}P^3$ is of maximal sectional curvature $\chi = \kappa$.

The principal curvatures of $M_r$ are

$$\mu_1 = \sqrt{\kappa}\tan(\sqrt{\kappa}r), \qquad \mu_2 = \frac{\sqrt{\kappa}}{2}\tan\frac{\sqrt{\kappa}r}{2}, \qquad \mu_3 = 0,$$

$$\hat{\mu}_1 = -\sqrt{\kappa}\cot(\sqrt{\kappa}r), \qquad \hat{\mu}_2 = -\frac{\sqrt{\kappa}}{2}\cot\frac{\sqrt{\kappa}}{2}$$

with multiplicities

$$m_1 = 5, \qquad m_2 = 8, \qquad m_3 = 1, \qquad \hat{m}_1 = 3, \qquad \hat{m}_2 = 8.$$

THEOREM 13. *A tube $M_r$ of radius $r$ around* AII(3) = SU(6)/Sp(3) *in* $E_6/F_4$ *is minimal if and only if*

$$r = 8\sqrt{3}\tan^{-1}\left(\frac{4-\sqrt{5}}{\sqrt{11}}\right).$$

*The only biharmonic tubes around* AII(3) *are minimal ones.*

PROOF. The mean curvature $H$ of a tube $M_r$ around AII(3) is computed as

$$\begin{aligned} 25H &= 5\mu_1 + 8\mu_2 + 3\hat{\mu}_1 + 8\hat{\mu}_2 \\ &= \sqrt{\kappa}\left(5\tan(\sqrt{\kappa}r) + \frac{8}{2}\tan\frac{\sqrt{\kappa}r}{2} - 3\cot(\sqrt{\kappa}r) - \frac{8}{2}\cot\frac{\sqrt{\kappa}}{2}\right) \end{aligned}$$

$$= \sqrt{\kappa}\left(\frac{5 \cdot 2t}{1-t^2} + 4t - \frac{3(1-t^2)}{2t} - \frac{4}{t}\right)$$

$$= \frac{\sqrt{\kappa}}{2t(1-t^2)}(20t^2 + 8t^2(1-t^2) - 3(1-t^2)^2 - 8(1-t^2))$$

$$= -\frac{\sqrt{\kappa}}{2t(1-t^2)}(11t^4 - 42t^2 + 11).$$

Thus $M_r$ is minimal if and only if

$$r = 8\sqrt{3}\tan^{-1}\left(\sqrt{\frac{21-8\sqrt{5}}{11}}\right) = 8\sqrt{3}\tan^{-1}\left(\frac{4-\sqrt{5}}{\sqrt{11}}\right) < \sqrt{6}\pi.$$

The square norm $|A|^2$ is

$$|A|^2 = 5\mu_1^2 + 8\mu_2^2 + 3\hat{\mu}_1^2 + 8\hat{\mu}_2^2$$

$$= \kappa\left\{\frac{20t^2}{(1-t^2)^2} + 2t^2 + \frac{3(1-t^2)^2}{4t^2} + \frac{2}{t^2}\right\}.$$

We see that the biharmonicity equation $|A|^2 = 1/2$ is same as the one in Section 11. Thus, it has no real solutions. □

## 14. Riemannian symmetric space of type G

In this section we consider the space $G_2/SO(4)$ of all quaternionic subalgebras of the Cayley algebra $\mathfrak{O}$ equipped with the Killing metric. Then the resulting homogeneous Riemannian space $G_2/SO(4)$ is an 8-dimensional quaternionic symmetric space of rank 2. This space has the same real homology as the quaternion projective plane $\mathbf{H}P^2$. With respect to the Killing metric, the maximal sectional curvature of $G_2/SO(4)$ is $1/4$.

Let us consider the singular orbit $\mathbf{C}P^2$ under the cohomogeneity one action of SU(3). The maximal sectional curvature of $\mathbf{C}P^2$ is $1/4$. Note that $G_2/SU(3) = \mathbf{S}^6$ is *not* Riemannian symmetric, but nearly Kähler 3-symmetric.

The principal curvatures of a tube $M_r$ of radius $r \in (0, \sqrt{3}\pi)$ around $\mathbf{C}P^2$ are computed explicitly by Verhóczki [19] (*cf.* García, Hullet [6]):

$$\lambda_1 = -\frac{1}{2\sqrt{3}}\cot\frac{r}{2\sqrt{3}},$$

$$\lambda_2 = 0,$$

$$\lambda_3 = \frac{1}{4\sqrt{3}}\left(-2\cot\frac{r}{\sqrt{3}} + \sqrt{4\cot^2\frac{r}{\sqrt{3}} + 3}\right),$$
$$\lambda_4 = \frac{1}{4\sqrt{3}}\left(-2\cot\frac{r}{\sqrt{3}} - \sqrt{4\cot^2\frac{r}{\sqrt{3}} + 3}\right)$$

with multiplicities

$$m_1 = 1, \qquad m_2 = m_3 = m_4 = 2.$$

By using this table, we obtain the following result:

THEOREM 14. *A tube* $M_r$ *around* $\mathbf{C}P^2$ *in* $\mathrm{G}_2/\mathrm{SO}(4)$ *is minimal if and only if its radius is*

$$r = 2\sqrt{3}\tan^{-1}\sqrt{\frac{3}{2}}.$$

*The only proper biharmonic tube* $M_r$ *around* $\mathbf{C}P^2$ *are tubes of radius*

$$r = \frac{2\pi}{\sqrt{3}} \qquad \text{or} \qquad r = 2\sqrt{3}\tan^{-1}\frac{1}{\sqrt{2}}.$$

PROOF. The mean curvature $H$ is computed as

$$\begin{aligned} 7H &= -\frac{1}{2\sqrt{3}}\cot\frac{r}{2\sqrt{3}} - \frac{2}{\sqrt{3}}\cot\frac{r}{\sqrt{3}} \\ &= \frac{1}{2\sqrt{3}\tan\frac{r}{2\sqrt{3}}}\left(2\tan^2\frac{r}{2\sqrt{3}} - 3\right). \end{aligned}$$

Hence $M_r$ is minimal if and only if $r = 2\sqrt{3}\tan^{-1}\sqrt{\frac{3}{2}} < \sqrt{3}\pi$.

Next the square norm $|A|^2$ is given by

$$\begin{aligned} |A|^2 &= \frac{1}{12}\left(\cot^2\frac{r}{2\sqrt{3}} + 4\cot^2\frac{r}{\sqrt{3}} + 4\cot^2\frac{r}{\sqrt{3}} + 3\right) \\ &= \frac{1}{12}\left(8\cot^2\frac{r}{\sqrt{3}} + \cot^2\frac{r}{2\sqrt{3}} + 3\right) \\ &= \frac{1}{12\tan^2\frac{r}{2\sqrt{3}}}\left(2\tan^4\frac{r}{2\sqrt{3}} - \tan^2\frac{r}{2\sqrt{3}} + 3\right). \end{aligned}$$

The biharmonicity equation $|A|^2 = 1/2$ becomes:

$$2t^4 - 7t^2 + 3 = 0,$$

where $t = \tan\frac{r}{2\sqrt{3}}$. Hence $t = \sqrt{3}$ or $t = 1/\sqrt{2}$. Thus we have

$$r = 2\sqrt{3}\tan^{-1}\sqrt{3} = \frac{2\pi}{\sqrt{3}} < \sqrt{3}\pi \quad \text{or} \quad r = 2\sqrt{3}\tan^{-1}\frac{1}{\sqrt{2}} < \sqrt{3}\pi < \sqrt{3}\pi. \quad \square$$

REMARK 3. For the classification of all totally geodesics submanifolds in $G_2/SO(4)$, we refer to [13].

## Concluding remark

The unit sphere $\mathbf{S}^n$ is a typical example of simply connected irreducible Riemannian symmetric space of compact type. Based on this fact, in Part I and this Part II, we have studied biharmonic homogeneous hypersurfaces in Riemannian symmetric spaces of compact type. Next, the odd-dimensional sphere $\mathbf{S}^{2n+1}$ is a standard example of *Sasakian space form* (see [3] and [4]). The Berger sphere is a typical example of Sasakian space form. Sasakian space forms are naturally reductive homogeneous Riemannian spaces. The odd-dimensional sphere is the only Riemannian symmetric Sasakian manifold. Berger spheres equipped with canonical Sasakian structure are normal homogeneous spaces which are not Riemannian symmetric. Thus the study on biharmonic hypersurfaces in compact normal homogeneous Riemannian spaces, *e.g.*, Berger spheres is another generalization of "biharmonic submanifold geometry in $\mathbf{S}^n$".

In [7], the first named author of the present paper classified proper biharmonic anti-invariant surfaces in 3-dimensional Sasakian space forms. Next, the second named author of this paper classified proper biharmonic Legendre surfaces in 5-dimensional Sasakian space forms [16].

To close this paper we propose the following problems.

PROBLEMS.
(1) Classify all biharmonic homogeneous hypersurfaces in simply connected irreducible Riemannian symmetric space of compact type.
(2) Construct explicit examples of proper biharmonic hypersurfaces in normal homogeneous Riemannian spaces.
(3) Classify all biharmonic homogeneous hypersurfaces in the Berger sphere.

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