

BIMINIMAL CURVES IN 2-DIMENSIONAL SPACE FORMS

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ABSTRACT. We study biminimal curves in 2-dimensional Riemannian manifolds of constant curvature.

Introduction

Elastic curves provide examples of classically known geometric variational problem. A plane curve is said to be an *elastic curve* if it is a critical point of the elastic energy, or equivalently a critical point of the total squared curvature [9].

In this paper, we study another geometric variational problem of curves in Riemannian 2-manifolds of constant curvature. The Euler-Lagrange equation studied in this paper is derived from the theory of biharmonic maps in Riemannian geometry.

A smooth map $\phi : (M, g) \rightarrow (N, h)$ between Riemannian manifolds is said to be *biharmonic* if it is a critical point of the bienergy functional:

$$E_2(\phi) = \int_M |\tau(\phi)|^2 dv_g,$$

where $\tau(\phi) = \text{tr } \nabla d\phi$ is the tension field of ϕ . Clearly, if ϕ is harmonic, *i.e.*, $\tau(\phi) = 0$, then ϕ is biharmonic. A biharmonic map is said to be *proper* if it is not harmonic.

Chen and Ishikawa [3] studied biharmonic curves and surfaces in semi-Euclidean space (see also [6]). Caddeo, Montaldo and Piu [1] studied biharmonic curves on Riemannian 2-manifolds. They showed that biharmonic curves on Riemannian 2-manifolds of non-positive curvature are geodesics. Proper biharmonic curves on the unit 2-sphere are small circles of radius $1/\sqrt{2}$.

Next, Loubeau and Montaldo introduced the notion of biminimal immersion [10].

An isometric immersion $\phi : (M, g) \rightarrow (N, h)$ is said to be *biminimal* if it is a critical point of the bienergy functional under all normal variations. Thus the biminimality is weaker than biharmonicity for isometric immersions, in general.

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In this paper we study biminimal curves on Riemannian 2-manifolds of constant curvature. We shall give natural equations for biminimal curves explicitly in terms of Jacobi's elliptic functions.

1. Preliminaries

1.1. Let (M^m, g) and (N^n, h) be Riemannian manifolds and $\phi : M \rightarrow N$ a smooth map. Then ϕ induces a vector bundle ϕ^*TN over M by

$$\phi^*TN = \bigcup_{p \in M} T_{\phi(p)}N,$$

where TN is the tangent bundle of N . The space of all smooth sections of ϕ^*TN is denoted by $\Gamma(\phi^*TN)$. A section of ϕ^*TN is called a *vector field along* ϕ .

The Levi-Civita connection ∇^h of (N, h) induces a unique connection ∇^ϕ of ϕ^*TN which satisfies the condition

$$\nabla_X^\phi(V \circ \phi) = (\nabla_{d\phi(X)}^h V) \circ \phi$$

for all $X \in \Gamma(TM)$ and $V \in \Gamma(\phi^*TN)$ (see [4, p. 4]).

The *second fundamental form* $\nabla d\phi$ is defined by

$$(\nabla d\phi)(X, Y) = \nabla_X^\phi d\phi(Y) - d\phi(\nabla_X Y), \quad X, Y \in \Gamma(TM).$$

Here ∇ is the Levi-Civita connection of (M, g) . The *tension field* $\tau(\phi)$ is a section of ϕ^*TN defined by

$$\tau(\phi) = \text{tr } \nabla d\phi.$$

A smooth map ϕ is said to be *harmonic* if its tension field vanishes. It is well known that ϕ is harmonic if and only if ϕ is a critical point of the *energy*:

$$E(\phi) = \frac{1}{2} \int |\text{d}\phi|^2 \, dv_g$$

with respect to all compactly supported variations.

Now let $\phi : M \rightarrow N$ be a harmonic map. Then the *Hessian* \mathcal{H}_ϕ of E is given by

$$\mathcal{H}_\phi(V, W) = \int h(J_\phi(V), W) \, dv_g, \quad V, W \in \Gamma(\phi^*TN).$$

Here the *Jacobi operator* \mathcal{J}_ϕ is defined by

$$J_\phi(V) := \bar{\Delta}_\phi V - \mathcal{R}_\phi(V), \quad V \in \Gamma(\phi^*TN),$$

$$\bar{\Delta}_\phi := - \sum_{i=1}^m (\nabla_{e_i}^\phi \nabla_{e_i}^\phi - \nabla_{\nabla_{e_i}^\phi e_i}^\phi), \quad \mathcal{R}_\phi(V) = \sum_{i=1}^m R^N(V, d\phi(e_i))d\phi(e_i),$$

where R^N and $\{e_i\}$ are the Riemannian curvature of N and a local orthonormal frame field of M , respectively. For general theory of harmonic maps, we refer to Urakawa's monograph [12].

Eells and Sampson [5] suggested to study *polyharmonic maps*. Polyharmonic maps of order 2 are frequently called *biharmonic maps*.

Definition 1.1. A smooth map $\phi : (M, g) \rightarrow (N, h)$ is said to be *biharmonic* if it is a critical point of the bienergy functional:

$$E_2(\phi) = \frac{1}{2} \int_M |\tau(\phi)|^2 dv_g,$$

with respect to all compactly supported variation.

The Euler-Lagrange equation of E_2 is

$$\tau_2(\phi) := -J_\phi(\tau(\phi)) = 0.$$

The section $\tau_2(\phi)$ is called the *bitension field* of ϕ . For more informations on biharmonic maps, we refer to a survey [11] by Montaldo and Oniciuc.

If ϕ is an isometric immersion, then $\tau(\phi) = m\mathbb{H}$, where \mathbb{H} is the mean curvature vector field. Hence ϕ is harmonic if and only if ϕ is a minimal immersion. As is well known, an isometric immersion $\phi : M \rightarrow N$ is minimal if and only if it is a critical point of the volume functional \mathcal{V} . The Euler-Lagrange equation of \mathcal{V} is $\mathbb{H} = 0$.

Motivated by this coincidence, the following notion was introduced by Loubeau and Montaldo:

Definition 1.2 ([10]). An isometric immersion $\phi : (M^m, g) \rightarrow (N^n, h)$ is called a *biminimal immersion* if it is a critical point of the bienergy functional E_2 with respect to all normal variation with compact support. Here, a normal variation means a variation $\{\phi_t\}$ through $\phi = \phi_0$ such that the variational vector field $V = d\phi_t/dt|_{t=0}$ is normal to M .

The Euler-Lagrange equation of this variational problem is $\tau_2(\phi)^\perp = 0$. Here $\tau_2(\phi)^\perp$ is the normal component of $\tau_2(\phi)$. Since $\tau(\phi) = m\mathbb{H}$, the Euler-Lagrange equation is given explicitly by

$$(1.1) \quad \{\bar{\Delta}_\phi \mathbb{H} - \mathcal{R}_\phi(\mathbb{H})\}^\perp = 0.$$

Obviously, every biharmonic immersion is biminimal, but the converse is not always true.

2. Biminimal curves

From now on we restrict our attention to unit speed curves in Riemannian 2-manifolds.

For a unit speed curve $\gamma(s)$ in a Riemannian 2-manifold M , its tension field is given by $\tau(\gamma) = \nabla_{\gamma'} \gamma'$. Thus the bienergy of γ is the elastic energy

$$E_2(\gamma) = \frac{1}{2} \int \kappa(s)^2 ds,$$

where $\kappa(s)$ is the signed curvature of γ .

Here we recall the following fundamental result.

Lemma 2.1 ([10]). *A unit speed curve $\gamma(s)$ in a Riemannian 2-manifold of Gaussian curvature K is biminimal if and only if its signed curvature $\kappa(s)$ satisfies:*

$$(2.1) \quad \kappa'' - \kappa^3 + \kappa K = 0.$$

Note that γ is biharmonic if and only if γ is biminimal and additionally satisfies $\kappa\kappa' = 0$. Thus non-geodesic biharmonic curves have constant curvature.

Corollary 2.1. *A non-geodesic curve in a Riemannian 2-manifold is biharmonic if and only if γ is a Riemannian circle of signed curvature κ satisfying $K = \kappa^2 > 0$. Thus proper biharmonic curves can exist only in constant positive curvature 2-manifolds.*

Remark 1. Let γ be a unit speed curve in Euclidean plane \mathbb{R}^2 . Then γ is an elastic curve if and only if its signed curvature satisfies

$$\kappa'' + \frac{1}{2}(\kappa^3 - \lambda\kappa) = 0$$

for some constant λ [9]. Thus the Euler-Lagrange equation of the biminimal curve is different from the elastic curve equation.

3. Biminimal curves on Euclidean plane

First, we investigate biminimal curves on the Euclidean plane \mathbb{R}^2 . In this case, the signed curvature $\kappa(s)$ is a solution to

$$\kappa''(s) - \kappa(s)^3 = 0.$$

Multiplying $2\kappa'(s)$ to both hand sides of this ordinary differential equation, we get

$$(\kappa')^2 = \frac{1}{2}(\kappa^4 + A)$$

for some constant A . Thus we obtain

$$\int \frac{d\kappa}{\sqrt{\kappa^2 + A}} = \pm \frac{1}{\sqrt{2}}(s - s_0).$$

The left hand side of this equation is an elliptic integral of the first kind. Hence the signed curvature $\kappa(s)$ can be represented by Jacobi's elliptic functions.

In our previous paper [8], we have solved the ordinary differential equation $\kappa'' = \kappa^3$. For our purpose, we recall the integration procedure given in [8].

Definition 3.1. For a positive constant k such that $0 < k < 1$, the Jacobi's sn-function sn of modulus k is defined by

$$\text{sn}^{-1}(x; k) = \int_0^x \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}, \quad -1 \leq x \leq 1.$$

The sn-function is defined on the interval $-K(k) \leq x \leq K(k)$, where $K(k)$ is the *complete elliptic integral of the first kind* defined by

$$K(k) := \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}.$$

The sn function is extended to the whole line \mathbb{R} as a periodic function with fundamental period $4K(k)$. The cn function is defined by

$$\text{cn}(x; k) = \sqrt{1 - \text{sn}(x; k)^2}.$$

One can check the following integral formulas.

$$(3.1) \quad \int_1^u \frac{du}{\sqrt{u^4 - 1}} = \frac{1}{\sqrt{2}} \text{cn}^{-1} \left(\frac{1}{u}; \frac{1}{\sqrt{2}} \right),$$

$$(3.2) \quad \int_1^u \frac{du}{\sqrt{u^4 + 1}} = K \left(\frac{1}{\sqrt{2}} \right) - \frac{1}{2} \text{cn}^{-1} \left(\frac{u^2 - 1}{u^2 + 1}; \frac{1}{\sqrt{2}} \right).$$

3.1. $A = 0$. A simple and particular case is $A = 0$. In this case, κ is an elementary function given explicitly by

$$(3.3) \quad \kappa(s) = \mp \frac{\sqrt{2}}{s - s_0}.$$

The plane curve determined by this signed curvature is a logarithmic spiral. This case was discussed in [10].

3.1.1. $A > 0$. In this case we express $A = a^2$ with $a > 0$. Put $\kappa = \sqrt{a}u$, then by (3.2), we have

$$\begin{aligned} \int_{\sqrt{a}}^{\kappa} \frac{d\kappa}{\sqrt{\kappa^4 + a^2}} &= \frac{1}{\sqrt{a}} \int_1^u \frac{du}{\sqrt{u^4 + 1}} \\ &= \frac{1}{\sqrt{a}} \left\{ K \left(\frac{1}{\sqrt{2}} \right) - \frac{1}{2} \text{cn}^{-1} \left(\frac{u^2 - 1}{u^2 + 1}; \frac{1}{\sqrt{2}} \right) \right\}. \end{aligned}$$

Thus we obtain

$$(3.4) \quad \kappa(s) = \pm \sqrt{a} \left(\frac{1 + \text{cn}(\nu(s); 1/\sqrt{2})}{1 - \text{cn}(\nu(s); 1/\sqrt{2})} \right)^{\frac{1}{2}},$$

where

$$\nu(s) = \mp \sqrt{2a}(s - s_0) + 2K(1/\sqrt{2}).$$

3.1.2. $A < 0$. In this case we express $A = -a^2$ with $a > 0$. Put $\kappa = \sqrt{a}u$ as before, then by (3.1) we get

$$\begin{aligned} \int_{\sqrt{a}}^{\kappa} \frac{d\kappa}{\sqrt{\kappa^4 - a^2}} &= \frac{1}{\sqrt{a}} \int_1^u \frac{du}{\sqrt{u^4 - 1}} \\ &= \frac{1}{\sqrt{a}} \left\{ \frac{1}{\sqrt{2}} \text{cn}^{-1} \left(\frac{1}{u}; \frac{1}{\sqrt{2}} \right) \right\}. \end{aligned}$$

From we get the following formula:

$$(3.5) \quad \kappa(s) = \frac{\sqrt{a}}{\operatorname{cn}\left(\sqrt{a}(s-s_0); \frac{1}{\sqrt{2}}\right)}.$$

Note that cn is an even function.

Theorem 3.1. *Let $\gamma(s)$ be a Frenet curve in Euclidean plane \mathbb{R}^2 . Then γ is biminimal if and only if it is determined by one of the following natural equations.*

(1)

$$\kappa(s) = \mp \frac{\sqrt{2}}{s-s_0}.$$

In this case γ is a logarithmic spiral.

(2)

$$\kappa(s) = \pm \sqrt{a} \left(\frac{1 + \operatorname{cn}(\nu(s); 1/\sqrt{2})}{1 - \operatorname{cn}(\nu(s); 1/\sqrt{2})} \right)^{\frac{1}{2}},$$

with $\nu(s) = \mp \sqrt{2a}(s-s_0) + 2K(1/\sqrt{2})$, or

(3)

$$\kappa(s) = \frac{\sqrt{a}}{\operatorname{cn}\left(\sqrt{a}(s-s_0); 1/\sqrt{2}\right)}.$$

4. Biminimal curves on the 2-sphere and the hyperbolic plane

In this section we study biminimal curves in space forms of curvature $c \neq 0$.

Multiplying $2\kappa'$ to the biminimal equation

$$(4.1) \quad \kappa''(s) - \kappa(s)^3 + c\kappa(s) = 0,$$

we obtain

$$(\kappa')^2 - \frac{1}{2}\kappa^4 + c\kappa^2 = d,$$

where d is a constant. From this equation, we have

$$\int \frac{d\kappa}{\sqrt{\kappa^4 - 2c\kappa^2 + 2d}} = \int \frac{ds}{\sqrt{2}} = \frac{1}{\sqrt{2}}(s-s_0).$$

The left hand side of this equation is an elliptic integral.

4.1. $c^2 - 2d > 0$. In this case, we can put $r = \sqrt{c^2 - 2d} > 0$. Then we have

$$\int \frac{d\kappa}{\sqrt{\kappa^4 - 2c\kappa + 2d}} = \int \frac{d\kappa}{\sqrt{(\kappa^2 - c + r)(\kappa^2 - c - r)}}.$$

In this case, the positivity of $(\kappa')^2$ implies

$$(4.2) \quad \kappa^2 > c + r \quad \text{or} \quad 0 < \kappa^2 < c - r.$$

We have three possibilities.

4.1.1. $c < 0$ and $d > 0$. Since $d > 0$, we can put

$$a^2 = -c + r > 0, \quad b^2 = -c - r > 0.$$

Equivalently, we have

$$a^2 + b^2 = -2c, \quad a^2b^2 = 2d.$$

Hence we get

$$\kappa^4 - 2c\kappa^2 + 2d = (\kappa^2 + a^2)(\kappa^2 + b^2).$$

Note that, in this case, the positivity condition $\kappa^2 > c + r$ is satisfied. By using the following integral formula

$$(4.3) \quad \int_0^x \frac{dx}{\sqrt{(x^2 + a^2)(x^2 + b^2)}} = \frac{1}{a} \operatorname{cn}^{-1} \left(\frac{b}{\sqrt{b^2 + x^2}}; \frac{\sqrt{a^2 - b^2}}{a} \right), \quad b \leq a,$$

we have

$$(4.4) \quad \kappa(s) = b \left\{ \frac{1 - \operatorname{cn}^2 \left(\frac{a(s-s_0)}{\sqrt{2}}; \frac{\sqrt{a^2 - b^2}}{a} \right)}{\operatorname{cn}^2 \left(\frac{a(s-s_0)}{\sqrt{2}}; \frac{\sqrt{a^2 - b^2}}{a} \right)} \right\}^{\frac{1}{2}}.$$

4.1.2. $c > 0$ and $d > 0$. In this case, we can put

$$a^2 = c + r > 0, \quad b^2 = c - r > 0.$$

Equivalently, we have

$$a^2 + b^2 = 2c, \quad a^2b^2 = 2d.$$

Hence we get

$$\kappa^4 - 2c\kappa^2 + 2d = (\kappa^2 - a^2)(\kappa^2 - b^2).$$

The positivity condition (3.4) is rewritten as

$$\kappa^2 > a^2 \text{ or } 0 < \kappa^2 < b^2.$$

Comparing this condition with the following integral formulas.

$$(4.5) \quad \int_x^\infty \frac{dx}{\sqrt{(x^2 - a^2)(x^2 - b^2)}} = \frac{1}{a} \operatorname{sn}^{-1} \left(\frac{a}{x}; \frac{b}{a} \right), \quad 0 < b < a \leq x,$$

$$(4.6) \quad \int_0^x \frac{dx}{\sqrt{(a^2 - x^2)(b^2 - x^2)}} = \frac{1}{a} \operatorname{sn}^{-1} \left(\frac{x}{b}; \frac{b}{a} \right), \quad 0 \leq |x| \leq b < a.$$

Then we obtain

$$(4.7) \quad \kappa(s) = \frac{a}{\operatorname{sn} \left(\frac{a(s-s_0)}{\sqrt{2}}; \frac{b}{a} \right)} \text{ or}$$

$$(4.8) \quad \kappa(s) = b \operatorname{sn} \left(\frac{a(s-s_0)}{\sqrt{2}}; \frac{b}{a} \right),$$

respectively.

4.1.3. $d < 0$. In this case, we can put

$$a^2 = -c + r, \quad b^2 = c + r.$$

Equivalently,

$$a^2 - b^2 = -2c, \quad a^2 b^2 = -2d.$$

Thus we have

$$\kappa^4 - 2c\kappa^2 + 2d = (\kappa^2 + a^2)(\kappa^2 - b^2).$$

By using the integral formula

$$(4.9) \quad \int_b^x \frac{dx}{\sqrt{(x^2 + a^2)(x^2 - b^2)}} = \frac{1}{\sqrt{a^2 + b^2}} \operatorname{cn}^{-1} \left(\frac{b}{x}; \frac{a}{\sqrt{a^2 + b^2}} \right), \quad b \leq x,$$

we get

$$(4.10) \quad \kappa(s) = \frac{b}{\operatorname{cn} \left(\frac{\sqrt{a^2 + b^2}}{\sqrt{2}}(s - s_0); \frac{a}{\sqrt{a^2 + b^2}} \right)}.$$

4.2. $c^2 - 2d = 0$. In this case, the ordinary equation is reduced to

$$\int \frac{d\kappa}{\kappa^2 - c} = \frac{1}{\sqrt{2}}(s - s_0).$$

Thus we obtain

$$(4.11) \quad \kappa(s) = -\sqrt{c} \tanh \left\{ \sqrt{c}(s - s_0)/\sqrt{2} \right\}, \quad c > 0,$$

$$(4.12) \quad \kappa(s) = \sqrt{-c} \tan \left\{ \sqrt{-c}(s - s_0)/\sqrt{2} \right\}, \quad c < 0.$$

4.3. $c^2 - 2d < 0$. Since $2d > c^2 > 0$, we may put $2d = \alpha^4$ ($\alpha > 0$). Then we can express c as

$$c = -\alpha^2 \cos(2\theta).$$

Because $|c/\sqrt{2d}| = |c/\alpha^2| \leq 1$. Then by using the integral formula

$$\int_0^x \frac{dx}{\sqrt{x^4 + 2\alpha^2 \cos(2\theta)x^2 + \alpha^4}} = \frac{1}{2\alpha} \operatorname{cn}^{-1} \left(\frac{\alpha^2 - x^2}{\alpha^2 + x^2}; \sin \theta \right),$$

we obtain

$$(4.13) \quad \kappa(s) = \alpha \left(\frac{1 - \operatorname{cn}(\sqrt{2}\alpha(s - s_0); \sin \theta)}{1 + \operatorname{cn}(\sqrt{2}\alpha(s - s_0); \sin \theta)} \right)^{\frac{1}{2}}.$$

Theorem 4.1. *Let $\gamma(s)$ be a unit speed curve in a Riemannian 2-manifold $M^2(c)$ of constant curvature $c \neq 0$. If γ is bimiminal and not a geodesic. Then the signed curvature κ of γ is given by (4.4), (4.7), (4.8), (4.10), (4.11), (4.12), or (4.13).*

5. Concluding remarks

Submanifolds with harmonic mean curvature ($\Delta\mathbb{H} = 0$) or normal harmonic mean curvature ($\Delta^\perp\mathbb{H} = 0$) have been studied extensively. Here Δ^\perp is the rough Laplacian of the normal bundle (and called the *normal Laplacian*). More generally, submanifolds with property $\Delta\mathbb{H} = \lambda\mathbb{H}$ or $\Delta^\perp\mathbb{H} = \lambda\mathbb{H}$ have been studied extensively by many authors (See references in [2], [7]). Analogously, we may generalize the notion of biminimal immersion to the following one:

Definition 5.1 ([10]). An isometric immersion $\phi : M \rightarrow N$ is called a λ -*biminimal immersion* if it is a critical point of the functional:

$$E_{2,\lambda}(\phi) = E_2(\phi) + \lambda E(\phi), \quad \lambda \in \mathbb{R}.$$

The Euler-Lagrange equation for λ -biminimal immersions is

$$\tau_2(\phi)^\perp = \lambda\tau(\phi).$$

More explicitly,

$$\{\bar{\Delta}_\phi\mathbb{H} - \mathcal{R}_\phi(\mathbb{H})\}^\perp = -\lambda\mathbb{H}$$

or equivalently

$$J_\phi(\mathbb{H})^\perp = -\lambda\mathbb{H}.$$

Corollary 5.1. *A non-geodesic curve γ in a Riemannian 2-manifold is λ -biminimal if and only if*

$$\kappa'' - \kappa^3 + \kappa(K - \lambda) = 0.$$

Thus, by replacing K by $c - \lambda$ in (2.1), one can obtain natural equations for λ -biminimal curves in the space form of curvature c .

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References

- [1] R. Caddeo, S. Montaldo, and P. Piu, *Biharmonic curves on a surface*, Rend. Mat. Appl. (7) **21** (2001), no. 1-4, 143–157.
- [2] B. Y. Chen, *A report on submanifolds of finite type*, Soochow J. Math. **22** (1996), no. 2, 117–337.
- [3] B. Y. Chen and S. Ishikawa, *Biharmonic surfaces in pseudo-Euclidean spaces*, Mem. Fac. Sci. Kyushu Univ. Ser. A **45** (1991), no. 2, 323–347.
- [4] J. Eells and L. Lemaire, *Selected Topics in Harmonic Maps*, CBMS Regional Conference Series in Mathematics, 50., American Mathematical Society, Providence, RI, 1983.
- [5] J. Eells and J. H. Sampson, *Variational theory in fibre bundles*, Proc. U.S.-Japan Seminar in Differential Geometry, pp. 22-3 Nippon Hyoronsha, Tokyo, 1966.
- [6] J. Inoguchi, *Biharmonic curves in Minkowski 3-space*, Int. J. Math. Math. Sci. **2003** (2003), no. 21, 1365–1368, part II, *ibid.* **2006** (2006), Article ID 92349, 4 pages.
- [7] ———, *Submanifolds with harmonic mean curvature vector field in contact 3-manifolds*, Colloq. Math. **100** (2004), no. 2, 163–179.
- [8] J. Inoguchi and J.-E. Lee, *Almost contact curves in normal almost contact 3-manifolds*, submitted.

- [9] J. Langer and D. A. Singer, *The total squared curvature of closed curves*, J. Differential Geom. **20** (1984), no. 1, 1–22.
- [10] E. Loubeau and S. Montaldo, *Biminimal immersions*, Proc. Edinb. Math. Soc. (2) **51** (2008), no. 2, 421–437.
- [11] S. Montaldo and C. Oniciuc, *A short survey on biharmonic maps between Riemannian manifolds*, Rev. Un. Mat. Argentina **47** (2006), no. 2, 1–22.
- [12] H. Urakawa, *Calculus of Variation and Harmonic Maps*, Transl. Math. Monograph. **132**, Amer. Math. Soc., Providence, 1993.

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