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BIHARMONIC HYPERSURFACES WITH THREE DISTINCT PRINCIPAL CURVATURES IN EUCLIDEAN SPACE

YU FU

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Abstract. The well known Chen's conjecture on biharmonic submanifolds states that a biharmonic submanifold in a Euclidean space is a minimal one ([10–13, 16, 18–21, 8]). For the case of hypersurfaces, we know that Chen's conjecture is true for biharmonic surfaces in $\mathbb{E}^3$ ([10], [24]), biharmonic hypersurfaces in $\mathbb{E}^4$ ([23]), and biharmonic hypersurfaces in $\mathbb{E}^m$ with at most two distinct principal curvature ([21]). The most recent work of Chen-Munteanu [18] shows that Chen's conjecture is true for $\delta(2)$-ideal hypersurfaces in $\mathbb{E}^m$, where a $\delta(2)$-ideal hypersurface is a hypersurface whose principal curvatures take three special values: λ_1, λ_2 and $\lambda_1 + \lambda_2$. In this paper, we prove that Chen's conjecture is true for hypersurfaces with three distinct principal curvatures in $\mathbb{E}^m$ with arbitrary dimension, thus, extend all the above-mentioned results. As an application we also show that Chen's conjecture is true for $O(p) \times O(q)$-invariant hypersurfaces in Euclidean space $\mathbb{E}^{p+q}$.

1. Introduction. Investigating the properties of biharmonic submanifolds in Euclidean spaces was initiated by B. Y. Chen in the middle of 1980s in his study on finite type submanifolds. At first, B. Y. Chen proved that biharmonic surfaces in Euclidean 3-spaces are minimal, which was also independently proved by G. Y. Jiang [24]. Later on, I. Dimitrić in his doctoral thesis [20] and his paper [21] proved that any biharmonic curve in Euclidean spaces $\mathbb{E}^m$ is a part of a straight line (i.e. minimal); any biharmonic submanifold of finite type in $\mathbb{E}^m$ is minimal; any pseudo umbilical submanifold M^n in $\mathbb{E}^m$ with $n \neq 4$ is minimal, and any biharmonic hypersurface in $\mathbb{E}^m$ with at most two distinct principal curvatures is minimal. Hence, based on these results B. Y. Chen [10] in 1991 made the following well known conjecture:

Every biharmonic submanifold of Euclidean spaces is minimal.

In 1995, the conjecture was proved by T. Hasanis and T. Vlachos [23] for hypersurfaces in Euclidean 4-spaces, (see also Defever's work[19] with a different proof). However, the conjecture remains open. The main difficulty is that the conjecture is a local problem and how to understand the local structure of submanifolds satisfying $\Delta \overrightarrow{H} = 0$. Nevertheless, the study of the conjecture is quite active nowadays. Recently, B. Y. Chen and M. I. Munteanu [18] proved that Chen's conjecture is true for $\delta(2)$-ideal and $\delta(3)$-ideal hypersurfaces of a Euclidean space with arbitrary dimension, where the principal curvatures of such hypersurfaces

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takes special values. Under the assumption of completeness, K. Akutagawa and S. Maeta [1] proved that biharmonic properly immersed submanifolds in Euclidean spaces are minimal.

On the other hand, from the view of k-harmonic maps, one can define a biharmonic map between Riemannian manifolds if it is a critical point of the bienergy functional. G. Y. Jiang in [24] showed that a smooth map is biharmonic if and only if its bitension field vanishes identically. In the past ten years, there exists a lot of remarkable work on biharmonic submanifolds in spheres or even in generic Riemannian manifolds (see, for instance [3, 5–9, 26–28]). Nowadays, investigating the properties of biharmonic submanifolds is becoming a very active field of study.

In contrast to the submanifolds in Euclidean spaces, Chen's conjecture is not always true for submanifolds in pseudo-Euclidean spaces. This fact was achieved by B. Y. Chen and S. Ishikawa [16, 17] who constructed several examples of proper biharmonic surfaces in 4-dimensional pseduo-Euclidean spaces $\mathbb{E}^4_s$ ($s = 1, 2, 3$). But for hypersurfaces in pseudo-Euclidean spaces, B. Y. Chen and S. Ishikawa proved in [16, 17] that biharmonic surfaces in pseudo-Euclidean 3-spaces are minimal, and A. Arvanitoyeorgos et al. [4] proved that biharmonic Lorentzian hypersurfaces in Minkowski 4-spaces are minimal.

As we know, Chen's conjecture for hypersurfaces in $\mathbb{E}^4$ and for hypersurfaces in $\mathbb{E}^m$ with two distinct principal curvatures were solved by Hasanis-Vlachos and Dimitrić, respectively. It is natural to study biharmonic hypersurfaces with three distinct principal curvatures as the next step. Following B. Y. Chen, I. Dimitrić, F. Defever et. al's techniques, we make further progress on the conjecture. In a previous work [22], we proved that Chen's conjecture is true for biharmonic hypersurfaces with three distinct principal curvatures in $\mathbb{E}^5$. In this paper, we are able to solve that general case. Precisely, we will prove that biharmonic hypersurfaces with at most three distinct principal curvatures in $\mathbb{E}^{n+1}$ with arbitrary dimension are minimal. As an immediate conclusion, we show that biharmonic $O(p) \times O(q)$-invariant hypersurfaces in Euclidean spaces $\mathbb{E}^{p+q}$ are minimal.

2. Preliminaries.

Let $x : M^n \to \mathbb{E}^{n+1}$ be an isometric immersion of a hypersurface M^n into $\mathbb{E}^{n+1}$. Denote the Levi-Civita connections of M^n and $\mathbb{E}^m$ by ∇ and $\tilde{\nabla}$, respectively. Let X and Y denote vector fields tangent to M^n and let ξ be a unite normal vector field. Then the Gauss and Weingarten formulas are given, respectively, by (cf. [11, 14, 15])

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y)\,, \tag{2.1}$$

$$\tilde{\nabla}_X \xi = -AX\,, \tag{2.2}$$

where h is the second fundamental form, and A is the shape operator. It is well known that the second fundamental form h and the shape operator A are related by

$$\langle h(X, Y), \xi \rangle = \langle AX, Y \rangle\,. \tag{2.3}$$

The mean curvature vector field $\overrightarrow{H}$ is given by

$$\overrightarrow{H} = \frac{1}{n}\text{trace}\, h\,. \tag{2.4}$$

The Gauss and Codazzi equations are given, respectively, by

$$R(X,Y)Z = \langle AY, Z\rangle AX - \langle AX, Z\rangle AY\,,$$
$$(\nabla_X A)Y = (\nabla_Y A)X\,,$$

where R is the curvature tensor and $(\nabla_X A)Y$ is defined by

$$(\nabla_X A)Y = \nabla_X(AY) - A(\nabla_X Y) \tag{2.5}$$

for all X, Y, Z tangent to M.

Let Δ be the Laplacian operator of a submanifold M. For an isometric immersion $x : M^n \to \mathbb{E}^m$, the mean curvature vector field $\overrightarrow{H}$ in $\mathbb{E}^m$ satisfies (see, for instance [11], p. 44)

$$\Delta x = -n\overrightarrow{H}\,.$$

DEFINITION 2.1. Let $x : M^n \to \mathbb{E}^m$ be an isometric immersion of a Riemannian n-manifold M into a Euclidean space $\mathbb{E}^m$. Then M^n is called a biharmonic submanifold in $\mathbb{E}^m$ if and only if $\Delta\overrightarrow{H} = 0$, or equivalently, $\Delta^2 x = 0$.

By Definition 2.1, it is clear that any minimal submanifolds in a Euclidean space $\mathbb{E}^m$ must be trivially biharmonic. A biharmonic submanifold in a Euclidean space $\mathbb{E}^m$ is called proper biharmonic if it is not minimal.

Let M^n be a hypersurface in $\mathbb{E}^{n+1}$. Assume that $\overrightarrow{H} = H\xi$. Note that H denotes the mean curvature. By identifying the normal and the tangent parts of the biharmonic condition $\Delta\overrightarrow{H} = 0$, we obtain necessary and sufficient conditions for M^n to be biharmonic in $\mathbb{E}^{n+1}$, namely

$$\Delta H + H \operatorname{trace} A^2 = 0\,, \tag{2.6}$$
$$2A \operatorname{grad} H + n H \operatorname{grad} H = 0\,, \tag{2.7}$$

where the Laplace operator Δ acting on scalar-valued function f is given by (e.g., [13])

$$\Delta f = -\sum_{i=1}^{n}(e_i e_i f - \nabla_{e_i} e_i f)\,. \tag{2.8}$$

Here, $\{e_1, \ldots, e_n\}$ is a local orthonormal tangent frame on M^n.

3. Biharmonic hypersurfaces with three distinct principal curvatures in $\mathbb{E}^{n+1}$.

From now on, we concentrate on biharmonic hypersurfaces M^n in a Euclidean space $\mathbb{E}^{n+1}$ with $n \geq 4$.

Assume that the mean curvature H is not constant.

Observe from (2.7) that grad H is an eigenvector of the shape operator A with the corresponding principal curvature $-\frac{n}{2}H$. Without loss of generality, we can choose e_1 such that e_1 is parallel to grad H, and therefore the shape operator A of M^n takes the following form with respect to a suitable orthonormal frame $\{e_1, \dots, e_n\}$.

$$A = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}, \tag{3.1}$$

where λ_i are the principal curvatures and $\lambda_1 = -\frac{n}{2}H$. Let us express grad$H$ as

$$\text{grad}H = \sum_{i=1}^{n} e_i(H)e_i\,.$$

Since e_1 is parallel to grad H, it follows that

$$e_1(H) \neq 0\,, \quad e_i(H) = 0\,, \quad i = 2, 3, \dots, n\,. \tag{3.2}$$

We write

$$\nabla_{e_i} e_j = \sum_{k=1}^{n} \omega_{ij}^k e_k\,, \quad i, j = 1, 2, \dots, n\,. \tag{3.3}$$

The compatibility conditions $\nabla_{e_k}\langle e_i, e_i\rangle = 0$ and $\nabla_{e_k}\langle e_i, e_j\rangle = 0$ imply respectively that

$$\omega_{ki}^i = 0\,, \quad \omega_{ki}^j + \omega_{kj}^i = 0\,, \tag{3.4}$$

for $i \neq j$ and $i, j, k = 1, 2, \dots, n$. Furthermore, it follows from (3.1) and (3.3) that the Codazzi equation yields

$$e_i(\lambda_j) = (\lambda_i - \lambda_j)\omega_{ji}^j\,, \tag{3.5}$$

$$(\lambda_i - \lambda_j)\omega_{ki}^j = (\lambda_k - \lambda_j)\omega_{ik}^j \tag{3.6}$$

for distinct $i, j, k = 1, 2, \dots, n$.

Since $\lambda_1 = -\frac{n}{2}H$, from (3.2) we get

$$[e_i, e_j](\lambda_1) = 0\,, \quad i, j = 2, \dots, n\,, \quad i \neq j\,,$$

which yields directly

$$\omega_{ij}^1 = \omega_{ji}^1\,, \tag{3.7}$$

for distinct $i, j = 2, \dots, n$.

Now we show that $\lambda_j \neq \lambda_1$ for $j = 2, \dots, n$. In fact, if $\lambda_j = \lambda_1$ for $j \neq 1$, by putting $i = 1$ in (3.5) we have that

$$0 = (\lambda_1 - \lambda_j)\omega_{j1}^j = e_1(\lambda_j) = e_1(\lambda_1)\,, \tag{3.8}$$

which contradicts the first expression of (3.2).

By the assumption, M^n has three distinct principal curvatures. Without loss of generality, we assume that

$$\lambda_2 = \lambda_3 = \cdots = \lambda_p = \alpha\,,$$
$$\lambda_{p+1} = \lambda_{p+2} = \cdots = \lambda_n = \beta\,,$$

$$\frac{n+1}{2} \leq p < n\,. \tag{3.9}$$

By the definition (2.4) of $\overrightarrow{H}$, we have $nH = \sum_{i=1}^{n} \lambda_i$. Hence

$$\beta = \frac{\frac{3}{2}nH - (p-1)\alpha}{n-p}. \tag{3.10}$$

Since $\lambda_j \neq \lambda_1$ for $i = 2, \ldots, n$, we obtain

$$\alpha \neq -\frac{n}{2}H\,, \quad \frac{3n}{2(n-1)}H\,, \quad \frac{n^2-(p-3)n}{2(p-1)}H\,. \tag{3.11}$$

The multiplicities of principal curvatures α and β are $p-1$ and $n-p$, respectively.

In the following, we will state a key conclusion for later use.

LEMMA 3.1. *Let M^n be a proper biharmonic hypersurface with three distinct principal curvatures in $\mathbb{E}^{n+1}$. Then $e_i(\lambda_j) = 0$ for $i = 2, \ldots, n$ and $j = 1, 2, \ldots, n$.*

PROOF. Consider the equation (3.5). Since $n \geq 4$, it follows from (3.9) that $p-1 \geq 2$. For $i, j = 2, \ldots, p$ and $i \neq j$ in (3.5), one has

$$e_i(\alpha) = 0\,, \quad i = 2, \ldots, p\,. \tag{3.12}$$

If the multiplicity of principal curvature β satisfies $n - p \geq 2$, then for $i, j = p+1, \ldots, n$ and $i \neq j$ in (3.5) we have

$$e_i(\beta) = 0\,, \quad i = p+1, \ldots, n\,. \tag{3.13}$$

Hence, the conclusion follows directly from (3.2), (3.10), (3.12) and (3.13).

If the multiplicity of principal curvature β is one, namely $p = n-1$, then from (3.12) we only need to show that $e_n(\alpha) = 0$.

Let us compute $[e_1, e_i](H) = \big(\nabla_{e_1}e_i - \nabla_{e_i}e_1\big)(H)$ for $i = 2, \ldots, n$. From the first expression of (3.4), we have $\omega_{i1}^1 = 0$. For $j = 1$ and $i \neq 1$ in (3.5), by (3.2) we have $\omega_{1i}^1 = 0$ $(i \neq 1)$. Hence we have

$$e_i e_1(H) = 0\,, \quad i = 2, \ldots, n\,. \tag{3.14}$$

By (3.12), a similar way can also show that

$$e_i e_1(\alpha) = 0\,, \quad i = 2, \ldots, n-1\,. \tag{3.15}$$

For $j = 1, k, i \neq 1$ in (3.6) we have

$$(\lambda_i - \lambda_1)\omega_{ki}^1 = (\lambda_k - \lambda_1)\omega_{ik}^1\,,$$

which together with (3.7) yields

$$\omega_{ij}^1 = 0\,, \quad i \neq j\,, \quad i, j = 2, \ldots, n\,. \tag{3.16}$$

Moreover, combining (3.16) with the second equation of (3.4) gives

$$\omega_{i1}^j = 0\,, \quad i, j = 2, \ldots, n\,, \quad j \neq i\,. \tag{3.17}$$

It follows from (3.5) that

$$\omega_{i1}^{i} = \frac{e_1(\lambda_i)}{\lambda_1 - \lambda_i}, \quad i = 2, \dots, n. \tag{3.18}$$

For $k = 2$ and $i = n$ in (3.6), we have

$$(\lambda_n - \lambda_j)\omega_{2n}^{j} = (\lambda_2 - \lambda_j)\omega_{n2}^{j},$$

which yields

$$\omega_{2n}^{j} = 0, \quad j = 3, \dots, n-1.$$

Hence, from the first expression of (3.4) and (3.16) we get

$$\omega_{2n}^{j} = 0, \quad j \neq 2. \tag{3.19}$$

Also, (3.5) yields

$$\omega_{2n}^{2} = \frac{e_n(\alpha)}{\lambda_n - \alpha}. \tag{3.20}$$

From the Gauss equation and (3.1) we have $R(e_2, e_n)e_1 = 0$. Recall the definition of Gauss curvature tensor

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{3.21}$$

It follows from (3.15), (3.17–20) and (3.4) that

$$\begin{aligned}
\nabla_{e_2}\nabla_{e_n} e_1 &= \frac{e_1(\lambda_n)e_n(\alpha)}{(\lambda_1 - \lambda_n)(\lambda_n - \alpha)} e_2, \\
\nabla_{e_n}\nabla_{e_2} e_1 &= e_n\left(\frac{e_1(\alpha)}{\lambda_1 - \alpha}\right)e_2 + \frac{e_1(\alpha)}{\lambda_1 - \alpha}\sum_{k=3}^{n}\omega_{n2}^{k} e_k, \\
\nabla_{[e_2,e_n]} e_1 &= \frac{e_n(\alpha)e_1(\alpha)}{(\lambda_n - \alpha)(\lambda_1 - \alpha)} e_2 - \frac{e_1(\alpha)}{\lambda_1 - \alpha}\sum_{k=3}^{n}\omega_{n2}^{k} e_k.
\end{aligned}$$

Hence we obtain

$$e_n\left(\frac{e_1(\alpha)}{\lambda_1 - \alpha}\right) = \left(\frac{e_1(\lambda_n)}{\lambda_1 - \lambda_n} - \frac{e_1(\alpha)}{\lambda_1 - \alpha}\right)\frac{e_n(\alpha)}{\lambda_n - \alpha}. \tag{3.22}$$

Note that $\lambda_1 = -\frac{n}{2}H$ and $\lambda_n = \beta = \frac{3}{2}nH - (n-2)\alpha$ in this case.

The equation (3.22) can be rewritten as

$$e_n e_1(\alpha) = \left\{ -\frac{e_1(\alpha)}{\lambda_1 - \alpha} + \left(\frac{e_1(\lambda_n)}{\lambda_1 - \lambda_n} - \frac{e_1(\alpha)}{\lambda_1 - \alpha}\right)\frac{\lambda_1 - \alpha}{\lambda_n - \alpha}\right\} e_n(\alpha). \tag{3.23}$$

By (3.23), we compute

$$\begin{aligned}
e_n\left(\frac{e_1(\lambda_n)}{\lambda_1 - \lambda_n}\right) &= -(n-2)\left(\frac{e_n e_1(\alpha)}{\lambda_1 - \lambda_n} + \frac{e_1(\lambda_n)e_n(\alpha)}{(\lambda_1 - \lambda_n)^2}\right) \\
&= -(n-2)\frac{e_n(\alpha)}{\lambda_1 - \lambda_n}\left(\frac{e_1(\lambda_n)}{\lambda_1 - \lambda_n} - \frac{e_1(\alpha)}{\lambda_1 - \alpha}\right)\frac{\lambda_1 + \lambda_n - 2\alpha}{\lambda_n - \alpha}. \qquad (3.24)
\end{aligned}$$

It follows from (3.5) and the second expression of (3.4) that

$$\omega_{ii}^{1} = -\omega_{i1}^{i} = -\frac{e_1(\lambda_i)}{\lambda_1 - \lambda_i}\,. \tag{3.25}$$

Now consider the equation (2.6). It follows from (2.8), (3.1) and (3.25) that

$$\begin{aligned} &- e_1 e_1(H) - \left(\frac{(n-2)e_1(\alpha)}{\lambda_1 - \alpha} + \frac{e_1(\lambda_n)}{\lambda_1 - \lambda_n}\right) e_1(H) \\ &\qquad + H[{\lambda_1}^2 + (n-2)\alpha^2 + {\lambda_n}^2] = 0. \end{aligned} \tag{3.26}$$

Differentiating (3.26) along e_n, by (3.22) and (3.24) we get

$$\left\{\frac{2}{\lambda_1 - \lambda_n}\left(\frac{e_1(\lambda_n)}{\lambda_1 - \lambda_n} - \frac{e_1(\alpha)}{\lambda_1 - \alpha}\right) e_1(H) + H\big(-3nH + 2(n-1)\alpha\big)\right\} e_n(\alpha) = 0\,.$$

If $e_n(\alpha) \neq 0$, then the above equation becomes

$$\frac{2}{\lambda_1 - \lambda_n}\left(\frac{e_1(\lambda_n)}{\lambda_1 - \lambda_n} - \frac{e_1(\alpha)}{\lambda_1 - \alpha}\right) e_1(H) + H\big(-3nH + 2(n-1)\alpha\big) = 0\,. \tag{3.27}$$

Differentiating (3.27) along e_n and using (3.22) and (3.24) again, one has

$$\begin{aligned} &\frac{2n(4-n)H + 2(n-2)(n-1)\alpha}{(\lambda_1 - \lambda_n)(\lambda_n - \alpha)}\left(\frac{e_1(\lambda_n)}{\lambda_1 - \lambda_n} - \frac{e_1(\alpha)}{\lambda_1 - \alpha}\right) e_1(H) \\ &\qquad + H\big((-7n + 10)nH + 4(n-1)(n-2)\alpha\big) = 0\,. \end{aligned} \tag{3.28}$$

Therefore, combining (3.28) with (3.27) gives

$$3(n-2)H[3nH - 2(n-1)\alpha]^2 = 0\,,$$

which implies that

$$\alpha = \frac{3n}{2(n-1)}H\,.$$

This contradicts to (3.11). Hence, we obtain $e_n(\alpha) = 0$, which completes the proof of Lemma 3.1. □

Now, we are ready to express the connection coefficients of hypersurfaces.

For $j = 1$ and $i = 2, \ldots, n$ in (3.5), by (3.2) we get $\omega_{1i}^{1} = 0$. Moreover, by the first and second expressions of (3.4) we have

$$\omega_{1i}^{1} = \omega_{11}^{i} = 0, \quad i = 1, 2, \ldots, n\,. \tag{3.29}$$

For $i = 1$, $j = 2, \ldots, n$ in (3.5), we obtain

$$\omega_{j1}^{j} = -\omega_{jj}^{1} = \frac{e_1(\lambda_j)}{\lambda_1 - \lambda_j}\,, \quad j = 2, \ldots, n\,. \tag{3.30}$$

For $i = p+1, \ldots, n$, $j = 2, \ldots, p$ in (3.5), by (3.2) we have

$$\omega_{ji}^{j} = -\omega_{jj}^{i} = 0\,. \tag{3.31}$$

Similarly, for $i = 2, \ldots, p$, $j = p+1, \ldots, n$ in (3.5), we also have

$$\omega_{ji}^{j} = -\omega_{jj}^{i} = 0\,. \tag{3.32}$$

For $i = 1$, by choosing $j, k = 2, \ldots, p$ or $k, j = p+1, \ldots, n$ $(j \neq k)$ in (3.6), we have

$$\omega_{k1}^{j} = \omega_{kj}^{1} = 0\,. \tag{3.33}$$

For $i = 2, \ldots, p$ and $j, k = p+1, \ldots, n$ $(j \neq k)$ in (3.6), we get

$$\omega_{ki}^{j} = \omega_{kj}^{i} = 0\,. \tag{3.34}$$

For $i = 2, \ldots, p$, $j = 1$ and $k = p+1, \ldots, n$ in (3.6), we have

$$(\alpha - \lambda_1)\omega_{ki}^{1} = (\beta - \lambda_1)\omega_{ik}^{1}\,,$$

which together with (3.7) and the second expression of (3.4) gives

$$\omega_{ki}^{1} = \omega_{ik}^{1} = \omega_{k1}^{i} = \omega_{i1}^{k} = 0\,. \tag{3.35}$$

For $i = 2, \ldots, p$, $k = 1$ and $j = p+1, \ldots, n$ in (3.6), we obtain

$$(\beta - \alpha)\omega_{1i}^{j} = (\lambda_1 - \alpha)\omega_{i1}^{j}\,,$$

which together with (3.35) yields

$$\omega_{1i}^{j} = \omega_{1j}^{i} = 0\,. \tag{3.36}$$

Combining (3.29–3.36) with (3.4) and summarizing, we have the following lemma.

LEMMA 3.2. *Let M^n be a biharmonic hypersurface with non-constant mean curvature in Euclidean space $\mathbb{E}^{n+1}$, whose shape operator given by* (3.1) *with respect to an orthonormal frame $\{e_1, \ldots, e_n\}$. Then we have*

$$\nabla_{e_1} e_1 = 0\,; \quad \nabla_{e_i} e_1 = \frac{e_1(\lambda_i)}{\lambda_1 - \lambda_i} e_i\,, \quad i = 2, \ldots, n\,;$$

$$\nabla_{e_i} e_j = \sum_{k=2, k\neq j}^{p} \omega_{ij}^{k} e_k\,, \quad i = 1, \ldots, n\,, \quad j = 2, \ldots, p,\ i \neq j\,;$$

$$\nabla_{e_i} e_i = -\frac{e_1(\lambda_i)}{\lambda_1 - \lambda_i} e_1 + \sum_{k=2, k\neq i}^{p} \omega_{ii}^{k} e_k\,, \quad i = 2, \ldots, p\,;$$

$$\nabla_{e_i} e_j = \sum_{k=p+1, k\neq j}^{n} \omega_{ij}^{k} e_k\,, \quad i = 1, \ldots, n\,, \quad j = p+1, \ldots, n\,, \quad i \neq j\,;$$

$$\nabla_{e_i} e_i = -\frac{e_1(\lambda_i)}{\lambda_1 - \lambda_i} e_1 + \sum_{k=p+1, k\neq i}^{n} \omega_{ii}^{k} e_k\,, \quad i = p+1, \ldots, n\,,$$

where $\omega_{ki}^{j} = -\omega_{kj}^{i}$ for $i \neq j$ and $i, j, k = 1, \ldots, n$.

Let us introduce two smooth functions A and B as follows

$$A = \frac{e_1(\alpha)}{\lambda_1 - \alpha}\,, \quad B = \frac{e_1(\beta)}{\lambda_1 - \beta}\,. \tag{3.37}$$

One can compute the curvature tensor R by Lemma 3.2 and apply the Gauss equation for different values of X, Y and Z. After comparing the coefficients with respect to the orthonormal basis $\{e_1, \dots, e_n\}$ we get the following:

- $X = e_1, Y = e_2, Z = e_1$,

$$e_1(A) + A^2 = -\lambda_1 \alpha\,; \tag{3.38}$$

- $X = e_1, Y = e_n, Z = e_1$,

$$e_1(B) + B^2 = -\lambda_1 \beta\,; \tag{3.39}$$

- $X = e_n, Y = e_2, Z = e_n$,

$$AB = -\alpha\beta\,. \tag{3.40}$$

Consider the equation (2.6) again. It follows from (2.8), (3.1), (3.37) and Lemma 3.2 that

$$-e_1 e_1(H) - \big[(p-1)A + (n-p)B\big]e_1(H) + H\big[\lambda_1^2 + (p-1)\alpha^2 + (n-p)\beta^2\big] = 0\,. \tag{3.41}$$

We will derive a key equation for later use.

LEMMA 3.3. *The functions A and B are related by*

$$\begin{aligned} &\big[(4-p)A + (3+p-n)B\big]e_1(H) + \frac{3n^2(n+6-p)}{4(n-p)}H^3 \\ &\qquad - \frac{3n(n-2+4p)}{2(n-p)}H^2\alpha + \frac{3n(p-1)}{n-p}H\alpha^2 = 0\,. \end{aligned} \tag{3.42}$$

PROOF. By (3.37), equations (3.38) and (3.39) further reduce to

$$e_1 e_1(\alpha) + 2Ae_1(\alpha) - Ae_1(\lambda_1) + \lambda_1\alpha(\lambda_1 - \alpha) = 0\,, \tag{3.43}$$

$$e_1 e_1(\beta) + 2Be_1(\beta) - Be_1(\lambda_1) + \lambda_1\beta(\lambda_1 - \beta) = 0\,. \tag{3.44}$$

Since α and β are related by $(n-p)\beta + (p-1)\alpha = \frac{3nH}{2}$, it follows from (3.37) that

$$e_1(\alpha) = \frac{3n}{2(p-1)}e_1(H) - \frac{n-p}{p-1}B(\lambda_1 - \beta)\,, \tag{3.45}$$

$$e_1(\beta) = \frac{3n}{2(n-p)}e_1(H) - \frac{p-1}{n-p}A(\lambda_1 - \alpha)\,. \tag{3.46}$$

Substituting (3.45) and (3.46) into (3.47) and (3.48), respectively, by (3.40) we have

$$\begin{aligned} &e_1 e_1(\alpha) + \left(\frac{3}{p-1} + \frac{1}{2}\right)nAe_1(H) \\ &\qquad + \frac{2(n-p)}{p-1}(\lambda_1 - \beta)\alpha\beta + \lambda_1\alpha(\lambda_1 - \alpha) = 0\,, \end{aligned} \tag{3.47}$$

$$\begin{aligned} &e_1 e_1(\beta) + \left(\frac{3}{n-p} + \frac{1}{2}\right)nBe_1(H) \\ &\qquad + \frac{2(p-1)}{n-p}(\lambda_1 - \alpha)\alpha\beta + \lambda_1\beta(\lambda_1 - \beta) = 0\,, \end{aligned} \tag{3.48}$$

where we use $\lambda_1 = -\frac{nH}{2}$. By using $(n-p)\beta + (p-1)\alpha = \frac{3nH}{2}$, we could eliminate $e_1e_1(H)$, $e_1e_1(\alpha)$ and $e_1e_1(\beta)$ from (3.41), (3.47) and (3.48). Consequently, we obtain the desired equation (3.42). □

Moreover, by using $(n-p)\beta + (p-1)\alpha = \frac{3nH}{2}$ and (3.37) we have

$$e_1(H) = -\left[\frac{p-1}{3}H + \frac{2(p-1)}{3n}\alpha\right]A + \left[-\frac{n+3-p}{3}H + \frac{2(p-1)}{3n}\alpha\right]B\,. \tag{3.49}$$

Substituting (3.49) into (3.42) and using (3.40), we get

$$\begin{aligned}(4-p)(p-1)(nH+2\alpha)A^2 &+ (3+p-n)[n(n+3-p)H - 2(p-1)\alpha]B^2\\ &= \frac{n(p-1)(-2p^2+2pn+11p+n-12)}{n-p}H\alpha^2\\ &\quad - \frac{2(p-1)^2(2p-n-1)}{n-p}\alpha^3 + \frac{9n^3(n+6-p)}{4(n-p)}H^3\\ &\quad + \frac{3n^2(p-1)(2p-2n-15)}{2(n-p)}H^2\alpha\,.\end{aligned} \tag{3.50}$$

Multiplying A and B successively on the equation (3.42), using (3.40) one gets respectively

$$\begin{aligned}&(4-p)A^2e_1(H) - (3+p-n)\alpha\beta e_1(H)\\ &\quad + \left[\frac{3n^2(n+6-p)}{4(n-p)}H^3 - \frac{3n(n-2+4p)}{2(n-p)}H^2\alpha + \frac{3n(p-1)}{n-p}H\alpha^2\right]A = 0\,,\end{aligned} \tag{3.51}$$

$$\begin{aligned}&(3+p-n)B^2e_1(H) - (4-p)\alpha\beta e_1(H)\\ &\quad + \left[\frac{3n^2(n+6-p)}{4(n-p)}H^3 - \frac{3n(n-2+4p)}{2(n-p)}H^2\alpha + \frac{3n(p-1)}{n-p}H\alpha^2\right]B = 0\,.\end{aligned} \tag{3.52}$$

Differentiating (3.42) along e_1, and using (3.38–39) and (3.41) we have

$$\begin{aligned}&\left[(4-p)\left(\frac{n}{2}H\alpha - A^2\right) + (3+p-n)\left(\frac{n}{2}H\beta - B^2\right)\right]e_1(H)\\ &\quad - \big[(4-p)A + (3+p-n)B\big]\big[(p-1)A + (n-p)B\big]e_1(H)\\ &\quad + \big[(4-p)A + (3+p-n)B\big]\left[\frac{n^2}{4}H^3 + (p-1)H\alpha^2 + (n-p)H\beta^2\right]\\ &\quad + \left[\frac{9n^2(n+6-p)}{4(n-p)}H^2 - \frac{3n(n-2+4p)}{n-p}H\alpha + \frac{3n(p-1)}{n-p}\alpha^2\right]e_1(H)\\ &\quad - \frac{3n(n-2+4p)}{2(n-p)}H^2e_1(\alpha) + \frac{6n(p-1)}{n-p}H\alpha e_1(\alpha) = 0\,.\end{aligned} \tag{3.53}$$

Substituting (3.51), (3.52), (3.42) into (3.53), and using the first expression of (3.37) we obtain

$$\begin{aligned}&\Big[\frac{3n^2(2n-2p+21)}{4(n-p)}H^2-\frac{3n(5p+1)}{n-p}H\alpha+\frac{(p-1)(2n+7)}{n-p}\alpha^2\Big]e_1(H)\\&+\Big[\frac{n^2(2pn-2p^2+7n+17p+30)}{4(n-p)}H^3-\frac{3n(3np+2p^2+4p-3n-6)}{2(n-p)}H^2\alpha\\&+\frac{(p-1)(2np-2n+p-4)}{n-p}H\alpha^2\Big]A+\Big[\frac{n^2\big(2(n-p)^2+15(n-p)+45\big)}{4(n-p)}H^3\\&-\frac{3n(n^2+np-2p^2+10p+n-8)}{2(n-p)}H^2\alpha\\&+\frac{(p-1)(2n^2-2np+7n-p-3)}{n-p}H\alpha^2\Big]B=0\,.\end{aligned}\tag{3.54}$$

From (3.49), the equation (3.54) further reduces to

$$\begin{aligned}&\Big[\frac{9}{4}n^3(3n-2p+17)H^3-\frac{3}{2}n^2(-6p^2+11np+43p-11n-37)H^2\alpha\\&+n(p-1)(4np-4n+26p+1)H\alpha^2-2(p-1)^2(2n+7)\alpha^3\Big]A\\&-\Big[\frac{9}{2}(2n-2p+3)H^3+\frac{9}{2}n^2(2p^2+n^2-3np-7p+n-3)H^2\alpha\\&-2n(p-1)(2n^2-2np+4n-13p-18)H\alpha^2-2(p-1)^2(2n+7)\alpha^3\Big]B=0\,.\end{aligned}\tag{3.55}$$

At this moment, we obtain all the desired equations (3.40), (3.50) and (3.55) concerning A and B.

In order to write handily, we introduce several notions: L, M denoting the coefficients of A and B respectively in (3.55), and N denoting the right-hand side of equal sign in the equation (3.50). Then (3.55) and (3.50) become

$$LA-MB=0\,,\tag{3.56}$$

$$\begin{aligned}&(4-p)(p-1)(nH+2\alpha)A^2\\&+(3+p-n)[n(n+3-p)H-2(p-1)\alpha]B^2=N\,.\end{aligned}\tag{3.57}$$

Multiplying LM on both sides of the equation (3.57), using (3.56) and (3.40) we can eliminate both A and B. Hence, we have

$$\begin{aligned}&(4-p)(p-1)(nH+2\alpha)M^2\alpha\frac{\frac{3}{2}nH-(p-1)\alpha}{n-p}\\&\quad+(3+p-n)[n(n+3-p)H-2(p-1)\alpha]L^2\alpha\frac{\frac{3}{2}nH-(p-1)\alpha}{n-p}\\&=LMN\,.\end{aligned}\tag{3.58}$$

In view of (3.58), we notice that the equation should have the form:

$$\begin{aligned} & a_9 H^9 + a_8 H^8 \alpha + a_7 H^7 \alpha^2 + a_6 H^6 \alpha^3 + a_5 H^5 \alpha^4 + a_4 H^4 \alpha^5 + a_3 H^3 \alpha^6 \\ & \quad + a_2 H^2 \alpha^7 + a_1 H \alpha^8 + a_0 \alpha^9 = 0 \end{aligned} \tag{3.59}$$

for constant coefficients a_i concerning n and p ($i = 0, \ldots, 9$). Since $p < n$, from (3.58), (3.55) and (3.50) we compute a_9:

$$a_9 = \frac{243 n^6 (n-p+6)(3n-2p+17)(2n-2p+3)}{16(n-p)} \neq 0\,. \tag{3.60}$$

Remark that $\alpha \neq 0$. In fact, if $\alpha = 0$, then (3.59) implies that

$$a_9 H^9 = 0\,,$$

which is impossible since H is non-constant and $a_9 \neq 0$.

Put $\Phi = \frac{H}{\alpha}$. Then (3.59) reduces to a non-trivial algebraic equation of ninth degree with respect to Φ:

$$\begin{aligned} & a_9 \Phi^9 + a_8 \Phi^8 + a_7 \Phi^7 + a_6 \Phi^6 + a_5 \Phi^5 + a_4 \Phi^4 + a_3 \Phi^3 \\ & \quad + a_2 \Phi^2 + a_1 \Phi + a_0 = 0\,. \end{aligned} \tag{3.61}$$

Clearly, the equation (3.61) shows that, even in case of the existence of a real solution, H is proportional to α, namely

$$H = c\alpha\,, \tag{3.62}$$

where c is a root of the equation (3.61) and has to be a nonzero constant.

At last, we will derive a contradiction. Substituting (3.62) into (3.37), and then applying on (3.38), (3.40) and (3.41) respectively, we have

$$e_1 e_1(\alpha) - \left(1 + \frac{2}{nc+2}\right) \frac{e_1^2(\alpha)}{\alpha} + \frac{nc}{2}\left(\frac{nc}{2} + 1\right)\alpha^3 = 0\,, \tag{3.63}$$

$$e_1^2(\alpha) + \frac{(nc+2)\big(nc(n-p) + 3nc - 2(p-1)\big)}{4(n-p)} \alpha^4 = 0\,, \tag{3.64}$$

$$\begin{aligned} & -e_1 e_1(\alpha) + \left[\frac{2(p-1)}{nc+2} + \frac{(n-p)\big(3nc - 2(p-1)\big)}{nc(n-p) + 3nc - 2(p-1)}\right] \frac{e_1^2(\alpha)}{\alpha} \\ & + \left[\frac{n^2 c^2}{4} + (p-1)^2 + \frac{(3nc - 2(p-1))^2}{4(n-p)}\right] \alpha^3 = 0\,. \end{aligned} \tag{3.65}$$

Substituting (3.64) into (3.63), we get

$$e_1 e_1(\alpha) + \frac{(nc+6)\big(nc(n-p) + 3nc - 2(p-1)\big)}{4(n-p)} \alpha^3 = 0\,. \tag{3.66}$$

Since $e_1(\alpha) \neq 0$, differentiating (3.64) along e_1 we obtain

$$e_1 e_1(\alpha) + \frac{2(nc+2)\big[nc(n-p) + 3nc - 2(p-1)\big]}{4(n-p)} \alpha^3 = 0\,. \tag{3.67}$$

Combining (3.67) with (3.66) gives

$$\frac{(nc-2)\big[nc(n-p)+3nc-2(p-1)\big]}{4(n-p)}\alpha^3 = 0\,. \tag{3.68}$$

Since $\alpha \neq 0$, we have either $nc = 2$ or $nc(n-p)+3nc-2(p-1) = 0$.

In the former case, substituting $nc = 2$ into (3.64) and (3.66), and then substituting (3.64) and (3.66) into (3.65) we have

$$(p-1)^2 + (p-1) - 1 + \frac{2(p-1)^2 - 13(p-1) + 21}{n-p} = 0\,,$$

which reduces to

$$(p-1)^2 + (p-1) - 1 + \frac{2\big[(p-1) - \frac{13}{4}\big]^2 - \frac{1}{8}}{n-p} = 0\,. \tag{3.69}$$

In this case, since $p < n$, $p \geq 2$ and $p \in \mathbb{Z}^+$, we have always

$$(p-1)^2 + (p-1) - 1 > 0\,,$$

and

$$2\left[(p-1) - \frac{13}{4}\right]^2 - \frac{1}{8} \geq 0\,. \tag{3.70}$$

Note that the equality in (3.70) holds if and only if $p = 4$. the above information shows that (3.69) gives a contradiction.

In the latter case, substituting $nc(n-p)+3nc-2(p-1) = 0$ into (3.64) and (3.66), respectively, we obtain

$$e_1e_1(\alpha) = e_1^2(\alpha) = 0\,,$$

which together with (3.65) yields a contradiction as well.

Consequently, we conclude that the mean curvature H must be constant. Therefore, biharmonic hypersurfaces with three distinct principal curvatures in $\mathbb{E}^{n+1}$ have to be minimal.

In conclusion, we can state the main theorem in the following.

THEOREM 3.4. *Every biharmonic hypersurface with three distinct principal curvatures in a Euclidean space with arbitrary dimension is minimal.*

REMARK 3.5. Remark that the approach in this paper is self-contained from a structural point of view, and it maybe provide better insight into the structure of biharmonic hypersurface. With this method, one could consider hypersurfaces with four distinct principal curvatures or the higher codimension cases of Chen's conjecture.

Finally, we give an application of the main theorem.

COROLLARY 3.6. *Every biharmonic $O(p) \times O(q)$-invariant hypersurface in a Euclidean space $\mathbb{E}^{p+q}$ is minimal.*

$O(p) \times O(q)$-invariant hypersurfaces, that is, invariant under the action of some isometry group $O(p) \times O(q)$, were studied in [2]. For an $O(p) \times O(q)$-invariant hypersurface M in Euclidean space $\mathbb{E}^{p+q}$, it can be parameterized by

$$\bar{x}\big(t, \phi_1, \ldots, \phi_{p-1}, \psi_1, \ldots, \psi_{q-1}\big) = \big(x(t)\Phi(\phi_1, \ldots, \phi_{p-1}), y(t)\Psi(\psi_1, \ldots, \psi_{q-1})\big),$$

where Φ and Ψ are orthogonal parameterizations of a unit sphere of the corresponding dimension. It is easy to check that M has at most three distinct principal curvatures, see details in [2]. Hence, by applying Theorem 3.4, we immediately obtain Corollary 3.6.

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SCHOOL OF MATHEMATICS AND QUANTITATIVE ECONOMICS
DONGBEI UNIVERSITY OF FINANCE AND ECONOMICS
DALIAN 116025
P. R. CHINA

E-mail address: yufudufe@gmail.com