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BINARY EVOLUTION IN STELLAR DYNAMICS

Douglas C. Heggie

Institute of Astronomy, Madingley Road, Cambridge CB3 0HA

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SUMMARY

Numerical experiments have demonstrated something of the importance of binaries in N-body systems, and this paper aims to provide a comprehensive theoretical picture of their behaviour. It begins by testing possible ' equilibrium ' distributions for binaries against the results of computational experiments, but is mainly concerned with the dynamics of encounters between binaries and other members of the system.

Using an impulsive approximation, it is shown that pairs with low binding energies, i.e. much less than the average kinetic energy of single cluster members, tend to be disrupted by encounters. The theory for energetic pairs is complicated by the considerably richer variety of possible phenomena, including distant encounters in which changes in eccentricity much exceed changes in energy, and exchange events in which an incoming star may replace one component of the binary. Especially difficult to treat are close encounters resulting in the formation of temporarily bound triple systems; a new hypothesis on the three-body problem deals successfully with this important case. The result of all types of encounter is that energetic pairs tend to become more energetic, at an average rate which is approximately independent of their binding energy.

1. INTRODUCTION

In a certain approximation it is possible to treat the dynamics of stars in our Galaxy as collisionless (Jeans 1929; Chandrasekhar 1942) and to describe the evolution of their distribution in terms of a Vlasov equation. However, in open star clusters, in globular clusters, and in clusters of galaxies, effects arising from encounters between pairs of constituent members operate sufficiently quickly to have a pronounced or even decisive influence on the dynamical evolution. On the assumption that encounters between three or more bodies are of negligible importance, an approximation, which, in particular, leads to neglect of interactions involving binaries, several authors (Larson 1970a, b; Hénon 1972a, b; Spitzer & Hart 1971a, b) have developed rapid computational methods for studying the evolution of clusters.

An independent test of this approximation is furnished by what is still a very popular method of studying small N-body systems, namely by direct numerical integration of the equations of motion. Some of the difficulties formerly attaching to its use have now been removed, and in particular the introduction of regularization (e.g. Aarseth 1972b; Bettis & Szebehely 1972) has much facilitated study of those stages of cluster evolution in which close encounters feature. It has been found for small clusters ($N \sim 10$–50: van Albada 1968a; Hayli 1972) and for those of moderate size ($N \sim 200$–500: Aarseth 1972a, 1974) that interactions with binaries play an important part in the ejection of particles, and so in the disruption of the system. That the role of binaries is not restricted to such phenomena is revealed by the observation (van Albada; Aarseth; *op. cit.*) that the total binding energy of binaries generally increases, becoming a large fraction of the binding energy of the cluster in a time much less than that required for disintegration of the system.

Despite their implicit neglect of binary phenomena, the fast approximate methods of Larson, Hénon and Spitzer & Hart (*op. cit.*) lead to results which suggest strongly that the appearance of energetic binaries is inescapable. They show that one effect of two-body encounters is the enhancement of spatial inhomogeneity, the system consisting after a time of a tenuous halo of stars, with predominantly radial velocities, surrounding a central core, where the velocity distribution is almost isotropic and the central density steadily increases. The mass of the core decreases with time and yet, macroscopically, we witness an inward flux of binding energy, progressively more energy being shared among the fewer and fewer members of the core. It seems difficult to avoid the conclusion that these processes can only end with the appearance of an energetic binary (Aarseth 1972a).

In the face of so much evidence for important three-body activity there is a strong need to establish whether rapid methods based on two-body encounters may nevertheless yield satisfactory information on most aspects of the evolution of clusters, as averred by Hénon (1972a) and by Spitzer & Hart (1971a). Before this can be done, however, it is necessary to study in detail the dynamics of individual encounters between binaries and the single stars in a stellar system, just as the (simpler) detailed study of two-body encounters is a prerequisite for investigation of the evolution of a system by collisional relaxation. It is the principal aim of the present paper to study three-body encounters from this viewpoint, and to deduce the rate at which encounters with a binary cause changes in its properties, especially its binding energy, by specified amounts. The actual application of these basic data to the question posed above will be found elsewhere (Heggie 1975), together with a discussion of several other phenomena in the evolution of clusters for which binaries are responsible. However, in the final section of the present paper we shall illustrate the simple application of our results by considering the origin and evolution of the visual binaries which are found in such abundance in the solar neighbourhood (Brosche 1962; van de Kamp 1971): is it possible, as suggested by several authors (Gurevich & Levin 1950; van Albada 1968b; Kumar 1972), that they have formed dynamically in clusters?

Although it is clear that encounters of binaries with other stars constitute the bulk of our investigation, we begin in the following section by enquiring how much may be learnt without detailed discussion of such events; that is, we discuss as far as possible important *equilibrium* properties of the distribution of binary stars in stellar systems, using both analytic and numerical techniques. The latter vividly highlight the limitations of the equilibrium approach, and so in Section 3 we construct a framework suitable for a description of the evolution of binaries. The dynamics of encounters between single stars and binaries of low energy is treated in Section 4, and Section 5 contains the considerably more difficult theory for energetic pairs.

It may be helpful to the reader to note that many of the results to be presented in this paper, but unencumbered with mathematical details in order that their significance may be clearer, are briefly discussed elsewhere (Heggie 1974b).

However, in the present paper each section and subsection begins with a brief survey of its contents, in the interests of readers who may wish to skim over some details.

2. BINARIES WITHOUT ENCOUNTERS

The study of binaries in clusters naturally begins with an effort to see how much can be learned about them without consideration of the dynamical processes by which they form and evolve; that is, in a sense, we begin with 'equilibrium' binary dynamics. In this section we shall also see how our conclusions compare with the results of numerical experiments.

2.1 *Energetics of clusters and binaries*

We first review some basic concepts of the dynamical theory of star clusters, such as the Virial Theorem (2.1), and then state some useful elementary considerations on the likely range of energies of binaries which might exist in clusters.

It will be assumed that all mass present is in the form of bodies with dimensions very small compared with their typical separations. They are accordingly treated as point masses and, velocities being assumed to be non-relativistic, Newton's equations of motion are assumed to hold, i.e. the system is treated as a classical gravitational N-body problem. For convenience we select units in which the constant of gravitation G is 1, and all masses are of order unity. Generally we shall consider only isolated systems of negative total energy whose angular momentum is small.

If the kinetic energy of a system with N particles of mean mass $\langle m\rangle$ is written as $\frac{3}{2}N\langle\beta^{-1}\rangle$, and its gravitational potential energy as $-0{\cdot}4\,N^2\langle m\rangle^2/R_h$ approximately (*cf.* Spitzer & Hart 1971a), then by the Virial Theorem (Chandrasekhar 1942, p. 199) we have

$$3\,N\langle\beta^{-1}\rangle \simeq \frac{0{\cdot}4\,N^2\langle m\rangle^2}{R_h} \tag{2.1}$$

if the second time-derivative of the moment of inertia of the system vanishes. The quantity R_h is a measure of the length scale of the system, and the mean crossing time (*cf.* Aarseth 1972c) t_{cr}, where

$$t_{\mathrm{cr}} \simeq 0{\cdot}19\,N\langle m\rangle^{5/2}\langle\beta^{-1}\rangle^{-3/2},$$

is a convenient unit of time for some purposes. However, it will be understood that, when the evolution of a cluster is well advanced, the system may be so inhomogeneous that the values of R_h, $\langle\beta^{-1}\rangle$, t_{cr}, N and even $\langle m\rangle$ used in these expressions may be inaccurate for describing conditions in the small dense core, where binary evolution is at its most active. In particular we define a *local* quantity β in such a way that the local mean kinetic energy per particle is $\frac{3}{2}\beta^{-1}$.

If the relative binding energy of a pair of stars with masses m_1 and m_2 is x, taken to be positive for a bound pair, their separation cannot exceed $2a$, where the semi-major axis, a, of their relative orbit is

$$a = \frac{m_1 m_2}{2x} \tag{2.2}$$

(Plummer 1918, p. 22). Considering only pairs in which this is at most comparable with R_h, from (2.1) we have typically

$$\beta x \gtrsim 7\cdot 5\, N^{-1}, \tag{2.3}$$

if β and the masses take roughly average values; while x cannot exceed the binding energy of the whole system, unless the binary is primordial, whence

$$\beta x \lesssim \tfrac{3}{2} N. \tag{2.4}$$

We shall later find it meaningful to divide this range into two, applying to those binaries satisfying the inequality $\beta x \leqslant 1$ the description 'soft', while those for which $\beta x > 1$ will be called 'hard'. We shall see that hard binaries are highly resilient to encounters with other stars, while soft binaries are somewhat fragile.

2.2 *Some binary distribution functions*

First we describe situations in which pairs of stars are uncorrelated, as implied by (2.8), and then the corresponding distributions of eccentricity (2.15) and binding energy (2.16) are computed. The appropriate results are also derived for a Boltzmann distribution, for which, however, there is less justification.

We define the pair distribution function $f(1, 2) \equiv f(\mathbf{q}_1, \mathbf{v}_1, \mathbf{q}_2, \mathbf{v}_2; m_1, m_2)$ to be the number density, per unit volume of pair phase space, of pairs with components which respectively have position vector $\mathbf{q}_i$, velocity vector $\mathbf{v}_i$ and mass m_i ($i = 1, 2$). If $f(1) \equiv f(\mathbf{q}_1, \mathbf{v}_1; m_1)$ is the analogous single-particle density, we can write in general

$$f(1, 2) \equiv f(1)f(2) + g(1, 2) \tag{2.5}$$

(*cf.* Gilbert 1968, 1971), where the correlation function g is non-zero whenever the distribution of one particle depends on our knowledge of the position of the other.

Using numerical techniques, Miller (1971) has shown that g is generally non-zero, but there are situations in which it may be neglected. Thus, suppose that the force on particle 1 due to particle 2 is negligible compared with that due to all the other $(N-2)$ particles in the system. In this case, $f(1\,|\,2)$, the distribution of particle 1 given information on particle 2, evolves in almost the same way as $f(1)$. Unless these particles were initially correlated, we conclude that

$$f(1) \simeq f(1\,|\,2) \equiv f(1, 2)/f(2),$$

whence $g \simeq 0$. The aforementioned condition on the forces holds whenever

$$r \gg N^{-1/2} R_h, \tag{2.6}$$

where the mean gravitational acceleration is taken to be of order $N\langle m\rangle R_h^{-2}$, r is the separation of the two particles, and their masses are of order $\langle m\rangle$.* If the stars are members of a binary, by (2.1) and (2.2) this relation is satisfied provided that

$$x \ll 3\, N^{-1/2}\langle\beta^{-1}\rangle \tag{2.7}$$

and the eccentricity is not very great.

* Thi latter asssumption will not be stated in future when such estimates are made.

Supposing now that this condition holds, then (2.5) becomes approximately

$$f(1, 2) = f(1) f(2). \tag{2.8}$$

We may infer from this equation that the processes responsible for the relaxation of the distribution of very soft pairs are the same as those by which the single-particle distribution evolves, i.e. by the action of collective effects and two-body encounters. In particular, we do not expect that a significant role will be played by interactions with a third body in which *both* components of a binary participate strongly. This we shall confirm in Section 4.3 when we discuss encounters with soft binaries.

Arguing by analogy with the statistical theory of simple reactions in chemistry or atomic physics, several authors (Jeans 1929, p. 302; Ambartsumian 1937; Gurevich & Levin 1950; Allen 1968; Lynden-Bell 1969) have proposed the alternative pair distribution function

$$f(1, 2) = f(1) f(2) \exp\left(\frac{\beta m_1 m_2}{r}\right), \tag{2.9}$$

where the single-particle distribution is Maxwellian. Detailed and rigorous justification is lacking, and certain difficulties associated with (2.9) will appear shortly, but it reduces to (2.8) when r is sufficiently large.

When (2.7) holds, the relative velocity of the two stars forming a binary is typically negligible in comparison with the velocity, $|\dot{\mathbf{Q}}|$, of their centre mass, and so we may write $\mathbf{v}_i \simeq \dot{\mathbf{Q}}$ in $f(\mathbf{q}_i, \mathbf{v}_i; m_i)$. Likewise, if

$$x \gg 8\, N^{-1}\langle\beta^{-1}\rangle, \tag{2.10}$$

then $r \ll R_h$; and, taking R_h as typically the density scale-height, we may substitute $\mathbf{q}_i \simeq \mathbf{Q}$, where $\mathbf{Q}$ is the position vector of the centre of mass. Hence (2.8) implies approximately that

$$f(1, 2) = f(\mathbf{Q}, \dot{\mathbf{Q}}; m_1) f(\mathbf{Q}, \dot{\mathbf{Q}}; m_2)\, h(x; \beta), \tag{2.11}$$

where $h \equiv 1$. It is to be noted that, if $f(1)$ is Maxwellian in the velocities, then (2.9) takes the same form, with $h \equiv \exp(\beta x)$, whenever (2.10) holds.

We now obtain from (2.11) the distribution of $\mathbf{Q}$ and $\dot{\mathbf{Q}}$, and of six new parameters selected to describe the relative orbit of the binary. These we choose to be x and e, its eccentricity, and four angles ω, Ω, i and M_0, as in celestial mechanics (Plummer 1918, p. 65), to describe its orientation and phase. Methods for finding the distribution of x and e are given by some of the authors mentioned above in connection with (2.9), although Jeans' calculation is unreliable, and also by Heggie (1972). The result may be combined with the distributions of ω, Ω, i and M_0, which are obvious, to yield

$$f(\mathbf{Q}, \dot{\mathbf{Q}}, x, e, \omega, \Omega, i, M_0) = f(\mathbf{Q}, \dot{\mathbf{Q}}; m_1) f(\mathbf{Q}, \dot{\mathbf{Q}}; m_2)\, h(x; \beta) \times \frac{1}{4\sqrt{2}} (m_1 m_2 M_{12})^{3/2}\, e x^{-5/2} \sin i, \tag{2.12}$$

where $M_{12} \equiv m_1 + m_2$. Hence the number-density of binaries with energy x, in the case when $h \equiv \exp(\beta x)$ and the single-particle function is Maxwellian in the velocities, is found to be

$$f(x, \mathbf{Q}) = \tfrac{1}{2} n_1(\mathbf{Q})\, n_2(\mathbf{Q}) (\pi\beta)^{3/2} (m_1 m_2)^3\, x^{-5/2} \exp(\beta x), \tag{2.13}$$

where n_i ($i = 1, 2$) is the number-density of stars with mass m_i.

The case of a rotating cluster was also treated by Gurevich & Levin (1950), the Maxwellian distribution (2.9) being modified in the usual way. The resulting distributions of x, e and i (taken to be the inclination to some plane normal to the total angular momentum vector of the cluster) are modified for pairs with large angular momentum.

When we adopt a Maxwellian velocity distribution for single particles, we note from (2.12) that $\dot{\mathbf{Q}}$ obeys such a distribution. Furthermore, if the single-particle distribution implies a spatial concentration towards the centre of the cluster, then the mass-centres of binaries are still more strongly centrally concentrated, as we observe in (2.13), for example.

When equal masses are being considered, an extra factor of one-half is required in equations for the pair distribution, otherwise each pair is counted twice. Therefore for Plummer's distribution (*cf.* Lynden-Bell & Sanitt 1969) the marginal distribution of x and e is

$$f(x, e) \simeq 71\, N^{-1} \langle \beta^{-1} \rangle^{3/2}\, e x^{-5/2} \qquad (2.14)$$

approximately, where we have taken $h = 1$, corresponding to (2.8). We see that, as N increases, the number of pairs at each energy decreases, but since the inequality (2.10) becomes more generous as N increases, their total number in the range in which (2.7) and (2.10) are valid will increase. Incidentally, the *form* of equation (2.14) holds for *any* homologous series of collisionless equilibrium models.

From (2.12) we observe that the eccentricity is distributed independently of the other variables, according to

$$f(e) = 2e, \qquad (2.15)$$

a form which we shall discuss again shortly. Finally, the energy is distributed according to

$$f(x) \propto x^{-5/2} \qquad (2.16)$$

for (2.8) and

$$f(x) \propto x^{-5/2} \exp(\beta x) \qquad (2.17)$$

for (2.11). The apparent divergence in each case for $x \downarrow 0$, which is caused by the great size of the volume of phase space available to a loosely-bound binary, is quashed by observance of (2.10), which embodies the spatial inhomogeneity of the system. The expression on the right of (2.17) also diverges for $x \uparrow \infty$ because of the exponential factor, and so the validity of the distribution (2.9) is by no means unrestricted.

2.3 *Comparison with experiment*

In the present section we shall temporarily leave the theoretical discussion and turn to numerical experiments. We shall describe the limited extent to which the theory is in harmony with the results of such work, and seek from it guidance on our subsequent course. We begin by discussing the spatial distribution of binaries, and then in Figs 1–3 the distribution of their eccentricities. This section then deals with the binding energies of wide pairs and finally, with the aid of Figs 5 and 6, relates the time-dependence of the energies of close binaries with the evolution of the whole cluster.

Remarks on the numerical techniques will be given in the Appendix, and here it suffices to say merely that the immediate aim of computational work is the numerical integration of the equations of motion for small N-body systems. We shall quote results for bound systems, at most very slowly rotating, where N lies between 25 and 250, and all masses are equal except where otherwise stated.

Those binaries present and satisfying the condition

$$x \geq 3\, N^{-2/3} \langle \beta^{-1} \rangle, \tag{2.18}$$

i.e. roughly speaking those with dimensions not exceeding the mean interparticle distance, were sampled roughly once each mean crossing time; save that a sample was rejected if there was at least one binary in common with the previous accepted sample. This was done to ensure, as far as possible, independence of the samples. Samples were also rejected for the first two or three mean crossing times in cases where the initial total kinetic energy of the system was given a value not consistent with equilibrium in the virial theorem.

It was seen in the previous section that at least very soft binaries are expected to be more centrally condensed in an N-body system than the single stars, and this is found to be true of the binaries sampled in the numerical experiments discussed here. We take the ' centre ' of the cluster to be the ' potential centre ', which is defined to be the location of that star for which the potential due to all other stars, except its nearest neighbour, is largest; had the nearest neighbour been included, the star with largest potential often would have been one component of a close binary. There is also evidence that the very softest pairs are less centrally condensed than the less soft.

In the discussion of the distribution of eccentricities, the pairs will be split up into three ranges by energy. When

$$3\, N^{-2/3} \langle \beta^{-1} \rangle \leq x \ll 3\, N^{-1/2} \langle \beta^{-1} \rangle \tag{2.19}$$

we expect (2.8), and so (2.15) and (2.16), to hold. By (2.16) the distribution of binding energies is decreasing so rapidly in this range, if $N \gg 1$, that we may expect (2.15) still to hold approximately when we relax (2.19) by writing

$$3\, N^{-2/3} \langle \beta^{-1} \rangle \leq x \leq 3\, N^{-1/2} \langle \beta^{-1} \rangle,$$

which is the lowest range of energies we consider. It will be seen later that the dynamics of hard binaries is quite different from that of soft pairs, and so the next division is taken at $x = \langle \beta^{-1} \rangle$; the intermediate range accordingly being $3\, N^{-1/2} \langle \beta^{-1} \rangle \leq x \leq \langle \beta^{-1} \rangle$, and the high-energy range embracing $x \geq \langle \beta^{-1} \rangle$.

Figs 1–3 respectively show the cumulative sample distribution functions for the three ranges, the binaries being sampled from five 25-body problems, and three each with $N = 50$ and $N = 100$. The smooth curves correspond to (2.15) and, by the Kolmogorov–Smirnov test, the fit is acceptable at the 20 per cent level in each case. As mentioned, agreement was to be expected for the low energy range, but only later, when we are discussing encounters with binaries, will explanations be offered for the observed distributions of eccentricity in the other ranges.

Van Albada (1968a), analysing numerical integrations of about 30 systems, each with 10 members initially, found that the distribution of eccentricities was consistent with (2.15). He chose, for each system, the most energetic binary remaining at the end of the computation, and these would be considered hard in the present classification. Aarseth & Hills (1972), considering a system with initial conditions especially conducive to the production of many binaries, also found results in harmony with (2.15) for a total sample consisting of 19 hard binaries.

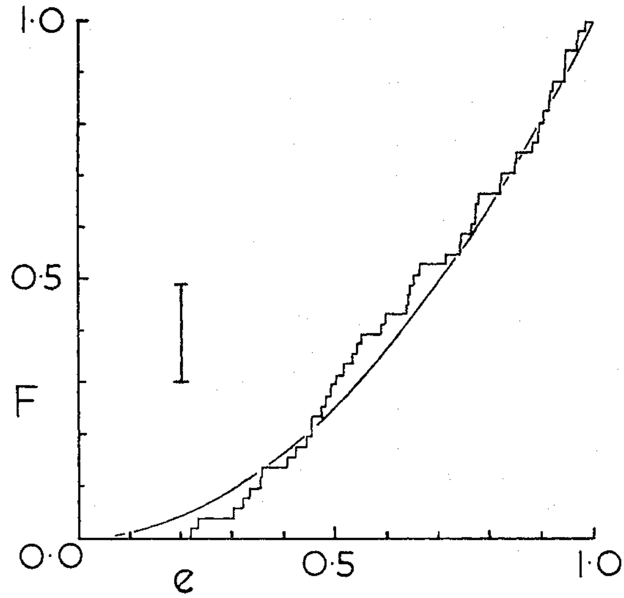

FIG. 1. *Cumulative distribution, F, of eccentricities, e, of low-energy binaries in 10 N-body systems. The bar in this figure and some others is the maximum allowable displacement, at the 5 per cent level, from the null hypothesis, according to the Kolmogorov–Smirnov test of goodness of fit. The smooth curve corresponds to* (2.15).

The distribution of eccentricities by itself says nothing about the functions on the right-hand side of (2.11) (*cf.* Ambartsumian 1937), and so we continue by discussing the distribution of binding energies, x. Since the aforementioned energy-ranges depend on N, it is no longer appropriate to accumulate the results from cases with different values of N. However, while discussing equation (2.14), we remarked that the total number of very soft pairs increases with N, and so we choose a case with large N. In fact the data we shall present were prepared from the results of a 250-body calculation computed by Dr S. J. Aarseth.

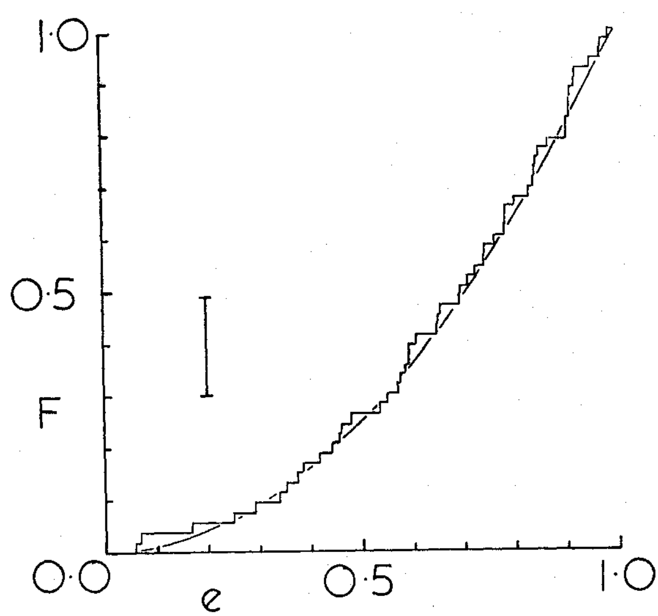

FIG. 2. *Cumulative distribution of eccentricities of medium-energy binaries in 10 N-body systems.*

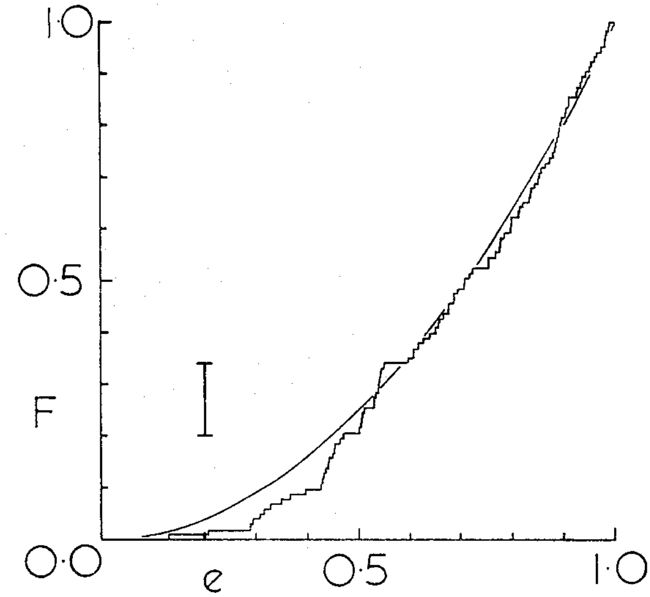

FIG. 3. *Cumulative distribution of eccentricities of high-energy (hard) binaries in* 10 *N-body systems.*

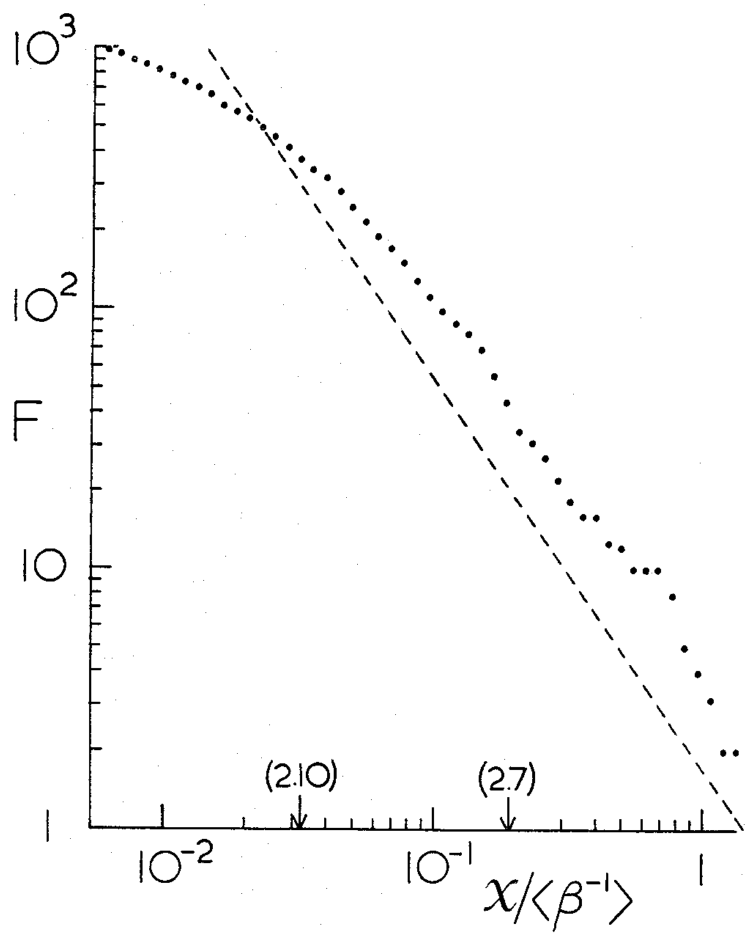

FIG. 4. *Cumulative distribution function, F, of binding energies, x, of binaries in a* 250*-body problem, plotted logarithmically. The dashed line has a slope corresponding to the relation* (2.20), *and the arrows indicate the limits imposed by* (2.7) *and* (2.10). *The data come from* 18 *samples.*

By (2.16) it is to be expected that the cumulative distribution

$$F(x) \equiv \int_x^\infty f(x')\, dx'$$

should approximately satisfy

$$F(x) \propto x^{-3/2} \tag{2.20}$$

for x satisfying (2.7) and (2.10). In Fig. 4 the experimental values of $\log F$ are plotted against $\log (x/\langle \beta^{-1} \rangle)$, the results from no less than 18 *consecutive* samples being accumulated. It is to be noted that successive samples were here retained even if there were binaries in common, but since we are now concerned mostly with very soft pairs which should have small mean lifetimes, it is not thought that correlations between successive samples are of any importance. The dashed line gives the result expected for 18 independent examples of a Plummer model, using (2.14), and a plot of any distribution of the form (2.20) would be parallel to this. It can be seen that the trend of the experimental results has approximately this same slope over at least a decade of energies. This is true even when (2.7) is not satisfied, which is an experimental indication that the conditions imposed in Section 2.2 for the validity of (2.16) were perhaps unnecessarily restrictive (*cf.* Heggie 1975).

Miller (1972) has shown that the pair energy distribution function can be represented empirically over a considerable range by the expression

$$f(x) \propto \exp\left(\frac{x}{\mu E_0}\right),$$

where μ is the reduced mass and E_0 is a constant. Although this form is different from (2.16), the logarithmic derivatives of the two functions are equal near the middle of the range, delimited by (2.7) and (2.10), over which (2.16) is valid.

None of the binaries for which data appear in Fig. 4 are hard, and we now turn to a discussion of such pairs. It was mentioned in Section 1 that the behaviour of energetic pairs is highly time-dependent, as we see for a certain case in Fig. 5: the energy of an individual pair may alter in response to an encounter with a third body, and the identity of one component may change. Further, the trend of binding energies is of great interest, for initially there are no hard pairs for two or three time units, whereafter the fraction of the energy of the cluster accounted for by the binding energy of hard pairs increases almost steadily, before long exceeding one half. The relation between binary evolution and that of the whole cluster is at least suggested by the fact that the most energetic escaper, which leaves at time $t \simeq 6$, does so when the vigorous phase of binary evolution is under way, but the connection emerges clearly from Fig. 6.

Violent relaxation is the process responsible for the early spread of binding energies in this diagram, but for most of the time it is collisional relaxation that is operative. Stars near the bottom of the diagram are loosely bound to the cluster, and the interval of time between successive visits to the centre of the cluster, which is the only site of reasonably rapid collisional relaxation, is typically quite long. Stars with considerably larger binding energies, however, are the members of the core, and relax vigorously all the time. If such a star once attains low binding energy it effectively stops relaxing, and so as time proceeds up to $t \sim 10$, we see in Fig. 6 that the number of low-energy stars comprising the 'halo' of the cluster steadily grows. By energy conservation, the binding energy of the system is mostly

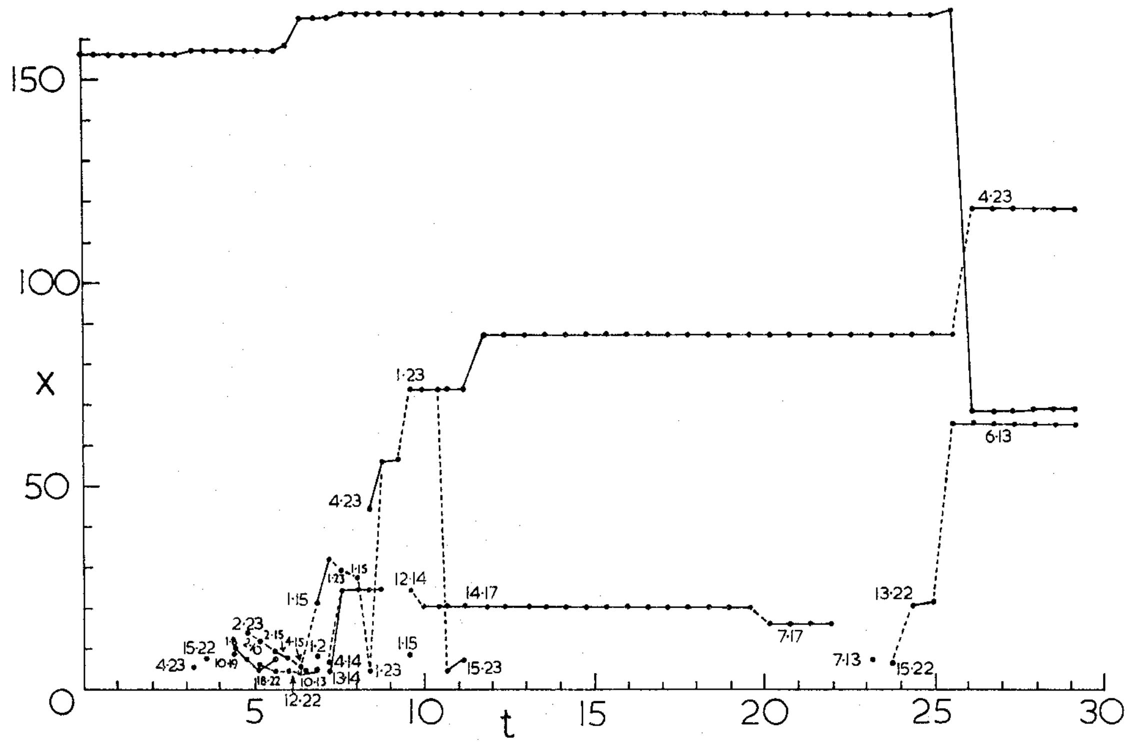

Fig. 5. *Binding energies, x, of all hard pairs in a 25-body problem, plotted against time, t. Solid lines connect the same pair, the components of each binary being indicated by two numbers, and dotted lines connect pairs linked by exchange of companion or by formation and disruption of a triple system. The uppermost line, which drops at about $t = 26$ because of the escape of the hard pair (4.23), is the total cluster energy, adjusted to exclude escaping particles, but in no case are these lines intended to show accurately the variations between sampling times. In the units of the abscissae, $t_{cr} \simeq 0{\cdot}57$; and in the units of the ordinates here and in Fig.* 6, $\langle\beta^{-1}\rangle \simeq 4{\cdot}3$.

shared amongst the steadily decreasing number of stars in the core. One may surmise (Aarseth 1972a; Hénon 1972a) that this process ends when the core has degenerated into a single binary with a binding energy which is a large fraction of that of the whole system.

These remarks successfully account in qualitative terms for the evolution of the system until about $t \sim 10$. With the arrival of a very energetic pair, collisional relaxation is not over, however, and over the next 12 or 15 time units one sees that the halo is itself behaving in much the same way as the whole cluster did previously. By $t \simeq 25$ most of the stars have very low energies indeed, the remaining binding energy having been donated in part to a new energetic pair and in part to the old binary.

The escape of the hardest binary at time $t \simeq 25$ is not typical, but in the features on which we have laid emphasis in this discussion, the case we have considered seems quite representative of all those so far studied by the author. Further confirmation of the generality of these phenomena will be found in the discussions of Aarseth (1972a, 1974), whose cases are all larger, with $N = 250$ or 500, than those considered here. Aarseth noted also that the evolution of hard pairs is substantially more rapid when not all stars have the same mass, and he stresses the fact that the components of the final most energetic pair are almost always amongst the two or three most massive members of the system. The link between the evolution of binaries and that of the core is strengthened when this fact is compared with the

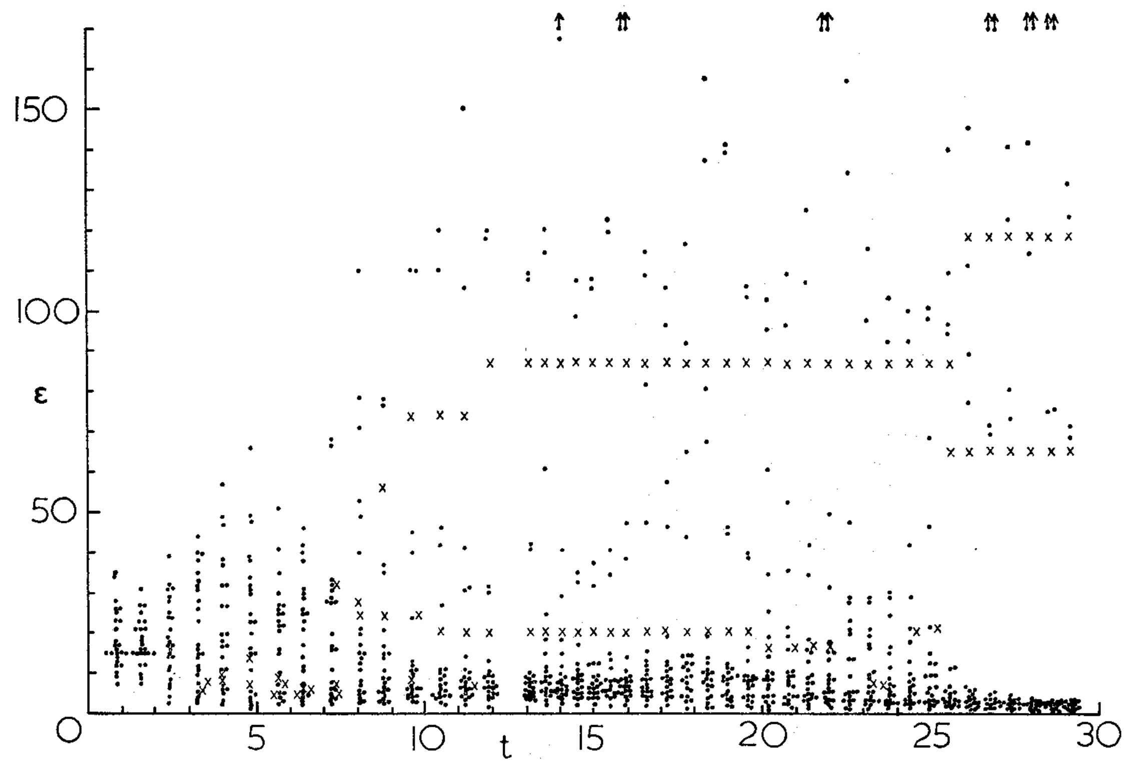

FIG. 6. *Binding energies, ϵ, of all bound stars (dots) and hard binaries (crosses) at each sample time, t, in the same 25-body problem as that displayed in Fig. 5. For a single star, $\epsilon = -m\phi - \frac{1}{2}mv^2$, where m is the mass of the particle, ϕ is the gravitational potential, and v is the velocity of the star relative to the centre of mass of the cluster. A number of points lie off the top of the plot.*

marked segregation of mass that is observed to occur (Aarseth 1966) in N-body systems. At the other end of the range in sizes, the energetics of binaries evolving in systems with a small number of members has been described by van Albada (1968a, b).

It is already clear that any attempt to find a time-independent distribution for hard pairs, such as those we discussed in the previous section, cannot succeed in describing the properties of those we find in N-body systems; for example, the distribution of binding energies certainly does not rise exponentially in accord with (2.17). Therefore, a complete understanding of the behaviour of hard pairs must proceed by way of a study of the mechanisms by which they form and evolve, and it is to this subject that we now turn.

3. GENERAL THEORY OF ENCOUNTERS

Our immediate purpose now is to provide a framework within which to discuss the effects of encounters on binary stars by dynamical interaction. From a classification of the types of encounter that may take place, we proceed to introduce some quantities useful in the discussion of the rates at which they occur. Finally, the 'detailed balance' relations impose certain general constraints.

3.1 *Classification of encounters*

In the work of Aarseth (1972a) there are indications that encounters between as many as four or five particles may be effective in the evolution of binaries, but these must be rarer than those involving three—a binary and a third body—to which we entirely confine our attention henceforth. In that case it is of importance to distinguish those encounters in which a third body passes a binary on a simple 'flyby' orbit from more complicated types. A suitable classification is provided by the following discussion.

Suppose a third body approaches a binary, whose initial binding energy is x, with a relative velocity of magnitude V_0 when their separation is still very large. During the ensuing encounter, let the binding energy of the pair change by an amount y, and, if the third body again escapes to infinity afterwards, let its final velocity be V_1. If all masses are equal to m, conservation of energy in the rest frame of the centre of mass of the three bodies implies that

$$\tfrac{1}{3}mV_1^2 = \tfrac{1}{3}mV_0^2 + y. \tag{3.1}$$

In classifying encounters, it will be instructive to adopt the nomenclature of atomic collision theory (Burgess & Percival 1968), to which the present problem bears a resemblance.

If $y > 0$, the third particle will escape, and the encounter results in a 'de-excitation' (Fig. 7). If $-\tfrac{1}{3}mV_0^2 < y < 0$, escape of the third body still occurs, and if in addition $-x < y$ then the binary survives the encounter, which is referred to as an 'excitation'. On the other hand, if $-\tfrac{1}{3}mV_0^2 < y < -x$, the binary is destroyed, the process known in atomic physics as 'ionization'. It may possibly happen in this case that the third body forms a new binary with one component of the old. Such an event, called an 'exchange' encounter, will occur in general when $y < -x$ and $y < -\tfrac{1}{3}mV_0^2$, for these conditions imply that the original binary is disrupted and yet, by (3.1), the third particle is unable to escape to infinity from the centre of mass of the old binary. If, finally, $-x < y < -\tfrac{1}{3}mV_0^2$, no particle escapes to infinity,

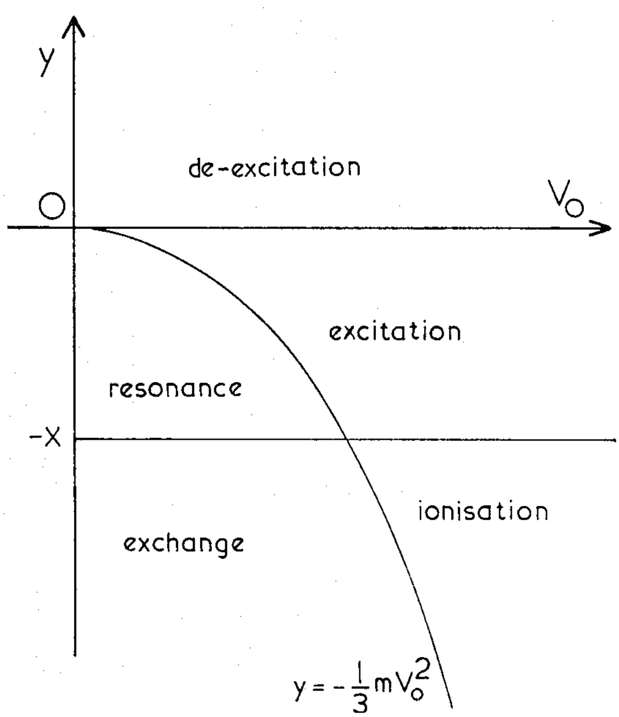

FIG. 7. *The classification of encounters between a binary and a third star. The symbols are defined in the text.*

at least not immediately, and the outcome of such a 'resonance' encounter is decided by further interactions. The precise boundaries between resonance, exchange and ionization on Fig. 7 are somewhat blurred, because the quantity y is ill-defined if the third body does not recede to infinity after the encounter and (3.1) is strictly inappropriate if the original binary is destroyed, but the concepts themselves are quite clearly distinguished. More information on this point will be found in Section 5.2.

The classification is not universal. For example, Yabushita (1971) refers under the name of 'capture' to events which are classified in our scheme as exchanges, while Harrington (1970) uses the same word 'capture' for encounters resulting in a resonant system. The classification is immediately extensible to encounters where not all masses are equal.

3.2 *Rate functions*

Now we define a set of functions, Q, which will be used to describe the rates at which various types of encounter occur. Granted certain assumptions on the distributions of single stars and of binaries, these rate functions are then expressed in terms of a 'cross-section', σ, defined in (3.11).

Let $f(x, \mathbf{R}, t)\, dx$ be the number-density, at time t and at the point with position vector $\mathbf{R}$, of binaries whose binding energies are in the small range dx about x. Then f may change with time as a result of two processes: by the spatial convection of binaries through the cluster, because of the motion of their centres of mass; and by encounters with other stars in which the binding energy, x, of a binary is altered.

The first mechanism is of limited interest as it leads to a change in the total number of binaries only if a binary escapes (*cf.* Fig. 5). With this uncommon exception its sole effect is the spatial redistribution of binaries. For example, the formation rate of new pairs is proportional to $n^3(\mathbf{R})$, where $n(\mathbf{R})$, the local number-density of single stars, increases markedly towards the centre of the cluster; therefore new binaries are initially much more strongly concentrated towards its centre than the single stars are. However, as a result of the velocity dispersion of their centres of mass, if their lifetimes are sufficiently long they soon convect into a spatial distribution broadly resembling that of the single stars, after a time of the order of one mean crossing time.

The rate at which $f(x, \mathbf{R}, t)$ evolves by the second mechanism may be expressed in terms of functions, Q, defined as follows: let $n^3(\mathbf{R})\, Q(x)\, dx$ be the rate of formation of new binaries with energies in the range $(x, x+dx)$, per unit volume; let $n(\mathbf{R}) f(x, \mathbf{R}, t)\, Q(x, -\infty)\, dx$ likewise be the rate of destruction of binaries with energies in the range $(x, x+dx)$, per unit volume; and let

$$n(\mathbf{R}) f(x, \mathbf{R}, t)\, Q(x, y)\, dx\, dy$$

be the rate, per unit volume, at which binaries with energies in the range $(x, x+dx)$ undergo encounters resulting in a change in the energy lying in the range $(y, y+dy)$. It should be noted that the three rate functions $Q(x)$, $Q(x, -\infty)$, $Q(x, y)$ thus defined have different physical dimensions. Note also that $Q(x, -\infty)$ is related to $Q(x, y)$ by

$$Q(x, -\infty) = \int_{-\infty}^{-x} Q(x, y)\, dy.$$

In these definitions we have implicitly assumed that the only encounters of importance for the evolution of f are those which occur at a distance much less than the typical length-scale of spatial inhomogeneities, otherwise the use of the local number-density, $n(\mathbf{R})$, is inappropriate. The extension to the case when a spectrum of masses are present is trivial.

In order to arrive at expressions for the rate functions Q, let us consider an encounter between a star and a binary initially separated by a great distance, when the velocities and coordinates of the binary components are $\mathbf{v}_1$, $\mathbf{q}_1$ and $\mathbf{v}_2$, $\mathbf{q}_2$, while $\mathbf{v}_3$, $\mathbf{q}_3$ are the corresponding initial conditions for the third body. Let $\mathbf{V}$ be the initial velocity of the third body relative to the centre of mass of the binary, called C; let Π be the plane through C and normal to $\mathbf{V}$; and let $\boldsymbol{\xi}$ be the position vector of that point at which the path of the third body, if unperturbed by the binary, would intersect Π. Then the rate of occurrence of encounters leading to a change in the binding energy of the pair from x to $x+y$ is obtainable from

$$Q(x,y)\,f(x,\mathbf{R},t)\,n(\mathbf{R},t) = \int d^3\mathbf{v}_1\, d^3\mathbf{v}_2\, d^3\mathbf{v}_3\, d^2\boldsymbol{\xi}\, d^3\mathbf{q}_1\, d^3\mathbf{q}_2\, f(\mathbf{q}_1,\mathbf{v}_1,\mathbf{q}_2,\mathbf{v}_2,t') \times f(\mathbf{q}_3,\mathbf{v}_3,t')\,|\mathbf{V}|\,\delta(x'-x)\,\delta(y'-y)\,\delta^3(\mathbf{R}'-\mathbf{R}), \quad (3.2)$$

where x', y' and $\mathbf{R}'$, the position vector of C, are to be regarded as functions of the variables of integration, and x, y and $\mathbf{R}$ are the values of interest isolated by the δ-functions. The functions f on the right-hand side are distribution functions, and t' is the time at which the initial conditions obtain.

In the right-hand side of (3.2) we have assumed that the third star and the binary are uncorrelated in distribution at time t', and so we certainly require that the interaction between the binary and the third body is negligible initially. Accordingly we suppose that $\langle m\rangle V^2 \gg \langle m\rangle^2/d$ and $d \gg a$ initially, where d and a are the initial separation of the third body from C, and the semi-major axis of the binary, respectively. Typically, therefore, we require

$$\left.\begin{aligned} &d \gg \langle m\rangle^2 \beta \\ \text{and}\qquad &d \gg \frac{\langle m\rangle^2}{x}. \end{aligned}\right\} \quad (3.3)$$

The vector $\mathbf{R}'$ is meant to be the position-vector of C at the time of encounter, but can be taken to be the position at t' if $d \ll R_h$, for then the distance moved by C during the encounter is much less than the length scale for spatial inhomogeneities. If we may set $\beta^{-1} \sim \langle\beta^{-1}\rangle$ in (2.1), this condition becomes of order

$$d \ll N\langle m\rangle^2 \beta. \quad (3.4)$$

We can also evaluate the distribution functions on the right-hand side of (3.2) at time t if the duration of the encounter, which is of order $d/|\mathbf{V}|$, $\sim (d/R_h)\,t_{cr}$, is negligible compared with the time scales for changes in the distributions. By (3.4) and the fact that the relaxation time for $f(\mathbf{q}_3, \mathbf{v}_3)$ is of order $(N/\log N)\,t_{cr}$, this condition holds for the single stars. The time scale for the evolution of the distribution of binaries cannot be stated so succinctly (Heggie 1975), but any circumstances in which our assumption is violated must be exceptional.

We now adopt specific forms for the distribution functions. For single particles we take a Maxwellian,

$$f(\mathbf{q}_3, \mathbf{v}_3) = n(\mathbf{q}_3)\left(\frac{m_3\beta(m_3)}{2\pi}\right)^{3/2} \exp\{-\tfrac{1}{2}\beta(m_3)\, m_3\mathbf{v}_3{}^2\}, \tag{3.5}$$

i.e. where we have allowed for a possible dependence of the mean kinetic energy on the mass, m_3, since it is known that equipartition of energies is not often obeyed in stellar systems (Spitzer & Hart 1971b; Aarseth 1973). This form is a reasonable approximation to the velocity distribution in the core of a cluster (Standish 1968), where most binary activity occurs because of the high particle density. However, it seriously overestimates the number of high-velocity single stars, which in N-body systems are very much depleted by escape processes. In future applications we shall take care to check whether this is of any importance.

We recorded in (2.11) the fact that the distribution of the centre of mass of a very soft pair is independent of that of the relative motion of its components. For a very hard pair, on the other hand, a change of order β^{-1} in the energy of the motion of C requires the approach of a third body to a distance of order $\langle m\rangle^2\,\beta$, but large changes in the binding energy, x, require much closer encounters, to a distance of order a, $\sim\langle m\rangle^2/x$. Since $\beta x \gg 1$ for a very hard pair, the distribution of the position and velocity of C relaxes much faster than, and is therefore approximately independent of, that of the internal motion. We therefore assume independence for all energies and, to be consistent with (3.5) in respect of the velocity, $\dot{\mathbf{R}}$, of the centre of mass, we take

$$f(\mathbf{q}_1, \mathbf{v}_1, \mathbf{q}_2, \mathbf{v}_2) \equiv g(x, \mathbf{R}) \exp\{-\tfrac{1}{2}\beta(M_{12})\, M_{12}\dot{\mathbf{R}}^2\}, \tag{3.6}$$

where $M_{12} \equiv m_1 + m_2$, the total mass of the binary, and g is some function. By the theory of Section 2.2 we see that we have imposed the form (2.15) on the distribution of eccentricity, and we shall offer some dynamical justification for this assumption in later sections; it is certainly consistent with the experimental data from Figs 1–3, and from Section 2.2 it is known to hold for very soft pairs. By (2.11) and (2.12) the function g can be related to $f(x, \mathbf{R})$ when we identify $\mathbf{Q}$ with $\mathbf{R}$, and (3.6) becomes

$$f(\mathbf{q}_1, \mathbf{v}_1, \mathbf{q}_2, \mathbf{v}_2) = \frac{1}{4\pi^{9/2}}\left(\frac{\beta(M_{12})}{m_1 m_2}\right)^{3/2} x^{5/2} \exp\{-\tfrac{1}{2}\beta(M_{12})\, M_{12}\dot{\mathbf{R}}^2\} f(x, \mathbf{R}). \tag{3.7}$$

The expressions (3.5) and (3.7) may be substituted into (3.2) and the variables transformed to the set $Z \equiv \beta(M_{12})\, M_{12}\dot{\mathbf{R}} + \beta(m_3)\, m_3\mathbf{v}_3$, $\mathbf{V} \equiv \mathbf{v}_3 - \dot{\mathbf{R}}$, $\mathbf{v} \equiv \mathbf{v}_1 - \mathbf{v}_2$, $\mathbf{R} \equiv (m_1\mathbf{q}_1 + m_2\mathbf{q}_2)/M_{12}$, $\mathbf{r} \equiv \mathbf{q}_1 - \mathbf{q}_2$ and $\boldsymbol{\xi}$. By (3.4) we may set $\mathbf{q}_3 \simeq \mathbf{R}$ and, after integrating out Z and $\mathbf{R}$, we obtain

$$Q(x, y) = \int \sigma|\mathbf{V}|\, f(\mathbf{V})\, d^3\mathbf{V}, \tag{3.8}$$

where, defining

$$m^*\beta^* \equiv \frac{m_3\beta(m_3)\, M_{12}\beta(M_{12})}{m_3\beta(m_3) + M_{12}\beta(M_{12})}, \tag{3.9}$$

we have written

$$f(\mathbf{V}) \equiv \left(\frac{m^*\beta^*}{2\pi}\right)^{3/2} \exp(-\tfrac{1}{2}m^*\beta^*\mathbf{V}^2) \tag{3.10}$$

and

$$\sigma = \frac{1}{\sqrt{2\pi^3}} \frac{1}{(m_1 m_2 M_{12})^{3/2}} \quad x^{5/2} \int \delta(x'-x)\, \delta(y'-y)\, d^2\boldsymbol{\xi}\, d^3\mathbf{r}\, d^3\mathbf{v}. \quad (3.11)$$

Since $f(\mathbf{V})$ is the distribution of $\mathbf{V}$, it is clear from the form of (3.8) that σ is to be interpreted as the cross-section for events of interest. An alternative form for σ can be obtained by transforming from $\mathbf{r}$ and $\mathbf{v}$ to x, e, ω, Ω, i and M_0, the Jacobian being obtained by comparing (2.11) and (2.12). The result is

$$\sigma = \frac{1}{8\pi^3} \int \delta(y'-y)\, e \sin i\, d^2\boldsymbol{\xi}\, de\, d\omega\, d\Omega\, di\, dM_0. \quad (3.12)$$

The conditions (3.3) and (3.4) are compatible only for binaries such that $\beta x \gg N^{-1}$, but by (2.3) binaries not satisfying this condition are of little interest. Also, since $|\boldsymbol{\xi}| \leqslant d$, (3.11) and (3.12) must be evaluated under the restriction $|\boldsymbol{\xi}| \ll N\langle m\rangle^2 \beta$. In these expressions for σ, y' must be entered as a function of the variables of integration, $\mathbf{V}$ and d, but the value of d used may be arbitrarily large, notwithstanding (3.4), since these expressions give the cross-sections for a binary exposed to a spatially uniform field of particles.

Even the condition $d \gg \langle m\rangle^2 \beta$ may be relaxed if we modify (3.11). If d_1 is a new value still obeying $d_1 \gg \langle m\rangle^2 x^{-1}$, then the relative motion of the third body and C is approximately Keplerian while their separation still exceeds the initial value d_1. If the origin of $\boldsymbol{\xi}$ is C, if $\boldsymbol{\xi}_1$ is the corresponding vector for d_1, and if $\mathbf{V}_1$ is the corresponding relative velocity, by conservation of energy and of angular momentum we may write $|\mathbf{V}_1|\ |\boldsymbol{\xi}_1| = |\mathbf{V}|\ |\boldsymbol{\xi}|$ and

$$\tfrac{1}{2}\mathbf{V}_1{}^2 - \frac{M_{123}}{d_1} = \tfrac{1}{2}\mathbf{V}^2, \quad (3.13)$$

where $M_{123} \equiv m_1 + m_2 + m_3$. Hence (3.11) can be written

$$\sigma = \frac{1}{\sqrt{2\pi^3}} \frac{1}{(m_1 m_2 M_{12})^{3/2}}\, x^{5/2} \int \left(1 + \frac{2M_{123}}{d_1 \mathbf{V}^2}\right) \delta(x'-x) \times \delta(y'-y)\, d^2\boldsymbol{\xi}_1\, d^3\mathbf{r}\, d^3\mathbf{v}, \quad (3.14)$$

and (3.12) may be altered similarly. The extra factor in (3.14) accounts for the gravitational ‘ focusing ’ of a beam of third particles, but this modification is useful only for hard binaries.

3.3 *Detailed balance relations*

In order to derive these useful constraints, (3.18) and (3.19), on the rate functions, we briefly consider the Boltzmann three-body distribution

$$f(\mathbf{q}_1, \mathbf{v}_1, \mathbf{q}_2, \mathbf{v}_2, \mathbf{q}_3, \mathbf{v}_3) = \exp(-\beta \mathscr{H}_3), \quad (3.15)$$

where $\mathscr{H}_3$ is the three-body Hamiltonian. This is a solution of the three-body Liouville equation, and since $\mathscr{H}_3$ is quadratic in the momenta, each encounter process is exactly balanced by its inverse. Hence we can write

$$n(\mathbf{R}) f(x, \mathbf{R})\, Q(x, y) = n(\mathbf{R}) f(x+y, \mathbf{R})\, Q(x+y, -y), \quad (3.16)$$

where $n(\mathbf{R})$ and $f(x, \mathbf{R})$ are the single-particle and pair distributions corresponding to (3.15) when the interaction between the binary and the third body is negligible, i.e.

$$f(\mathbf{q}_1, \mathbf{v}_1, \mathbf{q}_2, \mathbf{v}_2, \mathbf{q}_3, \mathbf{v}_3) \simeq \exp(-\beta \mathscr{H}_2) \exp(-\beta \mathscr{H}_1), \quad (3.17)$$

where $\mathscr{H}_1$ and $\mathscr{H}_2$ are, respectively, the one- and two-body Hamiltonians. Using (3.7) to obtain $f(x, \mathbf{R})$ we find that (3.16) implies that

$$x^{-5/2}Q(x, y) = (x+y)^{-5/2} \exp(\beta y)\, Q(x+y, -y). \tag{3.18}$$

Now the binary and single-particle distributions implicit in the right-hand side of (3.17) are proportional to those appearing in (3.7) and (3.5), respectively, if $\beta(m)$ does not vary with m. Equation (3.8) is independent of the factor of proportionality, and so the rate functions Q are the same for the two sets of distributions. In particular, the 'detailed balance relation' (*cf.* Fowler 1936) numbered (3.18) holds also for the rate functions when the distributions of single particles and binaries are given by (3.5) and (3.7) respectively.

There is a similar relation between the rates of formation and destruction, namely

$$Q(x) = Q(x, -\infty)\, \tfrac{1}{2}(\pi\beta)^{3/2}(m_1 m_2)^3\, x^{-5/2} \exp(\beta x), \tag{3.19}$$

and the theory of detailed balancing can even be refined to yield relations for the cross-sections, σ, corresponding to inverse processes. However, we shall use only (3.18) and (3.19), normally for the purpose of checking approximate analytic expressions for the rate functions.

4. SOFT BINARIES

Having constructed apparatus for the purpose in Section 3, we devote this section and the next to the calculation of the rate functions, Q. We begin with soft pairs for, although they are energetically much less significant than hard pairs, the discussion is easier, and their theory is a helpful preliminary to some aspects of the dynamics of hard pairs. Considering mainly the limit $\beta x \ll 1$, we progress from close, vigorous encounters to the milder but more numerous distant encounters, and, after discussing some implications of our results, compare them with numerical evidence.

4.1 *Close encounters*

We first consider encounters such that the third particle approaches one component of the binary to a distance much less than the initial semi-major axis of the binary, and such that the motion of the third star relative to this component is approximately unperturbed by the other component during the most important part of the encounter. Furthermore, the change in binding energy of the binary during the phases of the encounter both well before and well after the close approach are relatively negligible (Heggie 1972). Our expressions for the change in binding energy, for the cross-section, and for the rate functions, are given by (4.7), (4.10–11) and (4.12–14) respectively.

Suppose that the second component is the one which encounters the third body, and define $\mathbf{R} \equiv \mathbf{q}_3 - \mathbf{q}_2$, $\mathbf{r} \equiv \mathbf{q}_1 - \mathbf{q}_2$, where the vectors $\mathbf{q}_i$ ($i = 1, 2, 3$) are those defined in Section 3.2, except that they are not confined to the initial conditions. Note that the definition of $\mathbf{R}$ is different here. During the close approach, when $|\mathbf{R}| \ll |\mathbf{r}|$, the equations of motion may be written in the approximate forms

$$\ddot{\mathbf{r}} = -M_{12}\frac{\mathbf{r}}{|\mathbf{r}|^3} - m_3\frac{\mathbf{R}}{|\mathbf{R}|^3} \tag{4.1}$$

and

$$\ddot{\mathbf{R}} = -M_{23}\frac{\mathbf{R}}{|\mathbf{R}|^3}, \tag{4.2}$$

where m_i ($i = 1, 2, 3$) are the three masses and $M_{ij} \equiv m_i + m_j$. The first term on the right of (4.1) is negligible compared with the second, if the masses are comparable, whence

$$\dot{\mathbf{r}} \simeq \frac{m_3}{M_{23}}(\dot{\mathbf{R}} - \dot{\mathbf{R}}_0) + \dot{\mathbf{r}}_0 \tag{4.3}$$

if we write $\dot{\mathbf{R}}_0$, $\dot{\mathbf{r}}_0$ for the 'initial' values of $d\mathbf{R}/dt$, $d\mathbf{r}/dt$, respectively, i.e. those at the beginning of the close approach.

Defining the binding energy as usual by

$$x = \frac{m_1 m_2}{r} - \frac{1}{2}\frac{m_1 m_2}{M_{12}}\dot{\mathbf{r}}^2$$

we find from (4.1), (4.2) and (4.3) that it changes during the encounter by an amount

$$y = -\frac{m_1 m_2 m_3}{M_{12}M_{23}}\left\{\dot{\mathbf{r}}_0 + \frac{m_3}{2M_{23}}(\dot{\mathbf{R}}_1 - \dot{\mathbf{R}}_0)\right\}.(\dot{\mathbf{R}}_1 - \dot{\mathbf{R}}_0), \tag{4.4}$$

where $\dot{\mathbf{R}}_1$ is the value of $\dot{\mathbf{R}}$ at the end of the close approach. We may obtain $(\dot{\mathbf{R}}_1 - \dot{\mathbf{R}}_0)$ from (4.2); and, writing $V \equiv |\mathbf{V}|$, where $\mathbf{V}$ is the vector introduced in Section 3.2, we note that $|\dot{\mathbf{R}}_0| \simeq V$, since these vectors differ by an amount of order $|\dot{\mathbf{r}}_0|$, and $|\dot{\mathbf{r}}_0| \ll |\dot{\mathbf{R}}_0|$ typically for a soft pair. Using results for the two-body problem (*cf.* Chandrasekhar 1942, Appendix I) we find that

$$\dot{\mathbf{R}}_1 - \dot{\mathbf{R}}_0 = -2V\mathbf{k}\cos\phi, \tag{4.5}$$

where $\cos\phi$ is defined in terms of the initial impact parameter, p, of the third body relative to the second component of the binary by

$$\cos\phi = \left(1 + \frac{p^2V^4}{M_{23}^2}\right)^{-1/2}, \tag{4.6}$$

and $\mathbf{k}$ is a unit vector directed along the major axis of the hyperbolic orbit described by the vector $\mathbf{R}$, from the centre of attraction towards pericentre. Writing $\mathbf{v}$ for $\dot{\mathbf{r}}_0$, we find from (4.4) and (4.5) that

$$y = -\frac{2m_1 m_2 m_3}{M_{12}M_{23}}V\cos\phi\left(-\mathbf{v}.\mathbf{k} + \frac{m_3}{M_{23}}V\cos\phi\right). \tag{4.7}$$

We note in passing that this formula is expected to be valid provided that $p \ll a$, where a is the initial semi-major axis of the binary, otherwise (4.1) and (4.2) cease to be valid approximations. Since $|y|$ generally increases as p decreases, we shall suppose (4.7) to be valid provided that $|y|$ much exceeds the value it has when $p \sim a$, i.e. provided that

$$|y| \gg (\beta^* x)^{1/2}\, x \tag{4.8}$$

in order of magnitude.

We have now to obtain the cross-section using (3.11). Selecting spherical polar co-ordinates for $\mathbf{v}$ with $\mathbf{k}$ as polar axis we readily find that

$$\sigma = \frac{1}{\sqrt{2}\pi^2}\frac{M_{12}^{1/2}M_{23}}{(m_1m_2)^{7/2}}\frac{x^{5/2}}{m_3}\frac{}{V}\int \sec\phi\, d^3\mathbf{r}\, d^2\boldsymbol{\xi}$$

over the range

$$\left(y+\frac{2m_1m_2m_3^2}{M_{12}M_{23}^2}V^2\cos^2\phi\right)^2 \leqslant \frac{8m_1m_2m_3^2}{M_{12}M_{23}^2}V^2\cos^2\phi\left(\frac{m_1m_2}{r}-x\right),$$

where $r \equiv |\mathbf{r}|$. Selecting polar co-ordinates for $\mathbf{r}$ and $\boldsymbol{\xi}$, and noting that $p \equiv |\boldsymbol{\xi}|$ if the origin of $\boldsymbol{\xi}$ is suitably chosen, we obtain

$$\sigma = \frac{4\sqrt{2}}{3}\frac{M_{12}^{1/2}M_{23}}{(m_1m_2)^{1/2}m_3}\frac{x^{5/2}}{V}\int \sec\phi\left\{\frac{M_{12}M_{23}^2}{8m_1m_2m_3^2}\left(\frac{y}{V}\sec\phi+\frac{2m_1m_2m_3^2}{M_{12}M_{23}^2}\times V\cos\phi\right)^2+x\right\}^{-3} p\,dp.$$

Using (4.6) we may change the variable to $u \equiv \cos\phi$, and in case

$$|y| \ll \langle m\rangle V^2 \tag{4.9}$$

it is approximately valid to take the range of integration to be $0<u<\infty$. The result is then found to be

$$\sigma = \begin{cases} 4\pi\dfrac{m_1m_2m_3^2}{M_{12}}\dfrac{1}{V^2y^2}\left\{1-\dfrac{4}{3}\dfrac{x}{y}\right\} & (y<0) \quad (4.10) \\ 4\pi\dfrac{m_1m_2m_3^2}{M_{12}}\dfrac{1}{V^2y^2}\left\{\dfrac{4}{3}\dfrac{x}{y}+\dfrac{7}{3}\right\}\left(\dfrac{x}{x+y}\right)^{5/2} & (y>0) \quad (4.11) \end{cases}$$

after some labour, although an extra factor of 2 has been inserted because *either* component of the binary may act as a target for the third body.

The computation of Q, the rate function for this process, follows swiftly from (3.8), and leads to the result

$$Q(x,y) = 4\sqrt{2\pi}\frac{m_1m_2m_3^2m^{*1/2}}{M_{12}}\beta^{*1/2}y^{-2} \times \begin{cases} \left(1-\dfrac{4}{3}\dfrac{x}{y}\right) & \text{if } -1 \ll \beta^*y \ll -(\beta^*x)^{3/2} \\ \left(\dfrac{4}{3}\dfrac{x}{y}+\dfrac{7}{3}\right)\left(\dfrac{x}{x+y}\right)^{5/2} & \text{if } (\beta^*x)^{3/2} \ll \beta^*y \ll 1. \end{cases} \tag{4.12}$$

The conditions restricting the validity of these forms have been obtained from (4.8) and (4.9). It should be noted that the integration in (3.8) should be restricted in this case to exclude values of V sufficiently small that the third body becomes bound to the binary on completion of the encounter. However, by (4.9) this correction is negligible. It will be recalled from the discussion of Section 3.2 that the integration should also be restricted to velocities not much exceeding the local mean velocity of stars in the cluster, which is about $(\beta^*\langle m^*\rangle)^{-1/2}$. However, it is easy to see that the contribution to Q from the range $V \gg (\beta^*\langle m^*\rangle)^{-1/2}$ is relatively unimportant, and so these rate functions should be relatively accurate even if the high-energy tail of the Maxwellian is depleted by the escape of such fast particles. From (4.7) and the fact that $v \ll V$ in general, we see that $|y| \lesssim \langle m\rangle V^2$, and so the rates must drop off very quickly with increasing $|y|$ for energy changes numerically much larger than those covered by (4.12). Finally, the rates (4.12) obey the detailed balance relation (3.18) approximately.

We obtain the destruction rate by integrating the first of equations (4.12) over $y < -x$, and find that

$$Q(x, -\infty) = \frac{20\sqrt{2\pi}}{3} \frac{m_1 m_2 m_3{}^2 m^{*1/2}}{M_{12}} \beta^{*1/2} x^{-1} \qquad (4.13)$$

approximately. The formation rate may be computed by a calculation analogous to that by which (4.12) was obtained, except that the components are taken to be unbound initially. We obtain

$$Q(x) \simeq \frac{10\sqrt{2}}{3} \pi^2 (m_1 m_2)^4 m_3{}^{5/2} M_{12}{}^{1/2} \frac{(\beta_1\beta_2)^{3/2} \beta_3{}^{1/2}}{\left(\sum_1^3 m_i \beta_i\right)^{1/2} \sum_1^2 m_j \beta_j} x^{-7/2}, \qquad (4.14)$$

where $\beta_i \equiv \beta(m_i)$ for $i = 1, 2, 3$. In the case of equipartition, when β is independent of m, the detailed balance relation (3.19) is satisfied approximately by (4.13) and (4.14), since for the present we have $\beta x \ll 1$.

It is possible to calculate the destruction cross-section given the initial eccentricity, e_0, as well as the initial binding energy. The cross-section, and therefore the rate of destruction, turns out to be independent of e_0 except for very large e_0, and so a detailed balance argument leads us to conclude that new soft binaries already form in N-body systems with eccentricities nearly distributed according to (2.15). This observation helps to account for the success of this distribution as we noted in our discussion of Figs 1 and 2.

4.2 *Wide encounters*

Our attention is now directed to encounters such that the third body does not approach either component particularly closely. The change in binding energy is given by (4.19), and the cross-section and the rate function are given by (4.23), (4.24) and (4.26), respectively.

From (4.5) we observe that, if $\cos\phi$ is sufficiently small, then the third body is almost undeflected even by the closer of the components, and the sort of order-of-magnitude theory which led to the result (4.8) demonstrates that this is so when $p \gg (\beta^* x)\, a$, typically. Therefore, when this condition is satisfied we may assume that the vector $\mathbf{R}$, now defined to be the position vector of the third body with respect to the centre of mass of the binary, is approximately given by

$$\mathbf{R} = \mathbf{R}_0 + \dot{\mathbf{R}}_0 t, \qquad (4.15)$$

where $\mathbf{R}_0$ and $\dot{\mathbf{R}}_0$ are the initial values of $\mathbf{R}$ and $\dot{\mathbf{R}}$.

In the new notation, although $\mathbf{r}$ still has the meaning previously assigned, we have the equation of motion

$$\ddot{\mathbf{r}} = -M_{12} \frac{\mathbf{r}}{|\mathbf{r}|^3} + m_3 \frac{\mathbf{R} - (m_2/M_{12})\,\mathbf{r}}{|\mathbf{R} - (m_2/M_{12})\,\mathbf{r}|^3} - m_3 \frac{\mathbf{R} + (m_1/M_{12})\,\mathbf{r}}{|\mathbf{R} + (m_1/M_{12})\,\mathbf{r}|^3}, \qquad (4.16)$$

whence the change in binding energy is

$$y = -\frac{m_1 m_2 m_3}{M_{12}} \int \dot{\mathbf{r}} . \left\{ \frac{\mathbf{R} - (m_2/M_{12})\,\mathbf{r}}{|\mathbf{R} - (m_2/M_{12})\,\mathbf{r}|^3} - \frac{\mathbf{R} + (m_1/M_{12})\,\mathbf{r}}{|\mathbf{R} + (m_1/M_{12})\,\mathbf{r}|^3} \right\} dt. \qquad (4.17)$$

Now the duration of the encounter is effectively p/V, where $V \equiv |\dot{\mathbf{R}}_0|$ and p is here defined as the initial impact parameter relative to the centre of mass of the binary, and changes in $\mathbf{r}$ and $\dot{\mathbf{r}}$ during the encounter are negligible only if this is much less than the period of the binary. This condition is

$$p^2 \ll V^2 \left(\frac{m_1 m_2}{x}\right)^3 M_{12}^{-1} \tag{4.18}$$

in order of magnitude, and in that case (4.17) is approximately

$$y = -\frac{2m_1m_2m_3}{M_{12}V}\,\dot{\mathbf{r}}_0 \cdot \left\{\frac{\hat{\mathbf{R}}_0 - (m_2/M_{12})\,\hat{\mathbf{r}}_0}{|\hat{\mathbf{R}}_0 - (m_2/M_{12})\,\hat{\mathbf{r}}_0|^2} - \frac{\hat{\mathbf{R}}_0 + (m_1/M_{12})\,\hat{\mathbf{r}}_0}{|\hat{\mathbf{R}}_0 + (m_1/M_{12})\,\hat{\mathbf{r}}_0|^2}\right\}, \tag{4.19}$$

where $\mathbf{r}_0$ and $\dot{\mathbf{r}}_0$ are the initial values, a hat denotes the projection normal to $\dot{\mathbf{R}}_0$, and we have used (4.15). Note that $\hat{\mathbf{R}}_0$ is just the vector $\boldsymbol{\xi}$ of Section 3.2, with its origin located at the centre of mass of the binary, and that $p \equiv |\hat{\mathbf{R}}_0|$. Inserting typical values into (4.19), we can rewrite (4.18) as

$$|y| \gg (m_1m_2)^{-3/2}\, m_3 M_{12}^{1/2} x^{5/2} V^{-3} \tag{4.20}$$

apart from numerical factors, or, replacing V by its root mean square value under the distribution (3.10), as

$$|y| \gg \left(\frac{m^*}{m_1m_2}\right)^{3/2} m_3 M_{12}^{1/2} (\beta^* x)^{3/2}\, x. \tag{4.21}$$

Writing $\boldsymbol{\rho}_1 \equiv \hat{\mathbf{R}}_0 - (m_2/M_{12})\,\hat{\mathbf{r}}_0$, $\boldsymbol{\rho}_2 \equiv \hat{\mathbf{R}}_0 + (m_1/M_{12})\,\hat{\mathbf{r}}_0$, $\mathbf{r} \equiv \mathbf{r}_0$ and $\mathbf{v} \equiv \dot{\mathbf{r}}_0$, we proceed to the evaluation of σ using (3.11), and first select spherical polar co-ordinates for $\mathbf{v}$ with $(\rho_1^{-2}\boldsymbol{\rho}_1 - \rho_2^{-2}\boldsymbol{\rho}_2)$ as polar axis. Thus

$$\sigma = 2^{-1/2}\pi^{-2}(m_1m_2)^{-5/2}\, m_3^{-1} M_{12}^{-1/2} V x^{5/2} \int v\,\delta(x'-x)\,\rho_1\rho_2\hat{\mathbf{r}}^{-1}\, dv\, d^2\boldsymbol{\xi}\, d^3\mathbf{r}$$

over $|y| \leqslant 2m_1m_2m_3 v\hat{r}(M_{12}V\rho_1\rho_2)^{-1}$. We choose cartesian co-ordinates for $\boldsymbol{\xi} \equiv (\xi_1, \xi_2)$ in such a way that the position vectors of the components of the binary, projected on to the plane on which $\boldsymbol{\xi}$ is defined, are $\pm(0, \frac{1}{2}\hat{r})$. Then we define variables $(\zeta_1, \zeta_2) \equiv \boldsymbol{\zeta}$ by the transformation $\zeta_1 \equiv \xi_1^2 - \xi_2^2 + \frac{1}{4}\hat{r}^2$, $\zeta_2 \equiv 2\xi_1\xi_2$, whose significance is best appreciated by writing it in complex notation. Noting that the transformation is 2–1, we next choose plane polar coordinates for $\boldsymbol{\zeta}$ and find that

$$\sigma = 2^{1/2}\pi^{-2}(m_1m_2)^{-5/2}\, m_3^{-1} M_{12}^{-1/2} V x^{5/2}$$
$$\times \int v\,\delta(x'-x)\,\zeta^2\hat{r}^{-1}(\zeta + \tfrac{1}{4}\hat{r}^2)^{-1}\, K(\hat{r}^2\zeta(\zeta+\tfrac{1}{4}\hat{r}^2)^{-2})\, dv\, d\zeta\, d^3\mathbf{r},$$

where K is the usual complete elliptic integral of the first kind, and the integration is restricted to

$$\zeta \leqslant 2m_1m_2m_3 v\hat{r}(M_{12}V|\mathbf{y}|)^{-1},$$

$\equiv \zeta_0$, say. Using a known transformation (Abramowitz & Stegun 1965, equation 17.3.29) and applying a linear transformation to ζ, we may recast the result in the form

$$\sigma = 2^{-7/2}\pi^{-2}(m_1m_2)^{-5/2}\, m_3^{-1} M_{12}^{-1/2} V x^{5/2} \int v\,\delta(x'-x)\,\hat{r}^3\, dv\, d^3\mathbf{r} f\left(\frac{4\zeta_0}{\hat{r}^2}\right),$$

where

$$f(u) \equiv \begin{cases} \displaystyle\int_0^u t^2 K(t^2)\, dt & (u \leqslant 1) \\ \displaystyle f(1) + \int_1^u tK(t^{-2})\, dt & (u > 1). \end{cases}$$

To proceed further we shall resort to approximation of the integrand. Numerical evaluation of f is facilitated by using a formula for the derivative of E (Whittaker & Watson 1965, p. 521), where E is the complete elliptic integral of the second kind, to eliminate K, for E is non-singular. Thus it is found that the formulae

$$f(u) \simeq \begin{cases} \dfrac{\pi}{6} u^3(1+0 \cdot 3\, u^2) & (u < 1 \cdot 2) \\[2ex] \dfrac{\pi}{4}(u^2+0 \cdot 25) & (u \geqslant 1 \cdot 2) \end{cases} \qquad (4.22)$$

are in error by at most about 7 per cent and, furthermore, they exhibit the correct asymptotic behaviour for $u \downarrow 0$ and $u \uparrow \infty$. Taking spherical polar co-ordinates (r, θ, ϕ) for $\mathbf{r}$ with $\mathbf{V}$ as polar axis, we have $\hat{r} = r \sin \theta$ and find that

$$\sigma = 2^{-5/2}\pi^{-1}(m_1 m_2)^{-5/2}\, m_3{}^{-1} M_{12}{}^{-1/2} x^{5/2} V \int v\, \delta(x'-x)\, r^5 g(u_0)\, dv\, dr,$$

where

$$g(u) \equiv \int_0^{\pi} \sin^4\theta\, f\left(\frac{u}{\sin\theta}\right) d\theta$$

and

$$u_0 \equiv 8 m_1 m_2 m_3 v (M_{12} V |y|\, r)^{-1}.$$

By (4.22) we find that g itself may be approximated by the formulae

$$g(u) \equiv \begin{cases} \dfrac{\pi}{3} u^3(1+0 \cdot 18\, u^2) & (u < 1 \cdot 1) \\[2ex] \dfrac{\pi^2}{8}(u^2+0 \cdot 19), & (u \geqslant 1 \cdot 1) \end{cases}$$

where the total error is at most about 15 per cent and the asymptotic behaviour of g is preserved correctly.

The remaining integrations can be performed if we set $r \equiv (m_1 m_2/x) \sin^2 \phi$, and we obtain the asymptotic results

$$\sigma \simeq \begin{cases} \dfrac{16}{3}\pi \dfrac{m_1 m_2 m_3{}^2}{M_{12}} \dfrac{x}{V^2 |y|^3} & (|y| \uparrow \infty) \qquad (4.23) \\[2ex] \dfrac{\pi}{3\sqrt{2}} \dfrac{(m_1 m_2)^{3/2}\, m_3}{M_{12}{}^{1/2}} \dfrac{1}{x^{1/2} V y^2} & (|y| \downarrow 0). \qquad (4.24) \end{cases}$$

In general these results lie above the correct values, but if we always use the smaller of these two equations the error is at most a factor of 2, and in the asymptotic limits it is naturally much less. The result (4.23) is not true for arbitrarily large values of $|y|$, because the theory of this section is applicable only if the motion of the binary is only slightly affected by the encounter, whence necessarily $|y| \ll x$, but it will be noted that (4.23) agrees with (4.10) and (4.11) in this limit.

The result (4.24) holds if

$$|y| \ll 2^{9/2} m_3 (m_1 m_2 M_{12})^{-1/2}\, V^{-1} x^{3/2}, \qquad (4.25)$$

the expression on the right-hand side being the value of y at which (4.23) and (4.24) agree, but (4.24) is not expected to be valid for arbitrarily small values of $|y|$, because of (4.21). The evaluation of the corresponding reaction rate is very easy, and by (3.8) and (4.24) we obtain the approximate result

$$Q(x, y) = \frac{\pi}{3\sqrt{2}} \frac{(m_1 m_2)^{3/2}\, m_3}{M_{12}^{1/2}} \frac{1}{x^{1/2} y^2}, \quad ((\beta^* x)^{5/2} \ll \beta^* |y| \ll (\beta^* x)^{3/2}) \quad (4.26)$$

independent of the form of the function $\beta(m)$. The limits of validity have been obtained from (4.21) and (4.25) by ignoring mass-factors and taking orders of magnitude. As in Section 4.1, the occurrence of resonances and exchanges, and the loss of high-energy field stars, lead to only small errors.

When condition (4.18) is reversed the angular velocity of the third body is much less than that of the binary and the character of the dynamics changes. Such encounters are of importance for the study of hard binaries and will be studied in that context in Section 5.4, but here we merely remark that the corresponding energy changes must be extremely small.

4.3 *Discussion of the rates for soft binaries*

We have already remarked that the detailed balance theorem is approximately satisfied by the rate derived in Section 4.1, and the same is true, rather trivially, of that given by (4.26). Here we shall continue the discussion of these calculations by computing the mean energy change and the lifetime of a soft binary, and by comparing some of our results with those of previous workers.

The mean rate of change of binding energy, per unit density of free particles, is just

$$\frac{\langle \dot{x} \rangle}{n} = \int_0^\infty y\{Q(x, y) - Q(x, -y)\}\, dy, \quad (4.27)$$

where n is the number density of single particles. Now in the case of equipartition the detailed balance relation (3.18) implies that

$$Q(x, -y) = Q(x, y) - \frac{y}{f(x)} \frac{\partial}{\partial x} \{f(x)\, Q(x, y)\} \quad (\beta |y| \ll \beta x \ll 1) \quad (4.28)$$

approximately, where $f(x) \equiv x^{-5/2}$. Hence by (4.26)–(4.28) we find that, as a result of wide encounters only,

$$\frac{\langle \dot{x} \rangle}{n} \sim -\frac{\pi}{\sqrt{2}} \frac{(m_1 m_2)^{3/2}\, m_3}{M_{12}^{1/2}} \beta^{1/2}. \quad (4.29)$$

In performing the integration we have taken the range of validity of (4.26) to be $(\beta x)^{5/2} \ll |\beta y| \ll (\beta x)^{3/2}$, and so the result may be relied upon to order of magnitude only, especially in regard to its mass-dependence. Here, n is to be taken as the space density of single particles with mass m_3. It is interesting that this result is insensitive to the lower limit of integration, a result in marked contrast with the situation in two-body relaxation theory (Chandrasekhar 1942).

At this point we recall a remark made in Section 2.2, to the effect that the processes responsible for the evolution of very soft pairs must be the same as those effective in the relaxation of the single-particle distribution. Two-body relaxation may be treated by consideration of hyperbolic 'collisions' between pairs of stars, and we see that the encounters considered in Sections 4.1 and 4.2 may be regarded dynamically as simultaneous two-body ' collisions ' with both components, although one of these much dominates the other in the case of close encounters. In the case of distant encounters, the effect of the third body on one component almost cancels its effect on the other, which accounts for the relatively low importance of such encounters for the evolution of binaries.

No special device is required to obtain the rate of change of binding energy from (4.12), the result after a little labour being roughly

$$\frac{\langle \dot{x} \rangle}{n} = 8\sqrt{2\pi}\,\frac{m_1 m_2 m_3{}^2 m^{*1/2}}{M_{12}}\,\beta^{*1/2}\left(\ln\frac{\beta^* x}{2}+\frac{1}{3}\right), \tag{4.30}$$

where we have had to extend the range of validity of the expressions for the rate function to $(\beta^* x)^{3/2} \leqslant |\beta^* y| \leqslant 1$. It is noteworthy that the very largest changes contribute substantially to the logarithmic term. In the case of equipartition the contributions (4.29) and (4.30) may be added to obtain the total rate for close and wide encounters combined, and so the average effect of encounters with a soft binary is to decrease its binding energy as long as $\beta^* x \ll 1$ (*cf.* Heggie 1974b).

One problem which has motivated some previous studies of encounters with soft binaries is the question of their rate of destruction. Ambartsumian (1937), Gurevich & Levin (1950), Oort (1950) and Hills (1975), estimated the lifetime, t, essentially on the basis of

$$t = -\frac{x}{\langle \dot{x} \rangle}, \tag{4.31}$$

or an analogous formula based on the mean square energy change. Cruz-Gonzalez & Poveda (1971), however, noted that such a calculation might underestimate the lifetime if there were a small number of very large energy changes. Knowing the destruction rate, by (4.13), we see that t is in fact to be found from

$$t = \frac{1}{nQ(x, -\infty)}, \tag{4.32}$$

which was the type of formula used also by Öpik (1932, 1973). We deduce that (4.31) is indeed an underestimate for very soft binaries, although a slight one, for the error factor is only logarithmic. Apart from such logarithmic factors, the dependence of t on x and β^*, which one obtains from (4.13) and (4.32) (*cf.* Section 6.1), is shared by the expressions of Ambartsumian, Gurevich & Levin, Oort and Öpik (*op. cit.*).

Chandrasekhar (1944) estimated the lifetime by computing the average value of $(\mathbf{F}_1-\mathbf{F}_2).\mathbf{F}_1/|\mathbf{F}_1|$, where $\mathbf{F}_i$ $(i = 1, 2)$ are the perturbing forces on the two components, and computing the time needed to attain escape velocity. Such a method is open to an objection like that levelled against (4.31), and has been criticized on other grounds by Yabushita (1966). Furthermore, it is the component of $(\mathbf{F}_1-\mathbf{F}_2)$ parallel to $\mathbf{v}$, the relative velocity vector of the components, that gives the rate of change of the binding energy. The derivation of the lifetime by Cohen (1975) is in much the same spirit as Chandrasekhar's, but avoids some of its difficulties. It is concerned only with close binaries, however.

A radically different attack on this problem was made by Mansbach (1970), who applied the ' variational ' method of chemical and atomic physics to compute the rate of formation, R, of binaries with energies above a certain value x_0, assuming that all those with lower binding energies and all free particles satisfy a Boltzmann distribution. In our notation, the quantity computed is

$$R = n_1 n_2 n_3 \int_{x_0}^{\infty} Q(x)\, dx + n_3 \int_0^{x_0} dx f(x) \int_{x_0 - x}^{\infty} dy Q(x, y),$$

where n_i is the number-density of particles with mass m_i ($i = 1, 2, 3$), $f(x)$ is the Boltzmann distribution of binding energies given by (2.13), and different masses are in equipartition. Mansbach's semi-analytic result is, in our notation

$$R \simeq n_1 n_2 n_3\, 4\pi\sqrt{2}\, \frac{(m_1 m_2)^{9/2}\, m_3}{M_{12}^{1/2}}\, \beta^{3/2} x_0^{-3} \times \left\{0{\cdot}28 - 0{\cdot}24 \log_{10} \beta x_0 + 0{\cdot}24 \log_{10}\left(\frac{4}{3}\, \frac{m_1 m_2 M_{123}}{m_3 M_{12}^2}\right)\right\}, \tag{4.33}$$

although we have omitted a term in the curly bracket of order $(\beta x_0)^{0.8}$, which is negligible for sufficiently small energies.

We may evaluate R using our expressions for Q provided that, as before, we extend somewhat the range of $|y|$ for which they are valid. Here we shall do so with a little more care, transferring from the second of equations (4.12)–(4.26) when these are equal, i.e. at

$$y = \frac{32}{\sqrt{\pi}} (m_1 m_2 M_{123})^{-1/2} m_3^{3/2} \beta^{1/2} x^{3/2}$$

in the limit $\beta x \downarrow 0$. The same method may not be used to suggest a lower limit on $|y|$ for the applicability of (4.26), for we have not evaluated Q for $|y| \ll (\beta x)^{3/2} x$, but (4.21) implies that we should adopt

$$|y| \geqslant A (m_1 m_2 M_{123})^{-3/2}\, m_3^{5/2} M_{12}^2 \beta^{3/2} x^{5/2},$$

where A is some numerical factor. Accordingly we obtain contributions to R of the forms

$$n_1 n_2 n_3\, \frac{4\sqrt{2}}{3}\, \pi^2 \beta^2\, \frac{(m_1 m_2)^4\, m_3^{5/2}}{(M_{12} M_{123})^{1/2}}\, x_0^{-5/2} \tag{4.34}$$

$$n_1 n_2 n_3\, \frac{\pi^{5/2}}{6\sqrt{2}}\, \frac{(m_1 m_2)^{9/2}\, m_3}{M_{12}^{1/2}}\, \beta^{3/2} x_0^{-3} \tag{4.35}$$

and

$$n_1 n_2 n_3\, \frac{\pi^{5/2}}{6\sqrt{2}}\, \frac{(m_1 m_2)^{9/2}\, m_3}{M_{12}^{1/2}}\, \beta^{3/2} x_0^{-3} \left\{-\ln(\beta x_0) + \ln\left(\frac{32}{A\sqrt{\pi}}\, \frac{m_1 m_2 M_{123}}{m_3 M_{12}^2}\right)\right\} \tag{4.36}$$

from (4.14), (4.12) and (4.26) respectively, in the limit $\beta x_0 \downarrow 0$.

The dominant contributions are the last two, and their functional forms are exactly as in (4.33). The coefficients of the terms logarithmic in βx_0 agree to 10 per cent and a comparison of the remaining term in (4.33) is rendered impossible because of our ignorance of A. Its presence stems from our imposing an upper limit on the impact parameter of the third body, and Mansbach applied a comparable cut-off. The fact that a cut-off is required at all shows that R is dominated by distant encounters, i.e. by small energy changes, whereas Mansbach assumed that the most important contribution to R came from the formation of fresh pairs. Hence R cannot be used to estimate the formation rate, and by comparing (4.34) with (4.36) we see that doing so overestimates the rate by a factor of order $-\log(\beta x_0)/(\beta x_0)^{1/2}$.

4.4 *Numerical results*

In our comparison between theory and numerical experiments, we begin by testing (4.19), and then proceed in Figs 9–11 to investigate the extent of agreement with the theoretical cross-sections deduced in Sections 4.1 and 4.2.

Some impression of the accuracy of (4.19) may be obtained by computing three-body systems with simple geometries. Consider, for example, the third body initially approaching in the plane of the binary with velocity V on an orbit with $p = 1$ in some units, while the relative orbit of the binary components is a circle with unit radius. If all masses are unity, we find from (4.19) that

$$|y| \leqslant \frac{1}{V}\sqrt{\frac{2}{3}}, \tag{4.37}$$

the bound being attainable.

In Fig. 8 this limit, represented by a straight line, is compared with the results of computations, the two points for each value of V being obtained by two choices of the initial relative orientation of the binary components. For sufficiently large values of V (i.e. sufficiently soft binaries) the limit (4.37) is approximately observed, and broadly similar results have been achieved for other geometries.

We now describe the results of a series of numerical experiments which are designed to test σ almost directly. In preparation for this we may obtain expressions for the cumulative cross-section $\Sigma(y_0)$, where

$$\Sigma(y_0) \equiv \int_{-\infty}^{y_0} \sigma(y)\, dy \quad \text{for } y_0 < 0$$

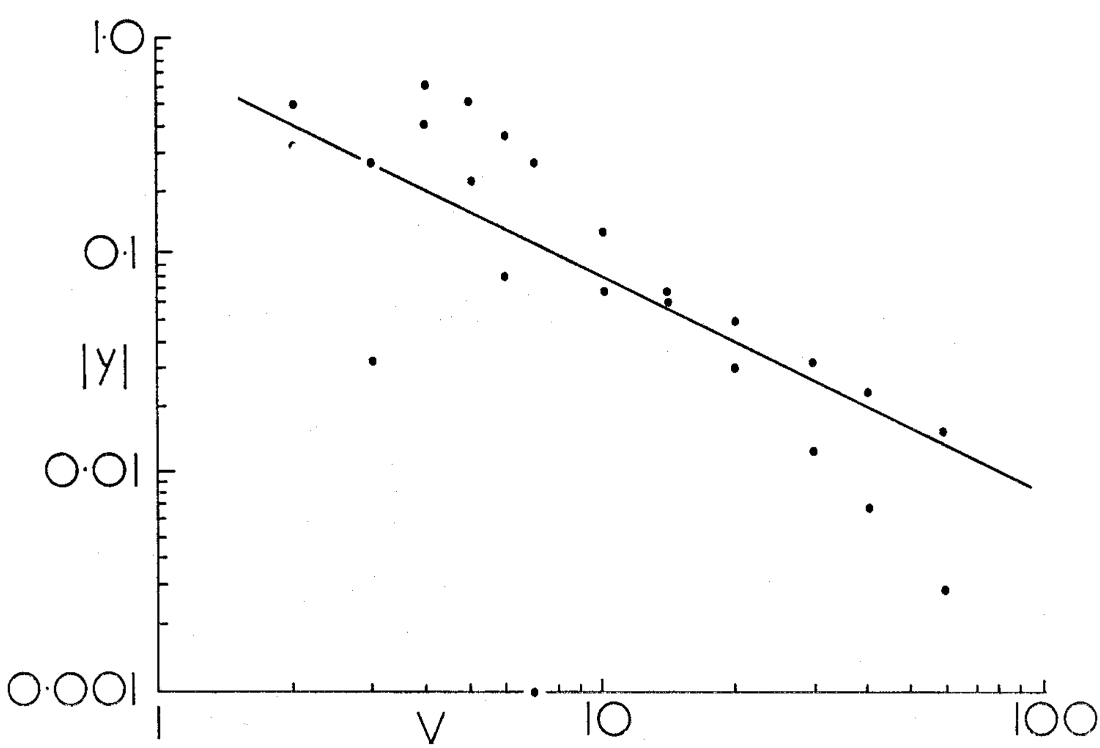

FIG. 8. *Energy changes, $|y|$, in a soft binary due to encounters with a third star initially travelling at velocity V, but always in the same orbital configuration.*

and

$$\Sigma(y_0) \equiv \int_{y_0}^{\infty} \sigma(y)\, dy \qquad \text{for } y_0 > 0,$$

by using (4.10), (4.11) and (4.24). Graphs of the function $\Sigma(y_0)$ thus obtained are represented in Figs 9 and 10 for the case $m_1 = m_2 = m_3 = 1 = V$, $x = 0{\cdot}01$. Here we have switched from (4.11) to (4.24) at that value of $|y|$ at which they are

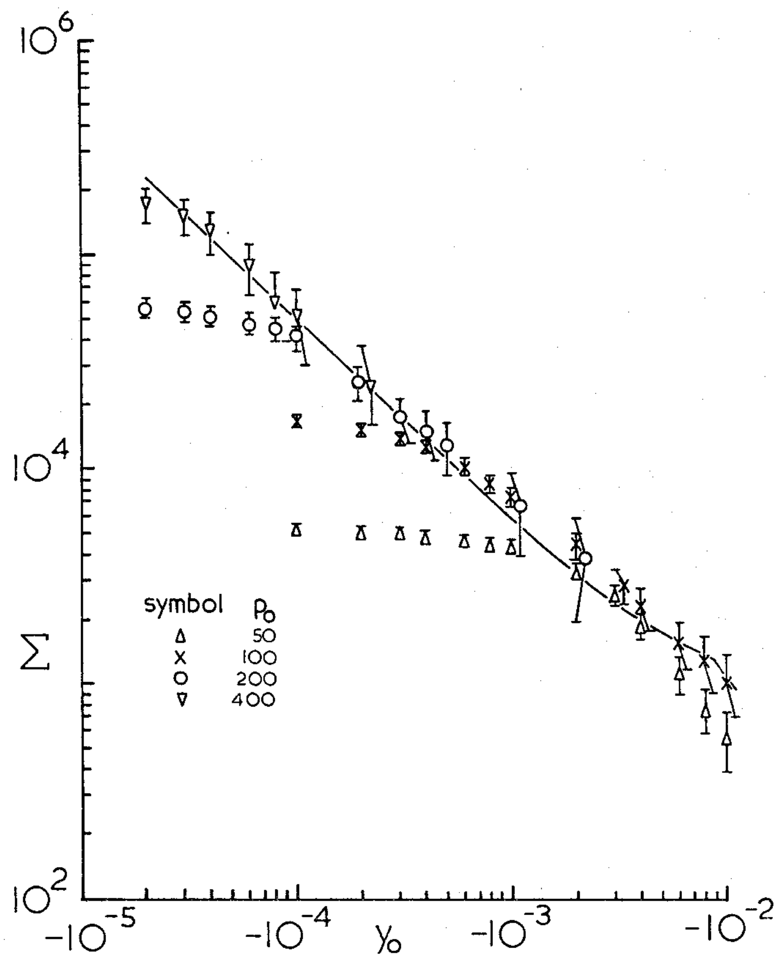

FIG. 9. *Cross-section, $\Sigma(y_0)$, in case $y_0 < 0$. The solid line is the theoretical result obtained from* (4.24) *and* (4.10), *and the plotted points are obtained from numerical data by* (A.1) *and* (A.2). *The error bars are* 95 *per cent confidence limits.*

equal. On the other hand, with these parameters (4.10) always exceeds (4.24), and we have simply switched from one expression to the other at the same value of $|y|$; since this results in a discontinuity in σ, the curve in Fig. 9 exhibits an abrupt change of slope.

Also plotted in these figures are experimentally determined values, obtained by the method described in the Appendix, but it should be noted that, being cumulative, the successive values are not independent in the sense of random variables. Different symbols correspond to different values of the maximum impact parameter, p_0, defined in the Appendix; as p_0 increases the data become complete for successively smaller values of $|y_0|$, but the uncertainty in Σ for larger values of $|y_0|$ deteriorates. There appears to be tolerable agreement between the curves and

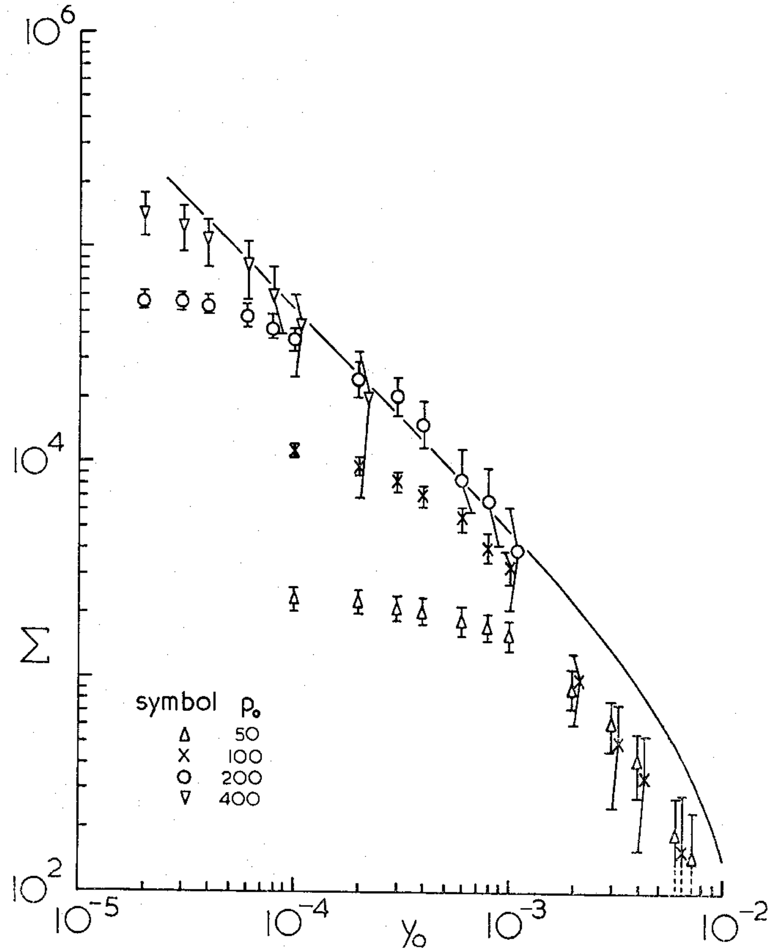

FIG. 10. *As Fig.* 13, *except that* $0 < y_0$ *and the solid curve is obtained from* (4.24) *and* (4.11).

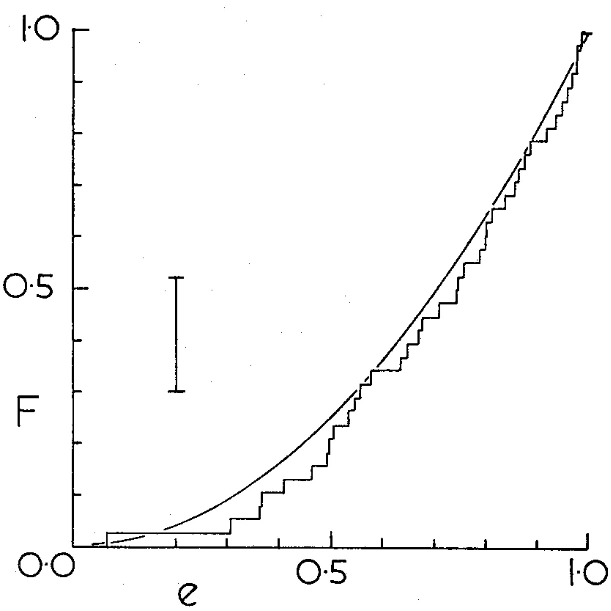

FIG. 11. *Cumulative distribution function, F, of initial eccentricities, e, of those binaries in a series of numerical experiments which were disrupted during the encounter. The smooth line represents the distribution from which the initial conditions were drawn. The bar gives the maximum deviation allowable by the Kolmogorov–Smirnov test at the* 95 *per cent level.*

the upper envelope of the plotted points, dependent though it is to some extent on the way in which the asymptotic expressions for σ are linked.

The agreement at $y_0 = -0{\cdot}01$, where Σ is the destruction cross-section, encourages us to investigate the distribution of initial eccentricities among these cases. The data shown in Fig. 11 confirm experimentally the result stated in Section 4.1, that the destruction rate is almost independent of eccentricity.

Concerning the accuracy of these integrations, it may be said that, of the first 300 cases of the series $p_0 = 50$, which contains the largest proportion of close and therefore difficult encounters, the mean absolute energy change in the integration, due to numerical errors, was 7×10^{-6}, and the largest value was only 3×10^{-5}.

Other numerical investigations already reported in the literature include those of Agekian, Anosova & Bezgubova (1969) and of Agekian & Anosova (1971). Although these deal with the formation of binaries, only in the second were initial conditions selected realistically; here the authors found that there were more high eccentricities among new binaries than would have been expected from (2.15), in contradiction with what was predicted in Section 4.1. However, an upper limit, p_0, was imposed on the initial impact parameters, and so any new binary with a semi-major axis much exceeding p_0 can only have high eccentricity. Other studies are described by Agekian & Anosova (1968a) and by Yabushita (1966), who found a significant mean decrease in the binding energy of a binary subject to random encounters, as we expect from the work of Section 4.3.

Certain workers, e.g. Yabushita (1972), motivated by an interest in cometary dynamics, have studied systems in which one component of the binary is massless. With certain obvious modifications, the work of Sections 4.1 and 4.2 is then equally applicable. Cruz-Gonzalez & Poveda (1971), beginning with binaries of this type in circular orbits and subjecting them to repeated encounters, found that the distribution of eccentricities was more or less uniform by the time a substantial fraction of them had been destroyed. This observation suggests that the success of (2.15), in the comparison with experimental data for soft pairs in Section 2.3, is due to the fact that newly-formed soft pairs already have this distribution, as remarked in Section 4.1. One cannot rely on their relaxing to this distribution before they are destroyed.

5. HARD BINARIES

Encounters with energetic, i.e. hard, binaries now become the subject of study. Again we proceed from close to distant encounters, but in either case—and especially in that of close encounters—there are considerable complications, which are dealt with separately in Sections 5.2 and 5.3. We mainly consider the limit $\beta x \gg 1$.

5.1 *Close encounters*

We begin with the case of simple encounters in which the third body approaches very close to the binary. We shall see how the results of Section 4.1 may be modified to deal, albeit roughly, with this situation, and how they lead to the expressions (5.3)–(5.6) for various reaction rates.

When we come to deal with hard pairs, the relative velocity of other stars is generally much less than that of the binary components, and the impulse approximation, which made the theory of soft pairs comparatively straightforward, is not directly applicable. Suppose, however, that the third body makes a close approach to the binary, whose semi-major axis, a, satisfies $x \equiv m_1 m_2/2a \gg 1/\beta$, since, by assumption, we now deal with hard pairs. Using (3.13), which will still be correct in order of magnitude when $d_1 \sim a$, and setting $V^2 \sim (\beta m)^{-1}$, where m is a typical mass, we find that

$$V_1 \sim \left(\frac{2M_{123}}{a}\right)^{1/2}, \tag{5.1}$$

where $M_{123} \equiv M_{12}+m_3$, and so V_1 generally exceeds the relative velocity of the components of the binary by a small numerical factor. Hence we may apply the theory of Section 4.1 provided we are prepared to accept results correct only in order of magnitude: prior to the close approach, the binary accelerates the third body so that the closest part of the encounter is slightly impulsive.

By (3.13) and (3.14) we may account for the ‘ focusing ’ of the beam of particles approaching the binary by multiplying σ by $V_1{}^2/V^2$. Since we must use the value of V *after* the initial approach to the binary has occurred, i.e. V_1, in (4.10) and (4.11), the effect of the factor is to restore the old form for σ! The computation of Q proceeds as before, except that it is now important to avoid exchange and resonant encounters by imposing the limit

$$y \geqslant -\frac{1}{2}\frac{M_{12}m_3}{M_{123}}V^2. \tag{5.2}$$

This restriction is responsible for the appearance of the exponential factor in the first of the expressions

$$Q(x,y) \sim \begin{cases} 4\sqrt{2\pi}\,\dfrac{m_1 m_2 m_3{}^2 m^{*1/2}}{M_{12}}\,\beta^{*1/2}y^{-2}\left\{1-\dfrac{4}{3}\dfrac{x}{y}\right\}\exp\left\{\dfrac{M_{123}m^*}{M_{12}m_3}\,\beta^* y\right\} & (y<0) \quad (5.3)\\[2ex] 4\sqrt{2\pi}\,\dfrac{m_1 m_2 m_3{}^2 m^{*1/2}}{M_{12}}\,\beta^{*1/2}y^{-2}\left\{\dfrac{4}{3}\dfrac{x}{y}+\dfrac{7}{3}\right\}\left(\dfrac{x}{x+y}\right)^{5/2}. & (y>0) \quad (5.4)\end{cases}$$

The conditions of validity given here are far too generous. By (4.9), with V replaced by V_1, we have $|y| \ll x$ by (5.1), and since the impact parameter, p, to one or other component of the binary must be much less than a, we can see by an argument similar to that leading to (4.8) that we require $|y| \gtrsim x$. The most we can expect, therefore, is that formulae (5.3) and (5.4) are correct in order of magnitude if $|y| \sim x$. However, since (5.3) is then held to be roughly correct for $y \sim -x$, we may obtain the formula for $y \gg x$ by detailed balance. Using (3.18) we find that this is precisely (5.4), which we therefore adopt for $y \gtrsim x$ in the case of equipartition, and not just for $y \sim x$.

The destruction rate is easily found from (5.3) to be

$$Q(x, -\infty) \sim \frac{28\sqrt{2\pi}}{3}\frac{m_1 m_2 m_3{}^3}{m^{*1/2}M_{123}}\,\beta^{*-1/2}x^{-2}\exp\left\{-\frac{M_{123}m^*}{M_{12}m_3}\,\beta^* x\right\} \tag{5.5}$$

whence, by (3.19), hard binaries form at the rate

$$Q(x) \sim \frac{14\sqrt{2}\pi^2}{3}\frac{(m_1 m_2)^4\, m_3{}^{5/2}}{(M_{12}M_{123})^{1/2}}\,\beta x^{-9/2} \tag{5.6}$$

if equipartition holds. Just as these results are held to be correct in order of magnitude, we may also expect, following the discussion towards the end of Section 4.1, that new hard pairs will be formed with a distribution of eccentricities not far removed from (2.15).

5.2 *Exchange*

Now we turn to the cases in which one component of the old binary is ejected immediately after the encounter, the third body becoming one component of a new pair. After some discussion of the mass dependence of this process, the rate functions are computed for the case of equal masses and found to be of the forms (5.17)–(5.19).

Let us, then, consider those encounters which have been excluded so far by the imposition of (5.2). Clearly, the third body cannot remain unbound to the binary at the end of the encounter, but the outcome also depends on what happens to the binary. The case in which it remains intact is considered in the next section, but here we suppose it is disrupted.

To summarize the whole event: we envisage the third body approaching initially with velocity V, and being accelerated towards the binary until there occurs a two-body encounter of the type considered in Sections 4.1 and 5.1. Thereafter the third body remains bound to the first component of the binary, while the second recedes, initially with velocity v_1 with respect to the first component, to an infinite distance from the new binary. In much the same way as before, our results will be valid in order of magnitude only.

As usual, we denote by $-y$ the decrease in the binding energy of the pair forming the original binary, and we let r be their separation during the encounter between the second component and the third body; this is approximately constant since the encounter is treated as being impulsive. Hence

$$\frac{1}{2}\frac{m_1 m_2}{M_{12}} v_1^2 - \frac{m_1 m_2}{r} = -x - y. \tag{5.7}$$

The quantity v_1 was assumed to be taken relative to the first component of the old binary, but it also gives, with sufficient accuracy, the initial relative velocity of the new escaper and the centre of mass of the new binary, as we now show. Since the new pair is bound, the relative velocity, v', between its components must satisfy

$$\frac{1}{2}\frac{m_1 m_3}{M_{13}} v'^2 \leqslant \frac{m_1 m_3}{r},$$

where $M_{13} \equiv m_1 + m_3$. Now the velocity of the receding third body relative to the centre of mass of the new pair is of order $|v_1 \pm v' m_3/M_{13}|$, and this must exceed $(2M_{123}/r)^{1/2}$ for escape. Thus $v' m_3/M_{13}$ will generally be substantially smaller than v_1.

With this approximation the condition for escape becomes

$$\frac{1}{2}\frac{m_2 M_{13}}{M_{123}} v_1^2 \geqslant \frac{m_2 M_{13}}{r}. \tag{5.8}$$

By conservation of energy we obtain

$$-z + \frac{1}{2}\frac{m_2 M_{13}}{M_{123}} v_1^2 - \frac{m_2 M_{13}}{r} = -x + \frac{1}{2}\frac{m_3 M_{12}}{M_{123}} V^2,$$

where z is written for the binding energy of the new pair. Using (5.7) we may eliminate v_1^2, and (5.8) becomes

$$y \leqslant -x - \frac{m_1 m_2 m_3}{M_{12} r}, \tag{5.9}$$

from which it is evident that changes in binding energy considerably exceeding x are required for exchange encounters. For this reason the boundary between exchange and resonance in Fig. 7 is not sharp, and depends a little on the detailed initial conditions.

The fact that substantial negative energy changes are needed also guides us to a simplified expression for y using (4.7). We note from this expression that, if $|y| \gtrsim x$, we require $mV^2 \cos^2 \phi \gtrsim x$, where m is a typical mass. Hence the second term in the bracket in (4.7) is at least comparable with the first, and it is always negative. Since we require only negative values of y, we shall neglect the first term, obtaining

$$y = -\frac{2 m_1 m_2 m_3^2}{M_{12} M_{23}^2} V^2 \cos^2 \phi, \tag{5.10}$$

where $\cos \phi$ is given by (4.6). We obtain a feeling for the error introduced with the above simplification by recomputing (4.10), the result being

$$\sigma = 4\pi \frac{m_1 m_2 m_3^2}{M_{12}} \frac{1}{V^2 y^2}, \quad (y < 0)$$

which is in error by a factor of about 2 at the most when $y \lesssim -x$. When (5.10) is applied to hard pairs, as before V must be replaced by its value, V_1, after initial acceleration towards the binary, and this applies also to the impact parameter.

By analogy with (3.14), the cross-section for events in which binaries of energy z form by exchange from those with energy x is

$$\sigma = \frac{1}{\sqrt{2}\pi^3} \frac{1}{(m_1 m_2 M_{12})^{3/2}} x^{5/2} \int \frac{V_1^2}{V^2} \delta(x - x')\, \delta(z - z')\, d^2\xi_1\, d^3\mathbf{r}\, d^3\mathbf{v},$$

although we must observe (5.9). Integration with respect to $\mathbf{v}$ removes the first delta-function, leaving

$$\sigma = \frac{4}{\pi^2} \frac{x^{5/2}}{(m_1 m_2)^3 V^2} \int d^3\mathbf{r}\, d^2\xi_1 V_1^2 \left(\frac{m_1 m_2}{r} - x\right)^{1/2}$$
$$\times \delta \left(\frac{m_2 m_3}{m_1 M_{123}} x + \frac{m_2 m_3 M_{13}}{M_{123}} \frac{1}{r} + \frac{M_{12} M_{13}}{m_1 M_{123}} y + \frac{1}{2} \frac{M_{12} m_3}{M_{123}} V^2 + z\right),$$

subject still to (5.9) and now $r \leqslant m_1 m_2 / x$. With plane polars, the ξ_1-integration is easy: using the corrected version of (5.10), and (4.6), it yields

$$\sigma = \frac{8}{\pi} \frac{x^{5/2}}{V^2} \frac{m_3^2 M_{13}}{M_{123} m_1^3 m_2^2} \int d^3\mathbf{r} \left(\frac{m_1 m_2}{r} - x\right)^{1/2}$$
$$\times \left(\frac{m_2 m_3}{m_1 M_{123}} x + \frac{m_2 m_3 M_{13}}{M_{123}} \frac{1}{r} + \frac{1}{2} \frac{M_{12} m_3}{M_{123}} V^2 + z\right)^{-2},$$

the conditions being $r \leqslant m_1 m_2 / x$ as before,

$$z \geqslant -\frac{1}{2} \frac{m_3 M_{12}}{M_{123}} V^2 + x \tag{5.11}$$

using (5.9), and, since $\cos^2\phi \leqslant 1$ in (5.10),

$$\frac{m_2 m_3}{m_1 M_{123}}\,x + \frac{m_2 m_3 M_{13}}{M_{123}}\,\frac{1}{r} + \frac{1}{2}\,\frac{M_{12} m_3}{M_{123}}\,V^2 + z \leqslant \frac{2 m_2 m_3^2 M_{13}}{M_{23}^2 M_{123}}\,V_1^2. \qquad (5.12)$$

The Keplerian approximation to the relative motion of the third body and the old binary is roughly valid until the distance of the third body is of order r, whence $V_1^2 \sim V^2 + 2M_{123}/r$ in the manner of (3.13), and so (5.12) becomes

$$z + \frac{m_2 m_3}{m_1 M_{123}}\,x + \frac{1}{2}\,\frac{m_3}{M_{123}}\,V^2\left(M_{12} - \frac{4 m_2 m_3 M_{13}}{M_{23}^2}\right) \leqslant \frac{m_2 m_3 M_{13}}{r}\left(\frac{4 m_3}{M_{23}^2} - \frac{1}{M_{123}}\right). \qquad (5.13)$$

This condition has interesting consequences. Thus, for fixed values of m_1 and m_2, we observe that exchange becomes impossible if m_3 is too small, for the left-hand side ultimately becomes positive and the right-hand side negative. In physical terms, the third body carries in too little energy for the disruption of the binary. Likewise, for fixed m_1 and m_3, exchange cannot occur if m_2 is too small, because, although it may be easy enough to disrupt the binary, so little energy is required that the third body remains unbound. From now on we restrict our discussion to the case of equal masses, with $m_1 = m_2 = m_3 = m$, noting simply that exchange is unlikely to lead to a great decrease in the masses of the components of a binary.

We select spherical polar coordinates (r, θ, χ) for $\mathbf{r}$ and find, setting

$$r \equiv m^2 x^{-1}(1+u^2)^{-1},$$

that

$$\sigma = 96\,\frac{m^3}{V^2 x^2}\int \frac{u^2\,du}{(1+u^2)^4\,(B+u^2)^2}, \qquad (5.14)$$

where

$$B \equiv \frac{3}{2}\left(1+\frac{z}{x}\right) + \frac{mV^2}{2x}$$

and we impose the conditions $u \geqslant 0$,

$$u^2 \geqslant \frac{3}{4}\left(\frac{z}{x} - 1\right)$$

by (5.13), and, from (5.11),

$$\tfrac{1}{3} m V^2 \geqslant x - z. \qquad (5.15)$$

The integral is fairly easy and, when $z < x$, it yields

$$\sigma = 3\pi\,\frac{m^3}{V^2 x^2}\,B^{-1/2}(1+\sqrt{B})^{-5}(B + 5\sqrt{B} + 8),$$

which is also approximately true if $0 \leqslant z - x \ll x$. If $z \gg x \gg \frac{1}{3} m V^2$, on the other hand, we find that

$$\sigma \simeq 96\,\frac{m^3 x^{5/2}}{V^2 z^{9/2}}\,\Gamma_1,$$

where

$$\Gamma_1 \simeq 0{\cdot}0614. \qquad (5.16)$$

We denote by $Q(x \to z)$ the rate, per unit densities of reactants, at which binaries of energy z are formed by exchange from those with energy x, and, by analogy with (3.8) in the case of equal masses and equipartition, we have

$$Q(x \to z) = 4\pi \left(\frac{\beta m}{3\pi}\right)^{3/2} \int V^3 \sigma \exp\left(-\tfrac{1}{3} m\beta V^2\right) dV,$$

though we must pay attention to the condition (5.15). Because the variation of the exponential is rapid in comparison with that of σV^2, this is approximately

$$Q(x \to z) = \begin{cases} 2\left(\dfrac{\beta m}{3\pi}\right)^{1/2} (\sigma V^2)_{mV^2/3 = x - z} \exp\{\beta(z-x)\} & \text{if } z < x, \\ 2\left(\dfrac{\beta m}{3\pi}\right)^{1/2} (\sigma V^2)_{V=0} & \text{if } z > x, \end{cases}$$

whence

$$Q(x \to z) = \begin{cases} \Gamma_2 m^{7/2} \beta^{1/2} x^{-2} \exp\{-\beta(x-z)\}, & (z \leqslant x) \quad (5.17) \\ \Gamma_2 m^{7/2} \beta^{1/2} x^{-2}, & (0 < z - x \ll x) \quad (5.18) \\ \dfrac{192\Gamma_1}{(3\pi)^{1/2}} m^{7/2} \beta^{1/2} z^{-9/2} x^{5/2}, & (x \ll z) \quad (5.19) \end{cases}$$

where $\Gamma_2 \simeq 0{\cdot}46$.

Where they are valid, the expressions (5.17) and (5.18) rather trivially obey the detailed balance relation (3.18), whereas from (5.17) and (5.19) we find that

$$\frac{Q(x \to z)}{Q(z \to x)} = \left(\frac{x}{z}\right)^{5/2} \exp\{\beta(z-x)\} \frac{192}{\pi\sqrt{3}} \frac{\Gamma_2}{\Gamma_1} \quad (z \gg x).$$

Thus the asymptotic *form* of the detailed balance relation is obtained, but the coefficients in the rate functions must be in error since the numerical factor here is about 8. This kind of error is to be expected because of the approximations used in the theory, and gives a useful estimate of their accuracy. In conclusion, it should be noted that the rates and cross-sections must be doubled because *either* component of the binary may be ejected.

5.3 *Resonance*

We must now deal with the type of encounter which is most difficult to treat by any analytical means, i.e. that in which the binary survives while the third body becomes bound to it. In deciding the outcome of such a system, approximate methods of the type we have used so far are useless, because the orbits of bound triple systems are so complicated (Szebehely & Peters 1967; Agekian & Anosova 1967; Szebehely 1972a). This very complication argues for the introduction of a kind of ergodic hypothesis, and although it cannot be justified with rigour, it leads to the expression (5.24) for the relative probability of different possible outcomes. Then it is an easy step to the rates (5.25).

There is clear numerical evidence (Agekian & Anosova 1967, 1968b; Anosova 1969; Szebehely 1971, 1972b) that the evolution of almost all bound triple systems of low angular momentum comes to an end with the ejection of one particle, leaving behind a binary. Indeed it has been proved (Gazaryan 1953) that the motion cannot remain bounded forever if it was previously unbounded for $t \downarrow -\infty$, although in principle ejection of one component is not the only possible motion satisfying this condition. In what follows we shall assume that ejection always happens.

Agekian & Anosova (1968b) found that the time for disruption is on average of the order of 60 crossing times for the triple system, in the case of equal masses, although this is reduced by a factor exceeding 3 when the masses are different. On the other hand, if we can apply Chandrasekhar's formula for relaxation time (Chandrasekhar 1942, p. 202) to such a small system it is found to be at most a few crossing times. We therefore conclude that a bound triple system whose components describe highly complicated orbits has sufficient time to relax before disruption, and, therefore, that the mode of disruption depends statistically only on the conserved quantities—binding energy, z, and angular momentum, H, in the rest frame of the centre of mass—while other details of the initial conditions are ' forgotten '. Incidentally a similar approximation, called the Bohr assumption, is often made in nuclear physics when discussing the disintegration of a compound nucleus (Blatt & Weisskopf 1952, p. 340).

Denoting by $Q(z, H \to x)$ the rate at which bound triple systems of energy z and angular momentum H disrupt to give binaries of energy x ($> z$), we find that the normalized distribution of final pair energies is

$$g(x|H, z) = \frac{Q(z, H \to x)}{\int Q(z, H \to x)\, dx}.$$

Now, by detailed balance of the formation and disruption of triple systems, this may be written as

$$g(x|H, z) = \frac{f(x)\, Q(x \to z, H)}{\int f(x)\, Q(x \to z, H)\, dx},$$

where $Q(x \to z, H)$ is the rate of the reaction inverse to that which we are considering, and $f(x)$ is the energy distribution for binaries under a Maxwellian pair distribution, i.e. (2.13). Note that the distribution of triple systems, which is introduced by the detailed balance relation, cancels out in the expression for g and so, fortunately, its calculation is unnecessary.

The function $g(x|H, z)$ clearly depends on H, if z is fixed. At very large H the pericentric distance of the receding third body must much exceed the semi-major axis of the binary, and so changes in the binding energy of the pair due to its perturbations will have been very small, as we shall see in detail in Section 5.4. Hence the escape energy of the third body must be much less than x; whence $x - z \ll x$ and $g(x)$ is strongly peaked near $x \approx z$. Much larger values of x are possible if H is sufficiently small that the three particles can interact strongly, but it is not expected that g depends critically on H for small values of H, except, perhaps, for $x \gg z$ (*cf.* Waldvogel 1975). Accordingly, if we consider only triple systems formed after close encounters, we can replace g by

$$g(x|z) \simeq \frac{f(x)\, Q(x \to z)}{\int f(x)\, Q(x \to z)\, dx}, \qquad (5.20)$$

where

$$Q(x \to z) \equiv \int Q(x \to z, H)\, dH,$$

integrated over small values of H.

In order to obtain the rate $Q(x \to x')$ at which binaries change binding energy from x to x' via resonant encounters, we write

$$Q(x \to x') = \int dz\, dH Q(x \to z, H)\, g(x'|z, H),$$
$$\simeq \int dz Q(x \to z)\, g(x'|z) \qquad (5.21)$$

over $z \leqslant \min(x, x')$. We now evaluate $Q(x \to z)$.

The total cross-section, Σ, for appropriate resonant encounters is obtained by integrating equation (4.10) over $-x \leqslant y \leqslant -\frac{1}{2}M_{12}m_3V^2/M_{123}$. These inequalities represent the conditions that the binary remain bound and that the third body become bound, although (5.9) indicates that our mathematical expression of the first condition is a little too restrictive, since both terms on the right-hand side of (5.9) are comparable. Now (4.10) is only roughly correct for $y \sim -x$ in the case of hard pairs, as we saw in Section 5.1, and so we may evaluate Σ roughly by setting $y = -x$ and multiplying by the range of integration, whence

$$\Sigma \sim \frac{28\pi}{3}\frac{m_1m_2m_3^2}{M_{12}}\frac{1}{V^2x^2}\left(x - \frac{1}{2}\frac{m_3M_{12}}{M_{123}}V^2\right). \qquad (5.22)$$

By analogy with (3.8), the rate itself is

$$Q(x \to z) \sim \int \Sigma|\mathbf{V}|\, f(\mathbf{V})\, \delta\left(z - x - \frac{1}{2}\frac{m_3M_{12}}{M_{123}}V^2\right) d^3\mathbf{V},$$
$$= \frac{28\sqrt{2\pi}}{3}\frac{m_1m_2m_3^{5/2}}{(M_{12}M_{123})^{1/2}}\beta^{3/2}x^{-2}z \exp\{-\beta(x-z)\}, \quad (z \leqslant x) \qquad (5.23)$$

if we take the function β in (3.9) to be independent of mass.

Again by detailed balance we may write

$$Q(z \to x) = \frac{n_3 f(x)}{f_3(z)}\, Q(x \to z),$$

where $f_3(z)$ is the ' equilibrium ' distribution of triple systems of energy z, and n_3 is the number-density of single stars with mass m_3. Using (2.13) and (5.23) we find that

$$Q(z \to x) \propto \frac{(m_1m_2m_3)^4}{f_3(z)\, M_{123}^{1/2}}\, m_3^{-3/2}M_{12}^{-1/2}zx^{-9/2} \exp(\beta z),$$

where the factor of proportionality is independent of x, z and the masses. Now m_3 is here the mass of the body ejected, and f_3 must be a symmetric function of the masses, and so we predict that a triple system is most likely to break up with the ejection of the lightest particle.

Returning to (5.20) we immediately find that

$$g(x|z) \simeq \tfrac{7}{2}z^{7/2}x^{-9/2} \qquad (5.24)$$

and so, by (5.21),

$$Q(x \to x') \simeq \begin{cases} \dfrac{98}{3}\sqrt{\dfrac{\pi}{3}}\, m^{7/2}\beta^{1/2}x^{-2} \exp\{-\beta(x - x')\}, & (1 \ll \beta x' < \beta x) \\ \dfrac{98}{3}\sqrt{\dfrac{\pi}{3}}\, m^{7/2}\beta^{1/2}x^{5/2}x'^{-9/2}, & (1 \ll \beta x < \beta x') \end{cases} \qquad (5.25)$$

in the case of equal masses. It will be noted that these rates satisfy the detailed balance relation (3.18), but it is clear from their construction that they must. Therefore, this should not be regarded as a significant check on their validity.

5.4 *Wide encounters*

When discussion turns to encounters with a hard binary such that the third body maintains a distance much exceeding the length of the semi-major axis, perturbation techniques again become of value. Following a discussion of the size of the first- and higher-order perturbations to the binding energy of the binary, the first-order term is found to be given by (5.42), whence the cross-section and rates are found in the forms (5.44) and (5.45), respectively.

Using the same notation as in Section 4.2, we find that the exact equations of motion can be expressed in the form

$$\mathbf{r} = -M_{12}\frac{\mathbf{r}}{|\mathbf{r}|^3}+m_3\frac{\partial\mathscr{R}}{\partial\mathbf{r}}, \tag{5.26}$$

$$\ddot{\mathbf{R}} = M_{123}\mu_1\mu_2\frac{\partial\mathscr{R}}{\partial\mathbf{R}}, \tag{5.27}$$

(*cf.* Lyttleton & Yabushita 1965) where $\mu_i \equiv m_i/M_{12}$ ($i = 1, 2$) and

$$\mathscr{R} \equiv \frac{1}{\mu_2|\mathbf{R}-\mu_2\mathbf{r}|}+\frac{1}{\mu_1|\mathbf{R}+\mu_1\mathbf{r}|}.$$

We can expand the perturbing function as

$$\mathscr{R} = \frac{1}{R}\sum_{n=0}^{\infty}\left(\frac{r}{R}\right)^n P_n\left(\frac{\mathbf{r}.\mathbf{R}}{rR}\right)\{\mu_2{}^{n-1}-(-\mu_1)^{n-1}\}, \tag{5.28}$$

where $r \equiv |\mathbf{r}|$, $R \equiv |\mathbf{R}|$ and P_n is the nth Legendre polynomial. The first few terms are

$$\mathscr{R} = \frac{1}{R}(\mu_2{}^{-1}+\mu_1{}^{-1})+\frac{1}{2}\frac{r^2}{R^3}\left(3\left[\frac{\mathbf{r}.\mathbf{R}}{rR}\right]^2-1\right)+O\left(\frac{r^3}{R^4}\right). \tag{5.29}$$

As usual, we are interested in the change, y, in the binding energy of the binary, which is

$$y = \frac{m_1m_2m_3}{M_{12}}\int_{-\infty}^{\infty}\dot{\mathbf{R}}.\frac{\partial\mathscr{R}}{\partial\mathbf{R}}\,dt \tag{5.30}$$

by (5.26), integration by parts, and the fact that $\mathscr{R}\to 0$ for $t\to\pm\infty$. We see from (5.28) that $\mathscr{R}$ is a series in powers of the components of $\mathbf{r}$ with coefficients which are functions of $\mathbf{R}$. If we substitute into (5.30) the unperturbed expressions for $\mathbf{r}$ and $\mathbf{R}$, found by retaining only the first term of (5.29) in (5.26) and (5.27), we may expand the components of $\mathbf{r}$ in Fourier series. Hence $\mathscr{R}$ contains terms of the form $f(\mathbf{R})\exp(imnt)$, where f is some function vanishing as $R\uparrow\infty$, m is an integer, and n is the mean motion for the binary. Terms with $m = 0$ contribute nothing to (5.30) since they give perfect differentials of functions vanishing as $R\uparrow\infty$. On the other hand, those with $m \neq 0$ give the integral of a rapidly oscillating harmonic function modulated by a very slowly varying function, and so the total integral is very small, as we shall see shortly in detail. We denote it by y_0.

In fact, into (5.30) we should substitute the *perturbed* expressions for $\mathbf{r}$ and $\mathbf{R}$, which presumably both contain rapidly oscillating terms as well as secular terms.

Then it is no longer clear that the integral (5.30) is very small, for there seems to be no reason why secular terms in this new integrand must integrate to give zero. The significance of this is that it is now not obvious that y_0 is, in order of magnitude, the leading term in y. However, it is reasonable to expect that, if any secular terms existed which did not integrate to 0, their contribution to y would be of order $x(a/q)^c$ for some positive number c, where a is the initial semi-major axis of the binary, q ($\gg a$) is the distance of closest approach by the third body, and x is, as usual, the original binding energy of the binary. We now show that this possibility may be discounted, and the result will be our justification, though it is not a rigorous one, for treating y_0 as a valid first approximation to y.

We write the Hamiltonian for the system as

$$\mathscr{H} = \frac{\beta}{2L^2}+\frac{M_{123}}{2M_{12}m_3}\mathbf{P}^2-\frac{m_3M_{12}}{R}+\frac{1}{R}(\mathscr{H}_2+\mathscr{H}_3+\ldots)$$

(*cf.* Harrington 1969), where we adopt Delaunay variables L, G, H, l, g, h for the rapid binary motion, $\mathbf{P}$ is the momentum conjugate to $\mathbf{R}$ and $\beta \equiv (m_1m_2)^3/M_{12}$; $\mathscr{H}_i/R$ ($i = 2, 3, \ldots$) are successive terms in the perturbing function, as in (5.28) except for a simple factor, and $\mathscr{H}_i$ is of order $(a/q)^i$. The anomaly l is the most rapidly varying quantity, and terms containing l may be removed from succeeding terms of the perturbing function by canonical transformations (see, for example, Born 1960, Section 41). It is not difficult to see that, if we denote the transform of L by L^*, this process can be carried out in such a way that $|L-L^*| \downarrow 0$ as $R \uparrow \infty$. However, if sufficient terms periodic in l^* have been removed that the leading one in the transformed Hamiltonian is of order $(a/q)^n$, for some n, we see that the total relative change in L^* is $O\,(a/q)^{n-1/2}$, whence the relative change in L is at most of the same order, and so is $|y|/x$. Since this is true for arbitrary n, we conclude that the total change of the binding energy decreases more quickly with increasing q than any power of a/q, which was to be proved.

From this discussion we take it that a valid approximation to (5.30) is obtained by substituting unperturbed expressions for $\mathbf{r}$ and $\mathbf{R}$. For the former we take

$$\mathbf{r} = a\hat{\mathbf{a}}(\cos E-e)+b\hat{\mathbf{b}}\sin E, \tag{5.31}$$

(*cf.* Plummer 1918, p. 23) where a, b are the lengths of the semi-axes of the orbit, $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$ are unit vectors parallel to them, $e \equiv (a^2-b^2)^{1/2}/a$ is the eccentricity, and E is the eccentric anomaly. It is related to the time, t, by the equation $n(t-t_0) = E-e \sin E$, where t_0 is a constant and

$$n^2a^3 = M_{12}. \tag{5.32}$$

If V is the relative velocity 'at infinity' between the third body and the centre of mass of the binary, the eccentricity, e', of the relative orbit, approximated to a Keplerian one, is given by $e' = 1+qV^2/M_{123}$. In order of magnitude, therefore, $e'-1 \sim q/(a\beta x)$. Since we deal with hard binaries, $\beta x \gg 1$, and if we take

$$1 \ll \frac{q}{a} \ll \beta x \tag{5.33}$$

we satisfy simultaneously the conditions that the third body remain well 'outside' the orbit of the binary, and that its motion relative to the binary be approximately parabolic. Hence

$$\mathbf{R} \simeq \hat{\mathbf{A}}q(1-\sigma^2)+\hat{\mathbf{B}}2q\sigma, \tag{5.34}$$

where $\hat{\mathbf{A}}$ and $\hat{\mathbf{B}}$ are orthogonal unit vectors in the plane of the relative motion, in the direction of the third body when the true anomaly in the relative orbit is 0, $\pi/2$, respectively, and σ is given in terms of t by the relation

$$\left(\frac{M_{123}}{2q^3}\right)^{1/2} t = \sigma + \tfrac{1}{3}\sigma^3 \tag{5.35}$$

(*cf.* Plummer 1918, p. 26). We have taken $t = 0$ to hold at the apse of the relative orbit, and deduce from these results that

$$\mathbf{R}.\dot{\mathbf{R}} = 2q^2(1+\sigma^2)\,\sigma\dot{\sigma}. \tag{5.36}$$

The largest contribution to the integrand in (5.30) comes from the $n = 2$ term in (5.28), and by (5.29) and (5.31) it can be written as a finite Fourier sum in E. Trigonometric functions of multiples of E are themselves expressible as Fourier series (Plummer 1918, pp. 37–8) in $n(t-t_0) \equiv M$, but only those terms as far as $\cos M$ and $\sin M$ need be retained, for later terms oscillate still more rapidly and will give yet smaller contributions to y_0. Terms independent of M may also be dropped, as it was proved above that they must be perfect differentials, and so the only term in $\mathscr{R}$ which we need retain is of the form

$$a^2R^{-5}(\cos M\{(\mathbf{R}.\hat{\mathbf{a}})^2 e_1 + \mathbf{R}.\hat{\mathbf{a}}\,\mathbf{R}.\hat{\mathbf{b}}e_2 + (\mathbf{R}.\hat{\mathbf{b}})^2 e_3\} + \sin M\{\mathbf{R}.\hat{\mathbf{a}}\,\mathbf{R}.\hat{\mathbf{b}}e_4\}), \tag{5.37}$$

where e_i ($i = 1, 2, 3, 4$) are certain functions of e.

From (5.30), (5.34) and (5.36), we see that we have to evaluate integrals of the form

$$\int_{-\infty}^{\infty} \exp(\pm int)\,\frac{\sigma^\alpha}{(1+\sigma^2)^6}\,\dot{\sigma}\,dt$$

for integral α satisfying $0 \leqslant \alpha \leqslant 5$. By (5.32) and (5.35), this is simply

$$I_\alpha \equiv \int_{-\infty}^{\infty} \exp\{\pm iK(\sigma + \tfrac{1}{3}\sigma^3)\}\,\frac{\sigma^\alpha\,d\sigma}{(1+\sigma^2)^6}, \tag{5.38}$$

where $K \equiv (2M_{12}q^3/M_{123}a^3)^{1/2}$, $\gg 1$, by assumption. The size of K and the form of the integrand invite us to use the method of steepest descents, the only complication being that the saddle points of the exponent are located at the poles of the integrand. Considering the upper sign, we displace the contour from $\mathscr{C}_1$ to $\mathscr{C}_2$ in Fig. 12,

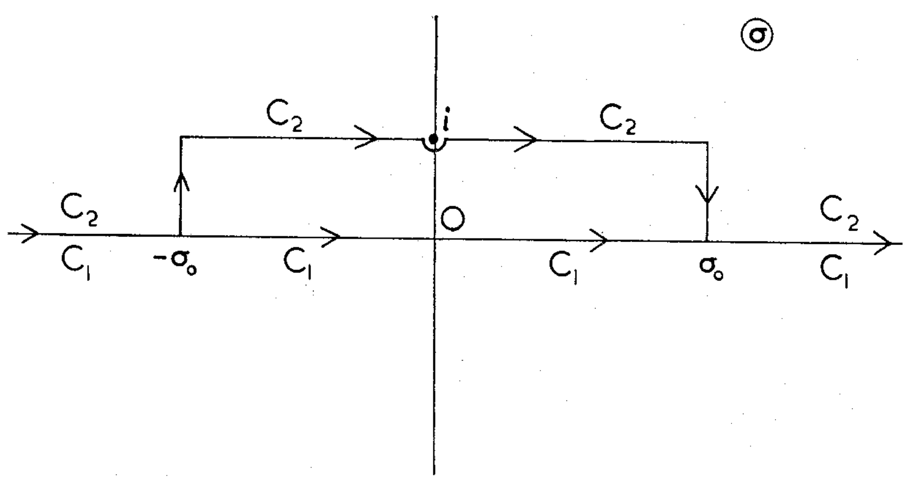

FIG. 12. *Deformation of the contour in the σ-plane for the approximate evaluation of* (5.38) *by the method of steepest descents. The limit $\sigma_0 \to \infty$ is considered in order that the contribution from the 'vertical' parts of $\mathscr{C}_2$ may be neglected.*

which shows part of the complex σ-plane, and after several partial integrations we find that the integral is just

$$\frac{K^{5/2}}{5!}\left(i\pi h^{(v)}(0)+\int_0^\infty \{h^{(v)}(\xi)-h^{(v)}(-\xi)\}\frac{d\xi}{\xi}\right),$$

where

$$h(\xi) \equiv \exp\left(-\tfrac{2}{3}K-\xi^2\right)\frac{(i+\xi/\sqrt{K})^\alpha}{(2i+\xi/\sqrt{K})^6}\exp\left(\frac{1}{3}i\frac{\xi^3}{\sqrt{K}}\right).$$

Retaining only the largest powers of K, we deduce that the contribution from the pole may be neglected, and the result is

$$I_\alpha \simeq \gamma K^{5/2}\exp\left(-\tfrac{2}{3}K\right)(\pm i)^\alpha, \quad (K\uparrow\infty) \tag{5.39}$$

where γ is some constant; we have given the answer for both signs in (5.38). Hence, substituting (5.37) into (5.30) and utilizing (5.39), we deduce that

$$y \simeq \frac{m_1m_2m_3}{M_{12}}a^2q^{-3}K^{5/2}\exp\left(-\tfrac{2}{3}K\right)(f_1\cos M_0+f_2\sin M_0), \tag{5.40}$$

in which $M_0 \equiv nt_0$ and f_1, f_2 are scalar functions of e and the orientations of the orbits, i.e. the vectors $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$, $\hat{\mathbf{A}}$, $\hat{\mathbf{B}}$.

Here it is of interest to estimate the error introduced by omitting higher terms in the perturbing function (5.28). The term after that which we have considered is smaller by a factor of order r/R, which is of order a/q. However, in place of integrals like (5.38) we have a set of integrals with $(1+\sigma^2)^7$ in the denominator of the integrand, and these would be larger than (5.39) by a factor of order $K^{1/2}$, $=O((q/a)^{3/4})$. Hence successive terms of y_0 decrease only as $(a/q)^{1/4}$. Terms in $\sin 2M$ would have increased the exponent by a factor of 2, and can be neglected safely. On the other hand, (5.39) is only the first term in an asymptotic expansion of I_α in powers of $K^{-1/2}$, and so the error thus introduced into (5.40) is less than that caused by neglect of higher terms in the perturbing function.

We now relate q to the impact parameter and velocity of the third body 'at infinity', using the equations for Keplerian motion, and obtain

$$q = \frac{p^2V^2}{2M_{123}} \tag{5.41}$$

approximately, if (5.33) holds. Reinstating the value of K in (5.40), we deduce that

$$y = Ap^{3/2}\exp(-Bp^3), \tag{5.42}$$

where A is $V^{3/2}x^{7/4}(f_1\cos M_0+f_2\sin M_0)$ times a function of the masses, and $B \equiv 3^{-1}M_{12}{}^{1/2}M_{123}{}^{-2}V^3(2x/m_1m_2)^{3/2}$.

The cross-section may be obtained from (3.12). After the integration with respect to $\boldsymbol{\xi}$, noting that $|\boldsymbol{\xi}| \equiv p$, we have

$$\sigma = \frac{1}{4\pi^2}\int\frac{e\sin i\,de\,d\omega\,d\Omega\,di\,dM_0}{3pB|y|\,|1-1/2Bp^3|},$$

where p is the solution of (5.42) and $Ay>0$. We have assumed throughout this section that $Bp^3 \gg 1$, whence the last factor of the denominator is approximately unity. From (5.42) we find that

$$\ln\frac{|y|}{x} = \ln\frac{|A|\,p^{3/2}}{x} - Bp^3,$$

where $|A|\,p^{3/2}/x$ is of order $(q/a)^{3/4}$, while $Bp^3 = O((q/a)^{3/2})$, and so the second logarithm may be neglected, giving approximately

$$p^3 = -\frac{1}{B}\ln\frac{|y|}{x}. \tag{5.43}$$

With these approximations,

$$\sigma \simeq \frac{1}{12\pi^2 B^{2/3}|y|\,|\ln(|y|/x)|^{1/3}}\int e\sin i\,de\,d\omega\,d\Omega\,di\,dM_0,$$

and since this does not contain A it is unlikely that the accuracy of the cross-section is significantly affected by the aforementioned slow convergence of the series for y in powers of $(a/q)^{1/4}$. The condition on A, i.e. $Ay>0$, clearly yields just half of the unrestricted integral, whence, restoring the value of B, we find

$$\sigma = \frac{\pi}{2\cdot 3^{1/3}}\frac{m_1m_2M_{123}{}^{4/3}}{M_{12}{}^{1/3}}\frac{1}{x|y|(-\ln\{|y|/x\})^{1/3}V^2}. \tag{5.44}$$

Now the reaction rate follows immediately from (3.8), the condition

$$y \geqslant -\tfrac{1}{2}M_{12}m_3V^2/M_{123}$$

being imposed to exclude resonances, which will be discussed in Section 5.5; we obtain

$$Q(x,y) = \frac{1}{3^{1/3}}\left(\frac{\pi}{2}\right)^{1/2} m_1m_2M_{12}{}^{-1/3}M_{123}{}^{4/3}m^{*1/2}\frac{\beta^{*1/2}}{x|y|(-\ln\{|y|/x\})^{1/3}} \times \begin{cases}\exp\left(\dfrac{M_{123}m^*}{M_{12}m_3}\beta^*y\right) & \text{if } y<0,\\ 1 & \text{if } y>0.\end{cases} \tag{5.45}$$

We recall that the theory is valid only if (5.33) holds, and deduce from (5.41) and (5.42) that this condition is of the form

$$(\beta x)^{3/4}\exp\{-(\beta x)^{3/2}\} \ll \frac{|y|}{x} \ll 1. \tag{5.46}$$

In the case of equipartition we see that (5.45) approximately, if rather trivially, satisfies the detailed balance relation (3.18).

5.5 *Hierarchical resonant systems*

Now we give our attention to resonances formed during wide encounters, i.e. we suppose that the third body becomes 'loosely' bound to the hard binary. In such a 'hierarchical' system (Aarseth 1972a) the third body undergoes successive, relatively wide encounters with the binary, each time slightly altering its binding energy, until it ultimately escapes. A useful estimate of the rate for this process is given by (5.64).

The initial binding energy of the third body to the centre of mass of the binary will be denoted by $E_0, \equiv -\tfrac{1}{2}M_{12}m_3V^2/M_{123}$, and after the first encounter the third body becomes bound to the binary, its energy now being $E_1>0$. Thereafter, successive encounters give the energy the positive values $E_2, \ldots, E_{n-1}$ until the particle escapes from the binary with energy $E_n<0$ after n encounters, say. Thus the total change in energy of the binary is

$$y \equiv E_0 - E_n. \tag{5.47}$$

While the third body is bound to the binary, its orbit is approximately parabolic except when it is so far away that perturbations are negligible, and so we may take successive values of $\Delta E_m \equiv E_m - E_{m-1}$ to be given by (5.42), where we write $-\Delta E_m$ for y. However, it is preferable to rewrite this in terms of h, where $h \equiv pV$ is the angular momentum, per unit reduced mass, for the relative motion of the third body and the binary. Thus we have

$$\Delta E_m \simeq G \sin (M_0 - F)$$

if we write

$$G \equiv Ch^{3/2} \exp(-Dh^3), \tag{5.48}$$

where C, D and F are functions of the masses, x, e, ω, Ω and i, with the properties $C>0$, $D \equiv BV^{-3}$. Perturbations are always light, and we can approximately take C, D, F and h to be constants during the lifetime of the resonant system, while M_0 is drawn from the distribution $f(M_0) = 1/2\pi$: the interval between successive returns to the binary is very long compared with the period of the binary, and so we assume that its phase at closest approach may be taken to be distributed uniformly.

The sequence of functions $f_m(E_m)$, which are defined to be the distributions of the variables E_m, is generated by the relation

$$f_m(z) = \int f_{m-1}(z - G \sin \theta_m) \frac{d\theta_m}{2\pi} \quad (G \sin \theta_m \leqslant z \text{ if } m \geqslant 2),$$

where θ_m is the value of $(M_0 - F)$ for the mth encounter and the restriction on θ_m is derived because only those systems in which the third body is bound after $(m-1)$ encounters will suffer another. By induction we find that, since the initial binding energy is E_0,

$$f_m(z) = \frac{1}{(2\pi)^m} \int d\theta_1 \ldots d\theta_m \, \delta\left(z - G \sum_1^m \sin \theta_r - E_0\right),$$

with the restrictions

$$z - G \sum_s^m \sin \theta_r \geqslant 0 \quad (s = 2, \ldots, m). \tag{5.49}$$

Hence

$$f_m(z) = \frac{2}{(2\pi)^m} \int d\theta_2 \ldots d\theta_m \left\{1 - \left(\frac{z - E_0}{G} - \sum_2^m \sin \theta_r\right)^2\right\}^{-1/2} G^{-1} \quad (m \geqslant 2), \tag{5.50}$$

provided that we impose the condition

$$\left| \frac{z - E_0}{G} - \sum_2^m \sin \theta_r \right| \leqslant 1 \tag{5.51}$$

in addition to (5.49). If $E_m < 0$, $f_m(E_m)$ is the probability that the third body escapes after m encounters, with energy E_m.

By analogy with (3.8) and (3.12), the rate of approach, R, of third bodies with given values of h, x, e, ω, Ω, i and E_0 is simply

$$\begin{aligned} R = (m^*\beta^*)^{3/2} (2\pi)^{-9/2} \int & \exp(-\tfrac{1}{2} m^*\beta^* \mathbf{V}^2) \, |\mathbf{V}| \, e' \sin i' \, \delta(h - pV) \\ & \times \delta(e - e') \, \delta(\omega - \omega') \, \delta(\Omega - \Omega') \, \delta(i - i') \\ & \times \delta\left(E_0 + \frac{1}{2} \frac{M_{12} m_3}{M_{123}} V^2\right) d^3\mathbf{V} \, d^2\boldsymbol{\xi} \, de' \, d\omega' \, d\Omega' \, di' \, dM_0, \end{aligned}$$

where $p \equiv |\xi|$. We easily find that

$$R = \left(\frac{m^*\beta^*}{2\pi}\right)^{3/2} \frac{M_{123}}{M_{12}m_3} 2eh \sin i \exp\left(\frac{M_{123}m^*}{M_{12}m_3}\beta^* E_0\right) \quad \text{for } E_0 < 0, \qquad (5.52)$$

and the rate of occurrence of those events in which the third body escapes after n encounters, and the binding energy of the binary changes by y, is just

$$Q_n(x, y) = \int_{E_0<0} R\, f_n(E_0 - y)\, dh\, de\, d\omega\, d\Omega\, di\, dE_0, \qquad (5.53)$$

using also (5.47). Note that we can obtain $Q(x, y)$, the rate for ordinary encounters, by taking $n = 1$.

We define a new variable

$$\kappa \equiv y/G \qquad (5.54)$$

and, using (5.48), we obtain

$$h^3 \simeq -\frac{1}{D}\ln\frac{y}{\kappa x},$$

in the manner of the derivation of (5.43), and

$$\left|\frac{d\kappa}{dh}\right| \simeq \frac{3Dh^{1/2}|y|}{C\exp(-Dh^3)}.$$

We also set

$$\epsilon_0 \equiv E_0/G \qquad (5.55)$$

and, by (5.50), (5.52) and (5.53), find that

$$Q_n(x, y) = 3^{-1/3}(2\pi)^{-n-3/2}\, m_1 m_2 m_3^{-1} M_{12}^{-4/3} M_{123}^{7/3} m^{*3/2} \beta^{*3/2} x^{-1}$$
$$\times \int \frac{2e \sin i \exp\{(M_{123}m^*/M_{12}m_3)\,\beta^* y \epsilon_0 \kappa^{-1}\}}{|\kappa|\,|\ln(y/\kappa x)|^{1/3}\{1 - (\kappa + \sum_2^n \sin\theta_r)^2\}^{1/2}}$$
$$\times\, d\theta_2 \ldots d\theta_n\, d\kappa\, de\, d\omega\, d\Omega\, di\, d\epsilon_0,$$

subject to the following restrictions:

$$0 \leqslant \frac{y}{\kappa} \ll x \qquad (5.56)$$

because $G > 0$ and our theory is only valid if the energy change during each of the n encounters is much less than x;

$$\epsilon_0 \leqslant 0; \qquad (5.57)$$

$$\epsilon_0 - \kappa - \sum_s^n \sin\theta_r \geqslant 0 \quad (s = 2, \ldots, n) \qquad (5.58)$$

from (5.49), (5.54) and (5.55);

$$\left|\kappa + \sum_2^n \sin\theta_r\right| < 1 \qquad (5.59)$$

by (5.51), etc.; and $\epsilon_0 - \kappa \leqslant 0$, since we assume that $E_n \leqslant 0$. The integrals with respect to e, ω, Ω, i and ϵ_0 are trivial, and we obtain

$$Q_n(x, y) = \frac{2}{3^{1/3}(2\pi)^{n-1/2}}\, m_1 m_2 M_{12}^{-1/3} M_{123}^{4/3} m^{*1/2} \beta^{*1/2} x^{-1} |y|^{-1}$$

$$\times \int \frac{d\theta_2 \ldots d\theta_n \, d\kappa}{|\ln (y/\kappa x)|^{1/3} \{1-(\kappa+\sum_2^n \sin\theta_r)^2\}^{1/2}}$$
$$\times \left[1-\exp\left\{\frac{M_{123}m^*}{M_{12}m_3}\beta^* y\left(1+\frac{1}{\kappa}\max_{2\leqslant s\leqslant n}\sum_s^n \sin\theta_r\right)\right\}\right]$$

in case $y \geqslant 0$, the integral being subject to the conditions (5.56), (5.59) and, by (5.57) and (5.58),

$$\kappa+\sum_s^n \sin\theta_r \leqslant 0 \quad (2\leqslant s\leqslant n). \tag{5.60}$$

We note that the factor including the exponential term is always less than unity, and so $Q_n(x, y)$ is always less than an expression whose dependence on y, x and the masses is very like that of (5.45). In fact the exponential term may be neglected provided that $\beta^* y \gg 1$. On the other hand, if $\beta^* y \ll 1$ its neglect is not permissible, and if the exponential is expanded we see that $Q_n(x, y)$ depends essentially logarithmically on y. At any rate it is then very much less than (5.45), the physical basis for this being that the great number of very wide encounters generally cannot lead to energy changes sufficiently large that the third body becomes bound.

In general, then, and if we neglect the dependence of the logarithm on κ which, by (5.59), is not a small or large parameter of the problem, we find that

$$Q_n(x,y) \leqslant \frac{2^{3/2}\pi^{1/2}}{3^{1/3}} m_1 m_2 M_{12}^{-1/3} M_{123}^{4/3} m^{*1/2} \beta^{*1/2} x^{-1} |y|^{-1} \left|\ln\frac{y}{x}\right|^{-1/3} I_n, \quad (y \geqslant 0)$$

where

$$I_n \equiv \frac{1}{(2\pi)^n}\int \frac{d\theta_2 \ldots d\theta_n \, d\kappa}{\{1-(\kappa+\sum_2^n \sin\theta_r)^2\}^{1/2}}$$

over (5.56), (5.59) and (5.60). In place of (5.56), which excludes only a very small part of the positive κ-axis if $|y| \ll x$, we may take

$$0 \leqslant \kappa, \tag{5.61}$$

and the error introduced is negligible.

It remains to sum over n, for which purpose the results of random-walk theory are useful, but since the application is slightly delicate it is given here in a little detail. Defining θ_1 by

$$\sin\theta_1 \equiv -\kappa - \sum_2^n \sin\theta_r \quad (0 \leqslant \theta_1 \leqslant 2\pi)$$

we find that $I_n \equiv \frac{1}{2} J_n(0)$, where, if $\theta \geqslant 0$,

$$J_n(\theta) \equiv (2\pi)^{-n} \int \exp\left(-\theta \sum_1^n \sin\theta_r\right) d\theta_1 \ldots d\theta_n \quad \text{over}$$

$$\begin{cases} \sum_1^s \sin\theta_r \geqslant 0, & 1 \leqslant s \leqslant n-1 \\ \sum_1^n \sin\theta_r \leqslant 0. \end{cases}$$

Defining $K_n(\theta)$ to be the same integral, except that the region of integration is

$$\sum_1^s \sin\theta_r \geqslant 0$$

for $1 \leqslant s \leqslant n$, we find that

$$J_n + K_n = K_{n-1} f^*, \tag{5.62}$$

where

$$f^*(\theta) \equiv \frac{1}{2\pi} \int_0^{2\pi} \exp(-\theta \sin \theta_n)\, d\theta_n;$$

this is one of the modified Bessel functions and $f^*(\theta) > 1$ if $\theta \neq 0$. From (5.62) we find that

$$\sum_2^N s^n J_n = sf^*(sK_1 - s^N K_N) - (1 - sf^*) \sum_2^N s^n K_n, \tag{5.63}$$

where $s > 0$. Putting $\theta = 0$, $s = 1$ and noting that $J_n > 0$, $K_n > 0$ we deduce that $\Sigma\, I_n$ converges. All that remains is to evaluate the sum.

Clearly $0 \leqslant J_n(\theta) \leqslant \exp(\theta)\, J_n(0)$, and so the left-hand side of (5.63) converges uniformly in the region $0 \leqslant s \leqslant 1$, $0 \leqslant \theta \leqslant 1$ (say). Each of the terms being continuous, the limit function is also continuous in this region. Setting $s = \{f^*(\theta)\}^{-1}$ in (5.63) we deduce that

$$\sum_2^\infty f^{*-n} J_n = f^{*-1} K_1 \quad (\theta > 0),$$

which is Wald's Identity for this particular random walk (*cf.* Cox & Miller 1965). By continuity we deduce that

$$\sum_2^\infty I_n = \tfrac{1}{2} K_1(0) = \tfrac{1}{4}.$$

Hence the total reaction rate for wide resonances, summed over the number of encounters, satisfies the bound

$$\sum_n Q_n(x, y) \leqslant 3^{-1/3} \left(\frac{\pi}{2}\right)^{1/2} m_1 m_2 M_{12}^{-1/3} M_{123}^{4/3} m^{*1/2} \beta^{*1/2} x^{-1} |y|^{-1} \left| \ln \frac{y}{x} \right|^{-1/3} \quad \text{if } y > 0. \tag{5.64}$$

Following our previous discussion we remark that this is probably a satisfactory estimate when $|\beta^* y| \geqslant 1$, but it is likely to be a great overestimate if $|\beta^* y| \ll 1$. A similar discussion of the case $y < 0$ leads merely to an extra factor

$$\exp(M_{123} m^* \beta^* y / M_{12} m_3).$$

5.6 *Discussion of encounters with hard binaries*

Having completed our investigation of encounters involving hard binaries, we shall summarize here our results for the case of equal masses. We shall comment on the average rate at which the energy of a hard binary changes, describe some extra results, and compare our conclusions with those of other authors. However, numerical studies are held over until Section 5.7.

From an inspection of Sections 5.1–5.3 it will be found that, within their respective ranges of validity, all of the reaction rates can be approximated by the simple formulae

$$Q(x, y) \sim \begin{cases} A m^{7/2} \beta^{1/2} x^{-2} \exp(\beta y) & (y < 0) \\ A m^{7/2} \beta^{1/2} x^{5/2} (x + y)^{-9/2} & (y > 0) \end{cases} \tag{5.65}$$

in the case of equal masses. Here A is a constant which is about 9 for the rates in Section 5.1; for exchange events, $A \simeq 1$ when $y<0$ or $0<y \ll x$, and $A \simeq 8$ when $x \ll y$; and $A \simeq 33$ for resonant encounters, which are therefore the dominant outcome of close encounters with a hard pair when all masses are equal. Therefore, the total rate function for this case is approximately represented by (5.65) if we set $A \simeq 45$.

The reaction rates for distant encounters are easily summarized. The results of Section 5.5 imply that we merely double the rates given by (5.45) when $|\beta y| \gg 1$, so as to include resonances.

An immediate consequence of these results is that the average rate at which the binding energy of a hard pair changes is positive (Heggie 1975), because the rate of occurrence of changes such that $\beta y \ll -1$ is exponentially small. For the reason stated in Section 3.2 our reaction rates are unreliable in this range, but our conclusion on the mean change in energy is unaffected because it is very insensitive to the value of Q for y in the stated range. Therefore the fact that hard pairs 'absorb' binding energy on average has little to do with the absence of high-velocity stars due to escape.

Gurevich & Levin (1950) used an equipartition argument to show that hard pairs should exhibit a tendency to harden, but they failed to take account of the initial acceleration of the third body, which makes the argument more involved (Heggie 1975, discussion). Jeans (1929, p. 309), also basing his reasoning on equipartition, reached erroneous conclusions.

For even a moderately hard binary, the lower limit on $|y|$ in (5.45) corresponds to such a small change in energy that there is little point in calculating y for even more distant encounters. The case when the third body is so distant that its relative motion is approximately rectilinear has been considered by Walters (1932), on the assumption that the orbit of the binary is nearly circular, and, in the context of atomic physics, by Percival & Richards (1967). This type of theory is also applicable to encounters with soft binaries even more distant than those considered in Section 4.2. Lyttleton & Yabushita (1965) treated the case when $m_2 = 0$ and e is small. They found no secular term in the variation of a due to the P_2 term in the analogue of (5.28), but made a mistake in the treatment of the P_3 term, finding the change in a to be of the form $\Delta a = \alpha_0 ea(a/q)^{5/2}$, where α_0 depends on the masses and the 'geometry' of the orbits, but not on their scale. In fact the changes are much less than this. Jeans (1929), Gurevich & Levin (1950) and Lynden-Bell (1969) all realized qualitatively that energy changes resulting from these distant encounters would be very small.

In contrast to the energy, the equation for changes in the angular momentum, and so of the eccentricity, has secular terms. For the situation analysed in Section 5.4 it is easy to show that the change in e is approximately

$$\Delta e \simeq \frac{15\pi}{4\sqrt{2}} \frac{m_3}{(M_{12}M_{123})^{1/2}} e\sqrt{1-e^2} \left(\frac{a}{q}\right)^{3/2} (\hat{\mathbf{a}}.\hat{\mathbf{A}}\hat{\mathbf{b}}.\hat{\mathbf{A}} + \hat{\mathbf{a}}.\hat{\mathbf{B}}\hat{\mathbf{b}}.\hat{\mathbf{B}}), \qquad (5.66)$$

and this is typically much bigger than the relative change in binding energy. Eccentricity therefore relaxes more rapidly than energy, as Hills (1975) found numerically, and since, as we noted in Section 5.1, new hard binaries will have a distribution of eccentricities not unlike (2.15), they should relax to this distribution before their energy has changed much. This may account for the success of the fit of (2.15) to experimental data on hard pairs, as described in Section 2.3.

5.7 *Numerical results*

A computational determination of the reaction rates for hard binaries is still more desirable than for soft binaries because so much of the theory, especially the results of Sections 5.1–5.3, is imprecise. However, the numerical work which is now to be discussed was much more time-consuming than that devoted to soft pairs, mainly because of the formation of resonant systems. From an investigation of cross-sections, in which the discussion is based on Fig. 13, we go on to discuss wide encounters in a little more detail. Then resonances take up our attention, and in Fig. 16 striking evidence is exhibited for the hypothesis introduced in Section 5.3.

With the exception of the initial binding energy, x, of the binary, and the upper bound, p_0, on the impact parameter for the third body, initial conditions were selected in the same way as in the experiments for which results are displayed in Figs 9–11. We define the cumulative cross-section slightly differently, as

$$\Sigma(y_0) \equiv \begin{cases} \displaystyle\int_{-\frac{1}{3}}^{y_0} \sigma(y)\, dy & \text{if } -\tfrac{1}{3} \leqslant y_0 < 0, \\ \displaystyle\int_{y_0}^{\infty} \sigma(y)\, dy & \text{if } 0 < y_0, \end{cases} \tag{5.67}$$

the lower limit on negative values arising because the initial binding energy of the triple system is $x - \frac{1}{2}M_{12}m_3V^2/M_{123}$. Here we suppose σ to be that appropriate to ordinary encounters, excluding resonances and exchange.

The results of a series of numerical experiments are shown in Fig. 13, which is a plot similar to Figs 9 and 10 for soft binaries. The continuous lines on these figures are plots of Σ obtained from (4.11) and (5.44), transfer from one expression to the other taking place at that value of y where they are equal.

The agreement with the experimental data for the larger maximum impact parameter p_0, at least for sufficiently small values of $|y_0|$, is quite acceptable. However, the data is not an adequate test of (4.11) for hard pairs, since, by the discussion of Section 5.1, (4.11) is not expected to be valid for values of y as low as those observed.

This series of experiments was performed with the aid of a regularization technique which is described elsewhere (Heggie 1973). The average absolute energy change due to numerical errors in the first 100 cases of the set with smallest p_0, which has the highest percentage of numerically difficult cases, was 6×10^{-3}, and the maximum 0·13. Fig. 13 shows that most of the observed energy changes in the binary exceed those attributable to errors.

From (5.40) we find that

$$\ln \frac{\langle |y| \rangle}{x} \simeq -B \left(\frac{q}{a}\right)^{3/2} \quad \text{if } a \ll q,$$

where B is some function of the masses and the average is taken with respect to everything except the masses, q and a. Likewise, from (5.66) we find that $\langle |\Delta e| \rangle = C(a/q)^{3/2}$, where C is some function of the masses. Thus

$$\langle |\Delta e| \rangle \simeq \frac{-D}{\log(\langle |y| \rangle / x)}, \tag{5.68}$$

where D is some positive constant for this series of experiments. The solid line in Fig. 14 is a plot of $|\Delta e| = -(\log(|y|/x))^{-1}$. Bearing in mind that (5.68) refers to

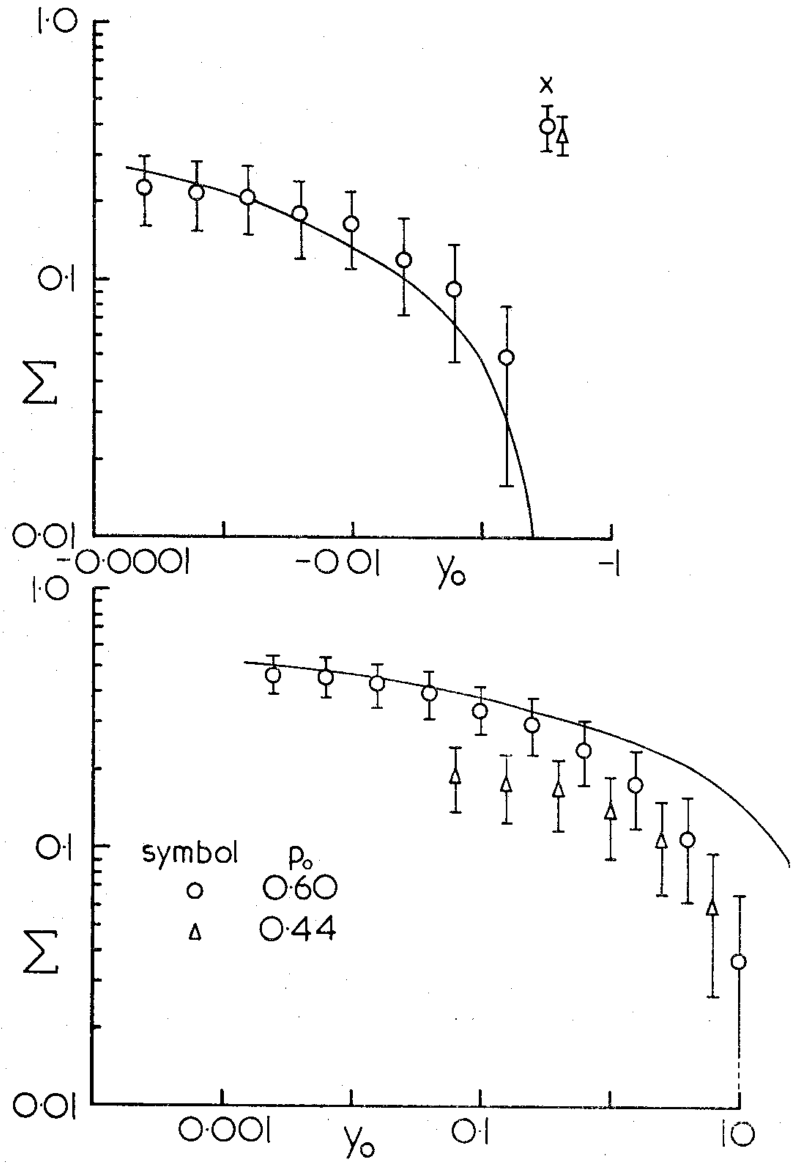

FIG. 13. *Cumulative cross-section, Σ, for energy changes such that $-\frac{1}{3} \leqslant y \leqslant y_0 < 0$ in the upper graph and $0 < y_0 \leqslant y$ in the lower, excluding resonances, which are plotted separately at $y_0 = -\frac{1}{3}$. The two symbols correspond to different values of the maximum impact parameter, p_0, but in both cases $x = 50$. The theoretical resonance cross-section, shown by a cross, and the theoretical result for ordinary encounters, represented by a smooth line, are derived as described in the text. Error bars illustrate 95 per cent confidence intervals.*

averages, and that it contains an uncomputed constant, we may be content that the curve at least follows the trend of the plotted points. Certainly, energy changes are relatively much smaller in general than those in eccentricity: the dashed line represents the equation $|\Delta e| = |y|/x$.

The experimental cross-section for exchange was $\Sigma = 0{\cdot}02$, with confidence limits of the same order, since only four such cases were obtained. From the theory of Section 5.2 we obtain the result $\Sigma = 0{\cdot}034$, which we may regard as being consistent with the experimental estimate.

The values of the cross-sections for resonances may now be compared. Equation (5.22) yields the theoretical result for close encounters. To this must be added the

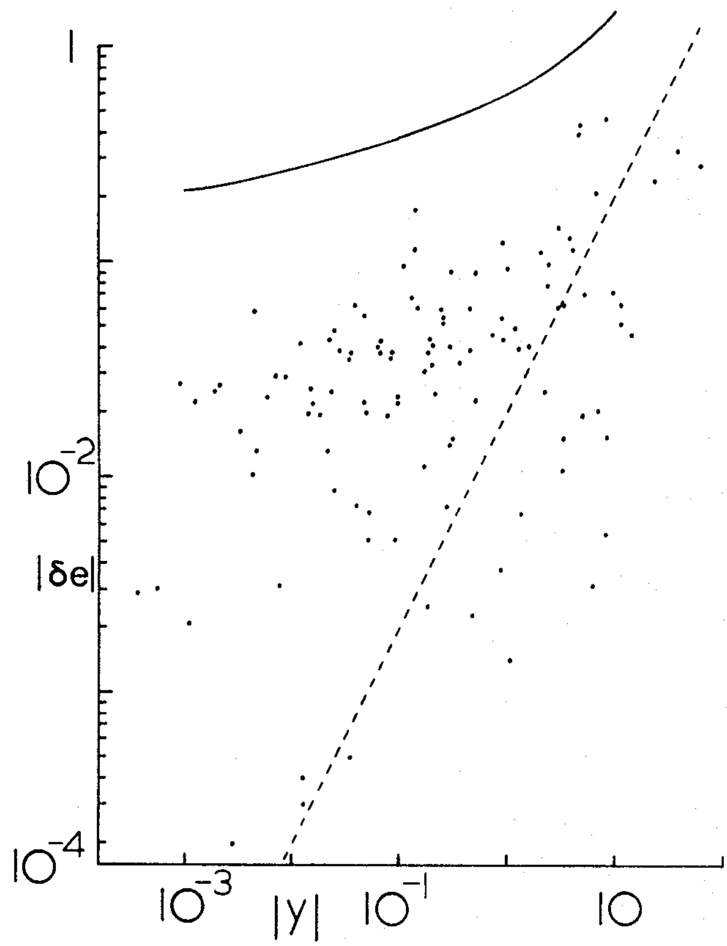

FIG. 14. *A plot on logarithmic scales of $|y|$ against the change in eccentricity $|\Delta e|$, for the ordinary encounters used to draw Fig.* 13. *The meaning of the two lines is discussed in the text.*

contribution from wide encounters, which may be obtained by integrating (5.44) over $-x \lesssim y \leqslant -\frac{1}{2}M_{12}m_3V^2/M_{123}$, and this yields

$$\Sigma \simeq \frac{9\pi}{2^{7/3}} x^{-1}(\ln x)^{2/3}.$$

Altogether, the cross-section is about 0·57 when $x = 50$, and 0·067 when $x = 500$. The former, indicated by a cross on Fig. 13, is not far above the upper experimental value, considering the very approximate nature of the theory. From a series of experiments with $x = 500$ the value of $0{\cdot}045 \pm 0{\cdot}008$ was obtained, and we now discuss the latter series of resonant systems in more detail.

One consequence of the hypothesis of Section 5.3—that resonant systems of low angular momentum can relax before disruption—is that, in the case of equal masses, the probabilities of ejection are the same for all three particles. Table I shows, for three ranges of the initial impact parameter, p, of the third body, the numbers of cases which resulted in ejection, respectively, of either component of the binary and of the original third body. Although the data are rather meagre, we see that the above prediction is roughly verified for close encounters, in the sense that about two-thirds of the systems decay with ejection of one or other component of the original pair. For resonances formed after very wide encounters, however,

TABLE I

Escaper identity

	p $\leqslant 0 \cdot 08$	$0 \cdot 08$–$0 \cdot 12$	$\geqslant 0 \cdot 12$
Binary component	12	8	3
Third body	7	11	16

ejection of the original body almost always occurs. The theory of very wide resonances was developed on this basis.

Resonances of low and high angular momentum are also distinguished by their lifetimes, whose average increases rapidly with increasing values of p. Furthermore, the distributions of the difference in energy, y, between the initial and final binaries are quite different, as we see in Fig. 15: the results from resonances of low angular momentum are consistent with the theoretical curve obtained from (5.24), itself a consequence of the relaxation hypothesis. As we would expect, resonances with large angular momentum lead, on the whole, to smaller changes in binding energy.

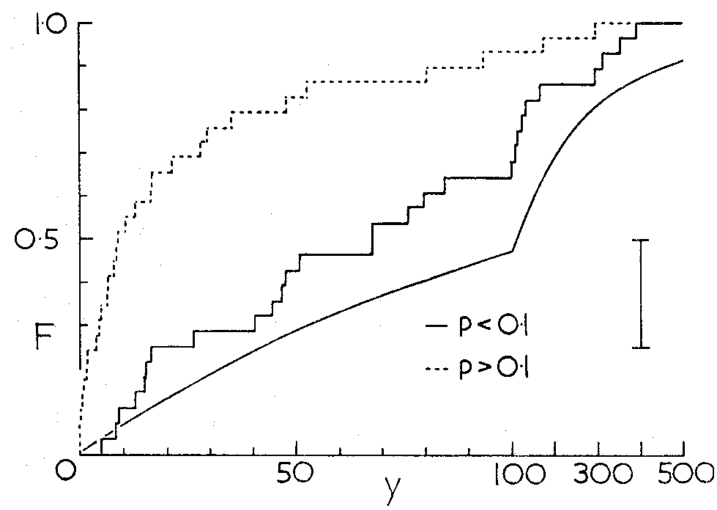

FIG. 15. *Cumulative distribution function, F, of energy differences, y, between initial and final binaries for resonances in case* $x = 500$. *Note the change in scale at* $y = 100$. *The solid experimental line gives results for the group with* $p < 0 \cdot 1$, *i.e. with low angular momentum, and the dotted line represents the cases for which* $p > 0 \cdot 1$. *The smooth line is derived from* (5.24) *and the bar is, as usual, the* 95 *per cent Kolmogorov–Smirnov acceptance limit.*

Because of the paucity of data in Fig. 15, the distribution (5.24) is compared in Fig. 16 with data published by Szebehely (1972b) on the disruption of a number of triple systems with vanishing angular momentum. The fit is acceptable even at the 20 per cent level.

It was noted in Section 5.4 that energy changes during wide encounters were generally much smaller, relatively, than those in eccentricity. Also, by (5.66) changes in e should be approximately independent of the 'phase' of the binary motion; now, in resonant systems formed after wide encounters, the third body makes repeated approaches to the binary along a slowly varying orbit, and so we expect that changes in the eccentricity of the binary may be systematic. These two statements are in fact verified on detailed study of the wide resonant systems which formed in the series of experiments discussed earlier in this section. Furthermore, Aarseth (1972a) noticed systematic evolution of the binary eccentricity in a triple system which formed in a real N-body problem, and previously van Albada (1968a) had drawn attention to this phenomenon. Again, detailed study of hard binaries in the N-body systems discussed in Section 2.3 above reveals that they are subject to much faster evolution of eccentricity than of energy.

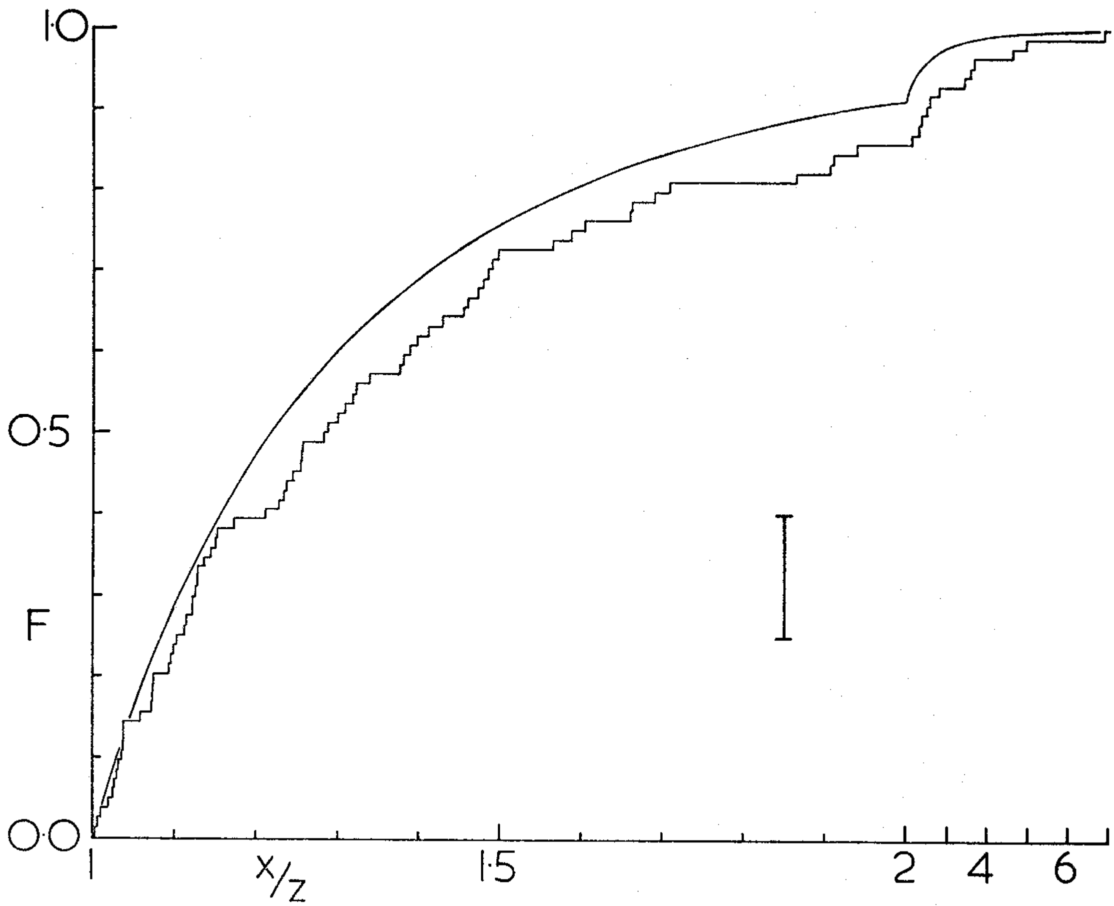

FIG. 16. *Cumulative distribution function, F, of (x/z) for* 84 *initially bound triple systems computed by Szebehely* (1972*b*), *where z is the binding energy of the system and x is that of the* final *binary. There is a change of scale at x/z = 2, and the curve is obtained from* (5.24).

Of the comparatively small number of numerical studies of hard binaries reported in the literature, mention may be made again of that by Lyttleton & Yabushita (1965), in case $e = 0$ initially and $m_2 = 0$, and that by Yabushita (1971), which was motivated more by galactic problems than by those in the solar system. Yabushita found cases of exchange in which the more massive component of the former binary was released to become a free particle; we saw in Section 5.2 that it would be unlikely for a very light component to escape. Agekian & Anosova (1968a) did not distinguish between exchange and resonance, and none of the initial binaries was very hard. Harrington's classification of his own results (1970), the initial orbit of the third body being parabolic, is rather vague, but it is clear that the outcome depends strongly on the distance of closest approach. This conclusion was also reached by Valtonen (1974), who noted in addition that the probability of exchange decreases with decreasing mass of the third body, as we expect from the discussion of Section 5.2. Hills' (1975) study showed that the mean change in energy satisfied $\langle y \rangle \simeq 0{\cdot}5x$ for the limit of hard binaries with all masses equal, and this agrees tolerably well with our theoretical result $\langle y \rangle \simeq 0.4\,x$, which is obtainable from (5.65).

Thorough studies of bound triple systems have been conducted by Agekian & Anosova (1967, 1968b) and by Anosova (1969). These were planar cases of vanishing or very small angular momentum, and the authors' observation that the outcome depends sensitively on the initial conditions may be adduced as further qualitative support for the relaxation hypothesis of Section 5.3. In the case of different masses, they showed that the escaper is most likely to be the particle with least mass, a conclusion confirmed by both Szebehely (1972b) and Valtonen (1974). This result is qualitatively consistent with the theory of Section 5.3, but the mass-dependence appears to be steeper than the theory would predict. Worrall (1967) considered a variety of initial conditions, but was often content to terminate the computation before disruption. Szebehely (1972a) and Valtonen (1974) noted that the lifetime tends to increase with the angular momentum.

6. APPLICATIONS

The theory of reaction rates given in Sections 3–5 was motivated in Sections 1 and 2 by a consideration of the role of binaries and three-body interactions in the dynamics of stellar systems. Its most important application is in this area, but has already been described elsewhere (Heggie 1974b, 1975), and so here we shall illustrate the use of the theory in a simpler problem, namely the lifetime of binaries in the galactic field. We close with some remarks on the theory that they may have formed in galactic clusters.

Almost half of the stars within 5·2 pc of the Sun are members of wide double or multiple systems (van de Kamp 1971), yielding a space density of 0·018 binaries per pc^3. Within 10 pc the density is 0·012 pc^{-3} (Brosche 1962), and only 11 of the 52 pairs in this volume of space have semi-major axes exceeding about 300 AU. The existence of still wider pairs has been pointed out by Miller (1967) and by Vandervoort (1968), though it is generally not known whether these are bound. On the other hand, the ' discovery ' of wide pairs among late dwarfs, reported by Lü & Upgren (1973), rests on a fallacious statistical procedure (Branch 1974).

Ambartsumian (1937) showed that the observed number density was in excess of that predicted by (2.13) by at least six orders of magnitude. He also found, for example, that the time scale for disruption of a binary with a semi-major axis even as large as 10^4 AU was about 5×10^9 yr. He used a formula of the type (4.31), which we have seen to be an underestimate, and furthermore he assumed a space density of 0·1 pc^{-3} for stars of mass 1 $M_\odot$. Since stars of this mass have much lower density, we note that again for this reason the time scale should be raised.

Our starting point for computing the lifetime is (4.32), using (4.13) modified in the obvious way to take account of the presence of a spectrum of masses m_3. Considering the large range of stellar masses, m, the dispersion of peculiar velocities, $\langle v^2(m) \rangle$, is remarkably insensitive to m (Gliese 1956), and it is comparable for single stars and for the centres of mass of binaries. Accordingly we may write approximately

$$t = \frac{1}{40}\sqrt{\frac{3}{2\pi}}\,\frac{M_{12}}{a}\,\{\langle v^2(\bar{m}_3)\rangle + \langle v^2(M_{12})\rangle\}^{1/2}\left(\int_0^\infty n(m_3)\,m_3^2\,dm_3\right)^{-1} \qquad (6.1)$$

where $\bar{m}_3$ is, roughly speaking, the mass contributing most to the integral. We have also neglected any anisotropy in the distribution of velocities. From data tabulated by Schmidt (1959) we find that the mass-integral is about $2{\cdot}7\times 10^{-2}\,M_\odot^2\,\mathrm{pc}^{-3}$, the greatest contribution coming from stars with mass of order $1\,M_\odot$, and so we shall adopt $\langle v^2\rangle^{1/2} = 40\ \mathrm{km\,s^{-1}}$. Equation (6.1) was calculated in units such that $G = 1$ and so, by dimensions, a factor G^{-1} is needed. Hence the e-folding time is approximately

$$t = 1{\cdot}7\times 10^{15}\,\frac{M_{12}}{a}\,\mathrm{yr},$$

where M_{12} is in solar units and a is in astronomical units. This agrees with Öpik's result (1973) to within a factor of about 2, and he also noticed that the stars of an intermediate mass contribute more than those of low or high mass. However, Ambartsumian's estimate, mentioned above, is too low by a factor exceeding 50.

The excessive number of wide pairs, noted by Ambartsumian, indicates that they are being disrupted at a far higher rate than that at which they are forming dynamically. Hence, an explanation for the observed abundance of binaries must be sought in phenomena accompanying stellar formation, i.e. fission or multiple condensation, or else in dynamical processes in operation in star clusters or associations. It may well prove possible to remove remaining objections to the fission theory (Lebovitz 1972), but Kumar (1972), who was troubled by difficulties with the theory of multiple condensation, and Gurevich & Levin (1950), have all been led to postulate that binaries are formed dynamically in clusters or associations, later evaporating into the field.

The most energetic pair that can form has about the binding energy of the cluster, whence its semi-major axis, a, satisfies $a \sim R_h m_1 m_2/N^2\langle m\rangle^2$, where m_i ($i = 1, 2$) are the masses of its components. We may take $N \sim 100$ and $R_h \sim 2$ pc (Hogg 1959), and since the hardest binary usually forms from the most massive stars, we may adopt $m_1 m_2/\langle m\rangle^2 \sim 100$, whence $a \sim 4\times 10^3$ AU, although it could be smaller for a richer cluster. However, only one or two binaries as 'close' as this can form from each cluster, which will also release into the field a large number of single stars, and it seems impossible thus to account for the large fraction of field stars which are observed to be members of binaries with semi-major axes of this size or less.

If dynamical processes are ruled out, then binaries must form during the formation or early evolution of stars, and we may therefore expect star clusters themselves to be richly endowed with binaries initially. Indeed their study has yielded evidence, direct and indirect, for the existence of perhaps a large population of wide binaries in some open clusters (Hogg 1959). The dynamics of a cluster richly endowed initially with binaries is a third problem in which results on three-body encounters have been applied (Hills 1975; Heggie 1975).

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APPENDIX

A1. NUMERICAL DETERMINATION OF CROSS-SECTIONS

The organization of the computer experiments mentioned in Sections 4.4 and 5.7 is described in this section, and we give a relation between the cross-section, for a certain class of events, and its probability, namely (A.2).

Units were so chosen that the initial velocity 'at infinity' of the third body relative to the centre of mass, C, of the binary was unity. Further, all masses were taken to be unity and, in each series of experiments, the initial binding energy of the binary was held fixed. The point C was located initially at rest at the origin of rectangular coordinates, the (x, y)-plane being the initial plane of motion of the binary and Cx being the line of apsides. The initial mean anomaly was selected randomly from a uniform distribution, and the initial eccentricity according to (2.15).

Each integration was started with the third body at a distance $r \gg a$ from C, where a was the initial semi-major axis of the binary. A maximum value, p_0, of the initial impact parameter was selected in such a way that, on the Keplerian approximation to the subsequent motion of the third star relative to C, the maximnm pericentric distance, q_0, between the two lay in the range $a \leqslant q_0 < 0{\cdot}2\, r$. Hence the initial relative perturbation on the binary was at most 1 per cent of that expected when the third particle was closest to C, and the neglect of perturbations during the part of the orbit prior to the commencement of the integration was justified to this accuracy. The initial velocity of the third particle was calculated from a formula like (3.13), and its initial position and velocity vectors were selected with random directions but in such a way as to generate the correct distribution of impact parameters. Hénon (1972c) has pointed out that insufficient care has sometimes been taken here in the past.

The integration was halted and a new case begun if at any time either of the following sets of conditions was met:

(i) the two particles closest to the centre of mass, P, of the triple system, form a binary with semi-major axis a_1, say; the distance of the remaining third particle from the centre of mass of this binary exceeds $(r/a)\, a_1$, where r and a are as defined above; and it is receding with a velocity sufficient, according to the Keplerian approximation, to escape 'to infinity';

(ii) the two particles closest to P are relatively unbound; the distance of the remaining third particle from their centre of mass exceeds r; and the same condition on the velocity holds.

Of the conditions in the first set, that on the distance was designed to ensure that perturbations in the succeeding, uncomputed parts of the orbit could be neglected. If the second set of conditions held, the binary was deemed to have been disrupted. In fact this is not strictly correct, for the pair could have become slightly bound subsequently as the third body receded. This could generally only happen if the separation of the first two bodies was not much less than r, for otherwise the perturbation on their relative motion would have been very small. Thus their final binding energy, ξ say, would have had to be much less than that of the original binary, x. The error in the experimentally determined destruction cross-section is accordingly a factor of order

$$\int_{-x}^{-x+\xi} \sigma(y)\, dy \Big/ \int_{-\infty}^{-x} \sigma(y)\, dy,$$

where $\sigma(y)\, dy$ is the cross-section for energy changes in the range $(y, y+dy)$, and $0 < \xi \ll x$. This error may be neglected.

Now it sometimes happened that all the conditions of the first set were met, save that the velocity of the third particle was insufficient to carry it to infinity. Following a suggestion of Mr M. J. Valtonen, the motion was then treated by the analytic formulae of Keplerian motion, on the one hand for the binary, and, on the other, for the third body relative to its centre of mass, C; until the third body was once more at a distance of $(r/a)\ a_1$ from C, but now approaching it. Thereafter numerical integration was resumed, and the previous criteria were again applied to the outcome of this new encounter. This sequence of events could happen many times in any one case.

Finally we relate the experimentally determined cross-section to that derived analytically. Suppose that a certain type of outcome, $\mathcal{O}$, is found in a fraction f of a series consisting of n trials. Then appropriate 95 per cent confidence limits for the probability, P, of $\mathcal{O}$ are given by (Lindgren 1960, p. 310).

$$P = f \pm 1{\cdot}96 \sqrt{\frac{f(1-f)}{n}}. \qquad \text{(A1)}$$

Bearing in mind that the maximum initial impact parameter was p_0, we may obtain an estimate of the cross-section, Σ, for $\mathcal{O}$ from

$$\Sigma = \pi p_0{}^2 P(1+6/r), \qquad \text{(A.2)}$$

the extra factor, as in (3.14), accounting for the focusing of the third particles in that part of the motion described prior to the commencement of the integration.

A2. COMPUTATIONAL TECHNIQUES

Here we make a few remarks on the integration methods used in these three-body experiments, and also in the larger N-body integrations quoted in Section 2.3.

Because of the r^{-2} singularity in the gravitational force, it is difficult to preserve the accuracy of an integration with Newton's equations of motion if a close approach occurs between two or more bodies. This difficulty can be mitigated somewhat by using a transformation of the independent variable, and such a method was used for some of the numerical work whose results appeared in Section 2.3 (Heggie 1971).

Since numerical errors arise mostly during close approaches of only two bodies, more powerful methods can be devised. The most satisfactory is KS-regularization (Stiefel & Scheifele 1971; Szebehely & Bettis 1971; Aarseth 1972b), removal of the r^{-2} singularity being achieved by transformation of both dependent and independent variables. Regularization is also possible, however, using a time-transformation only (Burdet 1967; Heggie 1973), and the method described in the second of these two papers was applied in the computation of the three-body systems for which results were discussed in Sections 4.4 and 5.7. This regularization and KS-regularization were used separately to obtain some of the numerical results described in Section 2.3. In future it should be advantageous for the computation of three-body systems to use one of the advanced regularization methods that have been developed recently (Aarseth & Zare 1974; Heggie 1974a).

The integration algorithm used was the force-polynomial fitting scheme described by Aarseth (1972b).