BIHARMONIC LEGENDRE CURVES
IN SASAKIAN SPACE FORMS

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ABSTRACT. Biharmonic Legendre curves in a Sasakian space form are studied. A non-existence result in a 7-dimensional 3-Sasakian manifold is obtained. Explicit formulas for some biharmonic Legendre curves in the 7-sphere are given.

1. Introduction

Especially in the last two decades, since the paper of G. Y. Jiang, [10], many mathematicians obtained important results related to biharmonic maps (see [12]). However, biharmonic submanifolds in a Sasakian space form have been studied only in the last few years and results were obtained in dimensions 3 and 5 (see for example [5], [7], [9], [14], [15]).

As in the general theory of contact manifolds, in the cited papers an important role is played by Legendre curves, which are one dimensional integral submanifolds. The aim of our work is to study such curves in $(2n+1)$-dimensional Sasakian space forms.

2. Preliminaries

2.1. Biharmonic maps

First we should recall some notions and results related to the harmonic and the biharmonic maps between Riemannian manifolds, as they are presented in [6], [10], [12] and in [18].

Let $f : M \to N$ be a smooth map between two Riemannian manifolds (M, g) and (N, h). Then the connection ∇ on the induced bundle $f^{-1}(TN) = \cup_{P \in M} T_{f(P)} N$ is defined as follows. For $X \in \chi(M), V \in \Gamma(f^{-1}(TN))$, define $\nabla_X V \in \Gamma(f^{-1}(TN))$ by $\nabla_X V = \nabla^N_{f_* X} V$.

The section $\tau(f) = \text{trace} \nabla df$ is called the tension field of f. A map f is said to be harmonic if $\tau(f)$ vanishes identically.

The bienergy of f is defined by $E_2(f) = \frac{1}{2}\int_M \|\tau(f)\|^2 * 1$. We say that f is a biharmonic map if it is a critical point of the bienergy, $E_2(f)$. It is proved in [10] that a map $f : M \to N$ is a biharmonic map if and only if it satisfies the equation

$$(1) \qquad \tau_2(f) = -\Delta\tau(f) - \text{trace}R^N(df(\cdot), \tau(f))df(\cdot) = 0,$$

where R^N denote the curvature tensor field on (N, h) and Δ is the rough Laplacian, defined by

$$\Delta = -\text{trace}_g(\nabla\nabla - \nabla_{\nabla^M}).$$

Note that any harmonic map is a biharmonic map and, moreover, an absolute minimum of the bienergy functional.

2.2. Sasakian manifolds

Concerning the Sasakian manifolds and the Legendre curves let us recall some notions and results as they are presented in [3].

Let M be an odd dimensional differentiable manifold and let (φ, ξ, η) be a tensor field of type $(1, 1)$, a vector field on M and an 1-form M, respectively. If $\varphi^2 = -I + \eta \otimes \xi$ and $\eta(\xi) = 1$, then (φ, ξ, η) is called an almost contact structure on M. On such a manifold, one obtains, by some algebraic computations, $\varphi\xi = 0$, $\eta \circ \varphi = 0$, $\varphi^3 + \varphi = 0$. If the tensor field S, of type $(1,2)$, defined by $S = N_\varphi + 2d\eta \otimes \xi$, where $N_\varphi(X, Y) = [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y] + \varphi^2[X, Y]$, is the Nijenhuis tensor field of φ, vanishes, then the almost contact structure is said to be normal (for more details see [3]). Let g be a (semi-)Riemannian metric on M. Then g is called an associated metric to the almost contact structure if $g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y)$, for any vector fields X, Y on M. Let Ω be the fundamental 2-form of the almost contact metric manifold $(M, \varphi, \xi, \eta, g)$, defined by $\Omega(X, Y) = g(X, \varphi Y)$. If $\Omega = d\eta$ then M is called a contact metric manifold. A normal contact metric manifold is called a Sasaki manifold. In [3] it is proved that an almost contact metric structure (φ, ξ, η, g) is Sasakian if and only if

$$(\nabla_X\varphi)Y = g(X, Y)\xi - \eta(Y)X,$$

where ∇ is the Levi-Civita connection of g. From this equation it can be easily obtained that $\nabla_X\xi = -\varphi X$.

If $\mathcal{D}$ is the contact distribution in a contact manifold (M, φ, ξ, η), defined by the subspaces $\mathcal{D}_m = \{X \in T_mM | \eta(X) = 0\}$, then a one-dimensional integral submanifold of $\mathcal{D}$ will be called a Legendre curve. A curve $\gamma : I \to M$, parametrized by its arc length is a Legendre curve if and only if $\eta(\gamma') = 0$.

A plane section in T_mM is called a φ-section if there exists a vector $X \in T_mM$ orthogonal to ξ such that $\{X, \varphi X\}$ span the section. The sectional curvature, $K(X, \varphi X)$, is called φ-sectional curvature. A Sasakian manifold of constant φ-sectional curvature c will be called a Sasakian space form and

denoted by $M(c)$. For such a manifold the curvature tensor is given by

$$(2) \quad \begin{aligned} R(X,Y)Z \\ &= \frac{c+3}{4}[g(Y,Z)X - g(X,Z)Y] \\ &+ \frac{c-1}{4}[\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi \\ &+ \Omega(Z,Y)\varphi X - \Omega(Z,X)\varphi Y + 2\Omega(X,Y)\varphi Z]. \end{aligned}$$

2.3. 3-Sasakian manifolds

If a manifold M admits three almost contact structures $(\varphi_a, \xi_a, \eta_a)$, $a = 1, 2, 3$, satisfying

$$\varphi_c = \varphi_a\varphi_b - \eta_b \otimes \xi_a = -\varphi_b\varphi_a + \eta_a \otimes \xi_b,$$

$$\xi_c = \varphi_a\xi_b = -\varphi_b\xi_a, \quad \eta_c = \eta_a \circ \varphi_b = -\eta_b \circ \varphi_a$$

for any even permutation $\{a, b, c\}$ of $\{1, 2, 3\}$, then the manifold is said to have an almost contact 3-structure. The dimension of such a manifold is of the form $4n + 3$. It is proved ([11]) that there exists an associated metric to each of this three structures. If all structures are Sasakian then we call the manifold M a 3-Sasakian manifold. It is proved that every 3-contact structure is 3-Sasakian, (see [3]). It is easy to see that if one of the φ_a-sectional curvatures, c_a, is constant then $c_a = 1$.

Concerning the Legendre curves, note that in case of a $(4n + 3)$-dimensional 3-Sasakian manifold the maximum dimension of an integral submanifold with respect to all three structures is n. Thus in dimension 7 these would be Legendre curves.

3. Biharmonic curves in a Sasakian space form

Let $\gamma : I \to M(c)$ be a curve defined on an open interval I and parametrized by its arc length, in a Sasakian space form $M(c)$, with dimension $2n + 1$ and structure tensors (φ, ξ, η, g). Let $\{T, N_1, \ldots, N_{2n}\}$ be the Frenet frame in $M(c)$, defined along γ, where $T = \gamma'$ is the unit tangent vector field of γ, N_1 is the unit normal vector field of γ, with the same direction as $\nabla_T T$ and the vectors $N_2, \ldots, N_{2n}$ are the unit vectors obtained from the Frenet equations for γ,

$$(1) \quad \begin{cases} \nabla_T T &= \chi_1 N_1, \\ \nabla_T N_1 &= -\chi_1 T + \chi_2 N_2, \\ \quad \vdots \quad\quad \vdots \quad\quad \vdots \\ \nabla_T N_k &= -\chi_k N_{k-1} + \chi_{k+1}N_{k+1}, \quad k = 2, \ldots, 2n - 1, \\ \quad \vdots \quad\quad \vdots \quad\quad \vdots \\ \nabla_T N_{2n} &= -\chi_{2n}N_{2n-1}, \end{cases}$$

where $\chi_1 = \|\nabla_T T\|$, and $\chi_2 = \chi_2(s), \ldots, \chi_{2n} = \chi_{2n}(s)$, are real valued functions, where s is the arc length of γ. If $\chi_k \in \mathbb{R}$, $k = 1, \ldots, 2n$, then γ is said to be a helix.

The biharmonic equation of γ is

$$(2) \qquad \tau_2(\gamma) = \nabla_T^3 T - R(T, \nabla_T T)T = 0.$$

Using the Frenet equations one obtains

$$(3) \quad \nabla_T^3 T = (-3\chi_1 \chi_1')T + (\chi_1'' - \chi_1^3 - \chi_1 \chi_2^2)N_1 + (2\chi_1' \chi_2 + \chi_1 \chi_2')N_2 + \chi_1 \chi_2 \chi_3 N_3.$$

From (2) we get

$$
\begin{aligned}
R(T, \chi_1 N_1)T = {} & -\frac{c+3}{4}\chi_1 N_1 + \frac{c-1}{4}[\chi_1(\eta(T))^2 N_1 - \chi_1 \eta(T)\eta(N_1)T \\
& + \chi_1 \eta(N_1)\xi - 3\chi_1 g(N_1, \varphi T)\varphi T] \\
& = \chi_1 \alpha_1 N_1 + \sum_{k=2}^{2n} \chi_1 \alpha_k N_k,
\end{aligned}
$$

(4)

where

$$
(5) \qquad
\begin{cases}
\alpha_1 = -\frac{c+3}{4} + \frac{c-1}{4}[(\eta(T))^2 + (\eta(N_1))^2 - 3(g(N_1, \varphi T))^2], \\[2mm]
\alpha_k = \frac{c-1}{4}[\eta(N_1)\eta(N_k) - 3g(N_1, \varphi T)g(N_k, \varphi T)], \quad k \geqslant 2.
\end{cases}
$$

Taking account of (3) and (4) in (2) we obtain the biharmonic equation of the curve γ

$$
\begin{aligned}
(6) \qquad \tau_2(\gamma) = {} & (-3\chi_1 \chi_1')T + (\chi_1'' - \chi_1^3 - \chi_1 \chi_2^2 - \chi_1 \alpha_1)N_1 \\
& + (2\chi_1' \chi_2 + \chi_1 \chi_2' - \chi_1 \alpha_2)N_2 + (\chi_1 \chi_2 \chi_3 - \chi_1 \alpha_3)N_3 \\
& - \sum_{k=4}^{2n} \chi_1 \alpha_k N_k = 0.
\end{aligned}
$$

Now, we can state

Theorem 3.1. *A curve, $\gamma : I \to M(c)$, parametrized by its arc length, is biharmonic and is not a geodesic if and only if*

$$
(7) \qquad
\begin{cases}
\chi_1 \in \mathbb{R} \setminus \{0\} \\
\chi_1^2 + \chi_2^2 = -\alpha_1 \\
\chi_2' = \alpha_2 \\
\chi_2 \chi_3 = \alpha_3 \\
\alpha_k = 0, \quad k \geqslant 4,
\end{cases}
$$

where α_k, $k \geqslant 1$, are given by (5).

From now on until the end we assume that γ is a Legendre curve in the Sasakian space form $M(c)$. Thus $\eta(\gamma') = \eta(T) = 0$. Differentiating along γ one obtains $g(\nabla_T T, \xi) + g(T, \nabla_T \xi) = 0$. Since $\nabla_T \xi = -\varphi T$, it follows that $g(N_1, \xi) = 0$, if γ is not a geodesic in $M(c)$.

First, from Theorem 3.1 it follows immediately.

Theorem 3.2. *A Legendre curve,* $\gamma : I \to M(1)$, *parametrized by its arc length, is biharmonic and is not a geodesic if and only if*

$$(8) \qquad \begin{cases} \chi_1 \in \mathbb{R} \setminus \{0\} \\ \chi_1^2 + \chi_2^2 = 1 \\ \chi_2 \in \mathbb{R} \\ \chi_2 \chi_3 = 0. \end{cases}$$

Theorem 3.1 and Theorem 3.2 suggest that it can be interesting to study Legendre curves in a Sasakian space form $M(c)$, with $\alpha_3 = 0$, where α_3 is given by (5).

We can state

Proposition 3.3. *If* γ *is a non-geodesic biharmonic Legendre curve in the Sasakian space form* $M(c)$, *with* $c \neq 1$, *then* $g(\nabla_{\gamma'} \gamma', \varphi \gamma') = a$, *where* a *is a constant and* χ_2 *is also a constant. Moreover, if* $\alpha_3 = 0$, *then* $\frac{1}{\chi_1} g(\nabla_{\gamma'} \gamma', \varphi \gamma') \in \{-1, 0, 1\}$, *that means* N_1 *is either orthogonal to* φT *or* $N_1 = \pm \varphi T$.

Proof. Suppose that $g(N_1, \varphi T) = a \notin \{-1, 0, 1\}$. Then we can consider the vector field E, defined by

$$E = \frac{1}{\sqrt{1-a^2}} N_1 - \frac{a}{\sqrt{1-a^2}} \varphi T.$$

It is easy to see that $g(E, E) = 1$, $g(E, T) = 0$, $g(E, \xi) = 0$ and $g(T, \varphi E) = 0$. Thus the vector fields $T, E, \varphi T$ and φE can be viewed as taking part of an orthonormal φ-basis in $M(c)$. Note that in this basis N_1 is given by $N_1 = \sqrt{1-a^2} E + a \varphi T$.

Differentiating $g(N_1, \varphi T) = a$ along γ one obtains

$$g(\nabla_T N_1, \varphi T) + g(N_1, \nabla_T \varphi T) = a'.$$

But, since $\nabla_T \varphi T = \xi + \varphi \nabla_T T$, the second term in the left side vanishes. Hence $\chi_2 g(N_2, \varphi T) = a'$.

Suppose that $\chi_2 \neq 0$. Then $g(N_2, \varphi T) = \frac{a'}{\chi_2}$. Since γ is biharmonic, from the third equation of (7), one obtains

$$\chi_2' = \alpha_2 = -3 \frac{c-1}{4} a \frac{a'}{\chi_2}.$$

It follows that $\chi_2^2 = -3 \frac{c-1}{4} a^2 + \mu$, where $\mu \in \mathbb{R}$ is a constant. From the second equation of (7) we have

$$\chi_1^2 + \chi_2^2 = \chi_1^2 - 3 \frac{c-1}{4} a^2 + \mu = -\alpha_1 = \frac{c+3}{4} + 3 \frac{c-1}{4} a^2.$$

Thus

$$\mu = \frac{c+3}{4} + 6 \frac{c-1}{4} a^2 - \chi_1^2,$$

and, since μ and χ_1 are constants, a is a real constant too. Hence χ_2 is a constant. If $g(N_1, \varphi T) = \in \{-1, 1\}$ it is easy to see, from Frenet equations and since M is a Sasakian space form, that $\nabla_T N_1 = \pm \nabla_T \varphi T = \pm \xi \pm \varphi \nabla_T T$.

Thus $N_2 = \pm\xi$ and $\chi_2 = 1$. If $g(N_1, \varphi T) = 0$, from Theorem 3.1 we have $\chi_2' = \alpha_2 = 0$, so χ_2 is a constant.

Now, assume again $g(N_1, \varphi T) = a \notin \{-1, 0, 1\}$, $\chi_2 \chi_3 = 0$ and $\chi_2 \neq 0$. We have

$$g(N_2, \varphi T) = \frac{a'}{\chi_2} = 0.$$

That means N_2 is orthogonal to φT and, moreover, N_2 is orthogonal to E. On the other hand, $g(N_1, \varphi T) \neq 0$ and γ is biharmonic, imply that $g(N_k, \varphi T) = 0$ and $g(N_k, E) = 0$ for any $k = 4, \ldots, 2n$. We just proved that

$$\{N_2, N_4, \ldots, N_{2n}\} \subset (\text{span}\{T, \varphi T, E\})^\perp.$$

Since every vector field in first set is orthogonal to each other, it follows that

$$\text{span}\{N_2, N_4, \ldots, N_{2n}\} = (\text{span}\{T, \varphi T, E\})^\perp.$$

Taking account of the fact that N_3 is orthogonal to T and to N_k for any $k = 2, \ldots, 2n$, $k \neq 3$, one obtains $N_3 \in \text{span}\{E, \varphi T\}$ and, since N_3 is a unitary vector field, from the expression of N_1, we have $N_3 = aE - \sqrt{1 - a^2}\varphi T$ or $N_3 = -aE + \sqrt{1 - a^2}\varphi T$. But, since $\alpha_3 = 0$ and $g(N_1, \varphi T) = a \neq 0$ it follows $g(N_3, \varphi T) = 0$ and, then $\pm\sqrt{1 - a^2} = 0$. Thus $a = \pm 1$ which is a contradiction.

Finally, suppose that $\chi_2 = 0$. Then, from the second Frenet equation, we have $\nabla_T N_1 = -\chi_1 T$. Thus $g(\nabla_T N_1, \xi) = 0$ and, since $g(\nabla_T N_1, \xi) = g(N_1, \varphi T) = a$, we conclude that $a = 0$, which, again, contradicts $a \notin \{-1, 0, 1\}$.

Hence, we proved that $a \in \{-1, 0, 1\}$. $\qquad\qquad\square$

Using Theorem 3.1 and Proposition 3.3 we can state

Theorem 3.4. *Let $\gamma : I \to M(c)$ be a Legendre curve, parametrized by its arc length, in Sasakian space form $M(c)$, $c \neq 1$, with structure tensors (φ, ξ, η, g). Then*

(i) *For $c \leqslant -3$, γ is biharmonic if and only if is a geodesic;*

(ii) *For $-3 \lesssim c \lesssim 1$ and $n = 1$, γ is biharmonic if and only if it is a geodesic;*

(iii) *For $-3 \lesssim c \lesssim 1$ and $n \gtrsim 1$, γ with $\chi_2 \chi_3 = 0$, is biharmonic and if and only if*

- *γ is a geodesic, or*
- *$g(\nabla_{\gamma'}\gamma', \varphi\gamma') = 0$ and* $\begin{cases} \chi_1 \in \mathbb{R} \setminus \{0\} \\ \chi_2 \in \mathbb{R} \\ \chi_1^2 + \chi_2^2 = \frac{c+3}{4}; \end{cases}$

(iv) *For $c \gtrsim 1$, γ with $\chi_2 \chi_3 = 0$, is biharmonic if and only if*

(1) *γ is a geodesic, or*

(2) *$g(\nabla_{\gamma'}\gamma', \varphi\gamma') = \pm\chi_1$ and $\chi_1 = \sqrt{c - 1}$, or*

(3) *$g(\nabla_{\gamma'}\gamma', \varphi\gamma') = 0$ and* $\begin{cases} \chi_1 \in \mathbb{R} \setminus \{0\} \\ \chi_2 \in \mathbb{R} \\ \chi_1^2 + \chi_2^2 = \frac{c+3}{4}. \end{cases}$

Proof. (i) It is easy to see that $\alpha_1 \geqslant 0$ and, then, from Theorem 3.1, it follows γ is biharmonic if and only if is a geodesic.

(ii) If $n = 1$ then $\chi_2 = 1$ and $g(N_1, \varphi T) = \pm 1$, (see [3]). Thus $\alpha_1 \geqslant 0$ and, if γ is biharmonic and non-geodesic, $\chi_1^2 = -\alpha_1 - 1 \leqslant 0$. Hence, in this case, only Legendre geodesics are biharmonic curves.

(iii) If $n \gneqq 1$ and $g(N_1, \varphi T) = \pm 1$, then $-\alpha_1 \leqslant 0$. Thus biharmonic Legendre curves are geodesics. Finally, the statement follows directly from Proposition 3.3 and Theorem 3.1.

(iv) Assume that $c \gneqq 1$. Let γ be a biharmonic non-geodesic Legendre curve with $\chi_2\chi_3 = 0$. First, suppose that $g(N_1, \varphi T) = \pm 1$. That means $N_1 = \pm\varphi T$. From Frenet equations one obtains $\nabla_T T = \pm\chi_1\varphi T$ and $\nabla_T N_1 = \pm\xi + \varphi\nabla_T T$. Hence $\chi_2^2 = 1$ and $N_2 = \pm\xi$. From Theorem 3.1, it follows $\chi_1 = \sqrt{c-1}$.

If $g(N_1, \varphi T) = 0$ then, from Theorem 3.1, it follows $\chi_1 \in \mathbb{R} \setminus \{0\}$, $\chi_1^2 + \chi_2^2 = \frac{c+3}{4}$, $\chi_2 \in \mathbb{R}$.

Conversely, suppose that γ is a Legendre curve, which is not a geodesic. If (iv)(2) holds one obtains easily, from the Frenet equations, that $\chi_2 = 1$, $\alpha_k = 0$, $k = 2, \ldots, 2n$, and $\chi_3 = 0$. Moreover, $\chi_1^2 + \chi_2^2 = c = -\alpha_1$. Thus γ is biharmonic by mean of Theorem 3.1. If (iv)(3) holds then $\alpha_k = 0$ for any $k = 1, \ldots, 2n$. Since $\chi_2\chi_3 = 0$, from Theorem 3.1 it follows γ is biharmonic. $\square$

Remark 3.5. From the proof of the Theorem 3.4 it is easy to see that in the case (iv)(2) the initial condition $\chi_2\chi_3 = 0$ it is not necessary.

4. Biharmonic Legendre curves in a 7-dimensional 3-Sasakian manifold

In this section we assume that M is a 7-dimensional 3-Sasakian manifold, with structure tensors $(\varphi_a, \xi_a, \eta_a, g)$, $a = 1, 2, 3$, and one of its three φ_a-sectional curvatures, let's say the third, being a constant, which we denote by c_3. Then $c_3 = 1$. As we shall see, choosing another curvature to be constant does not change the meaning of the main result.

Let $\gamma : I \to M(1)$ be a Legendre curve, with respect to all three Sasakian structures on $M(1)$, parametrized by its arc length. Moreover, assume that γ is not a geodesic. Since $M(1)$ is a 7-dimensional 3-Sasakian manifold it is easy to see that $\{T, \varphi_1 T, \varphi_2 T, \varphi_3 T, \xi_1, \xi_2, \xi_3\}$ is an orthonormal basis in $M(1)$, where we kept the notations from the previous section.

Differentiating $g(T, \xi_a) = 0$, for any $a = 1, 2, 3$, one obtains, using Frenet equations, $g(N_1, \xi_a) = 0$, $a = 1, 2, 3$. Assume that $g(N_1, \varphi_3 T) = 0$. Then $\nabla_T T = \lambda_1\varphi_1 T + \lambda_2\varphi_2 T$. Thus $\chi_1 = \sqrt{\lambda_1^2 + \lambda_2^2}$ and $N_1 = \frac{1}{\chi_1}[\lambda_1\varphi_1 T + \lambda_2\varphi_2 T]$. Taking account of the fact that structures on $M(c_3)$ are Sasakian, one obtains, after a straightforward computation

$$\nabla_T N_1 = \frac{1}{\chi_1}[(-\lambda_1^2 - \lambda_2^2)T + \lambda_1'\varphi_1 T + \lambda_2'\varphi_2 T + \lambda_1\xi_1 + \lambda_2\xi_2].$$

From the second Frenet equation we have

$$(1) \qquad \chi_2^2 = \frac{\lambda_1'^2 + \lambda_2'^2}{\chi_1^2} + 1$$

and $N_2 = \frac{1}{\chi_2}(\lambda_1 \xi_1 + \lambda_2 \xi_2 + \lambda_1' \varphi_1 T + \lambda_2' \varphi_2 T)$. A straightforward computation and the third Frenet equation give

$$(2) \qquad \begin{aligned} \nabla_T N_2 &= \frac{1}{\chi_1 \chi_2}[(-\lambda_1 \lambda_1' - \lambda_2 \lambda_2')T + (\lambda_1'' - \lambda_1)\varphi_1 T + (\lambda_2'' - \lambda_2)\varphi_2 T \\ &\quad + (\lambda_1' \lambda_2 - \lambda_2' \lambda_1)\varphi_3 T + 2\lambda_1' \xi_1 + 2\lambda_2' \xi_2] = -\chi_2 N_1 + \chi_3 N_3. \end{aligned}$$

Assume that γ is biharmonic. If $\chi_2 = 0$ then follows easily that $\chi_1 = 0$ and γ is a geodesic. If $\chi_2 \neq 0$, from Theorem 3.2 one obtains $\chi_3 = 0$ and, in the same way as above, $\chi_2 = 1$. Thus γ is biharmonic if and only if $\chi_1 = 0$, by the mean of Theorem 3.2. We can conclude that, in this case, only Legendre geodesics are biharmonic Legendre curves.

Obviously, we can take any of φ_a-sectional curvature instead of c_3 to be constant. In the same way it is easy to see that the previous result remains valid if $g(N_1, \xi_a) \neq 0$, $a = 1, 2, 3$ (see also [1]).

One obtains

Theorem 4.1. *Let M be a 7-dimensional 3-Sasakian manifold with constant φ_a-sectional curvature, for an $a = 1, 2, 3$, and let $\gamma : I \to M(1)$ be a Legendre curve, with respect to all three Sasakian structures on $M(1)$, parametrized by its arc length. Then γ is a biharmonic curve if and only if it is a geodesic.*

One of the most important example of a 3-Sasakian manifold is the unit sphere S^7 endowed with a structure obtained as follows (see [1]).

Consider the Euclidean space E^8 with three complex structures,

$$\mathcal{I} = \begin{pmatrix} 0 & -I_4 \\ I_4 & 0 \end{pmatrix}, \quad \mathcal{J} = \begin{pmatrix} 0 & 0 & 0 & I_2 \\ 0 & 0 & -I_2 & 0 \\ 0 & I_2 & 0 & 0 \\ -I_2 & 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{K} = -\mathcal{I}\mathcal{J},$$

where I_n denotes the $n \times n$ identity matrix. Let $\mathbf{x}$ denote the position vector of the unit sphere in E^8 and define three vector fields on S^7 by

$$\xi_1 = -\mathcal{I}\mathbf{x}, \quad \xi_2 = -\mathcal{J}\mathbf{x}, \quad \xi_3 = -\mathcal{K}\mathbf{x}.$$

The dual 1-forms η_i are three independent contact structures on S^7. The standard metric on S^7, g, of constant curvature 1, is an associated metric for all three considered structures (see [1], [2]). Then S^7 is an example of a 3-Sasakian manifold with φ_1-sectional curvature 1, where $(\varphi_a, \xi_a, \eta_a, g)$, $a = 1, 2, 3$, are the structure tensors (see [2]).

Using Theorem 4.1 we conclude that

Proposition 4.2. *A biharmonic Legendre curve with respect to all three Sasakian structures on S^7 is a geodesic.*

5. A class of biharmonic Legendre curves in S^7

In order to find examples of biharmonic Legendre curves which are not geodesics, let us consider again the Sasakian structure $(\varphi_1, \xi_1, \eta_1, g)$ on the 7-sphere, defined in the previous paragraph, and let us rename it simply by (φ, ξ, η, g). Now consider the deformed structure

$$\bar{\eta} = a\eta, \quad \bar{\xi} = \frac{1}{a}\xi, \quad \bar{\varphi} = \varphi, \quad \bar{g} = ag + a(a-1)\eta \otimes \eta,$$

where a is a positive constant. Such a deformation is called a $\mathcal{D}$-homothetic deformation, since the metrics restricted to the contact subbundle $\mathcal{D}$ are homothetic, and it was introduced in [16]. The deformed structure $(\bar{\varphi}, \bar{\xi}, \bar{\eta}, \bar{g})$ is still a Sasakian structure and $(S^7, \bar{\varphi}, \bar{\xi}, \bar{\eta}, \bar{g})$ is a Sasakian space form with constant $\bar{\varphi}$-sectional curvature $c = \frac{4}{a} - 3$, (see [2]). From now on we assume that $a \lesssim 1$ and then $c \gtrsim 1$.

Theorem 5.1. *Let $\gamma : I \to (S^7, \bar{\varphi}, \bar{\xi}, \bar{\eta}, \bar{g})$ be a biharmonic Legendre curve parametrized by its arc length such that $\bar{\nabla}_{\gamma'}\gamma' = \sqrt{c-1}\bar{\varphi}\gamma'$, where $\bar{\nabla}$ is the Levi-Civita connection on $(S^7, \bar{g})$. Then the equation of γ in the Euclidean space $E^8 = (\mathbb{R}^8, \langle,\rangle)$, is*

$$(1) \qquad \begin{aligned} \gamma(s) = &\sqrt{\frac{B}{A+B}}\cos(As)e_1 + \sqrt{\frac{B}{A+B}}\sin(As)e_2 \\ &+ \sqrt{\frac{A}{A+B}}\cos(Bs)e_3 + \sqrt{\frac{A}{A+B}}\sin(Bs)e_4, \end{aligned}$$

where $\{e_i\}$ are constant unit orthogonal vectors in E^8 such that $e_2 = \mathcal{I}e_1$, $e_4 = -\mathcal{I}e_3$ and

$$(2) \qquad \begin{cases} A = \sqrt{\dfrac{3-2a-2\sqrt{(a-1)(a-2)}}{a}} \\ B = \sqrt{\dfrac{3-2a+2\sqrt{(a-1)(a-2)}}{a}}. \end{cases}$$

Proof. Let us denote by ∇ and by $\widetilde{\nabla}$ the Levi-Civita connections on (S^7, g) and $(\mathbb{R}^8, \langle,\rangle)$, respectively.

Let $T = \gamma'$, $\bar{g}(T,T) = 1$, be the unitary tangent vector field. Since $\bar{g}(T, \bar{\xi}) = 0$ it is easy to see that $\bar{g}(\bar{\nabla}_T T, \xi) = 0$. According to Theorem 3.1 we have $\bar{\nabla}_T T = \sqrt{c-1}\varphi T$. Using the definition of the Levi-Civita connection, one obtains $\bar{g}(\bar{\nabla}_X X, Z) = ag(\nabla_X X, Z)$ for any $Z \in \chi(S^7)$ and $X \in (\text{span}\{\xi\})^\perp$. Taking T instead of X we have $\bar{g}(\bar{\nabla}_T T, Z) = ag(\nabla_T T, Z)$ for any $Z \in \chi(S^7)$. Now, it is easy to see that $\nabla_T T = \sqrt{c-1}\varphi T$.

From the equation of Gauss we get

$$\widetilde{\nabla}_T T = \nabla_T T - \langle T, T \rangle \gamma = \sqrt{c-1}\varphi T - \frac{1}{a}\gamma.$$

Thus, using again the equation of Gauss and the fact that (S^7, g) is a Sasakian space form, it follows

$$\widetilde{\nabla}_T \widetilde{\nabla}_T T = \sqrt{c-1}\,\widetilde{\nabla}_T \varphi T - \frac{1}{a}T$$

$$= \sqrt{c-1}\Big(\frac{1}{a}\xi - \sqrt{c-1}T\Big) - \frac{1}{a}T = -\Big(\frac{5}{a} - 4\Big)T + \frac{\sqrt{c-1}}{a}\xi.$$

In the same manner, one obtains

$$\widetilde{\nabla}_T \widetilde{\nabla}_T \widetilde{\nabla}_T T = -\Big(\frac{5}{a} - 4\Big)\widetilde{\nabla}_T T + \frac{\sqrt{c-1}}{a}\widetilde{\nabla}_T \xi$$

$$= -\Big(\frac{5}{a} - 4\Big)\widetilde{\nabla}_T T - \frac{\sqrt{c-1}}{a}\varphi T = -\Big(\frac{5}{a} - 4\Big)\gamma'' - \frac{1}{a}\Big(\gamma'' + \frac{1}{a}\gamma\Big).$$

Hence

(3) $$a^2 \gamma^{IV} + a(6 - 4a)\gamma'' + \gamma = 0,$$

which general solution is

$$\gamma(s) = \cos(As)c_1 + \sin(As)c_2 + \cos(Bs)c_3 + \sin(Bs)c_4,$$

where A, B are given by (2) and $\{c_i\}$ are constant vectors in E^8.

In $s = 0$ we have $\gamma = c_1 + c_3$, $\gamma' = Ac_2 + Bc_4$, $\gamma'' = -A^2 c_1 - B^2 c_3$, $\gamma''' = -A^3 c_2 - B^3 c_4$ and the following equations,

$$\langle \gamma, \gamma \rangle = 1,\ \langle \gamma', \gamma' \rangle = \frac{1}{a},\ \langle \gamma, \gamma' \rangle = 0,\ \langle \gamma', \gamma'' \rangle = 0,\ \langle \gamma'', \gamma'' \rangle = \frac{5 - 4a}{a^2},$$

$$\langle \gamma, \gamma'' \rangle = -\frac{1}{a},\ \langle \gamma', \gamma''' \rangle = -\frac{5 - 4a}{a^2},\ \langle \gamma'', \gamma''' \rangle = 0,\ \langle \gamma, \gamma''' \rangle = 0,$$

$$\langle \gamma''', \gamma''' \rangle = \frac{16a^2 - 44a + 29}{a^3},$$

becomes

(4) $$c_{11} + 2c_{13} + c_{33} = 1$$

(5) $$A^2 c_{22} + 2ABc_{24} + B^2 c_{44} = \frac{1}{a}$$

(6) $$Ac_{12} + Ac_{23} + Bc_{14} + Bc_{34} = 0$$

(7) $$A^3 c_{12} + AB^2 c_{23} + A^2 Bc_{14} + B^3 c_{34} = 0$$

(8) $$A^4 c_{11} + 2A^2 B^2 c_{13} + B^4 c_{33} = \frac{5 - 4a}{a^2}$$

(9) $$A^2 c_{11} + (A^2 + B^2)c_{13} + B^2 c_{33} = \frac{1}{a}$$

(10) $$A^4 c_{22} + (AB^3 + A^3 B)c_{24} + B^4 c_{44} = \frac{5 - 4a}{a^2}$$

(11) $$A^5 c_{12} + A^3 B^2 c_{23} + A^2 B^3 c_{14} + B^5 c_{34} = 0$$

$$(12) \qquad A^3 c_{12} + A^3 c_{23} + B^3 c_{14} + B^3 c_{34} = 0$$

$$(13) \qquad A^6 c_{22} + 2A^3 B^3 c_{24} + B^6 c_{44} = \frac{16a^2 - 44a + 29}{a^3},$$

where $c_{ij} = \langle c_i, c_j \rangle$.

Since the determinant of the system given by (6), (7), (11) and (12) is $-A^2 B^2 (A^2 - B^2)^4 \neq 0$ it follows that

$$c_{12} = c_{23} = c_{14} = c_{34} = 0.$$

The equations (4), (8) and (9) gives

$$c_{11} = \frac{B}{A+B}, \quad c_{13} = 0, \quad c_{33} = \frac{A}{A+B},$$

and, from (5), (10) and (13),

$$c_{22} = \frac{B}{A+B}, \quad c_{24} = 0, \quad c_{44} = \frac{A}{A+B}.$$

We obtained that $\{c_i\}$ are orthogonal vectors in E^8 with $\|c_1\| = \|c_2\| = \sqrt{\frac{B}{A+B}}$ and $\|c_3\| = \|c_4\| = \sqrt{\frac{A}{A+B}}$.

Finally, using the facts that γ is a Legendre curve and $\bar{\nabla}_{\gamma'} \gamma' = \sqrt{c-1} \bar{\varphi} \gamma'$ one obtains easily the expression of γ. $\qquad \square$

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