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BIHARMONIC SUBMANIFOLDS IN NON-SASAKIAN CONTACT METRIC 3-MANIFOLDS

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Abstract

In this paper, we characterize biharmonic Legendre curves in 3-dimensional (κ, μ, ν)-contact metric manifolds. Moreover, we give examples of Legendre geodesics in these spaces. We also give a geometric interpretation of 3-dimensional generalized (κ, μ)-contact metric manifolds in terms of its Legendre curves. Furthermore, we study biharmonic anti-invariant surfaces of 3-dimensional generalized (κ, μ)-contact metric manifolds with constant norm of the mean curvature vector field. Finally, we give examples of anti-invariant surfaces with constant norm of the mean curvature vector field immersed in these spaces.

1. Introduction

During the last decade important progress has been made in the study of both the geometry and the analytic properties of biharmonic maps. In differential geometry, special attention has been payed to the study of biharmonic submanifolds, i.e. submanifolds the inclusion map of which is a biharmonic map.

In [7], Caddeo et al. classified biharmonic curves and surfaces of the unit 3-sphere S^3. In fact, they found that these are circles, helices which are geodesics in the Clifford minimal torus, and small hyperspheres. The same authors in [8] constructed examples of nonharmonic biharmonic submanifolds of S^n, $n > 3$. In this case, the family of such submanifolds is much larger. In fact, any minimal submanifold of a certain parallel hypersphere of S^n is a nonharmonic biharmonic submanifold of S^n.

On the other hand, the odd dimensional spheres are typical examples of contact metric manifolds. More precisely, these are Sasakian space forms of constant ϕ-sectional curvature 1. In [11] J. Inoguchi classified biharmonic Legendre curves and Hopf cylinders in 3-dimensional Sasakian space forms. In particular, he proved that in Sasakian space forms of constant ϕ-sectional

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curvature $c \leq 1$ there are neither nonharmonic biharmonic Legendre curves nor nonharmonic biharmonic Hopf cylinders. On the contrary, for $c > 1$, there exist such submanifolds.

An extension of the Sasakian manifolds are the (κ, μ)-*contact metric manifolds* ([6]). We remind that a contact metric manifold $M(\eta, \xi, \phi, g)$ is called (κ, μ)-contact metric manifold if and only if the characteristic vector field ξ satisfies the curvature condition

$$R(X, Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY)$$

for every vector fields X, Y of M. Here κ, μ are constants and $h = \frac{1}{2}\mathscr{L}_\xi\phi$. If κ, μ are non-constant smooth functions on M, the manifold M is called *generalized* (κ, μ)-*contact metric manifold* [13]. In [14], the authors classified, at least locally, 3-dimensional generalized (κ, μ)-contact metric manifolds assuming additionally that the norm of the field grad κ is a constant. In [18], Perrone studied the harmonicity of the vector field ξ and introduced the notion of the H-contact metric manifolds (contact metric manifolds whose characteristic vector field is an harmonic vector field). In [15], the authors characterized the 3-dimensional H-contact metric manifolds in terms of (κ, μ, ν)-contact metric manifolds, which are defined by the following curvature condition:

$$\begin{aligned} R(X, Y)\xi = {} & \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY) \\ & + \nu(\eta(Y)\phi hX - \eta(X)\phi hY), \end{aligned}$$

where κ, μ, ν are smooth functions. Moreover, it is shown that if dim $M > 3$, then (κ, μ, ν)-contact metric manifolds are reduced to (κ, μ)-contact metric manifolds i.e. κ, μ are constants and ν is the zero function on M.

In [1], Arslan et al. classified biharmonic Legendre curves and biharmonic anti-invariant surfaces in 3-dimensional (κ, μ)-contact metric manifolds. Especially, they proved that biharmonic Legendre curves immersed in 3-dimensional (κ, μ)-contact metric manifolds are their geodesics or helices (curves with constant geodesic curvature and geodesic torsion). Later, Sasahara ([19]) pointed out that this characterization is true under the condition that $\nabla_{\gamma'}\gamma' \parallel \phi\gamma'$, where γ' is the unit tangent vector field of Legendre curve. In fact, if $\kappa \neq 1$, $\nabla_{\gamma'}\gamma'$ is not generally parallel to $\phi\gamma'$. On the other hand, a non-minimal anti-invariant surface of a 3-dimensional (κ, μ)-contact metric manifold is biharmonic if and only if it is of constant mean curvature vector field and $\kappa = 1$, i.e. the ambient space is a Sasakian manifold.

The main part of this paper is referred to the study of nonharmonic biharmonic curves and surfaces immersed in 3-dimensional (κ, μ, ν)-contact metric manifolds which are not Sasakian.

More explicitly, in Section 2 are contained some basic notions about contact metric manifolds and biharmonic maps.

In Section 3, we improve Proposition 4.1 of [1] omitting the assumption that $\nabla_{\gamma'}\gamma' \parallel \phi\gamma'$. More precisely, we prove the following Theorem:

THEOREM 1.1. *Let γ be a non-geodesic Legendre curve in a 3-dimensional (κ, μ)-contact metric manifold $M(\eta, \xi, \phi, g)$ with $\kappa < 1$. Then γ is biharmonic if and only if either $\nabla_{\gamma'}\gamma' \parallel \phi\gamma'$ and in this case γ is a helix satisfying $k_g^2 + \tau_g^2 = -(\kappa + \mu)$ or $\nabla_{\gamma'}\gamma' \parallel \xi$ and γ is a helix satisfying $k_g^2 + \tau_g^2 = \kappa + \delta\mu$, where the function $\delta = g(h\gamma', \gamma')$ is a constant.*

In the following, we classify biharmonic Legendre curves in 3-dimensional (κ, μ, ν)-contact metric manifolds. Especially, we prove the following Theorem:

THEOREM 1.2. *Let γ be a Legendre curve in a 3-dimensional (κ, μ, ν)-contact metric manifold $M(\eta, \xi, \phi, g)$ with κ a non-constant smooth function on M and $\nabla_{\gamma'}\gamma' \parallel \phi\gamma'$. Assuming that $\kappa < 1$ everywhere on M, then γ is biharmonic if and only if γ is a Legendre geodesic.*

Next we give examples of biharmonic Legendre curves of suitable (κ, μ, ν)-contact metric manifolds which are, indeed, their geodesics. Finally, by using the notion of Legendre curves, we characterize geometrically 3-dimensional generalized (κ, μ)-contact metric manifolds. In particular, we prove the following Theorem:

THEOREM 1.3. *Let $M(\eta, \xi, \phi, g)$ be a 3-dimensional (κ, μ, ν)-contact metric manifold with $\kappa < 1$ everywhere on M. If the ϕ-sectional curvature H of M is constant along every Legendre curve, then the characteristic vector field ξ defines an harmonic map $(\xi : (M, g) \mapsto (T_1M, g_S))$ where T_1M, is the unit tangent sphere bundle equipped with the Sasaki metric g_S. If it is assumed additionally that the function μ is constant on M and M is supposed to be complete, then M is locally isometric to one of the following Lie groups with a left invariant metric: $SU(2)$ (or $SO(3)$), $SL(2, \mathbf{R})$ (or $O(1, 2)$), $E(2)$, $E(1, 1)$.*

In Section 4, are studied biharmonic anti-invariant surfaces of 3-dimensional (κ, μ, ν)-contact metric manifolds. We exhibit the system of partial differential equations which describes these surfaces. Finally, we investigate biharmonic anti-invariant surfaces of 3-dimensional generalized (κ, μ)-contact metric manifolds with constant norm of the mean curvature vector field. The main result of this section is the following Theorem:

THEOREM 1.4. *Let $M(\eta, \xi, \phi, g)$ be a 3-dimensional generalized (κ, μ)-contact metric manifold with $\kappa < 1$. Let M^2 be an anti-invariant surface of M with constant norm of the mean curvature vector field equal to c. If M^2 is biharmonic, then M^2 is either minimal or it is locally flat and the functions κ and μ are constants on M^2. In the second case, there exists a coordinate system (u, v) defined in a neighborhood U_1 of any $p \in M^2$, such that the metric tensor g and the second fundamental form of M^2 are given on U_1 by*

$$g = (du)^2 + (dv)^2,$$

$$\sigma\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial u}\right) = 0,$$

$$\sigma\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right) = -\frac{\mu}{2}\phi\left(\frac{\partial}{\partial v}\right),$$

$$\sigma\left(\frac{\partial}{\partial v}, \frac{\partial}{\partial v}\right) = 2c\phi\left(\frac{\partial}{\partial v}\right).$$

Moreover, we give examples of anti-invariant surfaces with constant norm of the mean curvature vector field in these spaces.

2. Preliminaries

2.1. Contact metric manifolds. We start with some fundamental notions about contact Riemannian geometry. We refer to [5] for further details. All manifolds in the present paper are assumed to be connected and of class C^∞.

A differentiable $(2n+1)$-dimensional manifold is called a *contact manifold* if it admits a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere on M. It is known that a contact manifold admits an almost contact metric structure (η, ξ, ϕ, g), i.e. a global vector field ξ, which is called the *characteristic vector field* or the *Reeb vector field*, a tensor field ϕ of type $(1,1)$ and a Riemannian metric g (*associated metric*) such that

$$\eta(\xi) = 1, \quad \phi^2 = -Id + \eta \otimes \xi, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2.1}$$

for all vector fields X, Y on M. Moreover, the quadruple (η, ξ, ϕ, g) can be chosen so that $d\eta(X, Y) = g(X, \phi Y)$. The manifold M together with the structure tensors (η, ξ, ϕ, g) is called a *contact metric manifold* and is denoted by $M(\eta, \xi, \phi, g)$. Equations (2.1) imply

$$\phi(\xi) = 0, \quad \eta \circ \phi = 0. \tag{2.2}$$

Given a contact Riemannian manifold M, we define an operator h by $h = \frac{1}{2}(\mathscr{L}_\xi \phi)$, where $\mathscr{L}$ denotes Lie differentiation. The operator h is self-adjoint and satisfies

$$h\xi = 0, \quad h\phi = -\phi h \tag{2.3}$$

and

$$\nabla_X \xi = -\phi X - \phi h X. \tag{2.4}$$

A contact structure on M gives rise to an almost complex structure on the product $M \times \mathbf{R}$. If this structure is integrable, then the contact metric manifold is said to be *Sasakian*. Equivalently, a contact metric manifold is Sasakian if and only if

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y$$

for every vector fields X, Y on M.

A plane section at a point p of a contact metric manifold M is called a ϕ-section if there exists a vector $X \in T_pM$ orthogonal to ξ such that $\{X, \phi X\}$ span the section. The sectional curvature $K(X, \phi X)$, denoted by $H(X)$, is called *ϕ-sectional curvature*.

We remind now the notion of (κ, μ, ν)-contact metric manifolds.

Definition 2.1. A $(2n+1)$-dimensional contact metric manifold $M(\eta, \xi, \phi, g)$ is called (κ, μ, ν)-*contact metric manifold* if the curvature tensor satisfies the condition

$$R(X, Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY) + \nu(\eta(Y)\phi hX - \eta(X)\phi hY)$$

for every vector fields X, Y tangent to M and κ, μ, ν are smooth functions on M.

Moreover, on every 3-dimensional (κ, μ, ν)-contact metric manifold $M(\eta, \xi, \phi, g)$ the following relations are valid

$$h^2 = (\kappa - 1)\phi^2, \tag{2.5}$$

$$\xi(\kappa) = 2\nu(\kappa - 1), \tag{2.6}$$

$$r = 4\kappa + 2H, \tag{2.7}$$

$$\xi(H) = 2\nu(1 - \kappa), \quad \kappa < 1 \tag{2.8}$$

$$\xi(r) = 4(\kappa - 1)\nu, \quad \kappa < 1 \tag{2.9}$$

$$\begin{aligned} R(\xi, X)Y = {} & \kappa(g(X, Y)\xi - \eta(Y)X) + \mu(g(hX, Y)\xi - \eta(Y)hX) \\ & + \nu(g(\phi hY, X)\xi - \eta(Y)\phi hX) \end{aligned} \tag{2.10}$$

$$\begin{aligned} R(X, Y)Z = {} & \mu[g(hY, Z)X - g(hX, Z)Y + g(Y, Z)hX - g(X, Z)hY] \\ & + \nu[g(\phi hY, Z)X - g(\phi hX, Z)Y + g(Y, Z)\phi hX - g(X, Z)\phi hY] \\ & + \left(3\kappa - \frac{r}{2}\right)[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)]\xi \\ & + \left(3\kappa - \frac{r}{2}\right)[\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y] \\ & + \left(\frac{r}{2} - 2\kappa\right)[g(Y, Z)X - g(X, Z)Y], \quad \kappa < 1 \end{aligned} \tag{2.11}$$

$$\begin{aligned} (1 - \kappa)(\nabla_X h)Y = {} & -\frac{1}{2}g(hX, Y)\,\mathrm{grad}\,\kappa - \frac{1}{2}g(hX, \phi Y)\phi(\mathrm{grad}\,\kappa) \\ & + (1 - \kappa)[(1 - \kappa)g(X, \phi Y) + g(hX, \phi Y) - \nu g(hX, Y)]\xi \\ & + (1 - \kappa)\{\eta(Y)[(\kappa - 1)\phi X + h\phi X] \\ & + \eta(X)[\mu h\phi Y + \nu hY]\}, \end{aligned} \tag{2.12}$$

for every vector fields X, Y, Z tangent to M, where r denotes the scalar curvature of M. Formulas (2.5) and (2.6) occur in [15]. The proof of relations (2.7)–(2.12) is mainly based on the following Lemmas (see also [21] for the case $\nu = 0$):

Lemma 2.2 [13]. *Let $M(\eta, \xi, \phi, g)$ be a 3-dimensional (κ, μ, ν)-contact metric manifold. For every $p \in M$ with $\kappa(p) < 1$ there exists an open neighborhood W of p and orthonormal local vector fields $\{e, \phi e, \xi\}$ defined on W, such that*

$$he = \lambda e, \quad h\phi e = -\lambda \phi e, \quad h\xi = 0, \tag{2.13}$$

where $\lambda = \sqrt{1 - \kappa}$.

Lemma 2.3 ([9], [21]). *Let $M(\eta, \xi, \phi, g)$ be a 3-dimensional (κ, μ, ν)-contact metric manifold with $\kappa < 1$ everywhere on M. Then*

$$\nabla_e \xi = -(\lambda + 1)\phi e, \quad \nabla_{\phi e} \xi = (1 - \lambda)e, \tag{2.14}$$

$$\nabla_\xi e = -\frac{\mu}{2}\phi e, \quad \nabla_\xi \phi e = \frac{\mu}{2} e, \quad \nabla_e e = \frac{B}{2\lambda}\phi e, \quad \nabla_{\phi e} \phi e = \frac{A}{2\lambda} e, \tag{2.15}$$

$$\nabla_{\phi e} e = -\frac{A}{2\lambda}\phi e + (\lambda - 1)\xi, \quad \nabla_e \phi e = -\frac{B}{2\lambda} e + (\lambda + 1)\xi, \tag{2.16}$$

where $A = e(\lambda)$ and $B = \phi e(\lambda)$ and $\{e, \phi e, \xi\}$ the orthonormal basis of eigenvectors of h described on Lemma 2.2.

2.2. Biharmonic maps. Let (M^m, g), (N^n, h) be Riemannian manifolds and let $\varphi : (M^m, g) \mapsto (N^n, h)$ be a smooth map between them. We denote by ∇^φ the connection of the vector bundle $\varphi^{-1}TN$ induced from the Levi-Civita connection $\overline{\nabla}$ of (N, h) and ∇ the Levi-Civita connection of (M, g). Let Ω be a compact domain of M. The *energy* (*integral*) *of* ϕ over Ω is defined by

$$E_1(\varphi) = \frac{1}{2}\int_\Omega \|d\varphi\|^2 v_g,$$

where v_g is the volume element of M^m.

A smooth map $\varphi : M^m \mapsto N^n$ is said to be *harmonic* if it is a critical point of the energy functional for any compact subset $\Omega \subset M$. It is well known ([3]) that the map $\varphi : M^m \mapsto N^n$ is harmonic if and only if

$$\tau_1(\varphi) = \operatorname{tr}(\nabla^\varphi\, d\varphi) = \sum_{i=1}^m \{\nabla^\varphi_{e_i}\, d\varphi(e_i) - d\varphi(\nabla_{e_i} e_i)\} = 0,$$

where $\{e_i\}$ is a local orthonormal frame field of M^m. The equation $\tau_1(\varphi) = 0$ is called the *harmonic equation*.

A smooth map $\varphi : M^m \mapsto N^n$ is said to be *biharmonic* if it is a critical point of the bienergy functional:

$$E_2(\varphi) = \frac{1}{2}\int_\Omega \|\tau_1(\varphi)\|^2 v_g,$$

over every compact subset Ω of M. The Euler-Lagrange equation associated to the bienergy becomes more complicated and, it involves the Riemann curvature tensor of N. More precisely, a smooth map $\varphi : M^m \mapsto N^n$ is biharmonic if and only if it satisfies the following biharmonic equation ([12]):

$$\tau_2(\varphi) = -\mathscr{J}_\varphi(\tau_1(\varphi)) = 0.$$

The operator $\mathscr{J}_\varphi$ is the *Jacobi operator* of φ and is defined by

$$\mathscr{J}_\varphi(V) := \bar{\Delta}_\varphi V - \mathscr{R}_\varphi(V), \quad V \in \Gamma(\varphi^{-1}TN),$$

$$\bar{\Delta}_\varphi := -\sum_{i=1}^m \{\nabla^\varphi_{e_i}\nabla^\varphi_{e_i} - \nabla^\varphi_{\nabla_{e_i}e_i}\},$$

$$\mathscr{R}_\varphi(V) := \sum_{i=1}^m R^N(V, d\varphi(e_i))\, d\varphi(e_i),$$

where $\{e_i\}$ is a local orthonormal frame field of M^m and R^N is the curvature tensor of N. The section $\tau_2(\varphi)$ of $\varphi^{-1}TN$ is called the *bitension field* of φ. From the expression of the bitension field τ_2 it is clear that an harmonic map ($\tau_1 = 0$) is automatically a biharmonic map, in fact it is a minimum of the bienergy. However, the converse is not true in general. In fact, many examples of nonharmonic biharmonic maps have been obtained in [4], [7], [8] and [16]. Nonharmonic biharmonic maps are called *proper* biharmonic maps.

3. Biharmonic Legendre curves in 3-dimensional (κ, μ, ν)-contact metric manifolds

Let $M(\eta, \xi, \phi, g)$ be a 3-dimensional (κ, μ, ν)-contact metric manifold with $\kappa < 1$ everywhere on M. A curve $\gamma : I \subset \mathbf{R} \mapsto M$ parametrized by arclength is said to be a *Legendre curve* if and only if $\eta(\gamma') = 0$ where γ' is the tangential vector field on γ. In this section, we attempt to classify proper biharmonic Legendre curves on 3-dimensional (κ, μ, ν)-contact metric manifolds with $\kappa < 1$. Moreover, we characterize 3-dimensional generalized (κ, μ)-contact metric manifolds in terms of the biharmonicity of Legendre curves.

Let γ be a Legendre curve in M. Then, the Frenet-Sherret formulas for γ are given explicitly by

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & k_g & 0 \\ -k_g & 0 & \tau_g \\ 0 & -\tau_g & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \tag{3.1}$$

where k_g (res. τ_g) denotes the *geodesic curvature* (res. the *geodesic torsion*) of γ. We mention that every Legendre curve in Sasakian 3-manifolds has constant geodesic torsion equal to 1([2]).

A *helix* is a curve with constant geodesic curvature and geodesic torsion. *Geodesics* are curves with zero geodesic curvature.

Differentiating $g(\gamma', \xi) = 0$ along γ and using (2.4) we see that $g(\nabla_{\gamma'}\gamma', \xi) + g(\gamma', -\phi h\gamma') = 0$ and hence

$$\nabla_{\gamma'}\gamma' = a\xi + b\phi\gamma', \quad a = g(\gamma', \phi h\gamma'). \tag{3.2}$$

Thus the principal normal N is given by

$$N = \frac{1}{k_g}(a\xi + b\phi\gamma'). \tag{3.3}$$

On the other hand, decomposing the vector field $h\gamma'$ in terms of the orthonormal frame field $\{\gamma', \phi\gamma', \xi\}$, we get

$$h\gamma' = \delta\gamma' - a\phi\gamma'. \tag{3.4}$$

Differentiating N along γ and using the relations (3.1), we have ([5, page 135])

$$\tau_g B = \left\{-\frac{k_g'}{k_g^2}a + \frac{a'}{k_g} + \frac{b}{k_g}(1+\delta)\right\}\xi + \left\{-\frac{k_g'}{k_g^2}b + \frac{b'}{k_g} - \frac{a}{k_g}(1+\delta)\right\}\phi\gamma'. \tag{3.5}$$

From this and $k_g^2 = a^2 + b^2$, we obtain

$$\tau_g = \left|\frac{a'b - ab'}{k_g^2} + (1+\delta)\right|.$$

Now, we prove the following Theorem:

Theorem 3.1. *Let γ be a non-geodesic Legendre curve in a 3-dimensional (κ, μ)-contact metric manifold $M(\eta, \xi, \phi, g)$ with $\kappa < 1$. Then γ is biharmonic if and only if either $\nabla_{\gamma'}\gamma' \parallel \phi\gamma'$ and in this case γ is a helix satisfying $k_g^2 + \tau_g^2 = -(\kappa + \mu)$ or $\nabla_{\gamma'}\gamma' \parallel \xi$ and γ is a helix satisfying $k_g^2 + \tau_g^2 = \kappa + \delta\mu$, where the function $\delta = g(h\gamma', \gamma')$ is a constant.*

Proof. We assume that γ is a Legendre curve but not a geodesic with non-vanishing geodesic torsion. Furthermore, we can assume, without loss of generality, that the geodesic torsion is given by

$$\tau_g = \frac{a'b - ab'}{k_g^2} + 1 + \delta. \tag{3.6}$$

Setting $v = 0$ in (2.12) and using the fact that the functions κ and μ are constants with $\kappa < 1$, we compute the derivative of δ

$$\begin{aligned}\delta' &= g(\nabla_{\gamma'}h\gamma', \gamma') + g(h\gamma', \nabla_{\gamma'}\gamma') = g((\nabla_{\gamma'}h)\gamma', \gamma') + 2g(h\gamma', \nabla_{\gamma'}\gamma') \\ &= g(h\gamma', \phi\gamma')\eta(\gamma') + 2bg(h\gamma', \phi\gamma') = -2ab.\end{aligned} \tag{3.7}$$

Using the Frenet-Serret formulas, we compute the tension field of γ is given by

$$\tau_1(\gamma) = \nabla_T T = k_g N.$$

By direct computations, we deduce ([11])

$$-\bar{\Delta}_\gamma(\tau_1(\gamma)) = -3k_g k_g' T + (k_g'' - k_g^3 - k_g \tau_g^2)N + (2k_g'\tau_g + k_g\tau_g')B.$$

Since the ambient space $M(\eta, \xi, \phi, g)$ is a (κ, μ)-contact metric manifold, relations (2.2), (2.3), (2.10), (2.11), (3.3) and (3.4) give

$$\begin{aligned}\mathscr{R}_\gamma(\tau_1(\gamma)) &= k_g R(N, \gamma')\gamma' = aR(\xi, \gamma')\gamma' + bR(\phi\gamma', \gamma')\gamma' \\ &= a(\kappa + \delta\mu)\xi + \left(\frac{r}{2} - 2\kappa\right)\phi\gamma'\end{aligned}$$

where r denotes the restriction of the scalar curvature of M to the curve γ. On the other hand, the scalar curvature of a 3-dimensional (κ, μ)-contact metric manifold is given by ([6])

$$r = 2(\kappa - \mu).$$

As a consequence, we get

$$\mathscr{R}_\gamma(\tau_1(\gamma)) = a(\kappa + \delta\mu)\xi + b(-\kappa - \mu)\phi\gamma'.$$

Thus, the bitension field $\tau_2(\gamma)$ of γ is given by

$$\begin{aligned}\tau_2(\gamma) = -\bar{\Delta}_\gamma(\tau_1(\gamma)) + \mathscr{R}_\gamma(\tau_1(\gamma)) &= -3k_g k_g' T + (k_g'' - k_g^3 - k_g\tau_g^2)N \\ &+ (2k_g'\tau_g + k_g\tau_g')B + a(\kappa + \delta\mu)\xi + b(-\kappa - \mu)\phi\gamma'.\end{aligned} \tag{3.8}$$

Since γ is biharmonic, $\tau_2(\gamma) = 0$ or, equivalently,

$$g(\tau_2(\gamma), \gamma') = 0, \tag{3.9}$$

$$g(\tau_2(\gamma), \phi\gamma') = 0, \tag{3.10}$$

$$g(\tau_2(\gamma), \xi) = 0. \tag{3.11}$$

Combining (3.8), (3.9) and using the fact that γ is not a geodesic, we easily conclude that $k_g = c_1 = \text{const.} \neq 0$. Additionally, relations (3.3), (3.5), (3.8), (3.10) and (3.11) give

$$(-c_1^2 - \tau_g^2)a + \frac{\tau_g'}{\tau_g}(a' + b(1 + \delta)) + a(\kappa + \delta\mu) = 0, \tag{3.12}$$

$$(-c_1^2 - \tau_g^2)b + \frac{\tau_g'}{\tau_g}(b' - a(1 + \delta)) + b(-\kappa - \mu) = 0. \tag{3.13}$$

Moreover, differentiating the relation $a^2 + b^2 = c_1^2$ along γ, we get

$$aa' + bb' = 0. \tag{3.14}$$

On the other hand, on any 3-dimensional contact metric manifold $M(\eta, \xi, \phi, g)$ the following relation holds:

$$(\nabla_X \phi)Y = g(X + hX, Y)\xi - \eta(Y)(X + hX), \tag{3.15}$$

for every $X, Y \in \mathscr{X}(M)$. Differentiating (3.5) along γ and using relations (2.1), (2.4), (3.1), (3.2), (3.3), (3.4), (3.14) and (3.15), we have

$$\begin{aligned} c_1\tau_g' B + c_1\tau_g B' = c_1\tau_g' B - c_1\tau_g^2 N &= (a'' + b'(1+\delta) + b\delta')\xi + (a' + b(1+\delta))\nabla_{\gamma'}\xi \\ &\quad + (b'' - a'(1+\delta) - a\delta')\phi\gamma' + (b' - a(1+\delta))\nabla_{\gamma'}\phi\gamma' \\ &= \{a'' + b'(1+\delta) + b\delta' + (b' - a(1+\delta))(1+\delta)\}\xi \\ &\quad + \{b'' - a'(1+\delta) - a\delta' - (a' + b(1+\delta))(1+\delta)\}\phi\gamma' \\ &\qquad + \{-aa' - bb'\}\gamma', \end{aligned}$$

or, equivalently,

$$\frac{\tau_g'}{\tau_g}(a' + b(1+\delta)) - a\tau_g^2 = a'' + b'(1+\delta) + b\delta' + (b' - a(1+\delta))(1+\delta)$$

and

$$\frac{\tau_g'}{\tau_g}(b' - a(1+\delta)) - b\tau_g^2 = b'' - a'(1+\delta) - a\delta' - (a' + b(1+\delta))(1+\delta).$$

Using relations (3.12) and (3.13), the last two relations are transformed to

$$a'' + b'(1+\delta) + b\delta' + (b' - a(1+\delta))(1+\delta) = ac_1^2 - a(\kappa + \delta\mu), \tag{3.16}$$

and

$$b'' - a'(1+\delta) - a\delta' - (a' + b(1+\delta))(1+\delta) = bc_1^2 + b(\kappa + \mu). \tag{3.17}$$

In the sequel, we separately examine the cases for which the functions a or b are vanishing on γ. More precisely, the case for which $a = 0$ gives $\nabla_{\gamma'}\gamma' \parallel \phi\gamma'$. In this case, the function b is not vanishing on γ, since $a^2 + b^2 = c_1^2 > 0$. Moreover, using relations (3.6), (3.7) and (3.13), we easily deduce that the functions δ and τ_g are constants on γ and, therefore, γ is a helix satisfying additionally the relation $k_g^2 + \tau_g^2 = -(\kappa + \mu)$ ([1], [19]). Similarly, the case for which $b = 0$ gives $\nabla_{\gamma'}\gamma' \parallel \xi$. Moreover, using relations (3.6), (3.7) and (3.12), we easily deduce that the functions δ and τ_g are constants on γ and, therefore, γ is a helix satisfying additionally the relation $k_g^2 + \tau_g^2 = \kappa + \delta\mu$. In the following, we assume that the functions a and b are not vanishing on γ simultaneously. Multiplying relation (3.16) with b and relation (3.17) with a and substracting, we deduce

$$\frac{a''b - ab''}{c_1^2} + \delta' = \tau_g' = -\frac{ab}{c_1^2}(2\kappa + \mu(1+\delta)). \tag{3.18}$$

Substituting (3.18) in (3.12) and (3.13), we obtain

$$a\left\{-c_1^2-\tau_g^2-\frac{b}{\tau_g c_1^2}(2\kappa+\mu(1+\delta))(a'+b(1+\delta))+\kappa+\delta\mu\right\}=0,$$

$$b\left\{-c_1^2-\tau_g^2-\frac{a}{\tau_g c_1^2}(2\kappa+\mu(1+\delta))(b'-a(1+\delta))-\kappa-\mu\right\}=0.$$

Using (3.6), we easily observe that $-c_1^2-\tau_g^2-\frac{b}{\tau_g c_1^2}(2\kappa+\mu(1+\delta))(a'+b(1+\delta)) +\kappa+\delta\mu=-c_1^2-\tau_g^2-\frac{a}{\tau_g c_1^2}(2\kappa+\mu(1+\delta))(b'-a(1+\delta))-\kappa-\mu$. Since $a,b\neq 0$, we obtain

$$-c_1^2-\tau_g^2-\frac{a}{\tau_g c_1^2}(2\kappa+\mu(1+\delta))(b'-a(1+\delta))-\kappa-\mu=0. \tag{3.19}$$

On the other hand, using (3.14), multiplying relation (3.12) with a and (3.13) with b and summing, we get

$$-c_1^2-\tau_g^2=\frac{1}{c_1^2}\{b^2(\kappa+\mu)-a^2(\kappa+\delta\mu)\}. \tag{3.20}$$

Combining (3.19), (3.20) and using the fact that $a^2+b^2=c_1^2$, we have

$$a(2\kappa+\mu(1+\delta))\left(a+\frac{b'-a(1+\delta)}{\tau_g}\right)=0.$$

We assume that the function $2\kappa+\mu(1+\delta)$ is vanishing on γ. Differentiating the last relation along γ and using (3.7), we get $\mu ab=0$ i.e. $\mu=0$. Then, $\kappa=0$ and $\tau_g'=0$ (from (3.18)). In this case, relation (3.12) gives $a(-c_1^2-\tau_g^2)=0$, which is a contradiction. As a consequence, we get

$$b'=a(1+\delta-\tau_g). \tag{3.21}$$

Substituting (3.21) in (3.19), we get

$$c_1^4+c_1^2\tau_g^2-a^2(2\kappa+\mu(1+\delta))+(\kappa+\mu)c_1^2=0. \tag{3.22}$$

Differentiating (3.22) along γ and using relations (3.7), (3.14), (3.18) and (3.21), we have

$$(1+\delta-2\tau_g)(2\kappa+\mu(1+\delta))+a^2\mu=0. \tag{3.23}$$

Differentiating (3.23) along γ and using relations (3.7), (3.14), (3.18) and (3.21), we get

$$-(2\kappa+\mu(1+\delta))+\frac{(2\kappa+\mu(1+\delta))^2}{c_1^2}-\mu(2(1+\delta)-3\tau_g)=0. \tag{3.24}$$

Differentiating (3.24) along γ and using relations (3.7) and (3.18), we obtain

$$\mu\left(6 - 7\frac{2\kappa + \mu(1+\delta)}{c_1^2}\right) = 0.$$

We distinguish two cases:

- $\mu = 0$. In this case, relations (3.23) and (3.24) give

$$c_1^2 = 2\kappa, \quad 1 + \delta = 2\tau_g. \tag{3.25}$$

  Furthermore, relation (3.22) give

$$c_1^2 + \tau_g^2 - a^2 + \kappa = 0. \tag{3.26}$$

  On the other hand, using relations (2.5) and (3.4), we straightforward compute

$$a^2 + \delta^2 = g(h^2\gamma', \gamma') = 1 - \kappa. \tag{3.27}$$

  Combining relations (3.25), (3.26) and (3.27), we easily deduce that the function δ is a constant or, equivalently, $ab = 0$ which is a contradiction.
- $2\kappa + \mu(1+\delta) = \dfrac{6c_1^2}{7}$. Differentiating this relation along γ, we get $\mu\delta' = 0$ or $\mu ab = 0$ which is a contradiction.

Conversely, we assume that $\nabla_{\gamma'}\gamma' \parallel \phi\gamma'$ i.e. $a = 0$ and γ is a helix satisfying $k_g^2 + \tau_g^2 = -(\kappa + \mu)$. Then, we easily observe that relations (3.12) and (3.13) are satisfied. Hence, γ is a biharmonic Legendre curve. Similarly, we treat the case $\nabla_{\gamma'}\gamma' \parallel \xi$ and γ is a helix satisfying $k_g^2 + \tau_g^2 = \kappa + \delta\mu$. □

Remark 3.1. If $\tau_g = 0$, then γ is a Riemannian circle, since $k_g = c_1$. In this case, using relations (3.5), (3.12) and (3.13) we easily get that $\delta = -1$.

Considering κ and μ as functions on M, we get the following Theorem:

THEOREM 3.2. *Let γ be a Legendre curve in a 3-dimensional (κ, μ, ν)-contact metric manifold $M(\eta, \xi, \phi, g)$ with κ non-constant smooth function on M and $\nabla_{\gamma'}\gamma' \parallel \phi\gamma'$. Assuming that $\kappa < 1$ everywhere on M, then γ is biharmonic if and only if γ is a Legendre geodesic.*

Proof. Since $\nabla_{\gamma'}\gamma' \parallel \phi\gamma'$, we can choose a Frenet frame field such that $T = \gamma'$, $N = \phi\gamma'$ and $B = \xi$. In the sequel, using (2.1), (2.2), (2.3), (2.7) and (2.11), we easily get

$$\mathscr{R}_\gamma(\tau_1(\gamma)) = R(k_g N, T)T = k_g R(\phi T, T)T = k_g\left(\frac{r}{2} - 2\kappa\right)\phi T = k_g H N,$$

where r denotes the restriction of the scalar curvature of M to the curve γ. As a consequence, the Jacobi operator of γ is given by

$$\mathscr{J}_\gamma(\tau_1(\gamma)) = 3k_g k_g' T - (k_g'' - k_g^3 - k_g\tau_g^2 + k_g H)N - (2k_g'\tau_g + k_g\tau_g')B. \tag{3.28}$$

Since γ is biharmonic, $\mathscr{J}_\gamma(\tau_1(\gamma)) = 0$. Therefore, relation (3.28) gives

$$\begin{aligned} k_g k_g' &= 0, \\ 2k_g'\tau_g + k_g\tau_g' &= 0, \\ k_g'' - k_g^3 - k_g\tau_g^2 + k_g H &= 0. \end{aligned}$$

By the hypothesis γ is not a geodesic, therefore the first two relations give that k_g and τ_g are constants i.e. γ is an helix. Moreover, the third relation gives $k_g^2 + \tau_g^2 = H$ which implies that the ϕ-sectional curvature H is constant along γ. Combining now (2.1), (2.4), the Frenet-Serret formulas and the fact that γ is an helix, we have

$$\begin{aligned} B' = \nabla_T\xi &= -\phi T - \phi hT \\ &= -\phi T - \phi[g(hT,T)T + g(hT,\phi T)\phi T] \\ &= -(1 + g(hT,T))\phi T + g(hT,\phi T)T = -\tau_g\phi T, \end{aligned}$$

from which we conclude that $g(hT,\phi T) = 0$ and the expression $1 + g(hT,T)$ is constant along γ. Next, we consider an arbitrary point p of γ. Since $\kappa(p) < 1$, according to Lemma 4.1 there exists an open neighborhood W of p and orthonormal local vector fields $\{e,\phi e,\xi\}$ defined on W, such that the relations (2.13) are satisfied. In the sequel, we consider the arc Q of the curve γ involving p which lies in the open set W. Then, we decompose T in terms of the basis $\{e,\phi e,\xi\}$ as follows

$$T = \alpha e + \beta\phi e, \tag{3.29}$$

where α,β are smooth functions on Q. Relations (2.13) and the above decomposition, give

$$hT = \alpha\lambda e - \beta\lambda\phi e, \quad \phi T = \alpha\phi e - \beta e.$$

Using the last relations and (2.1), we obtain

$$0 = g(hT,\phi T) = -2\lambda\alpha\beta$$

from which we conclude that either $\alpha = 0$ or $\beta = 0$ on Q. Assume that the first case is valid. Since γ is a unit speed curve, we obtain that $\beta = \pm 1$ on Q. For the sake of simplicity, we assume that $\beta = 1$ on Q i.e. γ is an eigencurve of h. Using relations (2.16) and (3.1), we have

$$\begin{aligned} \nabla_T\phi T &= -\nabla_{\phi e}e \\ &= \frac{A}{2\lambda}\phi e - (\lambda - 1)\xi = -k_g T + \tau_g\xi. \end{aligned}$$

Since the terms k_g and τ_g are constants along γ, we derive that the function λ is a constant on Q which implies that $A = 0$. Then, relation (2.15) gives

$$\nabla_T T = \nabla_{\phi e}\phi e = \frac{A}{2\lambda}e = 0,$$

which means that the curve γ is a geodesic. As a consequence, the initial assertion leads to a contradiction. The case $\beta = 0$ works analogously. The converse is obvious. □

In the next Theorem, the harmonicity of the characteristic vector field ξ and geometrical properties of Legendre curves are connected. Specifically, we have

THEOREM 3.3. *Let $M(\eta, \xi, \phi, g)$ be a 3-dimensional (κ, μ, ν)-contact metric manifold with $\kappa < 1$ everywhere on M. If the ϕ-sectional curvature H of M is constant along every Legendre curve, then the characteristic vector field ξ defines an harmonic map $(\xi : (M, g) \mapsto (T_1M, g_S))$ where T_1M, is the unit tangent sphere bundle equipped with the Sasaki metric g_S. If it is assumed additionally that the function μ is constant on M and M is supposed to be complete, then M is locally isometric to one of the following Lie groups with a left invariant metric: $SU(2)$ (or $SO(3)$), $SL(2, \mathbf{R})$ (or $O(1,2)$), $E(2)$, $E(1,1)$.*

Proof. Let $p \in M$. Then, given a vector X on p orthogonal to ξ there exists a Legendre curve γ through p with X tangent to γ ([20]). By assumption, the ϕ-sectional curvature H of M is constant along γ. Hence by using (2.8) and $\kappa < 1$, we get that the function ν vanishes along γ. As a consequence, the function ν vanishes at the arbitrary point p i.e. $M(\eta, \xi, \phi, g)$ is a generalized (κ, μ)-contact metric manifold. On the other hand, Theorem 3.1 of [15] indicates that the vector field ξ defines an harmonic vector field. According to the Theorem 1.1 of [17], we deduce that the characteristic vector field ξ defines an harmonic map. If we assume additionally that the function μ is a constant, then Theorem 3.6 of [13] implies that the function κ is also a constant on M i.e. M is a (κ, μ)-contact metric manifold. The remaining part of the Theorem follows immediately from the classification of 3-dimensional (κ, μ)-contact metric manifolds in [6]. □

Remark 3.2. If we assume that the scalar curvature r is constant along every Legendre curve of $M(\eta, \xi, \phi, g)$, Theorem 3.3 is also valid (see (2.9)).

In the sequel, we construct some examples of 3-dimensional (κ, μ, ν)-contact metric manifolds. In these examples, we give the ordinary differential equations which satisfy their geodesics which additionally suppose to be Legendre curves.

Example 3.1. Let $M = \mathbf{R}^3$ with the cartesian coordinates (x, y, z). We define the following vector fields on $\mathbf{R}^3$:

$$e_1 = \frac{\partial}{\partial x}, \quad e_2 = \frac{\partial}{\partial y}, \quad e_3 = 2y\frac{\partial}{\partial x} + \left(\frac{1}{4}e^{2x} - y^2\right)\frac{\partial}{\partial y} + \frac{\partial}{\partial z}.$$

The vector fields e_1, e_2, e_3 are linearly independent at each point of M. We define a Riemannian metric g on M such that $g(e_i, e_j) = \delta_{ij}$, $i, j = 1, 2, 3$. We easily get that

$$[e_1, e_2] = 0, \quad [e_1, e_3] = \frac{e^{2x}}{2} e_2, \quad [e_2, e_3] = -2ye_2 + 2e_1. \tag{3.30}$$

Let η be the 1-form defined by $\eta(W) = g(W, e_1)$ for every $W \in \mathscr{X}(M)$. Then η is a contact form since $\eta \wedge d\eta \neq 0$ everywhere on M. Let ϕ be the tensor field of type $(1, 1)$, defined by $\phi e_1 = 0$, $\phi e_2 = e_3$, $\phi e_3 = -e_2$. Using the linearity of ϕ, $d\eta$ and g, we easily obtain that $\eta(e_1) = 1$, $\phi^2 Z = -Z + \eta(Z)e_1$, $d\eta(Z, W) = g(\phi Z, W)$ and $g(\phi Z, \phi W) = g(Z, W) - \eta(Z)\eta(W)$ for every vector fields Z, W on M. Hence $M(\eta, e_1, \phi, g)$ is a contact metric manifold. Let ∇ be the Levi-Civita connection corresponding to g and R the Riemann curvature tensor of g.

Setting $\xi = e_1$, $X = e_2$, $\phi X = e_3$, using the Koszul's formula

$$\begin{aligned} 2g(\nabla_Y Z, W) = {} & Yg(Z, W) + Zg(W, Y) - Wg(Y, Z) - g(Y, [Z, W]) \\ & - g(Z, [Y, W]) + g(W, [Y, Z]), \end{aligned}$$

and (3.30), we find

$$\begin{aligned} &\nabla_X \xi = \left(-\frac{e^{2x}}{4} - 1\right)\phi X, && \nabla_{\phi X} \xi = \left(1 - \frac{e^{2x}}{4}\right)X, && \nabla_\xi \xi = 0, \\ &\nabla_\xi X = \left(-\frac{e^{2x}}{4} - 1\right)\phi X, && \nabla_\xi \phi X = \left(1 + \frac{e^{2x}}{4}\right)X, && \nabla_X X = 2y\phi X, \\ &\nabla_X \phi X = -2yX + \left(\frac{e^{2x}}{4} + 1\right)\xi, && \nabla_{\phi X} X = \left(\frac{e^{2x}}{4} - 1\right)\xi, && \nabla_{\phi X} \phi X = 0. \end{aligned} \tag{3.31}$$

From the definition of the tensor field h and relations (3.31), we get that $h\xi = 0$ and

$$\begin{aligned} hX &= \frac{1}{2}(\mathscr{L}_\xi \phi)X = \frac{1}{2}\{[\xi, \phi X] - \phi[\xi, X]\} \\ &= \frac{e^{2x}}{4} X. \end{aligned} \tag{3.32}$$

Similarly, we easily obtain that

$$h\phi X = -\frac{e^{2x}}{4}\phi X. \tag{3.33}$$

Setting now, $\kappa = 1 - \frac{e^{4x}}{16}$, $\mu = 2\left(1 + \frac{e^{2x}}{4}\right)$, $\nu = 2$ and using the relations (3.31), (3.32) and (3.33), we easily deduce that

$$R(X,\xi)\xi = \frac{e^{2x}}{2}\phi X + \left(\frac{e^{2x}}{4} + 1\right)^2 X$$
$$= \kappa(\eta(\xi)X - \eta(X)\xi) + \mu(\eta(\xi)hX - \eta(X)h\xi) + \nu(\eta(\xi)\phi hX - \eta(X)\phi h\xi),$$
$$R(\phi X,\xi)\xi = \frac{e^{2x}}{2}X + \left(1 + \frac{e^{2x}}{4}\right)\left(1 - \frac{3e^{2x}}{4}\right)\phi X$$
$$= \kappa(\eta(\xi)\phi X - \eta(\phi X)\xi) + \mu(\eta(\xi)h\phi X - \eta(\phi X)h\xi) + \nu(\eta(\xi)\phi h\phi X - \eta(\phi X)\phi h\xi),$$

and

$$R(X,\phi X)\xi = 0$$
$$= \kappa(\eta(\phi X)X - \eta(X)\phi X) + \mu(\eta(\phi X)hX - \eta(X)h\phi X) + \nu(\eta(\phi X)\phi hX - \eta(X)\phi h\phi X).$$

Since $\{X, \phi X, \xi\}$ is a basis of $\mathbf{R}^3$, we easily obtain

$$R(Z,W)\xi = \kappa[\eta(W)Z - \eta(Z)W] + \mu[\eta(W)hZ - \eta(Z)hW] + \nu[\eta(W)\phi hZ - \eta(Z)\phi hW],$$

for all vector fields Z, W on $\mathbf{R}^3$. Hence, $\mathbf{R}^3$ is a (κ,μ,ν)-contact metric manifold. Let $\gamma(s) = (x(s), y(s), z(s)) \in \mathbf{R}^3$ a Legendre geodesic curve parametrized by its arc length. Then

$$T = \gamma' = x'\frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + z'\frac{\partial}{\partial z}$$
$$= [x' - 2yz']\xi + \left[y' - \left(\frac{e^{2x}}{4} - y^2\right)z'\right]X + z'\phi X$$

where "′" denotes derivative with respect to s. Since γ is a Legendre curve, we have $x' - 2yz' = 0$ and therefore,

$$\gamma' = \alpha X + \beta\phi X$$

where $\alpha = y' - \left(\frac{e^{2x}}{4} - y^2\right)z'$ and $\beta = z'$ with $\alpha^2 + \beta^2 = 1$. We set $\alpha = \cos\theta$, $\beta = \sin\theta$ for some function $\theta = \theta(s)$. On the other hand, using relations (3.31), we straightforward calculate

$$\nabla_{\gamma'}\gamma' = \nabla_{\gamma'}[\alpha X + \beta\phi X] = \alpha' X + \beta'\phi X + \alpha^2\nabla_X X + \alpha\beta\nabla_{\phi X}X + \alpha\beta\nabla_X\phi X + \beta^2\nabla_{\phi X}\phi X$$
$$= (\alpha' - 2y\alpha\beta)X + (\beta' + 2\alpha^2 y)\phi X + \alpha\beta\frac{e^{2x}}{2}\xi.$$

Since γ is a geodesic, $\nabla_{\gamma'}\gamma' = 0$ and therefore

$$\alpha' = 2y\alpha\beta, \quad \beta' + 2\alpha^2 y = 0, \quad \alpha\beta = 0.$$

The equation $\alpha\beta = 0$ is equivalent to $\sin(2\theta) = 0$. Differentiating the last relation, we easily get that the function θ is constant along γ and equals to $\frac{\rho\pi}{2}$ where $\rho \in \mathbf{Z}$. Hence, the functions α, β are also constants along γ. Taking ρ to be even or odd, we have that the constants α and β take the values 0, 1 or -1. We consider first the case for which $\alpha = \pm 1$ and $\beta = 0$. Then, we have $z' = 0$, $y = 0$ and $\alpha = 0$, which is a contradiction. On the other hand, if $\alpha = 0$ and $\beta = 1$, the definition of the functions α and β gives the following system of ordinary differential equations:

$$z' = 1, \quad y' - \left(\frac{e^{2x}}{4} - y^2\right)z' = 0, \quad x' = 2yz'$$

or, equivalently,

$$z = s + c_1, \quad y' + y^2 - \frac{e^{2x}}{4} = 0, \quad x' = 2y,$$

where c_1 is a constant. The last two relations of this system are reduced to the following second order ordinary differential equation:

$$2x'' + (x')^2 = e^{2x}. \tag{3.34}$$

Partial solutions of equation (3.34) are the functions $x(s) = \ln \frac{\sqrt{3}}{s+c}$ where c is a real constant. As a consequence, the corresponding Legendre geodesics are given by $\gamma(s) = \left(\ln \frac{\sqrt{3}}{s+c}, -\frac{1}{2(s+c)}, s + c_1\right)$.

Example 3.2. Let $M = \{(x, y, z) \in \mathbf{R}^3 \mid z \neq 0\}$, where (x, y, z) are the standard coordinates in $\mathbf{R}^3$. The vector fields

$$e_1 = \frac{\partial}{\partial x}, \quad e_2 = -2yz\frac{\partial}{\partial x} + \frac{2x}{z^3}\frac{\partial}{\partial y} - \frac{1}{z^2}\frac{\partial}{\partial z}, \quad e_3 = \frac{1}{z}\frac{\partial}{\partial y}$$

are linearly independent at each point of M. Let g be the Riemannian metric defined by $g(e_i, e_j) = \delta_{ij}$, $i, j = 1, 2, 3$ and η the dual 1-form to the vector field e_1. We define the tensor field ϕ of type $(1,1)$ by $\phi e_1 = 0$, $\phi e_2 = e_3$, $\phi e_3 = -e_2$. Following [13], we have that $M(\eta, e_1, \phi, g)$ is a generalized (κ, μ)-contact metric manifold with $\kappa = \frac{z^4 - 1}{z^4}$ and $\mu = 2\left(1 - \frac{1}{z^2}\right)$. We set $\xi = e_1$, $X = e_2$ and $\phi X = e_3$. Furthermore, we have that $hX = \frac{1}{z^2}X$ and $h\phi X = -\frac{1}{z^2}\phi X$ i.e. the vector fields X, ϕX are eigenvectors of the operator h. Let $\gamma(s) = (x(s), y(s), z(s)) \in \mathbf{R}^3$ a Legendre geodesic curve parametrized by its arc length. Then

$$\gamma' = [x' - 2yz'z^3]\xi - z'z^2 X + (y'z + 2xz')\phi X.$$

Since γ is a Legendre curve, we have $x' - 2yz'z^3 = 0$ and therefore,

$$\gamma' = \alpha X + \beta\phi X$$

where $\alpha = -z'z^2$ and $\beta = y'z + 2xz'$ with $\alpha^2 + \beta^2 = 1$. On the other hand, by using the relations (2.14)–(2.16), we get

$$\nabla_{\gamma'}\gamma' = \left(\alpha' + \frac{\beta^2}{z^3}\right)X + \left(\beta' - \frac{\alpha\beta}{z^3}\right)\phi X + \frac{2\alpha\beta}{z^2}\xi.$$

Since γ is a geodesic, we obtain

$$\alpha\beta = 0, \quad \alpha' + \frac{\beta^2}{z^3} = 0, \quad \beta' - \frac{\alpha\beta}{z^3} = 0.$$

We easily observe that the case in which $\alpha = 0$ leads to a contradiction. So, setting $\alpha = 1$ and $\beta = 0$, we get the following system of ordinary differential equations:

$$z'z^2 = -1, \quad y'z + 2xz' = 0, \quad x' = 2yz'z^3,$$

or, equivalently,

$$z'z^2 = -1, \quad y'z + 2xz' = 0, \quad x' = -2yz.$$

The first equation gives $z^3 = -3s + c_1$, hence $z = \sqrt[3]{-3s + c_1}$. The remainder two equations give the following second order differential equation:

$$x'' + \frac{x'}{-3s + c_1} + \frac{4x}{(-3s + c_1)^{2/3}} = 0.$$

Setting now $t = -3s + c_1$, the above equation is transformed to

$$\ddot{x} - \frac{1}{3t}\dot{x} + \frac{4}{9t^{2/3}}x = 0 \tag{3.35}$$

where $\dot{x}$ denotes $\dfrac{dx}{dt}$. We mention that the general solution of (3.35) is given by the expression

$$x(t) = t^{2/3}[J_1(t^{2/3})c_2 + Y_1(t^{2/3})c_3]$$

where J_1 is the first kind Bessel's function, Y_1 is the first kind spherical Bessel's function and c_2, c_3 are real constants.

4. Biharmonic anti-invariant surfaces

Let $M(\eta, \xi, \phi, g)$ be a $(2n+1)$-dimensional contact metric manifold and M^m $(m \leqq n+1)$ be an isometrically immersed submanifold of M tangent to ξ. If $\phi(TM^m) \subset T^{\perp}M^m$, then M^m is called an *anti-invariant* submanifold of M whereas if $\phi(TM^m) \subset TM^m$, then M^m is said to be an *invariant* submanifold of M (see [22]). Invariant submanifolds of M are *minimal* submanifolds (see [5]) and hence critical points of the bienergy functional. On the contrary, anti-invariant submanifolds are not critical points of the bienergy functional, generally. As a consequence, it is natural to study the class of non-minimal biharmonic anti-invariant submanifolds in contact metric manifolds. First, we summarize some basic notions from the geometry of submanifolds (see [10]).

Let $M(\eta, \xi, \phi, g)$ be a 3-dimensional (κ, μ, ν)-contact metric manifold with $\kappa < 1$ everywhere on M and M^2 be a non-minimal anti-invariant surface isometrically immersed in M by $x : M^2 \mapsto M$. Denote the Levi-Civita connection of M (res. M^2) by $\tilde{\nabla}$ (res. ∇). The Gauss and Weingarten formulas are given, respectively, by

$$\tilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y), \quad \tilde{\nabla}_X V = -A_V X + D_X V, \tag{4.1}$$

where $X, Y \in TM^2$, $V \in T^{\perp}M^2$. Here σ, A and D are the second fundamental form, the shape operator and the normal connection, respectively. The mean curvature vector $\mathbf{H}$ is given by $2\mathbf{H} = \operatorname{tr} \sigma$.

Since ξ is tangent to M^2, we consider the pair $\{e_1, \xi\}$ of orthonormal frame fields of M^2, where e_1 is a unit vector field tangent to M^2. Then, the triple $\{e_1, \xi, \phi e_1\}$ constitutes an orthonormal frame field of M. Furthermore, we assume that $\mathbf{H} = \alpha\phi e_1$, where α is a strictly positive smooth function of M^2. Then, we have

$$he_1 = g(he_1, e_1)e_1 + g(he_1, \phi e_1)\phi e_1.$$

Set $\beta = 1 + g(he_1, e_1)$ and $\gamma = g(he_1, \phi e_1)$. Using (2.4) and the relations (4.1), we obtain

$$\sigma(e_1, e_1) = 2\alpha\phi e_1, \quad \sigma(\xi, \xi) = 0, \quad \sigma(e_1, \xi) = -\beta\phi e_1, \tag{4.2}$$

$$\nabla_{e_1} e_1 = -\gamma\xi, \quad \nabla_{e_1}\xi = \gamma e_1, \quad \nabla_{\xi} e_1 = \nabla_{\xi}\xi = 0. \tag{4.3}$$

In the sequel, we give an example of a family of anti-invariant surfaces with constant norm of the mean curvature vector field, immersed in a 3-dimensional generalized (κ, μ)-contact metric manifold.

Example 4.1. Consider the 3-dimensional manifold $M = \{(x, y, z) \in \mathbf{R}^3 \mid z < 1\}$, where (x, y, z) are the cartesian coordinates in $\mathbf{R}^3$. The vector fields

$$e_1 = \frac{\partial}{\partial x}, \quad e_2 = \frac{\partial}{\partial y}, \quad e_3 = 2y\frac{\partial}{\partial x} + \left(2x\sqrt{1-z} - \frac{y}{4(z-1)}\right)\frac{\partial}{\partial y} + \frac{\partial}{\partial z}$$

are linearly independent at each point of M. Let g be the Riemannian metric defined by $g(e_i, e_j) = \delta_{ij}$, $i, j = 1, 2, 3$ and η the dual 1-form to the vector field e_1. We define the tensor field ϕ of type $(1,1)$ by $\phi e_1 = 0$, $\phi e_2 = e_3$, $\phi e_3 = -e_2$. Following [14], we have that $M(\eta, e_1, \phi, g)$ is a generalized (κ, μ)-contact metric manifold with $\kappa = z$ and $\mu = 2(1 + \sqrt{1-z})$. We set $\xi = e_1$, $X = e_2$ and $Y = \phi X = e_3$. Furthermore, we have

$$\operatorname{grad} \kappa = \xi(\kappa)\xi + X(\kappa)X + (\phi X)(\kappa)\phi X = \phi X = Y,$$

and, as a consequence, $\|\operatorname{grad} \kappa\| = 1$. Moreover, we have that $hX = \sqrt{1-z}X$ i.e. the vector field X is an eigenvector of the operator h. For every real constant $c < 1$, we consider the planes

$$M_c = \{(x, y, z) \in \mathbf{R}^3 \mid z = c\}$$

which are orthogonal to the z-axis at the points $(0, 0, c)$. The vector field $\operatorname{grad} \kappa$ is the unit normal to the surfaces M_c. Since $\xi(\kappa) = 0$, the pair $\{X, \xi\}$ constitutes an orthonormal frame field of M_c. We easily observe that M_c are anti-invariant surfaces. We denote by $\tilde{\nabla}$ (res. ∇) the Levi-Civita connection of M (res. M_c). Then, relation (2.15) gives

$$\tilde{\nabla}_X X = \frac{\phi X(\lambda)}{2\lambda}\phi X = \frac{1}{4(z-1)}\phi X,$$

where $\lambda = \lambda(z) = \sqrt{1-z}$, $z < 1$. On the other hand, the Gauss formula (4.1) and the last relation give

$$\begin{aligned}\tilde{\nabla}_X X &= \nabla_X X + \sigma(X, X)\\ &= \frac{1}{4(c-1)} \operatorname{grad} \kappa\end{aligned}$$

which implies that $\nabla_X X = 0$ and $\sigma(X, X) = \dfrac{1}{4(c-1)} \operatorname{grad} \kappa$. Then, the first two of the relations (4.2) give $\alpha = \dfrac{1}{8(1-c)}$, where it is supposed that $X = e_1$. It means that the planes M_c are surfaces with constant norm of the mean curvature vector field.

In the following, we give two basic lemmas for later use. Their proofs are mainly based on the equations of Gauss and Codazzi, formulas (2.11) and (2.12) and are similar to those given in [1].

Lemma 4.1. *For every non-minimal anti-invariant surface of a* 3*-dimensional* (κ, μ, ν)*-contact metric manifold* $M(\eta, \xi, \phi, g)$ *with* $\kappa < 1$, *the following relations are valid*:

$$2\xi(\alpha) = -e_1(\beta) - 2\alpha\gamma, \tag{4.4}$$

$$\xi(\beta) = \gamma(\mu - 2\beta) + \nu(\beta - 1), \tag{4.5}$$

$$\xi(\gamma) = (2\beta - \mu)(\beta - 1) + \nu\gamma, \tag{4.6}$$

$$e_1(\beta) = 4\alpha\gamma + \frac{1}{2(\kappa - 1)}(\beta - 1)e_1(\kappa) - \frac{1}{2(\kappa - 1)}\gamma\phi e_1(\kappa), \tag{4.7}$$

$$(\beta - 1)^2 + \gamma^2 = 1 - \kappa \tag{4.8}$$

Lemma 4.2.

$$\begin{aligned} -\bar{\Delta}_x\mathbf{H} = {} & [-6\alpha e_1(\alpha) + 2\alpha\beta\gamma + 2\xi(\alpha)\beta + \alpha\xi(\beta)]e_1 \\ & + [2\beta e_1(\alpha) + \alpha e_1(\beta) + 2\alpha^2\gamma]\xi \\ & + [e_1e_1(\alpha) + \xi\xi(\alpha) + \gamma\xi(\alpha) - \alpha(4\alpha^2 + 2\beta^2)]\phi e_1 \\ \mathscr{R}_x(\mathbf{H}) = {} & [\alpha\mu\gamma + \alpha\nu(\beta - 1)]e_1 \\ & + \left[-\alpha\mu(\beta - 1) + \alpha\nu\gamma + \alpha\left(\frac{r}{2} - \kappa\right)\right]\phi e_1 \end{aligned}$$

where x is the inclusion map $x : M^2 \mapsto M$.

We remind that an anti-invariant surface M^2 is biharmonic if and only if $\tau_2(\mathbf{H}) = -\bar{\Delta}_x\mathbf{H} + \mathscr{R}_x(\mathbf{H}) = 0$. Using Lemma 4.2, we have

Proposition 4.1. *The surface M^2 is biharmonic if and only if the following system of partial differential equations holds*:

$$e_1e_1(\alpha) + \xi\xi(\alpha) + \gamma\xi(\alpha) - \alpha(4\alpha^2 + 2\beta^2) + \alpha\left(\frac{r}{2} - \kappa\right) - \alpha\mu(\beta - 1) + \alpha\nu\gamma = 0, \tag{4.9}$$

$$6\alpha e_1(\alpha) - 2\alpha\beta\gamma - 2\xi(\alpha)\beta - \alpha\xi(\beta) - \alpha\mu\gamma - \alpha\nu(\beta - 1) = 0, \tag{4.10}$$

$$2\beta e_1(\alpha) + \alpha e_1(\beta) + 2\alpha^2\gamma = 0. \tag{4.11}$$

where $\alpha = \|\mathbf{H}\|$, $\beta = 1 + g(he_1, e_1)$ and $\gamma = g(he_1, \phi e_1)$.

In general, the problem of classifying biharmonic surfaces of 3-dimensional (κ, μ, ν)-contact metric manifolds is difficult because the above system of three partial differential equations involves six unknown functions $(\alpha, \beta, \gamma, \kappa, \mu, \nu)$ and the scalar curvature r. To this direction, we give the following Theorem:

Theorem 4.3. *Let $M(\eta, \xi, \phi, g)$ be a 3-dimensional generalized (κ, μ)-contact metric manifold with $\kappa < 1$. Let M^2 be an anti-invariant surface of M with constant norm of the mean curvature vector field equal to c. If M^2 is biharmonic, then M^2 is either minimal or it is locally flat and the functions κ and μ* ***are***

constants on M^2. *In the second case, there exists a coordinate system* (u, v) *defined in a neighborhood* U_1 *of any* $p \in M^2$, *such that the metric tensor* g *and the second fundamental form of* M^2 *are given on* U_1 *by*

$$g = (du)^2 + (dv)^2, \tag{4.12}$$

$$\sigma\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial u}\right) = 0, \tag{4.13}$$

$$\sigma\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right) = -\frac{\mu}{2}\phi\left(\frac{\partial}{\partial v}\right),$$

$$\sigma\left(\frac{\partial}{\partial v}, \frac{\partial}{\partial v}\right) = 2c\phi\left(\frac{\partial}{\partial v}\right).$$

Proof. We assume that M^2 is a biharmonic non-minimal surface with constant norm of the mean curvature vector field $\alpha = c = \text{const.} \neq 0$. Then, relation (4.11) gives

$$e_1(\beta) = -2c\gamma. \tag{4.14}$$

Furthermore, combining (4.5) and (4.10), we get

$$\begin{aligned} 0 &= -2c\beta\gamma - c\xi(\beta) - c\mu\gamma \\ &= -2c\beta\gamma - c[\gamma(\mu - 2\beta)] - c\mu\gamma \\ &= -2c\mu\gamma, \end{aligned}$$

or, equivalently,

$$\mu\gamma = 0. \tag{4.15}$$

First we assume that the function γ vanishes on M^2. In this case, by using (4.5), (4.8), (4.14) and the fact that $\kappa < 1$ we conclude that the function β is a constant on M^2 not equal to 1. Combining (2.6) and (4.7), we get $e_1(\kappa) = \xi(\kappa) = 0$ i.e. the function κ is a constant on M^2. Furthermore, (4.6) gives $\mu - 2\beta = 0$, i.e. the function μ is a constant on M^2. On the other hand, by using relations (4.3), we easily deduce that

$$[\xi, e_1] = 0.$$

As a consequence, for any $p \in M^2$ there exists an open neighborhood U_1 of p and a coordinate system (u, v) such that

$$\xi = \frac{\partial}{\partial u}, \quad e_1 = \frac{\partial}{\partial v}.$$

By now using the relations (4.2), we obtain that on U_1 the metric g and the second fundamental form σ of M^2 take the form (4.12) and (4.13), respectively. Furthermore, by using the relations (4.3), we get

$$\begin{aligned} R(\xi, e_1)e_1 &= \nabla_\xi \nabla_{e_1} e_1 - \nabla_{e_1} \nabla_\xi e_1 - \nabla_{[\xi, e_1]} e_1 \\ &= \nabla_\xi[-\gamma\xi] - \nabla_{\nabla_\xi e_1 - \nabla_{e_1}\xi} e_1 \\ &= (-\xi(\gamma) - \gamma^2)\xi. \end{aligned}$$

Since $\gamma = 0$, we deduce from the last relation that the Gauss curvature of M^2 is zero and so, M^2 is locally flat. Consider now the set $U_2 = \{p \in M^2 \mid \gamma(p) \neq 0\}$ which is an open subset of M^2. On U_2, we have that $\mu = 0$. In this case, (4.9) gives

$$\frac{r}{2} - \kappa - 4c^2 - 2\beta^2 = 0.$$

Differentiating the last relation with respect to ξ and using the relations (2.6) and (2.9) with $\nu = 0$, we get

$$\beta\xi(\beta) = 0. \tag{4.16}$$

If we suppose that $\beta = 0$ on U_2, then (4.14) gives $c\gamma = 0$, which is a contradiction. Consider now the open subset $U_3 = \{p \in U_2 \mid \beta(p) \neq 0\}$ of U_2. Then, (4.16) gives $\xi(\beta) = 0$ on U_3. In this case, (4.5) gives

$$\xi(\beta) = 0 = \gamma(\mu - 2\beta) = -2\beta\gamma$$

on U_3, which is also a contradiction and the proof of the Theorem has been completed. □

If $\kappa = \text{const.} < 1$, then the 3-dimensional (κ, μ, ν)-contact metric manifolds are reduced to (κ, μ)-contact metric manifolds ([15]). Hence, we have the following corollary:

Corollary 4.4. *Let M^2 be an anti-invariant surface of a 3-dimensional (κ, μ, ν)-contact metric manifold M with $\kappa = \text{const.} < 1$. Then M^2 is biharmonic if and only if M^2 is minimal.*

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