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Bijective Soft Matrix Theory and Multi-Bijective Linguistic Soft Decision System

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Abstract. In this paper, we firstly introduce bijective soft matrix theory and research its operations, properties and algebraic structures in detail. Also, we present a bijective soft decision system based on the bijective soft matrix theory. Moreover, we construct a multi-bijective linguistic soft decision system by employing the matrices corresponding to the bijective soft sets generated from the linguistic variable parameters. Finally, the system's decision algorithm and its application for a decision making problem are given. By using the algorithm, we determine both the linguistic variables according to the parameters and the parameters affecting the optimal choice according to the highest linguistic decision value.

1. Introduction

In order to describe and overcome the uncertainties meeting in everyday life, many mathematical theories such as fuzzy set [34], rough set [28], vague set [12], the interval mathematics [16] are developed. In recent years, soft set theory introduced by Molodtsov [27] has attracted the interest of some scholars, because there is no limited condition to the description of objects in soft set theory. Also, the convenience of description of objects provides various benefits for the solution according to the type of problem. The soft set theory is often used to overcome the uncertainties in various fields such as combined forecast, game theory, decision making, incomplete information and information systems. Many decision methods were developed according to the type of uncertainty presented in decision problems [2, 6, 8, 10, 17, 18, 24, 29, 32]. In addition to this, the decision methods based on linguistic variable parameters described in [35] were attracted interest not only to select the optimum object but also to determine the selection order of the objects [1, 9, 19–22, 30, 33]. Also, Gong et al. [13] introduced the concept of bijective soft set, and then constructed a bijective soft decision system. Immediately afterwards, many authors concentrated on this concept in handling the problems involving uncertainties in various fields [4, 14, 15, 25, 31].

Recently, the representations in the form of binary information table of soft sets [26] and matrix representations of soft sets [7] have been used for the solution of various decision making problems. Çağman and Enginoğlu introduced some products of soft matrices which are matrix representation of soft set, and they researched algebraic structures of these products for the set of all soft matrix in the same type. Atagün et al. [3] generalized products defined in [7] for soft matrices in different types and they presented two

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monoid for these generalized products. Also, they constructed a decision making method completing some deficiencies of decision method proposed in [7]. Basu et al. [5] introduced the soft matrices in different types with a new viewpoint. They emphasize that these matrices are more useful to solve some decision problems. Feng and Zhou [11] suggested a novel decision algorithm using the discernibility soft matrix which is a special form of the soft matrix.

In this study, we aim adapt to the computer using the matrix representation of the bijective soft set defined in [13]. Thus, we want to make the process of computational faster and easier. Also, we purpose to construct a multi-system that can be used for more than one decision maker by improving the bijective decision systems obtained by means of some products of the bijective soft sets. Moreover, two terms such as high sales and low sales are used in [13]. We aim to use more than two terms by using the notion of linguistic set.

In direction of these purposes, we firstly present the notion of bijective soft matrix and its operations such as restricted-And product and relaxed-And product. We also investigate the basic properties of these operations. At the same time, we show that the set of all bijective soft matrices is a monoid according to the And-product. Afterwards, the bijective soft decision system which forms the basis for future sections is established by using the concepts of max-row soft matrix and density measurement function. In the rest of the paper, we introduce the notions of linguistic set and linguistic-valued function. By using these concepts, we present the multi-bijective linguistic soft decision system and this decision system's algorithm with its application.

2. Preliminaries

The concept of the soft set was defined by Molodtsov [27] in the following manner:
Let U, E and $P(U)$ be an initial universe set, a set of parameters and the power set of U, respectively and let $A \subseteq E$.

Definition 2.1. *([27]) If there is a mapping given by $F : A \to P(U)$, then the pair (F, A) is said to be a* soft set *over U. A soft set (F, A) (or with another notation F_A) can be written as a set of ordered pair*

$$(F, A) = \{(e_j, F(e_j))|\ e_j \in E, F(e_j) \in P(U)\}$$

where $F(e_j) = \emptyset$ if $e_j \notin A$.

Hence, it is seen that a parameterized family of subset of U is a soft set over U.

Notation: $S(U)$ denotes the set of all soft sets over U.

Definition 2.2. *([7]) Let $U = \{u_1, u_2, ..., u_n\}$, $E = \{e_1, e_2, ..., e_m\}$, $A \subseteq E$ and let (F, A) be a soft set over U. If*

$$a_{ij} = \begin{cases} 1, & u_i \in F(e_j) \\ 0, & u_i \notin F(e_j) \end{cases}$$

then the matrix

$$[a_{ij}]_{n\times m} = \begin{bmatrix} a_{11} & a_{12} & . & . & . & a_{1m} \\ a_{21} & a_{22} & . & . & . & a_{2m} \\ . & . & . & & & . \\ . & . & & . & & . \\ . & . & & & . & . \\ a_{n1} & a_{n2} & . & . & . & a_{nm} \end{bmatrix}.$$

is called a soft matrix *of the soft set (F, A) over U.*

According to the definition of soft matrix, a soft set (F, A) is uniquely characterized by the matrix $[a_{ij}]$. It means that a soft set (F, A) is formally equal to its soft matrix $[a_{ij}]$.

Notation: $SM(U)$ denotes the set of all soft matrices corresponding to the soft sets in $S(U)$.

Example 2.3. *Let $U = \{u_1, u_2, u_3, u_4, u_5, u_6\}$ be a universal set and $E = \{e_1, e_2, e_3, e_4, e_5\}$ a set of parameters. If $A = \{e_1, e_3, e_5\}$ and $F : A \to P(U)$, $F(e_1) = \{u_1, u_3, u_4, u_6\}$, $F(e_3) = \{u_3\}$, $F(e_5) = \emptyset$, then we write a soft set*

$$(F, A) = \{(e_1, \{u_1, u_3, u_4, u_6\}), (e_3, \{u_3\}), (e_5, \emptyset)\}.$$

Then, the soft matrix $[a_{ij}] \in SM_{6\times5}$ of (F, A) is

$$[a_{ij}]_{6\times5} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Definition 2.4. *([7]) Let $[a_{ij}], [b_{ij}] \in SM_{n\times m}$.*

1. *If $a_{ij} = 0$ for all i, j, then the matrix $[a_{ij}]$ is said to be a zero soft matrix and it is denoted by $[0]$.*
2. *If $a_{ij} = 1$ for all i, j, then the matrix $[a_{ij}]$ is said to be a universal soft matrix and it is denoted by $[1]$.*
3. *If $a_{ij} \leq b_{ij}$ for all i, j, then the matrix $[a_{ij}]$ is said to be a soft submatrix of $[b_{ij}]$ and it is denoted by $[a_{ij}]\widetilde{\subseteq}[b_{ij}]$.*
4. *If $a_{ij} = b_{ij}$ for all i, j, then $[a_{ij}]$ and $[b_{ij}]$ are said to be equal matrices and it is denoted by $[a_{ij}]\widetilde{=}[b_{ij}]$.*
5. *If $c_{ij} = max\{a_{ij}, b_{ij}\}$ for all i, j, then the soft matrix $[c_{ij}]$ is said to be union of $[a_{ij}]$ and $[b_{ij}]$ and it is denoted by $[c_{ij}] = [a_{ij}]\widetilde{\cup}[b_{ij}]$.*
6. *If $c_{ij} = min\{a_{ij}, b_{ij}\}$ for all i, j, then the soft matrix $[c_{ij}]$ is said to be intersection of $[a_{ij}]$ and $[b_{ij}]$ and it is denoted by $[c_{ij}] = [a_{ij}]\widetilde{\cap}[b_{ij}]$.*
7. *If $c_{ij} = 1 - a_{ij}$ for all i, j, then the soft matrix $[c_{ij}]$ is said to be complement of $[a_{ij}]$ and it is denoted by $[c_{ij}] = [a_{ij}]^c$.*

Definition 2.5. *([3]) Let $U = \{u_1, u_2, ..., u_n\}$, $E = \{e_1, e_2, ..., e_m\}$, $A \subseteq E$ and let cardinality of A be m_1. Consider (F, A) is a soft set over U. If*

$$a_{ij} = \begin{cases} 1, & \text{if } e_j \in A \text{ and } u_i \in F(e_j) \\ 0, & \text{if } e_j \in A \text{ and } u_i \notin F(e_j) \end{cases}$$

then the matrix $[a_{ij}]_{n\times m_1}$ is called a reduced soft matrix *of the soft set (F, A) over U. Here $1 \leq m_1 \leq m$.*

In other words, let (F, A) be a soft set over U. Since $e_j \notin A$ implies $F(e_j) = \emptyset$, by eliminating parameters from $E\backslash A$, we can construct the soft matrix corresponding to the soft set (F, A) and then the type of this soft matrix will be $n \times m_1$.
We remark that if $A = E$, then the reduced soft matrix corresponding to the soft set (F, A) is equal to the soft matrix corresponding to the soft set (F, A). Since only the types of soft matrices and reduced soft matrices are different, we don't use a different display for the reduced soft matrices.

Example 2.6. *Consider the soft set (F, A) in Example 2.3. Then the reduced soft matrix $[a_{ij}] \in SM_{6\times3}$ of (F, A) is*

$$[a_{ij}]_{6\times3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

In this paper, from now on the universe U has been taken as $|U| = n$, i.e. U is a set consisting of n-elements. Also, all soft matrices will be taken the reduced soft matrices.

Let (F, A), $(F, B) \in S(U)$ and let the corresponding (reduced) soft matrices of (F, A) and (F, B) denoted by $[a_{ij}]$ and $[b_{ik}]$, respectively.

Definition 2.7. *([3]) Generalized And-product of $[a_{ij}]$ and $[b_{ik}]$, denoted by $\wedge$, is defined by*

$$\begin{array}{rcl} \wedge : SM_{n\times m_1} \times SM_{n\times m_2} & \longrightarrow & SM_{n\times m_1 m_2} \\ [a_{ij}],\ [b_{ik}] & \longrightarrow & [a_{ij}] \wedge [b_{ik}] = [c_{ip}] \end{array}$$

where $c_{ip} = min\{a_{ij}, b_{ik}\}$ such that $j = \alpha$, $p = (\alpha-1)m_2+k$ and α is the smallest positive integer which satisfies $p \leq \alpha m_2$.

Definition 2.8. *([3]) Generalized Or-product of $[a_{ij}]$ and $[b_{ik}]$, denoted by $\vee$, is defined by*

$$\begin{array}{rcl} \vee : SM_{n\times m_1} \times SM_{n\times m_2} & \longrightarrow & SM_{n\times m_1 m_2} \\ [a_{ij}],\ [b_{ik}] & \longrightarrow & [a_{ij}] \vee [b_{ik}] = [c_{ip}] \end{array}$$

where $c_{ip} = max\{a_{ij}, b_{ik}\}$ such that $j = \alpha$, $p = (\alpha - 1)m_2 + k$ and α is the smallest positive integer which satisfies $p \leq \alpha m_2$.

Definition 2.9. *([3]) Generalized And-Not-product of $[a_{ij}]$ and $[b_{ik}]$, denoted by $\bar{\wedge}$, is defined by*

$$\begin{array}{rcl} \bar{\wedge} : SM_{n\times m_1} \times SM_{n\times m_2} & \longrightarrow & SM_{n\times m_1 m_2} \\ [a_{ij}],\ [b_{ik}] & \longrightarrow & [a_{ij}\ \bar{\wedge}\ [b_{ik}] = [c_{ip}] \end{array}$$

where $c_{ip} = min\{a_{ij}, 1 - b_{ik}\}$ such that $j = \alpha$, $p = (\alpha - 1)m_2 + k$ and α is the smallest positive integer which satisfies $p \leq \alpha m_2$.

Definition 2.10. *([3]) Generalized Or-Not-product of $[a_{ij}]$ and $[b_{ik}]$, denoted by $\underline{\vee}$, is defined by*

$$\begin{array}{rcl} \underline{\vee} : SM_{n\times m_1} \times SM_{n\times m_2} & \longrightarrow & SM_{n\times m_1 m_2} \\ [a_{ij}],\ [b_{ik}] & \longrightarrow & [a_{ij}]\ \underline{\vee}\ [b_{ik}] = [c_{ip}] \end{array}$$

where $c_{ip} = max\{a_{ij}, 1 - b_{ik}\}$ such that $j = \alpha$, $p = (\alpha - 1)m_2 + k$ and α is the smallest positive integer which satisfies $p \leq \alpha m_2$.

Example 2.11. *Let $U = \{u_1, u_2, u_3, u_4\}$ be a universal set and $E = \{e_1, e_2, e_3, e_4, e_5\}$ be the set of all parameters. Assume that $A = \{e_1, e_2, e_3, e_5\}$ and $B = \{e_1, e_2, e_4\}$ are two subsets of E. Then we can write the following soft sets.*
$(F, A) = \{(e_1, \{u_1, u_2\}), (e_2, \{u_1, u_2, u_3\}), (e_3, U), (e_5, \{u_1\})\}$ and $(F, B) = \{(e_1, \{u_3\}), (e_2, \{u_4\}), (e_4, \{u_1, u_2, u_3\})\}$.
Hence, the soft matrices $[a_{ij}] \in SM_{4\times4}$ and $[b_{ik}] \in SM_{4\times3}$ corresponding to (F, A) and (F, B) are given as follows:

$$[a_{ij}] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \text{ and } [b_{ik}] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Then, we obtain

$$[c_{ip}] = [a_{ij}] \wedge [b_{ik}] = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Here the type of soft matrix $[c_{ip}]$ is 4×12.

Theorem 2.12. *([3]) The operation generalized And-product is associative, i.e. if $[a_{ij}] \in SM_{n\times m_1}$, $[b_{ik}] \in SM_{n\times m_2}$ and $[c_{il}] \in SM_{n\times m_3}$, then*

$$([a_{ij}] \wedge [b_{ik}]) \wedge [c_{il}] = [a_{ij}] \wedge ([b_{ik}] \wedge [c_{il}]).$$

Theorem 2.13. *([3]) The operation generalized Or-product is associative, i.e. if* $[a_{ij}] \in SM_{n\times m_1}$, $[b_{ik}] \in SM_{n\times m_2}$ *and* $[c_{il}] \in SM_{n\times m_3}$, *then*

$$([a_{ij}] \vee [b_{ik}]) \vee [c_{il}] = [a_{ij}] \vee ([b_{ik}] \vee [c_{il}]).$$

Notation: Let $[a_{ij}] \in SM_{n\times m}$ be a soft matrix. The sum of values in ith row of $[a_{ij}]$ will be denoted by $\nu_i([a_{ij}])$.

Lemma 2.14. *Let* $[a_{ij}] \in SM_{n\times m_1}$ $[b_{ik}] \in SM_{n\times m_2}$ *be two soft matrices. Then,* $min\{a_{ij}, b_{ik}\} = a_{ij}b_{ik}$ *for all* $i \in 1, 2, ..., n$.

Proof. Since the components of soft matrices a_{ij} and b_{ik} are made up of the numbers 0 and 1, the proof is seen easily.

Theorem 2.15. *Let* $[a_{ij}] \in SM_{n\times m_1}$ *and* $[b_{ik}] \in SM_{n\times m_2}$ *be two soft matrices. If* $[c_{ip}] = [a_{ij}] \wedge [b_{ik}]$, *then* $\nu_i([c_{ip}]) = \nu_i([a_{ij}])\nu_i([b_{ik}])$

Proof. Let $[a_{ij}] \in SM_{n\times m_1}$ and $[b_{ik}] \in SM_{n\times m_2}$ be two soft matrices, and let $[c_{ip}] = [a_{ij}] \wedge [b_{ik}]$. Since $[c_{ip}] \in SM_{n\times m_1 m_2}$,

$$\begin{aligned}
\nu_1([c_{ip}]) &= \sum_{p=1}^{m_1 m_2} c_{1p} = c_{11} + c_{12} + ... + c_{1(m_1 m_2)} \\
&= min\{a_{11}, b_{11}\} + min\{a_{11}, b_{12}\} + ... + min\{a_{11}, b_{1m_2}\} + min\{a_{12}, b_{11}\} \\
&\quad + min\{a_{12}, b_{12}\} + ... + min\{a_{12}, b_{1m_2}\} + ... + min\{a_{1m_1}, b_{11}\} \\
&\quad + min\{a_{1m_1}, b_{12}\} + ... + min\{a_{1m_1}, b_{1m_2}\} \\
&= a_{11}b_{11} + a_{11}b_{12} + ... + a_{11}b_{1m_2} + a_{12}b_{11} + a_{12}b_{12} + ... + a_{12}b_{1m_2} \\
&\quad + ... + a_{1m_1}b_{11} + a_{1m_1}b_{12} + ... + a_{1m_1}b_{1m_2} \\
&= a_{11}(b_{11} + b_{12} + ... + b_{1m_2}) + a_{12}(b_{11} + b_{12} + ... + b_{1m_2}) \\
&\quad + ... + a_{1m_1}(b_{11} + b_{12} + ... + b_{1m_2}) \\
&= (a_{11} + a_{12} + ... + a_{1m_1})(b_{11} + b_{12} + ... + b_{1m_2}) \\
&= \sum_{j=1}^{m_1} a_{1j} \sum_{k=1}^{m_2} b_{1k} \\
&= \nu_1([a_{ij}]).\nu_1([b_{ik}])
\end{aligned}$$

Similarly, it is shown that $\nu_i([c_{ip}]) = \nu_i([a_{ij}])\nu_i([b_{ik}])$ for $i = 2, 3, ..., n$. So the proof is complete.

3. Bijective Soft Matrix and Its Operations

In this part, we define bijective soft matrix and give an example of this concept. Also, we introduce operations of the bijective soft matrices and research algebraic structures for the bijective soft matrices using some operations.

Definition 3.1. *[13] Let* (F, B) *be a soft set over a common universe* U *where* F *is a mapping* $F : B \to P(U)$ *and* B *is nonempty parameter set. We say that* (F, B) *is a bijective soft set, if*
i) $\cup_{e\in B} F(e) = U$.
ii) *For any two parameters* $e_i, e_j \in B$, $e_i \neq e_j$, $F(e_i) \cap F(e_j) = \emptyset$.

Notation: $BS(U)$ denotes the set of all bijective soft sets over U.

Definition 3.2. *Let* $[a_{ij}] \in SM_{n\times m}$ *be a soft matrix.* $[a_{ij}]_{n\times m}$ *is called a* bijective soft matrix, *if* $\nu_i([a_{ij}]) = \sum_{j=1}^{m} a_{ij} = 1$ *for all* $i = 1, 2, ..., n$.

Example 3.3. *Assume that*

$$[a_{ij}]_{4\times 5} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Then, $[a_{ij}]_{4\times 5}$ *is a bijective soft matrix since* $\nu_1([a_{ij}]) = \nu_2([a_{ij}]) = \nu_3([a_{ij}]) = \nu_4([a_{ij}]) = 1$

Notation: $BSM(U)$ denotes the set of all bijective soft matrices corresponding to the bijective soft sets in $BS(U)$.

Notation: If $[F, A]$ denotes the soft matrix corresponding to the soft set (F, A).

Theorem 3.4. *(F, A) is a bijective soft set over U. $\Leftrightarrow$ $[F, A]$ is a bijective soft matrix.*

Proof. Assume that $[F, A] = [a_{ij}]_{n\times m}$ is a soft matrix. By Definitions 2.2 and 2.5, $a_{ij} = 1 \Leftrightarrow u_i \in f(e_j)$. Then by Definitions 2.2, 2.5 and 3.2, $\nu_i([a_{ij}]) = 1$ for all $i = 1, 2, ..., n \Leftrightarrow u_i \in f(e_j)$ implies that $u_i \notin f(e_k)$ for $j \neq k$. Again, it is seen that $\nu_i([a_{ij}]) = 1$ for all $i = 1, 2, ..., n \Leftrightarrow$ for each $u_i \in U$ there exist at least $e_j \in A$ such that $u_i \in f(e_j) \Leftrightarrow \bigcup_{e\in A} f(e) = U$. Therefore the proof is seen by Definition 3.1 and 3.2.

Theorem 3.5. *Let $[a_{ij}] \in BSM_{n\times m}$ be a bijective soft matrix and let $[0], [1] \in SM_{n\times m}$. Then,*

i) $[a_{ij}]\widetilde{\cap}[1]$ *is a bijective soft matrix.*
ii) $[a_{ij}]\widetilde{\cup}[0]$ *is a bijective soft matrix.*

Proof. Proof is clear by Definition 2.4.

Note: In [7], the authors prove that $(SM(U), \widetilde{\cap})$ and $(SM(U), \widetilde{\cup})$ are monoids (closure, associativity, identity). If $[a_{ij}] \neq [b_{ij}]$, then $[a_{ij}]\widetilde{\cap}[b_{ij}]$ and $[a_{ij}]\widetilde{\cup}[b_{ij}]$ are not bijective soft matrices. Therefore, we say that $(BSM(U), \widetilde{\cap})$ and $(BSM(U), \widetilde{\cup})$ are not semigroups and monoids.

Theorem 3.6. *Let $[a_{ij}] \in BSM_{n\times m_1}$ and $[b_{ik}] \in BSM_{n\times m_2}$ be two bijective soft matrices. Then $[a_{ij}] \wedge [b_{ik}]$ is a bijective soft matrix.*

Proof. Proof is clear by Theorem 2.15.

Lemma 3.7. *Let $U = \{u_1, u_2, ..., u_n\}$ be an initial universe set. Then*

$$[a_{ij}] \wedge [1]_{n\times 1} = [1]_{n\times 1} \wedge [a_{ij}] = [a_{ij}]$$

for each $[a_{ij}] \in BSM(U)$.

Theorem 3.8. *According to operation And-product, $BSM(U)$ is a monoid.*

Proof. By Theorem 2.12 and Theorem 3.6, $(BSM(U), \wedge)$ is a semigroup. Then $(BSM(U), \wedge)$ is a monoid by Lemma 3.7.

Note: In [3], the authors put forward that $(SM(U), \wedge)$ and $(SM(U), \vee)$ are monoids. Since $[a_{ij}] \vee [b_{ik}]$ is not a bijective soft matrix, we say that $(BSM(U), \vee)$ is not a semigroup and monoid.

Corollary 3.9. *If $(BSM(U), *)$ is a semigroup (monoid, group) then $(SM(U), *)$ is a semigroup (monoid, group), where $*$ is a binary operation, but not vice versa.*

Before introducing two new soft products, we need to give the following definition. We describe the comparison of bijective soft matrices using their columns by this definition.

Notation: Let $[a_{ij}] \in BSM_{n\times m}$ be a bijective soft matrix. The $(1 \leq q \leq j)$ qth column of $[a_{ij}]$ will be denoted by $|a_{ij}|_q$.

Definition 3.10. *Let* $[a_{ij}] \in BSM_{n\times m_1}$ *and* $[b_{ik}] \in BSM_{n\times m_2}$ *be bijective soft matrices. Then*

1. $|a_{ij}|_q$ *is called a (proper) sub-column of* $|b_{ik}|_r$*, denoted by* ($|a_{ij}|_q \widetilde{<} |b_{ik}|_r$*, if* $a_{iq} < b_{ir}$) $|a_{ij}|_q \widetilde{\leq} |b_{ik}|_r$*, if* $a_{iq} \leq b_{ir}$ *for all* $1 \leq i \leq n$ *where* $1 \leq q \leq m_1$ *and* $1 \leq r \leq m_2$.
2. $[a_{ij}]$ *is a bijective soft (proper) sub-column matrix of* $[b_{ik}]$*, denoted by* ($[a_{ij}] \widetilde{\sqsubset} [b_{ik}]$*, if* $|a_{ij}|_q \leq |b_{ik}|_r$ *for at least one term* $|a_{ij}|_q < |b_{ik}|_r$) $[a_{ij}] \widetilde{\sqsubseteq} [b_{ik}]$ *if there exist* $r \in \{1, 2, ..., m_2\}$ *providing* $|a_{ij}|_q \leq |b_{ik}|_r$ *for each* $1 \leq q \leq m_1$.
3. $|a_{ij}|_q$ *is called a equal-column of* $|b_{ik}|_r$*, denoted by* $|a_{ij}|_q = |b_{ik}|_r$*, if* $a_{iq} = b_{ir}$ *for all* $1 \leq i \leq n$*, where* $1 \leq q \leq m_1$ *and* $1 \leq r \leq m_2$*. Then,* $[a_{ij}]$ *and* $[b_{ik}]$ *are bijective soft equal-column matrices, denoted by* $[a_{ij}] \widetilde{=} [b_{ik}]$*, if there exist* $r \in \{1, 2, ..., m_2\}$ *providing* $|a_{ij}|_q = |b_{ik}|_r$ *for each* $1 \leq q \leq m_1$.

Example 3.11. i) *Assume that* $[a_{ij}]$ *and* $[b_{ik}]$ *are bijective soft matrices given as follows*

$$[a_{ij}]_{5\times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } [b_{ik}]_{5\times 2} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}.$$

Then $[a_{ij}]$ *is bijective soft proper sub-column matrix of* $[b_{ik}]$ *since* $|a_{ij}|_1 \widetilde{<} |b_{ik}|_1$, $|a_{ij}|_2 \widetilde{<} |b_{ik}|_1$ *and* $|a_{ij}|_3 = |b_{ik}|_2$.
ii) *Assume that* $[a_{ij}]$ *and* $[b_{ik}]$ *are bijective soft matrices given as follows*

$$[a_{ij}]_{5\times 2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} \text{ and } [b_{ik}]_{5\times 2} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}.$$

Then $[a_{ij}]$ *and* $[b_{ik}]$ *are bijective soft equal-column matrices since* $|a_{ij}|_1 = |b_{ik}|_2$ *and* $|a_{ij}|_2 = |b_{ik}|_1$.

Definition 3.12. *Let* $(F, A), (F, B) \in BS(U)$ *and let* $[a_{ij}]$ *and* $[b_{ik}]$*, their bijective soft matrices, respectively. Then,* restricted-And product *of* $[a_{ij}]$ *and* $[b_{ik}]$ *denoted by* $[a_{ij}] \underline{\curlywedge} [b_{ik}]$ *is defined by*

$$\begin{aligned} \underline{\curlywedge} : BSM_{n\times m_1} \times BSM_{n\times m_2} &\longrightarrow SM_{n\times m_1 m_2} \\ [a_{ij}],\ [b_{ik}] &\longrightarrow [a_{ij}] \underline{\curlywedge} [b_{ik}] = [c_{ip}] \end{aligned}$$

where the rth column of $[c_{ip}]$ *is given as*

$$|c_{ip}|_r = \begin{cases} |a_{ij}|_s; & \text{if } |a_{ij}|_s \leq |b_{ik}|_t \\ |0|; & \text{if } |a_{ij}|_s \nleq |b_{ik}|_t \end{cases}$$

for $1 \leq r \leq m_1 m_2$, $1 \leq s \leq m_1$ *and* $1 \leq t \leq m_2$ *such that* $s = \alpha$, $r = (\alpha - 1)m_2 + t$ *and* α *is the smallest positive integer which satisfies* $r \leq \alpha m_2$*. Here, the type of soft matrix* $[c_{ip}]$ *is* $n \times m_1 m_2$.

Definition 3.13. *Let* $(F, A), (F, B) \in BS(U)$ *and let* $[a_{ij}]$ *and* $[b_{ik}]$*, their corresponding bijective soft matrices,respectively. Then,* relaxed-And product *of* $[a_{ij}]$ *and* $[b_{ik}]$ *denoted by* $[a_{ij}] \widetilde{\curlywedge} [b_{ik}]$ *is defined by*

$$\begin{aligned} \widetilde{\curlywedge} : BSM_{n\times m_1} \times BSM_{n\times m_2} &\longrightarrow SM_{n\times m_1 m_2} \\ [a_{ij}],\ [b_{ik}] &\longrightarrow [a_{ij}] \widetilde{\curlywedge} [b_{ik}] = [c_{ip}] \end{aligned}$$

where the rth column of $[c_{ip}]$ *is given as*

$$|c_{ip}|_r = \begin{cases} |0|; & \text{if } |a_{ij}|_s \leq (|b_{ik}|_t)^c \\ |a_{ij}|_s; & \text{if } |a_{ij}|_s \nleq (|b_{ik}|_t)^c \end{cases}$$

for $1 \leq r \leq m_1 m_2$, $1 \leq s \leq m_1$ *and* $1 \leq t \leq m_2$ *such that* $s = \alpha$, $r = (\alpha - 1)m_2 + t$ *and* α *is the smallest positive integer which satisfies* $r \leq \alpha m_2$*, and the complement of* $|b_{ik}|_t$ *is denoted by* $(|b_{ik}|_t)^c$*. Here, the type of soft matrix* $[c_{ip}]$ *is* $n \times m_1 m_2$.

Example 3.14. *Assume that $[a_{ij}], [b_{ik}] \in BSM_{4\times 5}$ are bijective soft matrices given as follows*

$$[a_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \text{ and } [b_{ik}] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Then we obtain

$$[c_{ip}] = [a_{ij}]\underline{\wedge}[b_{ik}] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \text{ and } [d_{ip}] = [a_{ij}]\widetilde{\wedge}[b_{ik}] = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

Here the types of soft matrices $[c_{ip}]$ and $[d_{ip}]$ are 4×6.

Proposition 3.15. *Let $[a_{ij}] \in BSM_{n\times m_1}$ be a bijective soft matrix. Then,*

i) $[a_{ij}]\underline{\wedge}[1]_{n\times 1} = [a_{ij}]$

ii) $[a_{ij}]\widetilde{\wedge}[1]_{n\times 1} = [a_{ij}]$

iii) $[a_{ij}]\underline{\wedge}[a_{ij}] = [a_{ij}]\widetilde{\wedge}[a_{ij}]$

iv) $[a_{ij}]\underline{\wedge}[a_{ij}]^c = [a_{ij}]\widetilde{\wedge}[a_{ij}]^c$

Proof. i) Let $[a_{ij}] \in BSM_{n\times m_1}$ and $[c_{ip}] = [a_{ij}]\underline{\wedge}[1]_{n\times 1}$. Then $|c_{ip}|_r = |a_{ij}|_s$ by Definition 3.12 since $|a_{ij}|_s \le |1|$ for all $s \in \{1, 2, ..., m_1\}$. Also $r = s$ since $m_2 = 1$ and $t = 1$. Therefore $[c_{ip}] = [a_{ij}]$ since $|c_{ip}|_r = |a_{ij}|_r$ for all $r \in \{1, ..., m_1\}$.
ii) Let $[a_{ij}] \in BSM_{n\times m_1}$ and $[c_{ip}] = [a_{ij}]\widetilde{\wedge}[1]_{n\times 1}$. Then $|c_{ip}|_r = |a_{ij}|_s$ since

$$|c_{ip}|_r = \begin{cases} |0|; & \text{if } |a_{ij}|_s = |0| \\ |a_{ij}|_s; & \text{if } |a_{ij}|_s \neq |0| \end{cases}$$

by Definition 3.13. Also $r = s$ since $m_2 = 1$ and $t = 1$. Therefore $[c_{ip}] = [a_{ij}]$ since $|c_{ip}|_r = |a_{ij}|_r$ for all $r \in \{1, ..., m_1\}$.
iii) Let $[a_{ij}] \in BSM_{n\times m_1}$ be a bijective soft matrix and let $[c_{ip}] = [a_{ij}]\underline{\wedge}[a_{ij}]$ and $[d_{iq}] = [a_{ij}]\widetilde{\wedge}[a_{ij}]$. If $|a_{ij}|_s \le |a_{ij}|_t$ for all $s, t \in \{1, 2, ..., m_1\}$, then $|a_{ij}|_s \nleq (|a_{ij}|_t)^c$ since $[a_{ij}]$ is a bijective soft matrix. Hence

$$|c_{ip}|_r = |d_{iq}|_r = |a_{ij}|_s. \tag{1}$$

Also if $|a_{ij}|_s \nleq |a_{ij}|_t$ for all $s, t \in \{1, 2, ..., m_1\}$, then $|a_{ij}|_s \le (|a_{ij}|_t)^c$ since $[a_{ij}]$ is a bijective soft matrix. Hence

$$|c_{ip}|_r = |d_{iq}|_r = |0|. \tag{2}$$

Then, it is obtained $[a_{ij}]\underline{\wedge}[a_{ij}] = [a_{ij}]\widetilde{\wedge}[a_{ij}]$ from (1) and (2).
iv) Let $[a_{ij}] \in BSM_{n\times m_1}$ be a bijective soft matrix and let $[c_{ip}] = [a_{ij}]\underline{\wedge}[a_{ij}]^c$ and $[d_{iq}] = [a_{ij}]\widetilde{\wedge}[a_{ij}]^c$. If $|a_{ij}|_s \le (|a_{ij}|_t)^c$ for all $s, t \in \{1, 2, ..., m_1\}$, then $|a_{ij}|_s \nleq |a_{ij}|_t$ since $[a_{ij}]$ is a bijective soft matrix. Hence

$$|c_{ip}|_r = |d_{iq}|_r = |a_{ij}|_s. \tag{3}$$

Also if $|a_{ij}|_s \nleq (|a_{ij}|_t)^c$ for all $s, t \in \{1, 2, ..., m_1\}$ then $|a_{ij}|_s \le |a_{ij}|_t$ since $[a_{ij}]$ is a bijective soft matrix. Hence

$$|c_{ip}|_r = |d_{iq}|_r = |0|. \tag{4}$$

Then, it is obtained $[a_{ij}]\underline{\wedge}[a_{ij}]^c = [a_{ij}]\widetilde{\wedge}[a_{ij}]^c$ from (3) and (4).

4. Bijective Soft Decision System

In this part, we define density measurement function and calculate the density ratio of a bijective soft matrix on the other one. Also, we give concepts of bijective soft decision system and density of bijective soft decision system.

Definition 4.1. *([23]) Let $[c_{ij}] \in SM_{n\times m}$ be a soft matrix. Then* soft max-row function M_r *is defined as follows*

$$M_r : SM_{n\times m} \longrightarrow SM_{n\times 1},\ M_r([c_{ij}]) = [d_{i1}]$$

where $d_{i1} = \max_{j\in\{1,2,...,m\}} c_{ij}$.
The one column soft matrix $M_r([c_{ij}])$ is called max-row soft matrix.

Example 4.2. *For the soft matrix $[c_{ip}]$ given in Example 3.14, max-row soft matrix of $[c_{ip}]$ is*

$$M_r([c_{ip}]) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

Definition 4.3. *Let $[F,A]$,$[G,B]$ be two soft matrices corresponding to bijective soft sets (F,A) and (G,B) and also let $Mr([c_{ij}]) = [d_{i1}]$ be a max-row soft matrix of $[c_{ij}] = [F,A]\underline{\wedge}[G,B]$. Then* density measurement function *of the soft matrix $[F,A]$ on the soft matrix $[G,B]$ is defined by $d_f([d_{i1}])$ such that*

$$\begin{aligned} d_f : SM_{n\times 1} &\longrightarrow \mathbb{Z} \\ [d_{i1}] &\longrightarrow d_f([d_{i1}]) = \sum_{i=1}^{n} d_{i1} \end{aligned}$$

Definition 4.4. *Assume that $[F,A]$, $[G,B]$ are two bijective soft matrices corresponding to bijective soft sets (F,A) and (G,B) over the common universe U, where $A \cap B = \emptyset$. The τ* density ratio *of $[F,A]$ on $[G,B]$ is a number*

$$\tau = \delta([F,A];[G,B]) = \frac{d_f(Mr([F,A]\underline{\wedge}[G,B]))}{|U|}.$$

Here it is clear that $0 \le \tau \le 1$.

The concept of density is to describe effect degree of a bijective soft matrix on the other one.
If $\tau = 1$ we say $[F,A]$ is full density on $[G,B]$.
If $\tau = 0$ we say $[F,A]$ is null density on $[G,B]$.
To illustrate concept, will give following example.

Example 4.5. *For the bijective soft matrices $[F,A] = [a_{ij}]$ and $[G,B] = [b_{ik}]$ given in Example 3.14,*

$$\tau = \delta([F,A];[G,B]) = \frac{d_f(Mr([c_{ip}]))}{|U|} = \frac{1}{2}.$$

From now on, $[F_i,E_i]$ are r bijective soft matrices corresponding to bijective soft sets (F_i,E_i) for $(i = 1,2,...,r)$ over a common universe U, where any $E_i \cap E_j = \emptyset$ $(i,j = 1,2,...,r; i \neq j)$ and

$$[F,E] = [F_1,E_1] \wedge [F_2,E_2] \wedge ... \wedge [F_r,E_r] = \bigwedge_{i=1}^{r}[F_i,E_i].$$

Definition 4.6. *Let $[G,B]$ be a bijective soft matrix corresponding to soft set (G,B) over the common universe U such that $B \cap E_i = \emptyset$ for $(i = 1,2,...,r)$. Then $[G,B]$ is called the decision soft matrix and the triple $([F,E];[G,B] : U)$ is called a* bijective soft decision system *over a common universe U.*

Definition 4.7. *The density ratio of $[F,E]$ on $[G,B]$ is called* bijective soft decision system density *of $([F,E];[G,B] : U)$, formulated by*

$$\tau = \delta([F, E]; [G, B]).$$

Definition 4.8. *Let τ be the bijective soft decision density of* $([F, E]; [G, B] : U)$*. If $m < r$ and $\delta(\bigwedge_{i=1}^{m}[F_i, E_i]; [G, B]) = \tau$, then $(\bigwedge_{i=1}^{m}[F_i, E_i]; [G, B])$ is called a* reduction bijective soft decision system *of* $([F, E]; [G, B] : U)$.

By Definition 4.8, the soft matrices having no influence on decision making are obtained. Thus, elimination of the parameters generating these matrices is provided.

Example 4.9. *Assume that $[F_1, E_1], [F_2, E_2], [F_3, E_3]$ and $[G, B]$ are bijective soft matrices given as follows*

$$[F_1, E_1] = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, [F_2, E_2] = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, [F_3, E_3] = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } [G, B] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Then, we obtain $\tau = \delta([F, E]; [G, B]) = \frac{3}{5}$ and $\delta([F_1, E_1] \wedge [F_2, E_2]; [G, B]) = \frac{3}{5}$. Hence, $[F_1, E_1] \wedge [F_2, E_2]; [G, B] : U)$ is a reduction bijective soft decision system of $([F, E], [G, B], U)$.

As seen in Example 4.9, the soft matrix $[F_3, E_3]$ doesn't affect the density ratio of system. Thus, it appears that the parameter set E_3 doesn't affect the decision.

Notation: The tth column of matrix $[G, B]$ and the rth column of matrix $[H, C]$ will be denoted by $|G, B|_t$ and $|H, C|_r$, respectively.

Definition 4.10. *Let $[H, C] = \bigwedge_{i=1}^{m}[F_i, E_i]$ and let $([H, C]; [G, B] : U)$ be a reduction bijective soft decision system of $([F, E]; [G, B] : U)$. If $|H, C|_r \widetilde{\leq} |G, B|_t$ then $d_f(|H, C|_r)/d_f(|G, B|_t)$ is said to be a* decision component *induced by $[H, C]$ and denotes* the effect ratio of decision component.
Here, $t \leq |B|$ and $r \leq |C|$ such that $|B|$ and $|C|$ denote the element numbers of B and C, respectively..

Example 4.11. *Consider the bijective soft matrices $[F_1, E_1], [F_2, E_2], [F_3, E_3]$ and $[G, B]$ given in Example 4.9. Let $[H, C] = [F_1, E_1] \wedge [F_2, E_2]$. We obtain $d_f(|H, C|_2)/d_f(|G, B|_1) = \frac{1}{3}$ since $|H, C|_2 \widetilde{\leq} |G, B|_1$, and $d_f(|H, C|_6)/d_f(|G, B|_2) = \frac{1}{2}$ since $|H, C|_6 \widetilde{\leq} |G, B|_2$.*

5. Multi-Bijective Linguistic Soft Decision System

In this part, we introduce the notions of linguistic set and linguistic-valued function. Also, we built multi-bijective linguistic soft decision system. Afterwards, we give a formula calculating the decisive effect ratio of each column of the reduction bijective soft matrix. Finally, we construct a novel decision method using these concepts.

Definition 5.1. *([21]) Let $B = \{t_0, t_1, ..., t_{\ell-1}, t_\ell, t_{\ell+1}, ..., t_{2\ell}\}$ be a finite and fully ordered discrete term set where $\ell \in \mathbb{N}$. Then B is called a* linguistic set. *The following characteristics are required:*

1. $t_i \leq t_j \Leftrightarrow i \leq j$ *for* $i, j \in \{0, 1, ..., 2\ell\}$.
2. *The negation of element t_i for $i \in \{0, 1, ..., 2\ell\}$ is the element $t_{2\ell-i}$.*
3. $max(t_i, t_j) = t_i$ *if* $t_i \geq t_j$.
4. $min(t_i, t_j) = t_i$ *if* $t_i \leq t_j$.

Here the term t_ℓ is called a *middle term* of the set B.

Definition 5.2. *Let $B = \{t_0, t_1, ..., t_{\ell-1}, t_\ell, t_{\ell+1}, ..., t_{2\ell}\}$ be a linguistic set. Then*

$$f_L : [0, 1] \longrightarrow B$$
$$x \longrightarrow f_L(x) = \begin{cases} t_{[|2\ell x|]}, & \text{if } 0 \leq x < 0,5 \\ t_\ell, & \text{if } x = 0,5 \\ t_{(-[|-2\ell x|])}, & \text{if } 0,5 < x \leq 1 \end{cases}$$

is called a linguistic-valued function *related to B.*

Example 5.3. *Let $B = \{t_0 = very\ bad, t_1 = bad, t_2 = normal, t_3 = good, t_4 = very\ good\}$ for $\ell = 2$. Then B is a linguistic set. Also, $t_2 = normal$ is middle term of B. If we consider linguistic-valued function f_L for the set B, then we obtain $f_L(0) = t_0$, $f_L(0,2) = t_0$, $f_L(0,43) = t_1$, $f_L(0,5) = t_2$, $f_L(0,625) = t_3$, $f_L(0,75) = t_3$ and $f_L(1) = t_4$.*

Remark: From now on, the set B denotes a linguistic set.

Definition 5.4. *Let $[G_1, B], [G_2, B], ..., [G_\kappa, B]$ be bijective soft matrices corresponding to the bijective soft sets $(G_1, B), (G_2, B), ..., (G_\kappa, B)$ over the common universe U, respectively. Also let $B \cap E_i = \emptyset$ for $i = 1, 2, ..., r$ and let $([F, E]; [G_1, B] : U), ([F, E]; [G_2, B] : U), ..., ([F, E]; [G_\kappa, B] : U)$ be bijective linguistic soft decision systems over a common universe set. Then $[G_1, B], [G_2, B], ..., [G_\kappa, B]$ are called the linguistic multi-decision soft matrices and $([F, E]; [G_1, B], [G_2, B], ..., [G_\kappa, B] : U)$ is called a* multi-bijective linguistic soft decision system *over a common universe U.*

Definition 5.5. *Let $\tau_1 = \delta([F, E], [G_1, B])$, $\tau_2 = \delta([F, E], [G_2, B])$,...,$\tau_\kappa = \delta([F, E], [G_\kappa, B])$ be bijective linguistic soft decision system densities of $[F, E]$ on $[G_1, B]$, $[F, E]$ on $[G_2, B]$,...,$[F, E]$ on $[G_\kappa, B]$, respectively. Then density of $[F, E]$ on $[G_1, B], [G_2, B], ..., [G_\kappa, B]$ is said to be* multi-bijective linguistic soft decision system density *of $([F, E]; [G_1, B], [G_2, B], ..., [G_\kappa, B] : U)$ and formulated by*

$$\mathfrak{R} = \delta([F, E]; [G_1, B], [G_2, B], ..., [G_\kappa, B]) = \frac{\tau_1 + \tau_2 + ... + \tau_\kappa}{\kappa}.$$

Definition 5.6. *Let $([F, E]; [G_1, B], [G_2, B], ..., [G_\kappa, B] : U)$ be a multi-bijective linguistic soft decision system. If $m \leq r$ and $\delta([F, E]; [G_1, B]) = \delta(\bigwedge_{i=1}^{m}[F_i, E_i]; [G_1, B])$, $\delta([F, E]; [G_2, B]) = \delta(\bigwedge_{i=1}^{m}[F_i, E_i]; [G_2, B])$,...,$\delta([F, E]; [G_\kappa, B]) = \delta(\bigwedge_{i=1}^{m}[F_i, E_i]; [G_\kappa, B])$ then $(\bigwedge_{i=1}^{m}[F_i, E_i]; [G_1, B], [G_2, B], ..., [G_\kappa, B] : U)$ is called a* reduction multi-bijective linguistic soft decision system *of $([F, E]; [G_1, B], [G_2, B], ..., [G_\kappa, B] : U)$.*

Example 5.7. *Consider the bijective soft matrices $[F_1, E_1], [F_2, E_2]$ and $[F_3, E_3]$ given in Example 4.9. Let $B = \{t_0 = small, t_1 = medium, t_2 = large\}$ be a linguistic set and let $[G_1, B], [G_2, B]$ be bijective soft matrices given as follows*

$$[G_1, B] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \text{ and } [G_2, B] = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then, we obtain $([F_1, E_1] \wedge [F_2, E_2]; [G_1, B], [G_2, B]; U)$ is a reduction multi-bijective linguistic soft decision system of $([F, E]; [G_1, B], [G_2, B]; U)$ since $\delta([F, E]; [G_1, B]) = \delta([F_1, E_1] \wedge [F_2, E_2]; [G_1, B]) = \frac{3}{5}$ and $\delta([F, E]; [G_2, B]) = \delta([F_1, E_1] \wedge [F_2, E_2]; [G_2, B]) = 1$.

Notation: The t_εth column of $[G_\lambda, B]$ for $\lambda \in \{1, 2, ..., \kappa\}$ is denoted by $|G_\lambda, B|_{t_\varepsilon}$.

Definition 5.8. *Let $[H, C] = \bigwedge_{i=1}^{m}[F_i, E_i]$ and let $([H, C]; [G_1, B], [G_2, B], ..., [G_\kappa, B] : U)$ be a reduction multi-bijective linguistic soft decision system of*
$([F, E]; [G_1, B], [G_2, B], ..., [G_\kappa, B] : U)$.
For $I = \{1 \leq \lambda \leq \kappa : |H, C|_r \widetilde{\leq} |G_\lambda, B|_{t_\varepsilon}$ for any $0 \leq \varepsilon < \ell\}$,

$$\eta_r = \sum_{\lambda \in I} (\ell - \varepsilon) d_f(|H, C|_r)/d_f(|G_\lambda, B|_{t_\varepsilon})$$

(where ε is a variable depending on λ) is called negative decision component *induced by $[H, C]$ and denotes* the negative effect ratio of decision component *for rth column.*
For $J = \{1 \leq \lambda \leq \kappa : |H, C|_r \widetilde{\leq} |G_\lambda, B|_{t_\varepsilon}$ for any $\ell \leq \varepsilon \leq 2\ell\}$,

$$\sigma_r = \sum_{\lambda \in J} (\varepsilon - \ell) d_f(|H, C|_r)/d_f(|G_\lambda, B|_{t_\varepsilon})$$

(where ε is a variable depending on λ) is called positive decision component *induced by $[H, C]$ and denotes* the positive effect ratio of decision component *for rth column.*
Here, $t_\varepsilon \leq |B|$ and $r \leq |C|$ such that $|B|$ and $|C|$ denote the element numbers of B and C, respectively.

Definition 5.9. *Consider* $([F,E];[G_1,B],[G_2,B],...,[G_\kappa,B]:U)$*,* η_r *and* σ_r *given in Definition 5.8. Then,*

$$\mathcal{P}_r = \frac{\sigma_r-\eta_r}{2\ell(|I|_r+|J|_r)}$$

is called a decisive effect ratio *for rth column of* $[H,C]$ *where* $|I|_r$ *and* $|J|_r$ *denote the element numbers of I and J for rth column, respectively.*
Here, it means that for the linguistic decision value $x = \frac{1}{2} + \mathcal{P}_r$

- *if* $\mathcal{P}_r > 0$*, then the parameters generating rth column of* $[H,C]$ *affect the linguistic parameter corresponding to* $f_L(x) = t_{(-[|-2\ell x|])}$.
- *if* $\mathcal{P}_r = 0$*, then the parameters generating rth column of* $[H,C]$ *affect the linguistic parameter corresponding to* $f_L(x) = t_\ell$*.*
- *if* $\mathcal{P}_r < 0$*, then the parameters generating rth column of* $[H,C]$ *affect the linguistic parameter corresponding to* $f_L(x) = t_{[|2\ell x|]}$.

Example 5.10. *Consider the bijective soft matrices* $[F_1,E_1],[F_2,E_2],[F_3,E_3]$ *given in Example 4.9 and* $[G_1,B],[G_2,B]$ *given in Example 5.7. We know that* $([F_1,E_1]\wedge[F_2,E_2];[G_1,B],[G_2,B]:U)$ *is a reduction multi-bijective linguistic soft decision system of* $([F,E];[G_1,B],[G_2,B]:U)$ *by Example 5.7. Let* $[H,C]=[F_1,E_1]\wedge[F_2,E_2]$*. Then, the negative decision component is* $\eta_2 = d_f(|H,C|_2)/d_f(|G_1,B|_1) = \frac{1}{3}$ *since* $|H,C|_2\widetilde{\leq}|G_1,B|_1$ *for* $r=2$ *and the positive decision component is* $\sigma_2 = d_f(|H,C|_2)/d_f(|G_2,B|_3) = 1$ *since* $|H,C|_2\widetilde{\leq}|G_2,B|_3$ *for* $r=2$*. Therefore,*

$$\mathcal{P}_2 = \frac{\sigma_2-\eta_2}{2.1.2} = \frac{1.1-1.\frac{1}{3}}{4} = \frac{1}{6}.$$

By Definition 5.9, we can say that the parameters generating 2. column of $[H,C]$ *effect the linguistic parameter (large) corresponding to* $t_{(-[|-2.1.\frac{2}{3}|])} = t_2$ *for the linguistic decision value* $x=\frac{2}{3}$ *since* $\mathcal{P}_2 > 0$*.*

Now, we can construct a novel decision method by the following algorithm.

Algorithm

Step 1. Construct the bijective soft sets (F_i,E_i), (G_1,B), (G_2,B),...,(G_κ,B).

Step 2. Construct the bijective soft matrices $[F,E]$, $[G_1,B]$, $[G_2,B]$,...,$[G_\kappa,B]$.

Step 3. Calculate each bijective linguistic soft decision system density of $([F,E];[G_1,B]:U)$, $([F,E];[G_2,B]:U)$, ..., $([F,E];[G_\kappa,B]:U)$.

Step 4. Find a reduction multi-bijective linguistic soft decision system of $([F,E];[G_1,B],[G_2,B],...,[G_\kappa,B]:U)$, if possible.

Step 5. Obtain positive and negative decision components and calculate the decisive effect ratios and the linguistic decision values.

Step 6. Determine the linguistic parameters with linguistic decision value.

6. An Application of Multi-Bijective Linguistic Soft Decision System

Assume that a company wants to export cell phones to the countries K, L and M. For this aim, the company examines the features and customer satisfaction rates of the latest version of cell phones of seven companies which already exports cell phones to these countries. After this examination, the company wants to produce cell phones after taking these satisfaction rates into account.
$U=\{u_1,u_2,u_3,u_4,u_5,u_6,u_7\}$ denotes the alternative set which describes the latest version of cell phones that seven companies export to these countries. E denotes the parameter set, $E=E_1\times E_2\times E_3$. E_1 describes the cell phone's camera quality, E_2 describes the weight and thickness of the cell phone and E_3 describes

the cell phone's processor speed. These parameter sets are $E_1 = \{x_1 = normal, x_2 = good, x_3 = very\ good\}$, $E_2 = \{y_1 = light - thin, y_2 = heavy - thick\}$ and $E_3 = \{z_1 = speedy, z_2 = extremely\ speedy\}$, respectively. Also, the linguistic set B denotes customer satisfaction rates of the cell phones such that $B = \{t_0 = very\ low, t_1 = low, t_2 = slightly\ low, t_3 = moderate, t_4 = slightly\ high, t_5 = high, t_6 = very\ high\}$.

Step 1. The company has the following bijective soft sets

$(F_1, E_1) = \{(x_1, \{u_4, u_7\}), (x_2, \{u_3, u_6\}), (x_3, \{u_1, u_2, u_5\})\}$,

$(F_2, E_2) = \{(y_1, \{u_1, u_4, u_5, u_6\}), (y_2, \{u_2, u_3, u_7\})\}$,

$(F_3, E_3) = \{(z_1, \{u_1, u_2, u_5\}), (z_2, \{u_3, u_4, u_6, u_7\})\}$,

$(G_1, B) = \{(t_1, \{u_4\}), (t_3, \{u_3, u_7\}), (t_4, \{u_2, u_6\}), (t_5, \{u_1, u_5\})\}$ for the country K,

$(G_2, B) = \{(t_1, \{u_7\}), (t_2, \{u_3, u_4\}), (t_3, \{u_2\}), (t_4, \{u_5\}), (t_5, \{u_6\}), (t_6, \{u_1\})\}$ for the country L,

$(G_3, B) = \{(t_0, \{u_7\}), (t_3, \{u_4\}), (t_4, \{u_3, u_6\}), (t_6, \{u_1, u_2, u_5\})\}$ for the country M.

Step 2. The company construct the bijective soft matrices as follows

$$[F,E] = \begin{bmatrix} 0&0&0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&1&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0&0&0 \end{bmatrix}, [G_1,B] = \begin{bmatrix} 0&0&0&0&0&1&0 \\ 0&0&0&0&1&0&0 \\ 0&0&0&1&0&0&0 \\ 0&1&0&0&0&0&0 \\ 0&0&0&0&0&1&0 \\ 0&0&0&0&1&0&0 \\ 0&0&0&1&0&0&0 \end{bmatrix},$$

$$[G_2,B] = \begin{bmatrix} 0&0&0&0&0&0&1 \\ 0&0&0&1&0&0&0 \\ 0&0&1&0&0&0&0 \\ 0&0&1&0&0&0&0 \\ 0&0&0&0&1&0&0 \\ 0&0&0&0&0&1&0 \\ 0&1&0&0&0&0&0 \end{bmatrix} \text{ and } [G_3,B] = \begin{bmatrix} 0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&1 \\ 0&0&0&0&1&0&0 \\ 0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&1 \\ 0&0&0&0&1&0&0 \\ 1&0&0&0&0&0&0 \end{bmatrix}.$$

Step 3. The company obtains each bijective linguistic soft decision system density as

$\tau_1 = \delta([F,E];[G_1,B]) = \frac{d_f(Mr([F,E]\underline{\wedge}[G_1,B]))}{|U|} = 1$,

$\tau_2 = \delta([F,E];[G_2,B]) = \frac{d_f(Mr([F,E]\underline{\wedge}[G_2,B]))}{|U|} = \frac{5}{7}$,

$\tau_3 = \delta([F,E];[G_3,B]) = \frac{d_f(Mr([F,E]\underline{\wedge}[G_3,B]))}{|U|} = 1$.

Step 4. It is seen that $([F_1,E_1] \wedge [F_2,E_2];[G_1,B],[G_2,B],...,[G_v,B] : U)$ is a reduction multi-bijective linguistic soft decision system since

$\delta([F_1,E_1] \wedge [F_2,E_2];[G_1,B]) = 1$, $\delta([F_1,E_1] \wedge [F_2,E_2];[G_2,B]) = \frac{5}{7}$ and $\delta([F_1,E_1] \wedge [F_2,E_2];[G_3,B]) = 1$.

Step 5. It is obtained

$\mathcal{P}_1 = -\frac{5}{36}$, then $x = \frac{13}{36}$ and $t_{[|2.3.\frac{13}{36}|]} = t_2$. $\mathcal{P}_2 = -\frac{5}{18}$, then $x = \frac{2}{9}$ and $t_{[|2.3.\frac{2}{9}|]} = t_1$.

$\mathcal{P}_3 = \frac{1}{6}$, then $x = \frac{2}{3}$ and $t_{(-[|-2.3.\frac{2}{3}|])} = t_4$. $\mathcal{P}_4 = 0$, then $x = \frac{1}{2}$ and $t_\ell = t_3$.

$\mathcal{P}_5 = \frac{1}{3}$, then $x = \frac{5}{6}$ and $t_{(-[|-2.3.\frac{5}{6}|])} = t_5$. $\mathcal{P}_6 = \frac{1}{12}$, then $x = \frac{7}{12}$ and $t_{(-[|-2.3.\frac{7}{12}|])} = t_4$.

Step 6. The company can determine the features of the cell phones which will be produced in the near future using the following:

1. If the cell phone's camera quality is normal and the weight-thickness is light-thin, then slightly low customer satisfaction($\frac{13}{36}$).
2. If the cell phone's camera quality is normal and the weight-thickness is heavy-thick, then low customer satisfaction($\frac{2}{9}$).
3. If the cell phone's camera quality is good and the weight-thickness is light-thin, then slightly high customer satisfaction($\frac{2}{3}$).
4. If the cell phone's camera quality is good and the weight-thickness is heavy-thick, then moderate customer satisfaction($\frac{1}{2}$).

5. If the cell phone's camera quality is very good and the weight-thickness is light-thin, then high customer satisfaction($\frac{5}{6}$).
6. If the cell phone's camera quality is very good and the weight-thickness is heavy-thick, then slightly high customer satisfaction($\frac{7}{12}$).

For instance, ($\frac{5}{6}$) means that five of six customers can be satisfied if the company produces cell phone whose camera quality is very good and the weight-thickness is light-thin.

7. Conclusion

Decision making problems can have some alternatives that do not correspond to the parameters of the decision makers, in general. If there is exactly one parameter for every alternative then the presented methods to solve this kind of problems is not sufficient. A new solution method which employs bijective soft sets was given in [13]. However, since this method was built upon the soft set structure, the calculations take very long time. In addition, it is not efficient when the number of alternatives and parameters are too many. Relatedly, we introduced the bijective soft matrix theory in this study. Thus, we increased the computation speed by ensuring transfer to the computer via matrix. Also, we created a new decision method, which employs the bijective soft matrix, by improving the method presented in [13].

Appendix

Scilab Algorithms:
We give Scilab codes of $\wedge$-product, $\underset{\sim}{\wedge}$-product and $\widetilde{\wedge}$-product. We can use the following algorithms in terms of convenience for multi-bijective linguistic decision system involving a large number of bijective soft matrices.

for $\wedge$-product and Multi-$\wedge$-product

```
function c=andprod(a,b)
[n,m1]=size(a);
[n,m2]=size(b);
c=zeros(n,m1*m2);
 for i=1:n
  for j=1:m1
   for k=1:m2
    p=(j-1)*m2+k;
    c(i,p)=a(i,j)*b(i,k);
   end
  end
 end
endfunction

function f=andprodmulti(varargin)
r=argn(2);
X=varargin(1);
 for i=2:r
  X=andprod(X,varargin(i));
 end
  f=X
endfunction
```

for $\underset{\sim}{\wedge}$-product

```
function R=restand(f,g)
[n,m1]=size(f);
[n,m2]=size(g);
R=zeros(n,m1*m2);
 for j=1:m1
  for k=1:m2
   p=(j-1)*m2+k;
    if f(:,j)<=g(:,k)
     R(:,p)=f(:,j);
    else
     R(:,p)=zeros('c')
    end
  end
 end
endfunction
```

for $\widetilde{\wedge}$-product

```
function Rl=reland(f,g)
[n,m1]=size(f);
[n,m2]=size(g);
Rl=zeros(n,m1*m2);
 for j=1:m1
  for k=1:m2
   p=(j-1)*m2+k;
    if f(:,j)<=1-g(:,k)
     Rl(:,p)=zeros('c');
    else
     Rl(:,p)=f(:,j)
    end
  end
 end
endfunction
```

Note that a, b, f, g in this table represent bijective soft matrices.

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