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**BIHOLOMORPHIC MAPPINGS BETWEEN WEAKLY
PSEUDOCONVEX DOMAINS**

JOHN ERIK FORNAESS

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Assume we have a biholomorphic mapping between weakly pseudoconvex domains. It is an old question whether this extends to a diffeomorphism between their closures. The well known theorem of Fefferman states that this is true for strongly pseudoconvex domains. We will show that if the map has a smooth extension to the boundary, then it cannot map an analytic disc in the boundary to a single point.

This is an immediate consequence of the following theorem.

THEOREM. *Assume Ω, W are bounded pseudoconvex sets with \mathcal{C}^2 boundary in C^n , and assume $\Phi: \Omega \rightarrow W$ is a biholomorphic map with a \mathcal{C}^2 -extension $\bar{\Phi}: \bar{\Omega} \rightarrow \bar{W}$. Then $\bar{\Phi}$ is a \mathcal{C}^2 -diffeomorphism between $\bar{\Omega}$ and \bar{W} .*

This theorem generalizes a similar result for strongly pseudoconvex domains by the author [2], see also Pinchuk [3].

The theorem is false in general for \mathcal{C}^1 -domains and maps Φ with \mathcal{C}^1 -extensions. To see this, let $\Omega = \{z \in C; |z + 1| < 1\}$ and let $\Phi(z) = z/\text{Log } z$ and $W = \Phi(\Omega)$.

Also the theorem is false in general for proper holomorphic maps. For example, let $\Omega = \{(z, w) \in C^2; |z|^2 + |w|^4 < 1\}$, $W = \{(z, w); |z|^2 + |w|^2 < 1\}$ and $\Phi(z, w) = (z, w^2)$.

Proof of the Theorem. Let us fix a point $p \in b\Omega$. We want to show that $\bar{\Phi}'(p)$ is a nonsingular linear transformation. The proof is complete if this is true for all p in the boundary of Ω .

To simplify the computations, we will make affine complex changes of coordinates such that p becomes the origin in C^n with variables (z_1, \dots, z_n) and $\bar{\Phi}(p)$ becomes the origin in C^n with variables (w_1, \dots, w_n) . We may arrange this such that

$$\begin{aligned} \Omega &= \{z = (z_1, \dots, z_n); \rho(z) = \text{Re } z_1 + R(z_1, \dots, z_n) < 0\} \text{ and} \\ W &= \{w = (w_1, \dots, w_n); \sigma(w) = \text{Re } w_1 + S(w_1, \dots, w_n) < 0\} \end{aligned}$$

where R, S are \mathcal{C}^2 -functions which vanish to at least second order at the origin.

From Diederich, Fornæss [1] it follows that there exists a \mathcal{C}^2 -defining function $\hat{\sigma}$ of W such that $-(-\hat{\sigma})^{2/3}$ is strictly plurisubharmonic in W near the origin. It follows that $-(-\hat{\sigma})^{2/3} \circ \bar{\Phi}$ is strictly

plurisubharmonic in Ω near the origin.

We apply the Hopf lemma to points in Ω of the form $(t, 0, \dots, 0)$ with $-1 \ll t < 0$. There must exist a $K > 0$ such that

$$-(-\hat{\sigma})^{2/3}(\Phi(t, 0, \dots, 0)) \leq Kt.$$

Since σ is of the same order of magnitude as $\hat{\sigma}$, we obtain for a possibly different K , that

$$-\operatorname{Re} \varphi_1(t, 0, \dots, 0) - S(\Phi(t, 0, \dots, 0)) \geq K|t|^{3/2}$$

where we have written $\Phi = (\varphi_1, \dots, \varphi_n)$. Hence $(\partial\varphi_1/\partial z_1)(0, \dots, 0) > 0$.

Consider the \mathcal{E}^2 -map $A(z_1, \dots, z_n) = (\varphi_1(z_1, \dots, z_n), z_2, \dots, z_n)$. Then $A'(0)$ is invertible. Therefore A maps the germ of Ω at the origin to the germ of a pseudoconvex set U with \mathcal{E}^2 boundary at the origin. We can describe U by

$$U = \{\eta = (\eta_1, \dots, \eta_n); \tau(\eta) = \operatorname{Re} \eta_1 + T(\eta_1, \dots, \eta_n) < 0\}$$

for some \mathcal{E}^2 -function T vanishing to at least second order at the origin.

We will study the map $\Psi = \Phi \circ A^{-1}$, $\Psi = (\psi_1, \dots, \psi_n)$. It suffices to show that $\Psi'(0)$ is a nonsingular linear map. We can describe Ψ by

$$\Psi(\eta_1, \dots, \eta_n) = (\eta_1, \psi_2(\eta_1, \dots, \eta_n), \dots, \psi_n(\eta_1, \dots, \eta_n)).$$

If $\Psi'(0)$ is singular, then we may assume, after a linear change in the (w_2, \dots, w_n) - and (η_2, \dots, η_n) - direction, that

$$\Psi'(0) = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ \frac{\partial\psi_2}{\partial\eta_1}(0) & 0 & \frac{\partial\psi_2}{\partial\eta_3}(0) & \dots & \frac{\partial\psi_2}{\partial\eta_n}(0) \\ \vdots & \vdots & \vdots & & \vdots \\ \frac{\partial\psi_n}{\partial\eta_1}(0) & 0 & \frac{\partial\psi_n}{\partial\eta_3}(0) & \dots & \frac{\partial\psi_n}{\partial\eta_n}(0) \end{bmatrix}.$$

Next we consider points $t_1 = (t, 0, \dots, 0)$ and $t_2 = (t, t, 0, \dots, 0)$ in U with $-1 \ll t < 0$. We then have the estimates, for $\tau_i = \Psi(t_i)$, $i = 1, 2$:

$$\tau_i = \left(t, \frac{\partial\psi_2}{\partial\eta_1}(0) \cdot t + 0(t^2), \dots, \frac{\partial\psi_n}{\partial\eta_1}(0) \cdot t + 0(t^2) \right),$$

$i = 1, 2$.

Define W_i to be the set

$$W_i = \{(w_1, \dots, w_n) \in W; w_1 = t\}.$$

There exists some $\delta > 0$ independent of t such that

$$W_t \supset \{(t, w_2, \dots, w_n); \|(w_2, \dots, w_n)\| < \delta |t|^{1/2}\} = \tilde{W}_t.$$

Let us write $\Phi^{-1}: W \rightarrow \Omega$ as $\Phi^{-1} = (\mu_1, \dots, \mu_n)$. We then have that for some constant $K > 0$, independent of t , that

$$|\mu_2(t, w_2, \dots, w_n)| \leq K$$

for all points in \tilde{W}_t , since Ω is bounded. The points τ_1, τ_2 are in \tilde{W}_t and satisfy the estimates $\|\tau_1\| \leq K_1 |t|$, $\|\tau_2\| \leq K_1 |t|$ and $\|\tau_1 - \tau_2\| \leq K_1 |t|^2$ for some constant K_1 independent of t . It follows from Schwarz's lemma that for some constant K_2 , independent of t , we have

$$|\mu_2(\tau_1) - \mu_2(\tau_2)| \leq K_2 |t|^{3/2}.$$

However, by construction we know that $|\mu_2(\tau_1) - \mu_2(\tau_2)| = |t|$ which is a contradiction for all small enough $|t|$.

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