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Binary systems: higher order gravitational radiation damping and wave emission

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SUMMARY

The paper treats the motion of binary systems under the back-reaction of the gravitational radiation generated by the quasi-elliptic and quasi-hyperbolic post-Newtonian motions of the binaries. The angular momentum losses are calculated and, together with the already known energy losses, the changes of the eccentricities and semimajor axes are derived, allowing the determination of a detailed picture of the radiation damping in binary systems up to the 3.5 post-Newtonian order. The waveforms of the higher order gravitational radiation are presented. In particular, their dependencies on the periastron shifts are made explicit. Various cases of the motion of the binaries and of the waveforms are shown graphically.

1 INTRODUCTION

In regard of several aspects – the now excellent monitoring of the motion of pulsars in binary systems (see e.g. Taylor & Weisberg 1989; Damour & Taylor 1991), the future detection of gravitational waves (see e.g. Thorne 1987), the astrophysics of coalescing binaries (see e.g. Clark & Eardley 1977; Krolak & Schutz 1987; Redmount & Rees 1989) – a precise evaluation of the motion of close (relativistic) binary systems with compact components is of great interest, although, at present, there are no binary systems known for which our calculations will be important from an observational point of view.

In a recent paper by Blanchet & Schäfer (1989) the rate of emission of gravitational energy from orbiting or scattering binary systems with point-like components has been computed with an accuracy which goes one order of magnitude beyond the standard quadrupole formula [post-Newtonian (PN) or higher order gravitational wave emission]. Hereof the 3.5 PN rate of decrease of the orbital period and the semimajor axis of a binary system could be determined. However, the knowledge of the energy emission alone is not sufficient to derive the full change in time of these quantities.

In this paper we shall compute the higher order rate (3.5 PN approximation) of angular momentum loss in a binary system. Together with the corresponding rate of energy loss we are then able to calculate the changes in time of the orbital period, the periastron advance, the semimajor axis and the eccentricities of binary systems, generalizing the Newtonian works by Peters (1964) and Hansen (1972, for a correction see Turner 1977), in the cases of bound and unbound orbits, respectively. [Note that already in the post-Newtonian approximation the notions of 'semimajor axis' and 'eccentricity' become coordinate dependent. Also, in a given coordinate system the motion can no longer be described by one eccentricity alone; instead, one needs three of them (Damour & Deruelle 1985) in the cases where the eccentricities are different from zero (circular motion) or one (parabolic motion).] We also improve upon a recent paper by Lincoln & Will (1990) in that we treat the radiation damping at a higher order level.

Our treatment of the coalescence of binary systems, in particular, will reveal the limit of validity of radiation damping calculations in post-Newtonian frameworks, assuming that the internal structure of the bodies can be neglected (point-like masses). When the binaries contain two neutron stars this assumption breaks down near a separation of the two bodies of (in harmonic or isotropic coordinates) about $(10\text{–}7)GM/c^2$ (M, G and c denote the total mass of the system, the Newtonian gravitational constant and the velocity of light, respectively) where tidal stripping becomes important (Lincoln & Will 1990). For black holes the assumption should be valid until a separation near $2MG/c^2$ is reached.

We shall further determine the higher order gravitational waveforms which are generated by the post-Newtonian motion of the binaries. Besides getting different amplitudes for reasonably high velocities of the bodies compared to the Newtonian wave generation problem, our waveforms nicely show the influence of the periastron shift on the phase of the waves. For unbound orbits our developments generalize Newtonian results obtained by Turner (1977) and post-Newtonian bremsstrahlung (large orbital eccentricity) results worked out by Turner & Will (1978). Quite recently, Lincoln & Will (1990) have given post-Newtonian gravitational waveforms resulting from 'circular' orbits which decay by the emission of gravitational waves at leading order (quadrupole radiation damping). Our representation of the waveforms will be different in the sense that we give analytic expressions for a set of irreducible multipole moments, valid for arbitrary eccentricities, and that we describe the decay of the orbits by the secular variations (higher order radiation damping) of the orbital parameters.

The paper is organized as follows. In Section 2 we derive a higher order angular momentum loss formula for a binary system, and we deduce a lowest order linear momentum loss expression which is valid, also, for weakly self-gravitating binaries. In Section 3 the higher order (post-Newtonian) decay of a binary system is worked out. In particular, the time evolution of the second post-Newtonian periastron shift will be given. Section 4 is devoted to the higher order radiation damping in scattering processes. The smallest impact parameter for scattering processes will be determined. In Section 5 multipole components of the gravitational radiation for a bound and a scattering orbit are presented. Appendix A gives some improved recoil calculations, and in Appendix B a collection of the relevant pure-spin tensor harmonics can be found.

2 GRAVITATIONAL WAVE GENERATION IN BINARY SYSTEMS

In the general relativity theory the gravitational waves emitted by isolated systems in asymptotically flat space–times are expected to exist in the asymptotic rest frame of the source in the form (Thorne 1980)

$$h_{jk}^{TT}=\frac{G}{c^4 r}P_{jkim}(\boldsymbol{n})\sum_{l=2}^{\infty}\left[\left(\frac{1}{c}\right)^{l-2}\left(\frac{4}{l!}\right)\overset{(l)}{\mathscr{I}}_{imA_{l-2}}(t-r/c)N_{A_{l-2}}+\left(\frac{1}{c}\right)^{l-1}\left(\frac{8l}{(l+1)!}\right)\varepsilon_{pq(i}\overset{(l)}{\mathscr{J}}_{m)pA_{l-2}}(t-r/c)n_q N_{A_{l-2}}\right], \tag{1}$$

where $\mathscr{I}_{A_l}$ and $\mathscr{J}_{A_l}$ are symmetric and tracefree (STF) radiative mass and current multipole moments which parametrize the radiation field in a Cartesian coordinate basis. A_l denotes a multi-index of length l, i.e. $A_l=a_1a_2\ldots a_l$, where the a's run over 1, 2, 3. $N_{A_l}:=n_{a_1}n_{a_2}\ldots n_{a_l}$, and n_a is the unit normal in the direction of the radial vector $\boldsymbol{r}$, pointing from the source to the observer. The parentheses around indices mean to take the symmetric part of the corresponding tensor. $\overset{(l)}{\mathscr{I}}:=d^l\mathscr{I}/dt^l$ and $\overset{(l)}{\mathscr{J}}:=d^l\mathscr{J}/dt^l$, and $P_{ijkl}(\boldsymbol{n})$ denotes the transverse traceless projection orthogonal to $\boldsymbol{r}$ acting on symmetric tensors:

$$P_{ijkl}(\boldsymbol{n}):=(\delta_{ik}-n_in_k)(\delta_{jl}-n_jn_l)-\tfrac{1}{2}(\delta_{ij}-n_in_j)(\delta_{kl}-n_kn_l). \tag{2}$$

ε_{ijk} is the usual antisymmetric tensor of Levi-Civita, and $r=|\boldsymbol{r}|$ is the Cartesian observer–source distance.

Assuming balance equations between the radiation field and its source, one obtains the following expressions for the energy, angular momentum and linear momentum losses in terms of the STF-multipole moments (Thorne 1980)

$$\frac{d\tilde{E}}{dt}=-\frac{G}{c^5}\sum_{l=2}^{\infty}\left[\left(\frac{1}{c}\right)^{2(l-2)}\frac{(l+1)(l+2)}{l(l-1)l!(2l+1)!!}\overset{(l+1)}{\mathscr{I}}_{A_l}\overset{(l+1)}{\mathscr{I}}_{A_l}+\left(\frac{1}{c}\right)^{2(l-1)}\frac{4l(l+2)}{(l-1)(l+1)!(2l+1)!!}\overset{(l+1)}{\mathscr{J}}_{A_l}\overset{(l+1)}{\mathscr{J}}_{A_l}\right], \tag{3}$$

$$\frac{d\tilde{J}_j}{dt}=-\frac{G}{c^5}\sum_{l=2}^{\infty}\left[\left(\frac{1}{c}\right)^{2(l-2)}\frac{(l+1)(l+2)}{(l-1)l!(2l+1)!!}\varepsilon_{jpq}\overset{(l)}{\mathscr{I}}_{pA_{l-1}}\overset{(l+1)}{\mathscr{I}}_{qA_{l-1}}+\left(\frac{1}{c}\right)^{2(l-1)}\frac{4l^2(l+2)}{(l-1)(l+1)!(2l+1)!!}\varepsilon_{jpq}\overset{(l)}{\mathscr{J}}_{pA_{l-1}}\overset{(l+1)}{\mathscr{J}}_{qA_{l-1}}\right], \tag{4}$$

$$\frac{d\tilde{P}_j}{dt}=-\frac{G}{c^7}\sum_{l=2}^{\infty}\left[\left(\frac{1}{c}\right)^{2(l-2)}\frac{2(l+2)(l+3)}{l(l+1)!(2l+3)!!}\overset{(l+2)}{\mathscr{I}}_{jA_l}\overset{(l+1)}{\mathscr{I}}_{A_l}+\left(\frac{1}{c}\right)^{2(l-1)}\frac{8(l+3)}{(l+1)!(2l+3)!!}\overset{(l+2)}{\mathscr{J}}_{jA_l}\overset{(l+1)}{\mathscr{J}}_{A_l}\right.$$
$$\left.+\left(\frac{1}{c}\right)^{2(l-2)}\frac{8(l+2)}{(l-1)(l+1)!(2l+1)!!}\varepsilon_{jpq}\overset{(l+1)}{\mathscr{I}}_{pA_{l-1}}\overset{(l+1)}{\mathscr{J}}_{qA_{l-1}}\right]. \tag{5}$$

In contrast to the result we would get from the paper by Thorne (1980) we do not write average brackets because in a Hamiltonian formulation of the matter-field dynamics the expressions (3)–(5) emerge naturally; see the equations (5.24)–(5.27) in the paper by Blanchet, Damour & Schäfer (1990).

Our interest in the post-Newtonian wave generation calls for the computation of the moments $\mathscr{I}_{A_l}$ and $\mathscr{J}_{A_l}$ with $l=2$, 3, 4 and 2, 3, respectively. The most simple and efficient (and at the same time mathematically well defined) representation of the 1PN moment $\mathscr{I}_{A_2}$ was given by Blanchet & Damour (1989). Specialized to a two-body system with its centre of energy at rest in a harmonic coordinate system (at the level of our approximation the harmonic and standard post-Newtonian coordinate systems are both isotropic and they differ only by a simple transformation of the time coordinate which does not affect our formulae) we get the explicit forms (see Blanchet & Schäfer 1989)

$$\mathscr{I}_{ij}=\mu R_{\langle ij\rangle}\left[1+\frac{29}{42}(1-3\nu)\frac{V^2}{c^2}-\frac{1}{7}(5-8\nu)\frac{GM}{c^2R}\right]+\frac{\mu(1-3\nu)}{21c^2}[-12(RV)R_{\langle i}V_{j\rangle}+11R^2V_{\langle ij\rangle}], \tag{6}$$

$$\mathscr{I}_{ijk}=-\mu\sqrt{1-4\nu}R_{\langle ijk\rangle}, \tag{7}$$

$$\mathscr{I}_{ijkl}=\mu(1-3\nu)R_{\langle ijkl\rangle}, \tag{8}$$

$$\mathscr{J}_{ij}=-\mu\sqrt{1-4\nu}\,\varepsilon_{ab\langle i}R_{j\rangle a}V_b, \tag{9}$$

$$\mathscr{J}_{ijk}=\mu(1-3\nu)\varepsilon_{ab\langle k}R_{ij\rangle a}V_b, \tag{10}$$

where R_i and V_i denote the relative position and velocity vectors of the two bodies, respectively ($\boldsymbol{R}$ is pointing from body 2 to body 1), $\mu := m_1 m_2/M$ is the reduced mass, and $\nu := \mu/M$ $(0 \leq \nu \leq 1/4)$. Also, the definitions $R_{\langle ij \rangle} = R_{\langle i} R_{j \rangle}$, etc. have been introduced where the brackets mean that the symmetric and tracefree part of the tensor has to be taken.

The computation of the time derivatives of the radiative multipole moments needs the post-Newtonian equations for the relative motion

$$\frac{d\boldsymbol{V}}{dt} = -\frac{GM}{R^3}\boldsymbol{R} + \frac{GM}{c^2R^3}\left\{\boldsymbol{R}\left[\frac{GM}{R}(4+2\nu) - V^2(1+3\nu) + \frac{3}{2}\nu\frac{(RV)^2}{R^2}\right] + (4-2\nu)\boldsymbol{V}(\boldsymbol{R}\cdot\boldsymbol{V})\right\}. \tag{11}$$

For the time derivatives of the above multipole moments which appear in the expressions for the radiation field (equation 1), we obtain

$$\overset{(2)}{\mathscr{I}}_{ij} = 2\mu V_{\langle ij\rangle}\left[1 + \frac{V^2}{c^2}\frac{9}{14}(1-3\nu) - \frac{GM}{c^2R}\frac{1}{21}(25-54\nu)\right] + 2\mu R_{\langle i}V_{j\rangle}\left[\frac{GM}{c^2R^3}(RV)\frac{1}{7}(25+9\nu)\right]$$
$$-\mu R_{\langle ij\rangle}\frac{GM}{R^3}\left[2 + \frac{V^2}{c^2}\frac{1}{21}(61+48\nu) - \frac{(RV)^2}{R^2c^2}\frac{2}{7}(1-3\nu) - \frac{GM}{c^2R}(10-\nu)\right], \tag{12}$$

$$\overset{(3)}{\mathscr{I}}_{ijk} = -3\mu\sqrt{1-4\nu}\left[3\frac{GM(RV)}{R^5}R_{\langle ijk\rangle} - 7\frac{GM}{R^3}V_{\langle i}R_{jk\rangle} + 2V_{\langle ijk\rangle}\right], \tag{13}$$

$$\overset{(4)}{\mathscr{I}}_{ijkl} = 4\mu(1-3\nu)\left[6V_{\langle ijkl\rangle} - 48\frac{GM}{R^3}R_{\langle ij}V_{kl\rangle} + 42\frac{GM}{R^5}(RV)R_{\langle ijk}V_{l\rangle} + \frac{GM}{R^3}R_{\langle ijkl\rangle}\left(7\frac{GM}{R^3} - 15\frac{(RV)^2}{R^4} + 3\frac{V^2}{R^2}\right)\right], \tag{14}$$

$$\overset{(2)}{\mathscr{J}}_{ij} = \mu\sqrt{1-4\nu}\frac{GM}{R^3}\varepsilon_{ab\langle i}R_{j\rangle}R_aV_b, \tag{15}$$

$$\overset{(3)}{\mathscr{J}}_{ijk} = 2\mu(1-3\nu)\frac{GM}{R^3}\varepsilon_{ab\langle k}\left[3\frac{(RV)}{R^2}R_{ij\rangle} - 4V_iR_{j\rangle}\right]R_aV_b. \tag{16}$$

By the aid of three additional time derivatives [see the equations (3.38) to (3.40) in the paper by Blanchet & Schäfer (1989)] we can straightforwardly derive explicit loss expressions for energy, angular momentum and linear momentum, using the equations (3) to (5) in our approximation. They turn out to be

$$\frac{d\tilde{E}}{dt} = -\frac{8}{15}\frac{G^3M^2\mu^2}{c^5R^4}\left\{12V^2 - 11\frac{(RV)^2}{R^2} + \frac{1}{28c^2}\left[(785-852\nu)V^4 - 2(1487-1392\nu)\frac{(RV)^2}{R^2}V^2 + 3(687-620\nu)\frac{(RV)^4}{R^4}\right.\right.$$
$$\left.\left.-160(17-\nu)\frac{GMV^2}{R} + 8(367-15\nu)\frac{(RV)^2}{R^2}\frac{GM}{R} + 16(1-4\nu)\frac{G^2M^2}{R^2}\right]\right\}, \tag{17}$$

$$\frac{d\tilde{J}_j}{dt} = -\frac{8}{5}\frac{G^2M\mu^2}{R^3c^5}\varepsilon_{jpq}R_pV_q\left\{2V^2 - 3\frac{(RV)^2}{R^2} + 2\frac{GM}{R} + \frac{1}{84c^2}\left[(307-548\nu)V^4 - 6(74-277\nu)\frac{(RV)^2}{R^2}V^2 + 15(19-72\nu)\frac{(RV)^4}{R^4}\right.\right.$$
$$\left.\left.-4(58+95\nu)\frac{GM}{R}V^2 + 2(372+197\nu)\frac{GM}{R}\frac{(RV)^2}{R^2} - 2(745-2\nu)\frac{G^2M^2}{R^2}\right]\right\}, \tag{18}$$

$$\frac{d\tilde{P}_j}{dt} = -\frac{8}{105}\frac{m_1-m_2}{M}\frac{G^3M^2\mu^2}{c^7R^6}\left\{R_j\left[55V^2(RV) + 12\frac{GM}{R}(RV) - 45\frac{(RV)^3}{R^2}\right] + V_j[38(RV)^2 - 50R^2V^2 - 8GMR]\right\}. \tag{19}$$

The expression (17) is already known from Blanchet & Schäfer (1989) and Wagoner & Will (1976), as corrected by an erratum (1977).

3 DECAY OF QUASI-ELLIPTIC ORBITS

The explicit solution of the equations of motion (11) has found several representations in the literature. The most Keplerian-like representation of the solution was obtained by Damour & Deruelle (1985): in usual polar coordinates, $\boldsymbol{R} = R(\cos\Theta, \sin\Theta, 0)$ associated with the harmonic coordinates, the quasi-elliptic motion can be put into the form

$$n \cdot t = u - e_t \sin u, \tag{20}$$

$$R = a_R(1 - e_R\cos u), \tag{21}$$

$$\Theta = 2(k+1)\arctan\left[\left(\frac{1+e_\Theta}{1-e_\Theta}\right)^{1/2}\tan\frac{u}{2}\right], \tag{22}$$

where u is some 'eccentric anomaly' which parametrizes the motion. The constants n, k, a_R, e_t, e_R and e_Θ are the mean motion ($n=2\pi/P$, where P denotes the orbital period), the periastron advance, some 'semimajor axis', and some 'time, radial and angular eccentricity', respectively. The three eccentricities are related through

$$e_t = e_R\left[1-(3\nu-8)\frac{E}{c^2}\right], \tag{23}$$

$$e_\Theta = e_R\left[1-\nu\frac{E}{c^2}\right], \tag{24}$$

where E denotes a reduced 1PN conserved energy of the system, $E := \tilde{E}/\mu$, and the other parameters are given by

$$a_R = -\frac{GM}{2E}\left[1-\frac{1}{2}(\nu-7)\frac{E}{c^2}\right], \tag{25}$$

$$n = \frac{(-2E)^{3/2}}{GM}\left[1-\frac{1}{4}(\nu-15)\frac{E}{c^2}\right], \tag{26}$$

$$e_R^2 = 1+2Eh^2+[2(\nu-6)+5(\nu-3)Eh^2]\frac{E}{c^2}, \tag{27}$$

$$k = \frac{3}{h^2c^2}, \tag{28}$$

where $h := \tilde{J}/GM\mu$ is a reduced 1PN conserved angular momentum of the system. k measures the (fractional) angle of precession of the periastron per revolution: $\Delta\Theta = 2\pi k$. Note that in contrast to the eccentricities e_t, e_Θ, e_R and the semimajor axis a_R the period P, the periastron advance k, the energy E and the angular momentum h are coordinate independent quantities (e.g. see Damour & Schäfer 1988). For example, the relations between the radial eccentricities and the semimajor axes in isotropic and Schwarzschild coordinates at the 1PN level read $(e_R)_S = e_R(1+2E/c^2)$ and $(a_R)_S = a_R(1-2E/c^2)$, where S refers to the Schwarzschild coordinates.

The quasi-elliptic motion is periodic in the time t with period P. Therefore, we reasonably average quantities over one period. Denoting the average by '$\langle\ \rangle$', after rather long but straightforward calculations we end up with the following expressions for the averaged angular momentum, where the equations (20)–(28) have been taken into account:

$$\left\langle\frac{d\tilde{J}}{dt}\right\rangle = -\frac{4}{5}\frac{c^2\nu\mu}{(1-e_R^2)^3}\left(\frac{GM}{a_Rc^2}\right)^{7/2}\left\{(8+7e_R^2)(1-e_R^2)-\frac{1}{336}\frac{GM}{a_Rc^2}[8(2423+588\nu)+8(2210+1841\nu)e_R^2-(14279-3388\nu)e_R^4]\right\}, \tag{29}$$

or, as a function of E and h,

$$\left\langle\frac{d\tilde{J}}{dt}\right\rangle = -\frac{4}{5}\nu\mu\frac{(-2E)^{3/2}}{c^5h^4}\left\{15+14Eh^2+\frac{1}{336}\frac{1}{h^2c^2}[35(1077-940\nu)+84(760-763\nu)Eh^2+4(8693-4270\nu)(Eh^2)^2]\right\}. \tag{30}$$

We can pick up the expression for the energy loss, corresponding to equation (29), from the paper by Blanchet & Schäfer (1989). It reads

$$\left\langle\frac{d\tilde{E}}{dt}\right\rangle = -\frac{\nu\mu}{15GMc^5}\frac{(-2E)^{3/2}}{h^7}\left\{425+732Eh^2+148(Eh^2)^2+\frac{1}{h^2c^2}\left[\left(\frac{40341}{8}-\frac{5635}{2}\nu\right)+\left(10065-\frac{32225}{4}\nu\right)Eh^2\right.\right.$$
$$\left.\left.+\left(\frac{85047}{14}-5415\nu\right)(Eh^2)^2+\left(\frac{6278}{7}-481\nu\right)(Eh^2)^3\right]\right\}. \tag{31}$$

The transformation of the expression (31) into the variables a_R and e_R yields

$$\left\langle\frac{d\tilde{E}}{dt}\right\rangle = -\frac{1}{15}\frac{c^3}{GM}\frac{c^2\nu\mu}{(1-e_R^2)^{9/2}}\left(\frac{GM}{a_Rc^2}\right)^5\left\{(96+292e_R^2+37e_R^4)(1-e_R^2)-\frac{1}{56}\frac{GM}{a_Rc^2}[16(2927+420\nu)+8(24833+4704\nu)e_R^2\right.$$
$$\left.-6(2555-4676\nu)e_R^4-(12753-2072\nu)e_R^6]\right\}. \tag{32}$$

The special case of circular orbits, $e_R=0\ (=e_t=e_\Theta)$, results in

$$\frac{d\tilde{E}}{dt}=\omega\frac{d\tilde{J}}{dt}, \tag{33}$$

where ω denotes the angular velocity of the relative motion

$$\omega=n(1+k). \tag{34}$$

With the aid of the equations (26), (27), (29) and (32) a complete set of equations for the decay of quasi-elliptic orbits can be derived in terms of orbital elements [another complete set consists of the equations (30) and (31)]. We obtain

$$\left\langle\frac{da_R}{dt}\right\rangle=-\frac{2c}{15}\frac{\nu}{(1-e_R^2)^{9/2}}\left(\frac{GM}{a_Rc^2}\right)^3\left\{(96+292e_R^2+37e_R^4)(1-e_R^2)-\frac{1}{28}\frac{GM}{a_Rc^2}[(14008+4704\nu)+(80124+21560\nu)e_R^2+(17325+10458\nu)e_R^4-\tfrac{1}{2}(5501-1036\nu)e_R^6]\right\}, \tag{35}$$

$$\left\langle\frac{de_R}{dt}\right\rangle=-\frac{1}{15}\frac{\nu c^3}{GM}\left(\frac{GM}{a_Rc^2}\right)^4\frac{e_R}{(1-e_R^2)^{7/2}}\left\{(304+121e_R^2)(1-e_R^2)-\frac{1}{56}\frac{GM}{a_Rc^2}[8(16705+4676\nu)+12(9082+2807\nu)e_R^2-(25211-3388\nu)e_R^4]\right\}. \tag{36}$$

The special case of circular orbits takes the simple form

$$\left\langle\frac{da_R}{dt}\right\rangle=-\frac{64c\nu}{5}\left(\frac{GM}{a_Rc^2}\right)^3\left[1-\frac{GM}{a_Rc^2}\frac{1}{336}(1751+588\nu)\right]. \tag{37}$$

In view of a more precise picture of the decay of binary orbits we adapt from the paper by Damour & Schäfer (1988) (see also Damour & Schäfer 1987) the periastron shift at 2PN approximation:

$$k=\frac{3}{h^2c^2}\left[1+\frac{1}{2}(5-2\nu)\frac{E}{c^2}+\frac{5}{4}(7-2\nu)\frac{1}{h^2c^2}\right]. \tag{38}$$

The evaluation of this expression along the decaying orbits, with and without the last two terms, will give us useful information about the relevance and the validity of our post-Newtonian approximation calculations (see Fig. 4).

In the test body limit ($\nu\to 0$) the exact formula of the perihelion shift has been obtained by Damour & Schäfer (1988) in a very compact and explicit form. In the circular case, this formula reads:

$$k=\left(1-\frac{12}{h^2c^2}\right)^{-1/4}-1. \tag{39}$$

The last stable circular orbit is known to have the values $h^2c^2=12$ and

$$-\frac{E}{c^2}\left(1+\frac{1}{2}\frac{E}{c^2}\right)=\frac{1}{18},$$

for $\nu=0$. Equation (39) shows that there the perihelion shift tends to infinity. The post-Newtonian expansion of equation (39), i.e. the expansion in $1/c^2$, obviously loses its meaning if $hc\to 2\sqrt{3}$. This in particular indicates that any truncated expansion, e.g. the 2PN approximation, will badly describe the orbits near the last stable circular orbit. One should keep this in mind when one is discussing the decay of orbits in the two-body case.

In the following we shall present several graphs for making the dynamical evolution more explicit. [As the influence of the mass ratio ν on to the time evolution is mainly determined by the (ν-)prefactors in the equations (30), (31) and (35), (36), which can be absorbed into the time-coordinate, we may restrict ourselves to the equal-mass value $\nu=0.25$. Furthermore, we restrict the values of the eccentricities to zero, because this is the most important case for the late-time evolution of our system.]

A complete set of variables are E and h. Figs 1 and 2 show their Newtonian and post-Newtonian time evolutions for two different initial values of the energy. In Fig. 2, near $4\times10^4 GM/c^3$, the limit of validity of our post-Newtonian approximation is reached (there the relative difference of the Newtonian and post-Newtonian angular momentum values exceeds 30 per cent). This corresponds to a 1PN radius of $10GM/c^2$. For the same initial conditions as in Fig. 2, Figs 3 and 4 respectively present orbital periods and periastron advances as functions of time. Here it is interesting to point out that, compared to the 2.5PN radiation damping, the 3.5PN radiation damping slows down the acceleration of the 2PN periastron shift. Only the 2.5PN damping has been treated by Lincoln & Will (1990). Fig. 5 shows time evolutions of the radius of a circular orbit. Adjusted to a 1PN

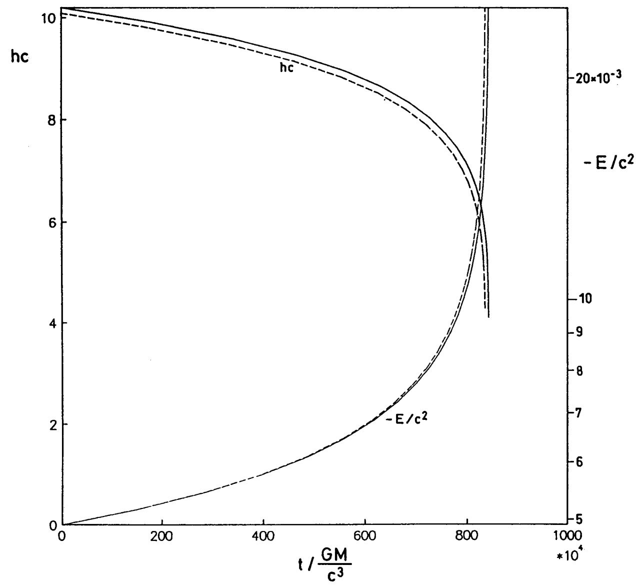

Figure 1. Angular momentum (hc) and energy $(-E/c^2)$ in dimensionless units as functions of time for circular orbits $(e_R=0)$, equal masses $(\nu=0.25)$ and initial energy $E/c^2=-0.004916$, corresponding to a 1PN radius of $a_R=100GM/C^2$. Dashed (full) line: 2.5PN (3.5PN) radiation damping of the Newtonian (post-Newtonian) expressions.

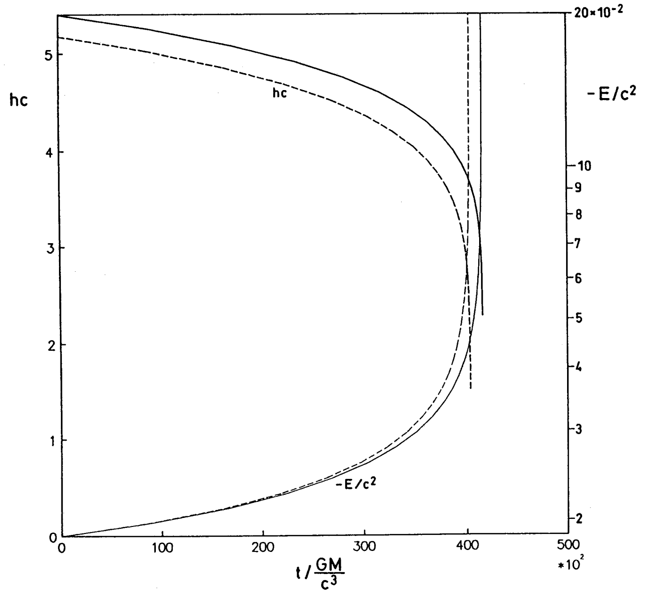

Figure 2. As Fig. 1 but with initial energy $E/c^2=-0.01865$, corresponding to a 1PN radius of $a_R=25GM/c^2$.

system with 3.5PN radiation damping are two Newtonian systems: one, which starts with the same initial energy, and the other, which has the same initial radius. In the first case the lifetimes are much more similar to each other than in the second case.

4 RADIATION DAMPING IN SCATTERING PROCESSES

The solutions for the quasi-hyperbolic motions can be obtained simply from the quasi-elliptic orbits by the substitutions $a_R=-\bar{a}_R$, $n=-i\bar{n}$ and $u=iv$, considering $\bar{a}_R$, $\bar{n}$ and v as new real variables, $-\infty<v<+\infty$, and taking $E>0$ and e_R, e_t and $e_\Theta>1$. Useful quantities for a characterization of scattering processes are the impact parameter, say b, and the absolute value of the relative velocity at infinity, V_∞. For these two quantities the post-Newtonian dynamics gives the expressions

$$b=\frac{hGM}{\sqrt{2E}}\left[1-\frac{1}{4}(1-3\nu)\frac{E}{c^2}\right], \tag{40}$$

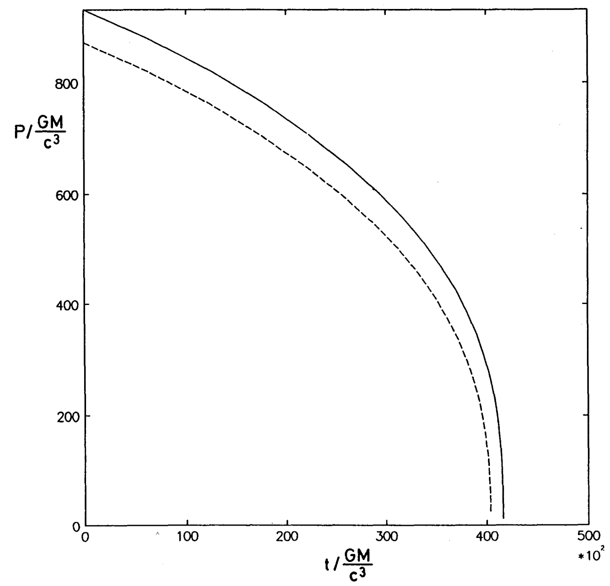

Figure 3. Orbital period as a function of time for circular orbits, equal masses and initial energy $E/c^2=-0.01865$. Dashed (full) line: 2.5PN (3.5PN) radiation damping of the Newtonian (post-Newtonian) expression.

$$V_\infty=\sqrt{2E}\left[1-\frac{3}{4}(1-3\nu)\frac{E}{c^2}\right], \tag{41}$$

($bV_\infty=|\boldsymbol{R}\times \boldsymbol{V}|$ for $R\to\infty$). With the aid of the analytic continuation procedure introduced by Blanchet & Schäfer (1989) we derive the expression for the total (integrated over the entire orbit) angular momentum loss, $\Delta\tilde{J}$, in case of quasi-hyperbolic orbits. The formula we obtain reads

$$\Delta\tilde{J}=-\frac{8}{5}\frac{GM\mu}{c}\frac{\nu}{(e_R^2-1)^3}\left(\frac{GM}{\bar{a}_Rc^2}\right)^2\left[\!\!\left[(e_R^2-1)\left[\arccos\left(-\frac{1}{e_R}\right)(8+7e_R^2)+\sqrt{e_R^2-1}\,(13+2e_R^2)\right]\right.\right.$$
$$-\frac{1}{1008}\frac{GM}{\bar{a}_Rc^2}\left\{\arccos\left(-\frac{1}{e_R}\right)[24(911-999\nu)+24(2399+5330\nu)e_R^2-3(3695+11828\nu)e_R^4]\right.$$
$$\left.\left.\left.+\sqrt{e_R^2-1}\,[2(24267+17668\nu)+(27021+32116\nu)e_R^2-144(50-7\nu)e_R^4]\right\}\right]\!\!\right], \tag{42}$$

or, as a function of E and h,

$$\Delta\tilde{J}=-\frac{8}{5}\frac{GM\mu\nu}{c^5h^4}\left[\!\!\left[\arccos\left(-\frac{1}{\sqrt{1+2Eh^2}}\right)(15+14Eh^2)+\sqrt{2Eh^2}(15+4Eh^2)\right.\right.$$
$$+\frac{1}{1008}\frac{1}{h^2c^2}\left\{\arccos\left(-\frac{1}{\sqrt{1+2Eh^2}}\right)[105(1077-940\nu)+252(535-748\nu)Eh^2+12(4283+10064\nu)(Eh^2)^2]\right.$$
$$\left.\left.\left.+\frac{\sqrt{2Eh^2}}{1+2Eh^2}[105(1077-940\nu)+224(1275-1429\nu)Eh^2+4(42711-61600\nu)(Eh^2)^2+288(109-35\nu)(Eh^2)^3]\right\}\right]\!\!\right]. \tag{43}$$

The lowest order part of equation (42) has been derived by Hansen (1972). In his formula, however, our factor 2 in the last term is erroneously replaced by 1.

The total energy loss, $\Delta\tilde{E}$, has been obtained by Blanchet & Schäfer (1989) in the form

$$\Delta\tilde{E}=-\frac{2}{15c^5}\frac{\mu\nu}{h^7}\left[\!\!\left[\arccos\left(-\frac{1}{e_R}\right)(96+292e_R^2+37e_R^4)+\tfrac{1}{3}\sqrt{e_R^2-1}(602+673e_R^2)\right.\right.$$
$$+\frac{1}{840}\frac{1}{h^2c^2}\left\{15\arccos\left(-\frac{1}{e_R}\right)[(52624-9408\nu)+(117288-61936\nu)e_R^2+(94542-78148\nu)e_R^4+(17933-8288\nu)e_R^6]\right.$$
$$\left.\left.\left.+\sqrt{e_R^2-1}\,[(1516596-312200\nu)+(1447788-1251460\nu)e_R^2+(1271421-803040\nu)e_R^4]\right\}\right]\!\!\right]. \tag{44}$$

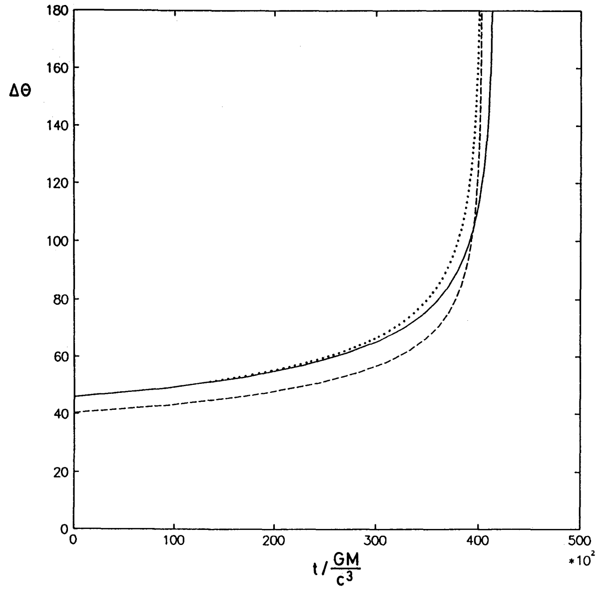

Figure 4. Periastron advance (in degrees) per orbital revolution ($\Delta\Theta = 360° \times k$) as a function of time for circular orbits, equal masses and initial energy $E/c^2 = -0.01865$. Dashed (full) line: 2.5PN (3.5PN) radiation damping of the 1PN (2PN) expression. Dotted line: 2PN expression (equation 38) plotted with 2.5PN radiation damping.

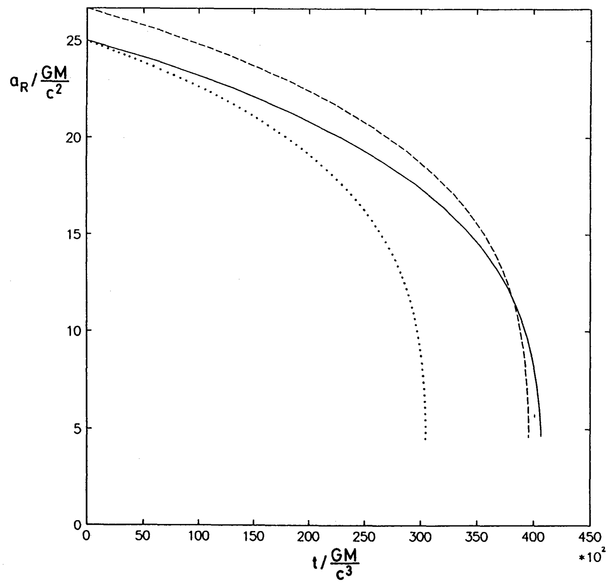

Figure 5. Radius as a function of time for circular orbits and equal masses. Dashed (full) line: 2.5PN (3.5PN) radiation damping of the Newtonian (post-Newtonian) expression with initial energy $E/c^2 = -0.01865$, corresponding to a 1PN radius of $a_R = 25GM/c^2$. Dotted line: 2.5PN radiation damping of the Newtonian expression with initial Newtonian radius $a_R = 25GM/c^2$.

As a function of E and h, equation (44) takes the form

$$\Delta\tilde{E} = -\frac{2}{15c^5}\frac{\mu\nu}{h^7}\Bigg[\arccos\left(-\frac{1}{\sqrt{1+2Eh^2}}\right)[425+732Eh^2+148(Eh^2)^2]+\tfrac{1}{3}\sqrt{2Eh^2}\,(1275+1346Eh^2)$$

$$+\frac{1}{840}\frac{1}{h^2c^2}\Bigg\{\arccos\left(-\frac{1}{\sqrt{1+2Eh^2}}\right)[735(5763-3220\nu)+450(15813-14840\nu)Eh^2$$

$$+180(15539-24416\nu)(Eh^2)^2+120(2393-3108\nu)(Eh^2)^3]+\frac{\sqrt{2Eh^2}}{1+2Eh^2}[105(40341-22540\nu)$$

$$+70(182337-140480\nu)Eh^2+4(2506431-3009160\nu)(Eh^2)^2+24(89907-156380\nu)(Eh^2)^3]\Bigg\}\Bigg]. \quad (45)$$

The lowest order part of equation (44) coincides with the expression found by Turner (1977) which corrected an earlier result by Hansen (1972). One gets the corresponding results for the (hypothetical) quasi-parabolic motion simply by putting $E=0$ in the results for the quasi-hyperbolic motion.

As a first application let us determine the smallest impact parameter where no radiation capture of the two bodies is taking place. The condition for the radiation capture reads

$$0 \le \mu E < |\Delta \tilde{E}|. \tag{46}$$

The insertion of the expansion of, e.g., equation (45) around $E=0$ (while holding h fixed) into equation (46) yields

$$0 \le E < \frac{170\pi\nu}{3c^5 h^7}\left[1+\frac{1}{h^2c^2}\frac{7}{3400}(5763-3220\nu)\right]. \tag{47}$$

The elimination of h between the equations (40) and (47), in our approximation, results in

$$b_{\rm min}=\frac{GM}{\sqrt{2}c^2}\left(\frac{E}{c^2}\right)^{-9/14}\left(\frac{170\pi\nu}{3}\right)^{1/7}\left[1+\left(\frac{3}{170\pi\nu}\frac{E}{c^2}\right)^{2/7}\frac{1}{3400}(5763-3220\nu)\right]. \tag{48}$$

Fig. 6 shows the ratio $b_{\rm min}/(2GM/c^2)$ as a function of E/c^2 for different values of m_1/m_2.

Let us now discuss the change of the impact parameter in a scattering process due to gravitational radiation damping at the 1PN approximation. As this will be a very small effect, we may apply the formula

$$\delta b=\frac{\partial b}{\partial E}\frac{\Delta\tilde{E}}{\mu}+\frac{\partial b}{\partial h}\frac{\Delta\tilde{J}}{GM\mu} \tag{49}$$

for the evaluation of this effect, with $\Delta\tilde{E}$ and $\Delta\tilde{J}$ given by equations (45) and (43), respectively. We find

$$\begin{aligned}\delta b=\frac{2\nu}{15c^5}\frac{GM}{h^6(2E)^{3/2}}\Bigg[&\arccos\left(-\frac{1}{\sqrt{1+2Eh^2}}\right)[425+372Eh^2-188(Eh^2)^2]+\tfrac{1}{3}\sqrt{2Eh^2}\,[1275+284Eh^2-288(Eh^2)^2]\\ &+\frac{1}{168}\frac{1}{h^2c^2}\Bigg\{\arccos\left(-\frac{1}{\sqrt{1+2Eh^2}}\right)[147(5763-3220\nu)+210(4708-4735\nu)Eh^2\\ &+36(1833-7294\nu)(Eh^2)^2-24(5326+25777\nu)(Eh^2)^3]\\ &+\frac{1}{5}\frac{\sqrt{2Eh^2}}{(1+2Eh^2)}[105(40341-22540\nu)+70(151302-116105\nu)Eh^2+4(1405414-1672895\nu)(Eh^2)^2\\ &+8(112099-10990\nu)(Eh^2)^3+11520(51-7\nu)(Eh^2)^4]\Bigg\}\Bigg].\end{aligned}$$

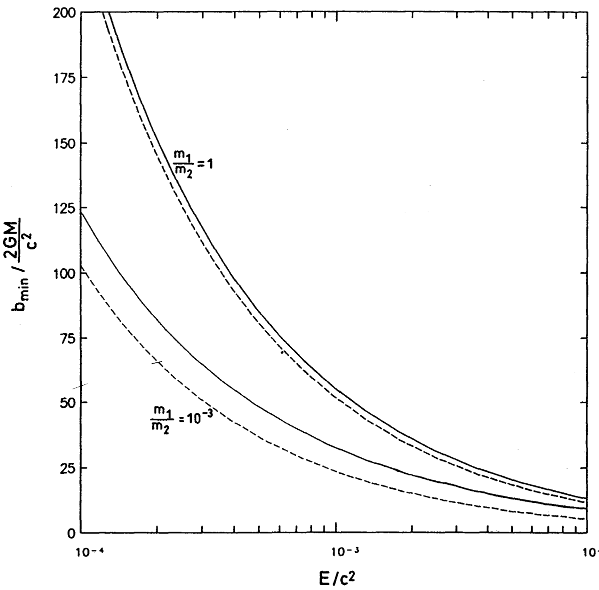

Figure 6. Minimal impact parameter for scattering as a function of energy, plotted for two different mass ratios. Dashed (full) lines: 2.5PN (3.5PN) radiation damping of the Newtonian (post-Newtonian) expression.

Similarly, we get for the total change of the relative velocity of the bodies

$$\delta V_\infty = -\frac{2}{15c^5}\frac{\nu}{h^7\sqrt{2E}}\left[\arc\cos\left(-\frac{1}{\sqrt{1+2Eh^2}}\right)[425+732Eh^2+148(Eh^2)^2]+\tfrac{1}{3}\sqrt{2Eh^2}\,[1275+1346Eh^2]\right.$$
$$+\frac{1}{840}\frac{1}{h^2c^2}\left\{\arc\cos\left(-\frac{1}{\sqrt{1+2Eh^2}}\right)[735(5763-3220\nu)+450(14028-9485\nu)Eh^2\right.$$
$$+180(7853-1358\nu)(Eh^2)^2+120(62+3885\nu)(Eh^2)^3]$$
$$+\frac{\sqrt{2Eh^2}}{(1+2Eh^2)}[105(40341-22540\nu)+70(170862-106055\nu)Eh^2+4(1892811-1168300\nu)(Eh^2)^2$$
$$\left.\left.+24(19242+55615\nu)(Eh^2)^3]\right\}\right]. \tag{50}$$

5 GRAVITATIONAL WAVEFORMS

The insertion of the expressions for the solutions of the equations of motion (see Sections 3 and 4) into the equations (6)–(10) and (1) yields an explicit expression for the gravitational radiation field in terms of the parameters of the orbital motion. If our dynamical system showed no preferred axis this representation of the radiation field would be optimally adapted to our matter system. However, its angular momentum defines a preferred axis and thus the source of the radiation field shows more structure than is made explicit by the STF-multipole moments, $\mathscr{I}_{A_l}$ and $\mathscr{J}_{A_l}$. The mass and current moments which are irreducibly defined with respect to the axis of angular momentum, say I^{lm} and S^{lm}, with $m=-l,\ldots,+l$, are obviously better adapted to our problem. The relation between both classes of multipole moments reads

$$I^{lm}(t)=\frac{16\pi}{(2l+1)!!}\left[\frac{(l+1)(l+2)}{2(l-1)l}\right]^{1/2}\mathscr{I}_{A_l}(t)\,Y^{lm*}_{A_l}, \tag{51}$$

$$S^{lm}(t)=-\frac{32\pi l}{(l+1)(2l+1)!!}\left[\frac{(l+1)(l+2)}{2(l-1)l}\right]^{1/2}\mathscr{J}_{A_l}(t)\,Y^{lm*}_{A_l}, \tag{52}$$

where

$$Y^{lm}_{A_l}:=(-1)^m(2l-1)!!\left[\frac{2l+1}{4\pi(l-m)!(l+m)!}\right]^{1/2}(\delta^1_{\langle i_1}+i\delta^2_{\langle i_1})\cdots(\delta^1_{i_m}+i\delta^2_{i_m})\delta^3_{i_{m+1}}\cdots\delta^3_{i_l\rangle}\quad\text{for}\quad m\geq 0, \tag{53}$$

$$Y^{lm}_{A_l}:=(-1)^m Y^{l|m|*}_{A_l}\quad\text{for}\quad m<0.$$

Thus,

$$I^{lm*}=(-1)^m I^{l-m},\qquad S^{lm*}=(-1)^m S^{l-m}, \tag{54}$$

holds.

Taking into account the equations which are inverse to (51)–(52) (see e.g. Thorne 1980) the radiation field (1) can be put into the form

$$h^{TT}_{jk}=\frac{G}{c^4 r}\sum_{l=2}^{\infty}\sum_{m=-l}^{l}\left[\left(\frac{1}{c}\right)^{l-2}\overset{(l)}{I^{lm}}(t-r/c)T^{E2,lm}_{jk}(\theta,\phi)+\left(\frac{1}{c}\right)^{l-1}\overset{(l)}{S^{lm}}(t-r/c)T^{B2,lm}_{jk}(\theta,\phi)\right], \tag{55}$$

where $T^{E2,lm}_{jk}$ and $T^{B2,lm}_{jk}$ are the so-called pure-spin tensor-spherical harmonics of electric [parity $(-1)^l$] and magnetic [parity $(-1)^{l+1}$] types, respectively. These harmonics are orthonormal on the unit sphere and they obey the following relations under complex conjugation:

$$T^{E/B2,lm*}=(-1)^m T^{E/B2,l-m}. \tag{56}$$

In our approximation of post-Newtonian wave generation the radiation field (55) takes the form

$$h^{TT}_{ij}=\frac{G}{c^4 r}\left\{\sum_{m=-2}^{2}\overset{(2)}{I^{2m}}(t-r/c)T^{E2,2m}_{ij}+\frac{1}{c}\left[\sum_{m=-2}^{2}\overset{(2)}{S^{2m}}(t-r/c)T^{B2,2m}_{ij}+\sum_{m=-3}^{3}\overset{(3)}{I^{3m}}(t-r/c)T^{E2,3m}_{ij}\right]\right.$$
$$\left.+\frac{1}{c^2}\left[\sum_{m=-3}^{3}\overset{(3)}{S^{3m}}(t-r/c)T^{B2,3m}_{ij}+\sum_{m=-4}^{4}\overset{(4)}{I^{4m}}(t-r/c)T^{E2,4m}_{ij}\right]\right\}. \tag{57}$$

The tensor-spherical harmonics needed here are explicitly listed in Appendix B. The multipole moments of equation (57) have the simple structure $\overset{(l)}{I}{}^{lm} = \mathrm{e}^{-mi\Theta} f_{lm}(t)$ and $\overset{(l)}{S}{}^{lm} = \mathrm{e}^{-mi\Theta} g_{lm}(t)$, where Θ denotes the azimuthal angle of the binary system (see e.g. equation 22) and where $f_{lm}(t)$ and $g_{lm}(t)$ reduce to constants for circular motions [there, $\Theta = \omega t$, with ω given by equation (34)].

In the case of quasi-elliptic motion of the bodies we find for the multipole moments the expressions

$$\overset{(2)}{I}{}^{22} = 8\left(\frac{2\pi}{5}\right)^{1/2} \mu E\, \mathrm{e}^{-2i\Theta} \Bigg[1 - \frac{1}{A(u)} + \frac{2(1-e_R^2)}{A(u)^2} + 2i\frac{e_R\sqrt{1-e_R^2}\sin u}{A(u)^2} + \frac{1}{42}\frac{E}{c^2}\Bigg\{ 9(3\nu-1) - \frac{3(51\nu-115)}{A(u)} + \frac{42(8\nu-25)-18(3\nu-1)e_R^2}{A(u)^2}$$
$$- \frac{4(111\nu-254)(1-e_R^2)}{A(u)^3} + 2i\frac{e_R\sin u}{\sqrt{1-e_R^2}A(u)^3}[-171\nu+253-3(23\nu-87)e_R\cos u+(213\nu-505)e_R^2$$
$$+9(3\nu-1)e_R^3\cos u]\Bigg\}\Bigg], \tag{58}$$

$$\overset{(2)}{I}{}^{21} = 0, \tag{59}$$

$$\overset{(2)}{I}{}^{20} = -16\left(\frac{\pi}{15}\right)^{1/2} \mu E \left\{ 1 - \frac{1}{A(u)} + \frac{1}{14}\frac{E}{c^2}\left[3(3\nu-1) - \frac{51\nu-115}{A(u)} + \frac{2(19\nu-4)}{A(u)^2} + 4(\nu-26)\frac{1-e_R^2}{A(u)^3} \right]\right\}, \tag{60}$$

$$\overset{(2)}{S}{}^{22} = 0, \tag{61}$$

$$\overset{(2)}{S}{}^{21} = \frac{32}{3}\left(\frac{\pi}{5}\right)^{1/2} \nu(m_1-m_2)(-E)^{3/2}\, \mathrm{e}^{-i\Theta}\frac{\sqrt{1-e_R^2}}{A(u)^2}, \tag{62}$$

$$\overset{(2)}{S}{}^{20} = 0, \tag{63}$$

$$\overset{(3)}{I}{}^{33} = 8\left(\frac{2\pi}{21}\right)^{1/2} \nu(m_1-m_2)\mathrm{e}^{-3i\Theta}(-E)^{3/2}\left[-\frac{e_R\sin u}{A(u)}\left(1+\frac{4(1-e_R^2)}{A(u)^2}\right) + i\frac{\sqrt{1-e_R^2}}{A(u)}\left(3-\frac{5/2}{A(u)}+\frac{4(1-e_R^2)}{A(u)^2}\right)\right], \tag{64}$$

$$\overset{(3)}{I}{}^{32} = 0, \tag{65}$$

$$\overset{(3)}{I}{}^{31} = 8\left(\frac{2\pi}{35}\right)^{1/2} \nu(m_1-m_2)\mathrm{e}^{-i\Theta}(-E)^{3/2}\left[\frac{e_R\sin u}{A(u)} - i\frac{\sqrt{1-e_R^2}}{A(u)}\left(1-\frac{5/6}{A(u)}\right)\right], \tag{66}$$

$$\overset{(3)}{I}{}^{30} = 0, \tag{67}$$

$$\overset{(3)}{S}{}^{33} = 0, \tag{68}$$

$$\overset{(3)}{S}{}^{32} = \frac{8}{3}\left(\frac{2\pi}{7}\right)^{1/2} \mu(1-3\nu)\mathrm{e}^{-2i\Theta}E^2\frac{\sqrt{1-e_R^2}}{A(u)^3}[e_R\sin u - 4i\sqrt{1-e_R^2}], \tag{69}$$

$$\overset{(3)}{S}{}^{31} = 0, \tag{70}$$

$$\overset{(3)}{S}{}^{30} = -16\left(\frac{\pi}{105}\right)^{1/2} \mu(1-3\nu)E^2\frac{\sqrt{1-e_R^2}\,e_R\sin u}{A(u)^3}, \tag{71}$$

$$\overset{(4)}{I}{}^{44} = \frac{4}{9}\left(\frac{2\pi}{7}\right)^{1/2} \mu(1-3\nu)\mathrm{e}^{-4i\Theta}E^2\left\{ 6 - \frac{6}{A(u)} + \frac{43-48e_R^2}{A(u)^2} - \frac{27(1-e_R^2)}{A(u)^3} + \frac{48(1-e_R^2)^2}{A(u)^4} + 6i\frac{e_R\sqrt{1-e_R^2}\sin u}{A(u)^2}\left[4+\frac{1}{A(u)}+\frac{8(1-e_R^2)}{A(u)^2}\right]\right\}, \tag{72}$$

$$\overset{(4)}{I}{}^{43} = 0, \tag{73}$$

$$\overset{(4)}{I}{}^{42} = \frac{8}{63}(2\pi)^{1/2}\mu(1-3\nu)E^2\mathrm{e}^{-2i\Theta}\left[-6 + \frac{6}{A(u)} - \frac{7-12e_R^2}{A(u)^2} + \frac{3(1-e_R^2)}{A(u)^3} - 3i\frac{e_R\sqrt{1-e_R^2}\sin u}{A(u)^2}\left(4+\frac{1}{A(u)}\right)\right], \tag{74}$$

$$\overset{(4)}{I}{}^{41}=0, \tag{75}$$

$$\overset{(4)}{I}{}^{40}=\frac{8}{21}\left(\frac{\pi}{5}\right)^{1/2}\mu(1-3\nu)E^2\left[6-\frac{6}{A(u)}-\frac{5}{A(u)^2}+\frac{5(1-e_R^2)}{A(u)^3}\right], \tag{76}$$

where $A(u):=1-e_R\cos u$ and where, as a function of E, e_R and u,

$$\Theta(u)=2\left(1-\frac{E}{c^2}\frac{6}{1-e_R^2}\right)\arctan\left[\left(\frac{1+e_R}{1-e_R}\right)^{1/2}\left(1-\frac{E}{c^2}\frac{e_R\nu}{1-e_R^2}\right)\tan\frac{u}{2}\right] \tag{77}$$

holds. To obtain the moments as functions of the time t one has to use the equation (*cf.* equation 20):

$$\frac{t}{P}=\frac{1}{2\pi}\left\{u-e_R\sin u\left[1-\frac{E}{c^2}(3\nu-8)\right]\right\}. \tag{78}$$

By the respective multiplication of the expressions (58)–(76) with e^{imknt} the influence of the periastron advance onto the radiative multipole moments drops out, leaving back functions periodic in the time t with the period P of the radial motion.

Figs 7(a)–(j) show the multipole components of equations (58)–(76) for a specific bounded orbit. The non-periodicity of the curves results from the periastron advance. The 'Newtonian' curves in Figs 7(a) and 7(b) are the 1PN curves with Newtonian amplitudes. Although the expressions $\overset{(2)}{S}{}^{21}$, $\overset{(3)}{I}{}^{31}$, and $\overset{(3)}{I}{}^{33}$ are zero for the considered case ($\nu=0.25$), for being more general, we have plotted them in reduced form in Figs 7(c)–7(e).

In the case of quasi-hyperbolic motion we get for the multipole moments the expressions

$$\begin{aligned}\overset{(2)}{I}{}^{22}=8\left(\frac{2\pi}{5}\right)^{1/2}\mu E\,e^{-2i\Theta}\Bigg\{1&+\frac{1}{B(v)}-\frac{2(e_R^2-1)}{B(v)^2}-2i\frac{e_R\sqrt{e_R^2-1}\,\sinh v}{B(v)^2}+\frac{1}{42}\frac{E}{c^2}\Bigg[9(3\nu-1)+\frac{3(51\nu-115)}{B(v)}\\&+\frac{42(8\nu-25)-18(3\nu-1)e_R^2}{B(v)^2}-\frac{4(111\nu-254)(e_R^2-1)}{B(v)^3}-2i\frac{e_R\sinh v}{\sqrt{e_R^2-1}\,B(v)^3}\{-171\nu+253-3(23\nu-87)e_R\cosh v\\&+(213\nu-505)e_R^2+9(3\nu-1)e_R^3\cosh v\}\Bigg]\Bigg\},\end{aligned} \tag{79}$$

$$\overset{(2)}{I}{}^{21}=0, \tag{80}$$

$$\overset{(2)}{I}{}^{20}=-16\left(\frac{\pi}{15}\right)^{1/2}\mu E\left\{1+\frac{1}{B(v)}+\frac{1}{14}\frac{E}{c^2}\left[3(3\nu-1)+\frac{51\nu-115}{B(v)}+\frac{2(19\nu-4)}{B(v)^2}+4(\nu-26)\frac{e_R^2-1}{B(v)^3}\right]\right\}, \tag{81}$$

$$\overset{(2)}{S}{}^{22}=0 \tag{82}$$

$$\overset{(2)}{S}{}^{21}=\frac{32}{3}\left(\frac{\pi}{5}\right)^{1/2}\nu(m_1-m_2)E^{3/2}\,e^{-i\Theta}\frac{\sqrt{e_R^2-1}}{B(v)^2}, \tag{83}$$

$$\overset{(2)}{S}{}^{20}=0, \tag{84}$$

$$\overset{(3)}{I}{}^{33}=8\left(\frac{2\pi}{21}\right)^{1/2}\nu(m_1-m_2)\,e^{-3i\Theta}E^{3/2}\left[\frac{e_R\sinh v}{B(v)}\left(1-\frac{4(e_R^2-1)}{B(v)^2}\right)-i\frac{\sqrt{e_R^2-1}}{B(v)}\left(3+\frac{5/2}{B(v)}-\frac{4(e_R^2-1)}{B(v)^2}\right)\right], \tag{85}$$

$$\overset{(3)}{I}{}^{32}=0, \tag{86}$$

$$\overset{(3)}{I}{}^{31}=8\left(\frac{2\pi}{35}\right)^{1/2}\nu(m_1-m_2)\,e^{-i\Theta}E^{3/2}\left[-\frac{e_R\sinh v}{B(v)}+i\frac{\sqrt{e_R^2-1}}{B(v)}\left(1+\frac{5/6}{B(v)}\right)\right], \tag{87}$$

$$\overset{(3)}{I}{}^{30}=0, \tag{88}$$

$$\overset{(3)}{S}{}^{33}=0, \tag{89}$$

$$\overset{(3)}{S}{}^{32}=\frac{8}{3}\left(\frac{2\pi}{7}\right)^{1/2}\mu(1-3\nu)\,\mathrm{e}^{-2i\Theta}E^{2}\,\frac{\sqrt{e_R^2-1}}{B(v)^3}\,[e_R\sinh v-4i\sqrt{e_R^2-1}], \tag{90}$$

$$\overset{(3)}{S}{}^{31}=0, \tag{91}$$

$$\overset{(3)}{S}{}^{30}=-16\left(\frac{\pi}{105}\right)^{1/2}\mu(1-3\nu)E^{2}\,\frac{\sqrt{e_R^2-1}\,e_R\sinh v}{B(v)^3}, \tag{92}$$

$$\overset{(4)}{I}{}^{44}=\frac{4}{9}\left(\frac{2\pi}{7}\right)^{1/2}\mu(1-3\nu)\,\mathrm{e}^{-4i\Theta}E^{2}\left\{6+\frac{6}{B(v)}+\frac{43-48e_R^2}{B(v)^2}-\frac{27(e_R^2-1)}{B(v)^3}+\frac{48(e_R^2-1)^2}{B(v)^4}\right.$$
$$\left.+6i\,\frac{e_R\sqrt{e_R^2-1}\,\sinh v}{B(v)^2}\left[-4+\frac{1}{\mathrm{B}(v)}+\frac{8(e_R^2-1)}{B(v)^2}\right]\right\}, \tag{93}$$

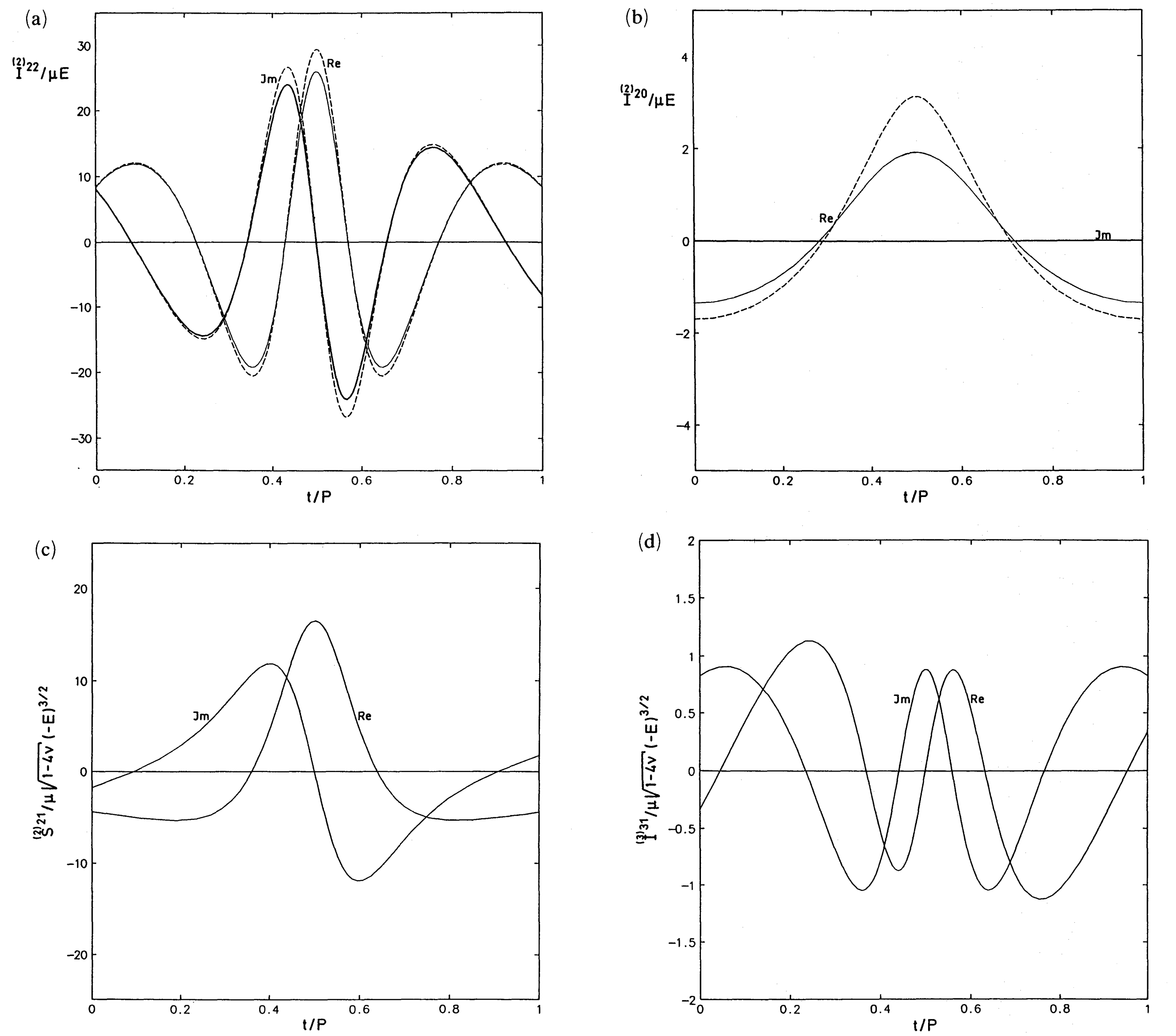

Figure 7. Gravitational waveforms (real and imaginary parts of time derivatives of the multipole moments) as functions of time for one 1PN orbital period. Eccentric orbit ($e_R=0.3$) with energy $E/c^2=-0.01865$, corresponding to a 1PN radius of $25GM/c^2$ for equal masses. Phase Θ and period P are always treated according to equations (77) and (78) with $\nu=0.25$. The periastron passage is located at $t/P=0.5$. Full lines: post-Newtonian wave generation with $\nu=0.25$. Dashed lines: 'Newtonian' generation of the amplitudes.

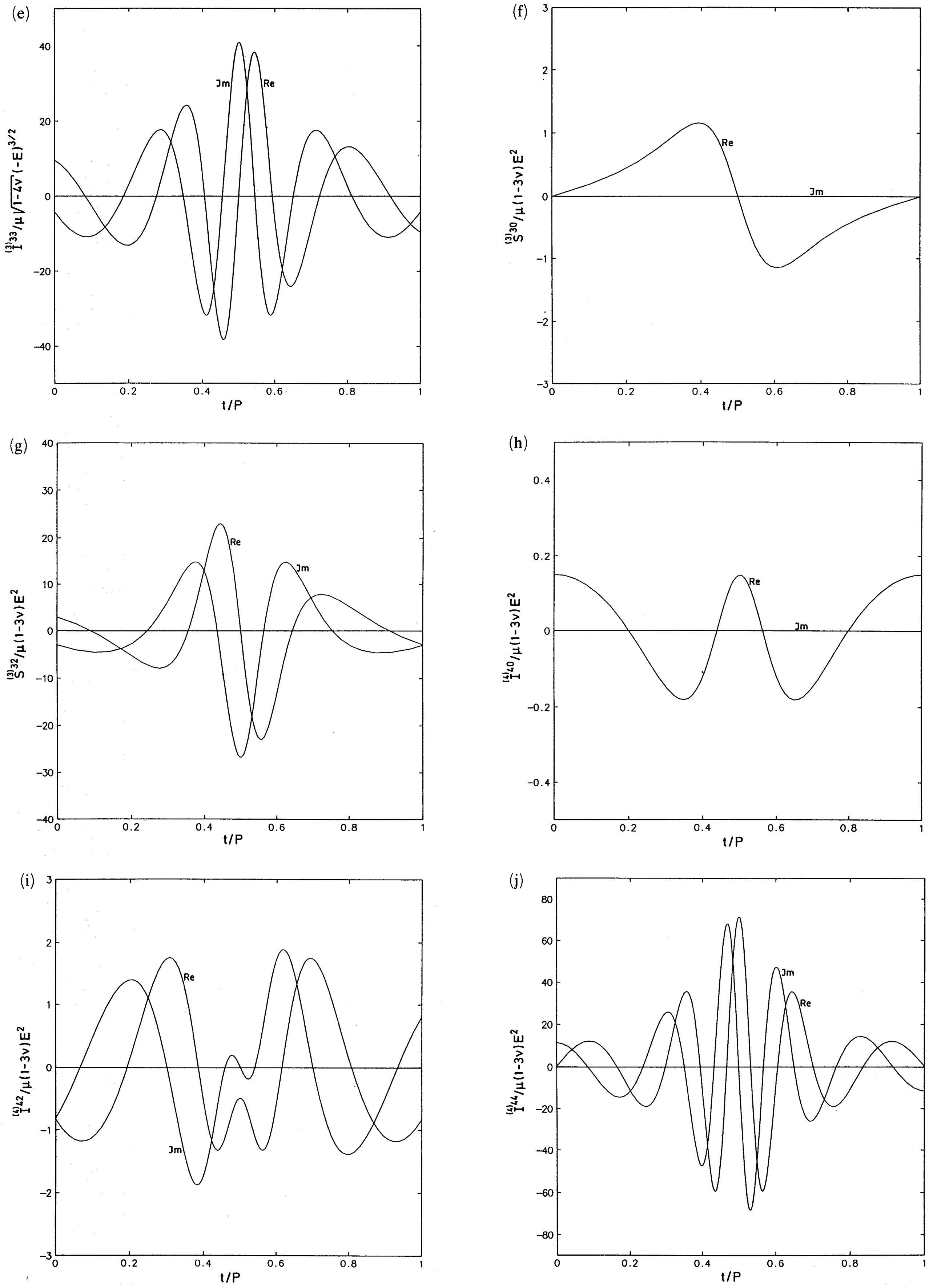

Figure 7 – *continued*

$$\overset{(4)}{I}{}^{43}=0, \tag{94}$$

$$\overset{(4)}{I}{}^{42}=\frac{8}{63}(2\pi)^{1/2}\mu(1-3\nu)E^2\mathrm{e}^{-2i\Theta}\left[-6-\frac{6}{B(v)}-\frac{7-12e_R^2}{B(v)^2}+\frac{3(e_R^2-1)}{B(v)^3}-3i\,\frac{e_R\sqrt{e_R^2-1}\,\sinh v}{B(v)^2}\left(-4+\frac{1}{B(v)}\right)\right], \tag{95}$$

$$\overset{(4)}{I}{}^{41}=0, \tag{96}$$

$$\overset{(4)}{I}{}^{40}=\frac{8}{21}\left(\frac{\pi}{5}\right)^{1/2}\mu(1-3\nu)E^2\left[6+\frac{6}{B(v)}-\frac{5}{B(v)^2}+\frac{5(e_R^2-1)}{B(v)^3}\right], \tag{97}$$

where $B(v):=e_R\cosh v-1$ and where as function of E, e_R and v

$$\Theta(v)=2\left(1+\frac{E}{c^2}\frac{6}{e_R^2-1}\right)\arctan\left[\left(\frac{e_R+1}{e_R-1}\right)^{1/2}\left(1+\frac{E}{c^2}\frac{e_R\nu}{e_R^2-1}\right)\tanh\frac{v}{2}\right] \tag{98}$$

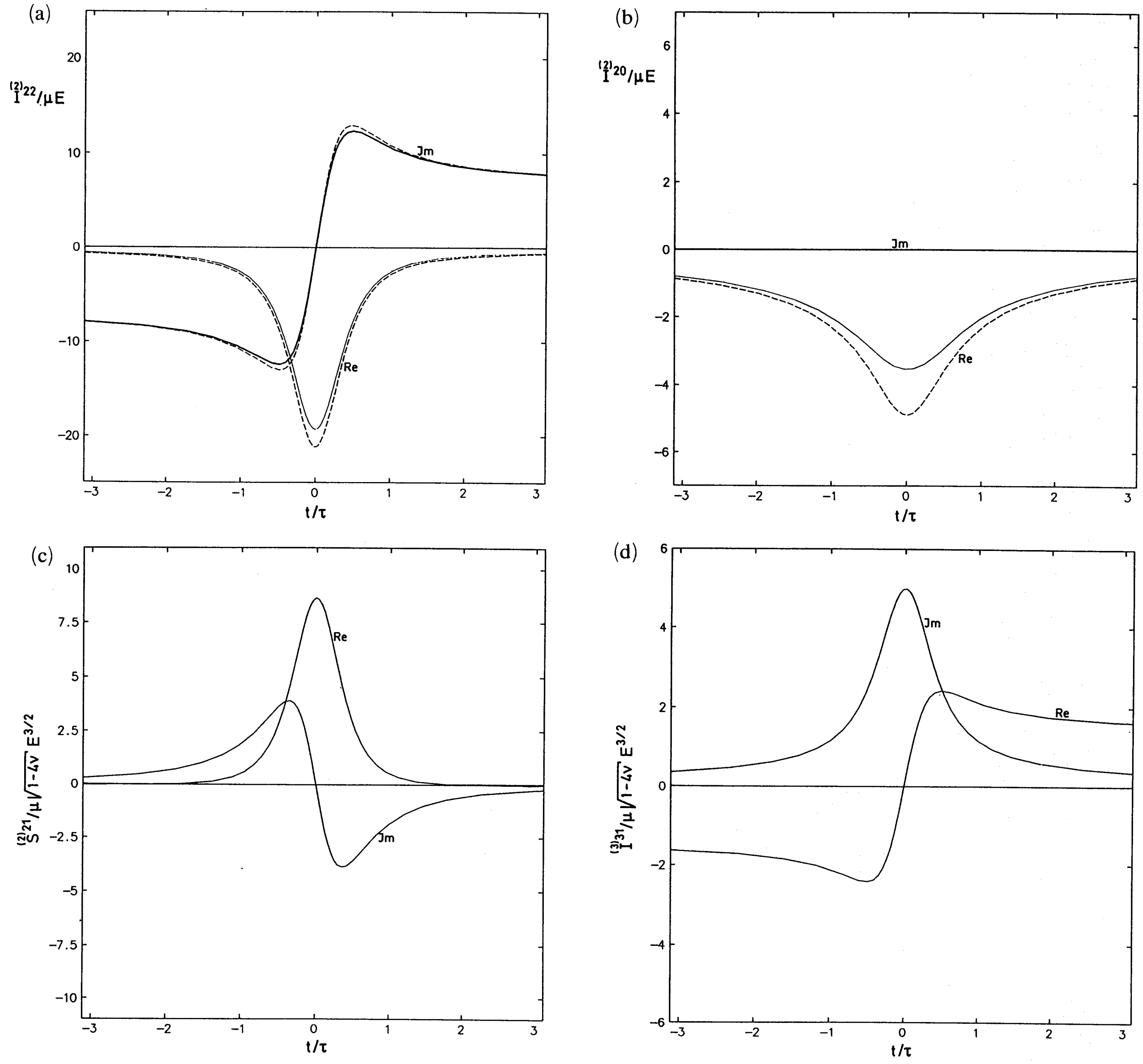

Figure 8. Gravitational waveforms (real and imaginary parts of time derivatives of the multipole moments) as functions of time for a scattering binary system with nearest passage at $t=0$. Eccentricity $e_R=2.5$ and relative velocity at infinity $V_\infty/c=0.15$. Phase Θ and encounter time τ are always treated according to equations (98) and (100) with $\nu=0.25$. Full lines: post-Newtonian wave generation with $\nu=0.25$. Dashed lines: 'Newtonian' generation of the amplitudes.

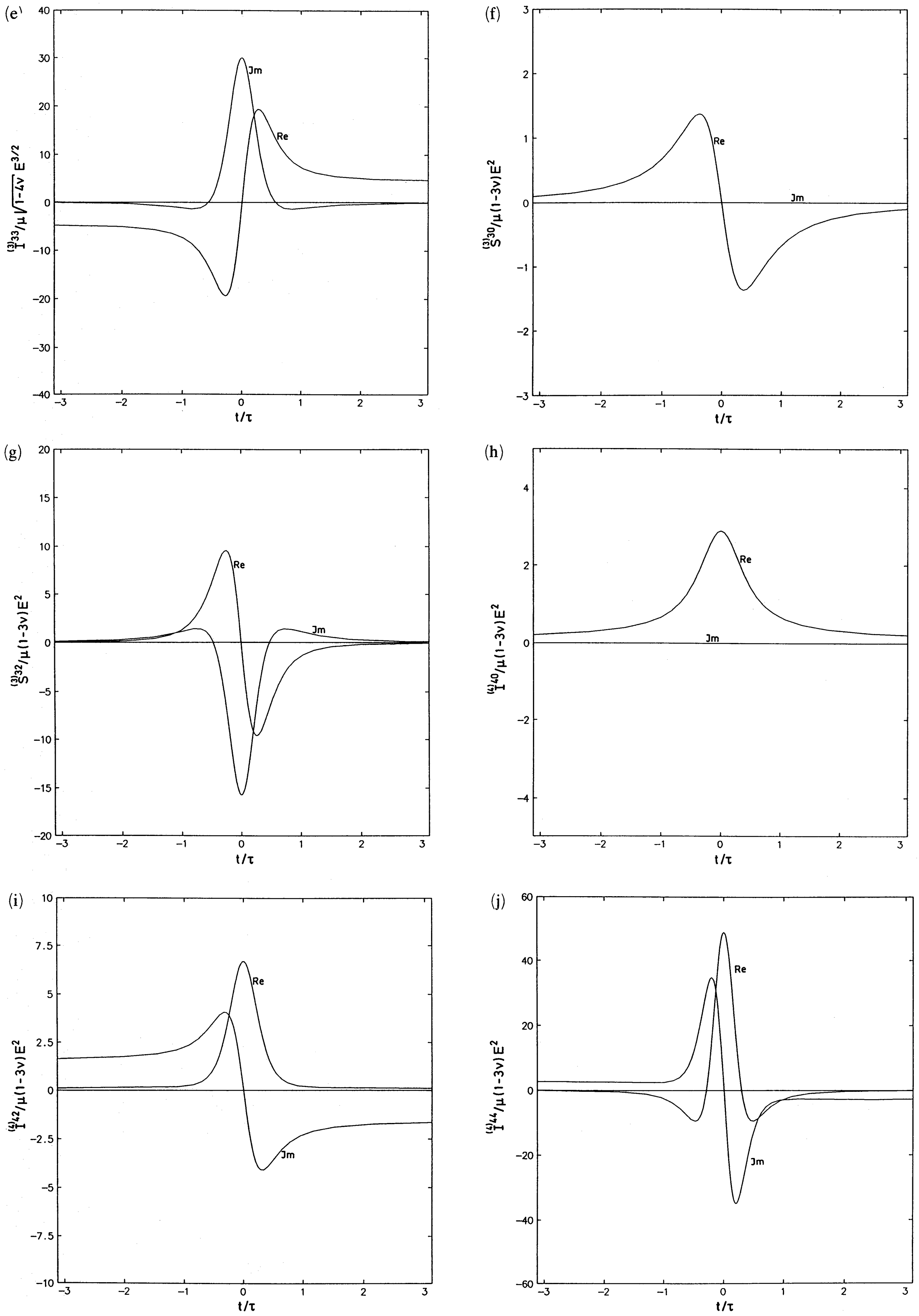

Figure 8 – *continued*

holds. With the aid of the characteristic encounter time τ, defined through the equation

$$\tau := \frac{b}{V_\infty} = \sqrt{e_R^2 - 1}\,\frac{GM}{(2E)^{3/2}}\left[1 + \frac{1}{4}(17 - 11\nu)\frac{E}{c^2} + \frac{2(6-\nu)}{e_R^2 - 1}\frac{E}{c^2}\right] \tag{99}$$

[for b and V_∞ see equations (40) and (41), respectively], we can write down the following formula relating the time t with the parameter v:

$$\frac{t}{\tau} = (e_R^2 - 1)^{-1/2}\left[e_R \sinh v\left(1 - \frac{E}{c^2}\frac{6-\nu}{e_R^2 - 1}\right) - v\left(1 + \frac{E}{c^2}(3\nu - 8) - \frac{E}{c^2}\frac{6-\nu}{e_R^2 - 1}\right)\right\}. \tag{100}$$

Figs 8(a)–(j) show the multipole components of equations (79)–(97) for a specific scattering process. The 'Newtonian' curves in Figs 8(a) and (b) are the 1PN curves with Newtonian amplitudes. Although the expressions $\overset{(2)}{S}{}^{21}$, $\overset{(3)}{I}{}^{31}$, and $\overset{(3)}{I}{}^{33}$ are zero for the considered case ($\nu = 0.25$), to be more general, we have plotted them in reduced form, Figs 8(c)–(e). The non-vanishing difference in the wave amplitudes at $t = \pm\infty$ has been coined the 'memory effect' by Braginsky & Grishchuk (1986).

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APPENDIX A: RECOIL CONSIDERATIONS

Recoil calculations at the level of approximation we are interested in have already been performed by Fitchett (1983). However, his treatment was based on a formula for the gravitational wave emission (corresponding to our equation 5) which was derived by Bekenstein (1973) for linearized gravity only, i.e. in the case where the gravitational interaction between and in the interior of bodies vanishes. Within our improved formalism we are able to put on a solid footing the results obtained by Fitchett (1983).

Equations (19) and (5) are valid in the centre of energy (mass) frame where $\boldsymbol{P} = 0$ holds. From the equation

$$\boldsymbol{P} = \frac{E_0/c^2}{\sqrt{1 - V_{\mathrm{CE}}^2/c^2}}\,\boldsymbol{V}_{\mathrm{CE}}, \tag{101}$$

where $E_0 = Mc^2 + \tilde{E}$ denotes the energy in the centre of energy system and $\boldsymbol{V}_{\mathrm{CE}}$ the centre of energy velocity [this equation, at least, holds up to the order c^{-6} (2PN approximation in $\tilde{E}$)], we obtain for the centre of energy acceleration the equation

$$\frac{d\boldsymbol{P}}{dt} = \frac{d}{dt}\left(\frac{E_0/c^2}{\sqrt{1 - V_{\mathrm{CE}}^2/c^2}}\right)\boldsymbol{V}_{\mathrm{CE}} + \frac{E_0/c^2}{\sqrt{1 - V_{\mathrm{CE}}^2/c^2}}\frac{d\boldsymbol{V}_{\mathrm{CE}}}{dt}. \tag{102}$$

If restricted to the case $V_{CE}=0$, the expression (19) may be substituted into equation (102). Furthermore, in our approximation, we may substitute E_0 by Mc^2 in the second term on the right-hand side of equation (102).

By performing similar calculations as in the main part of this paper we find the following expressions for the changes of the centre of energy velocities

$$\left\langle \frac{dV_{CE}}{dt} \right\rangle = -\frac{\pi c}{210}\left(\frac{2GM}{ac^2}\right)^4 \frac{1}{(1-e^2)^4} f(m_1/m_2)\frac{1}{P}\left(273e+399e^3+\frac{259}{8}e^5\right)\boldsymbol{e}_y \tag{103}$$

(elliptic motion, $0 \leq e < 1$, time average taken over one orbital period);

$$\Delta V_{CE} = -\frac{c}{210}\left(\frac{2GM}{R_{Per}c^2}\right)^4 \frac{1}{(1+e)^4} f(m_1/m_2)\left[\arccos\left(-\frac{1}{e}\right)\left(273e+399e^3+\frac{259}{8}e^5\right)+\frac{\sqrt{e^2-1}}{24e}(448+10514e^2+5943e^4)\right]\boldsymbol{e}_y \tag{104}$$

(hyperbolic motion, $e>1$), where

$$f(m_1/m_2) := \left(1+\frac{m_2}{m_1}\right)^{-2}\left(1+\frac{m_1}{m_2}\right)^{-3}\left(1-\frac{m_1}{m_2}\right).$$

P is the orbital period in the Newtonian approximation, $P=2\pi a^{3/2}/(GM)^{1/2}$, $R_{Per}:=\bar{a}(e-1)$ is the distance between the centre of energy and the periastron, and $\boldsymbol{e}_y$ denotes the unit vector pointing in the direction of the y-axis of a Cartesian (x, y, z)-coordinate system which has its origin at the centre of energy and where the z- and x-axes point in the directions of the angular momentum of the system and the periastron of body 1, respectively. In the derivation of equations (103)–(104) only Newtonian orbits have been used.

For equal masses ($\nu=0.25$) and for circular orbits the change of the centre of energy velocity (recoil) vanishes. In the other cases the change points into the direction of sign$[(m_1-m_2)/M]\,\boldsymbol{e}_y$.

For bound orbits, the recoil velocity (103) seems to achieve large values, starting from zero velocity. However, the periastron advance destroys this cumulative effect, and leads to a maximal centre of energy velocity of 'only'

$$V_{CE}^{max} = |V_{CE}^{max}|(1, -\pi k, 0)\,\text{sign}\left(\frac{m_2-m_1}{M}\right), \tag{105}$$

where

$$|V_{CE}^{max}| = \frac{c}{315}\frac{e}{(1-e^2)^3}\left(\frac{2GM}{ac^2}\right)^3 |f(m_1/m_2)|\left(273+399e^2+\frac{259}{8}e^4\right), \tag{106}$$

(Fitchett 1983), which, however, is still two orders of magnitude in $1/c$ bigger than the acceleration (103). Note that because of the smallness of $|V_{CE}^{max}|$ one was allowed to apply the equation (103) also in the case where V_{CE} was different from zero.

APPENDIX B: THE TENSOR HARMONICS

In this Appendix we explicitly give the pure-spin tensor harmonics which appear in the radiation field, equation (57). Since they obey the relations (56) it is sufficient to show only those for which $m \geq 0$. We take them from Turner & Will (1978). For $l=2$ they are

$$T^{E2,22} = \left(\frac{5}{128\pi}\right)^{1/2}[(1+\cos^2\theta)(\hat{\boldsymbol{\theta}}\otimes\hat{\boldsymbol{\theta}}-\hat{\boldsymbol{\phi}}\otimes\hat{\boldsymbol{\phi}})+2i\cos\theta(\hat{\boldsymbol{\theta}}\otimes\hat{\boldsymbol{\phi}}+\hat{\boldsymbol{\phi}}\otimes\hat{\boldsymbol{\theta}})]\,e^{2i\phi},$$

$$T^{E2,20} = \left(\frac{15}{64\pi}\right)^{1/2}\sin^2\theta(\hat{\boldsymbol{\theta}}\otimes\hat{\boldsymbol{\theta}}-\hat{\boldsymbol{\phi}}\otimes\hat{\boldsymbol{\phi}}),$$

$$T^{B2,21} = -\left(\frac{5}{32\pi}\right)^{1/2}[i\sin\theta(\hat{\boldsymbol{\theta}}\otimes\hat{\boldsymbol{\theta}}-\hat{\boldsymbol{\phi}}\otimes\hat{\boldsymbol{\phi}})-\sin\theta\cos\theta(\hat{\boldsymbol{\theta}}\otimes\hat{\boldsymbol{\phi}}+\hat{\boldsymbol{\phi}}\otimes\hat{\boldsymbol{\theta}})]\,e^{i\phi},$$

for $l=3$ they are

$$T^{E2,33} = -\left(\frac{21}{256\pi}\right)^{1/2}\sin\theta[(1+\cos^2\theta)(\hat{\boldsymbol{\theta}}\otimes\hat{\boldsymbol{\theta}}-\hat{\boldsymbol{\phi}}\otimes\hat{\boldsymbol{\phi}})+2i\cos\theta(\hat{\boldsymbol{\theta}}\otimes\hat{\boldsymbol{\phi}}+\hat{\boldsymbol{\phi}}\otimes\hat{\boldsymbol{\theta}})]\,e^{3i\phi},$$

$$T^{E2,31} = \left(\frac{35}{256\pi}\right)^{1/2}\sin\theta[(3\cos^2\theta-1)(\hat{\boldsymbol{\theta}}\otimes\hat{\boldsymbol{\theta}}-\hat{\boldsymbol{\phi}}\otimes\hat{\boldsymbol{\phi}})+2i\cos\theta(\hat{\boldsymbol{\theta}}\otimes\hat{\boldsymbol{\phi}}+\hat{\boldsymbol{\phi}}\otimes\hat{\boldsymbol{\theta}})]\,e^{i\phi},$$

$$T^{B2,32} = -\left(\frac{7}{128\pi}\right)^{1/2} [2i(2\cos^2\theta - 1)(\hat{\boldsymbol{\theta}}\otimes\hat{\boldsymbol{\theta}} - \hat{\boldsymbol{\phi}}\otimes\hat{\boldsymbol{\phi}}) - \cos\theta(3\cos^2\theta - 1)(\hat{\boldsymbol{\theta}}\otimes\hat{\boldsymbol{\phi}} + \hat{\boldsymbol{\phi}}\otimes\hat{\boldsymbol{\theta}})]\,e^{2i\phi},$$

$$T^{B2,30} = \left(\frac{105}{64\pi}\right)^{1/2} \cos\theta\sin^2\theta(\hat{\boldsymbol{\theta}}\otimes\hat{\boldsymbol{\phi}} + \hat{\boldsymbol{\phi}}\otimes\hat{\boldsymbol{\theta}}),$$

and, finally, for $l=4$

$$T^{E2,44} = \left(\frac{63}{512\pi}\right)^{1/2} \sin^2\theta[(1+\cos^2\theta)(\hat{\boldsymbol{\theta}}\otimes\hat{\boldsymbol{\theta}} - \hat{\boldsymbol{\phi}}\otimes\hat{\boldsymbol{\phi}}) + 2i\cos\theta(\hat{\boldsymbol{\theta}}\otimes\hat{\boldsymbol{\phi}} + \hat{\boldsymbol{\phi}}\otimes\hat{\boldsymbol{\theta}})]\,e^{4i\phi},$$

$$T^{E2,42} = \left(\frac{9}{128\pi}\right)^{1/2} [(7\cos^4\theta - 6\cos^2\theta + 1)(\hat{\boldsymbol{\theta}}\otimes\hat{\boldsymbol{\theta}} - \hat{\boldsymbol{\phi}}\otimes\hat{\boldsymbol{\phi}}) + i\cos\theta(7\cos^2\theta - 5)(\hat{\boldsymbol{\theta}}\otimes\hat{\boldsymbol{\phi}} + \hat{\boldsymbol{\phi}}\otimes\hat{\boldsymbol{\theta}})]\,e^{2i\phi},$$

$$T^{E2,40} = -\left(\frac{45}{256\pi}\right)^{1/2} (7\cos^4\theta - 8\cos^2\theta + 1)(\hat{\boldsymbol{\theta}}\otimes\hat{\boldsymbol{\theta}} - \hat{\boldsymbol{\phi}}\otimes\hat{\boldsymbol{\phi}}).$$

$\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\phi}}$ are the unit basis vectors corresponding to the spherical coordinates θ, ϕ (denoting the location of the observer), and $(\hat{\boldsymbol{\theta}}\otimes\hat{\boldsymbol{\theta}} - \hat{\boldsymbol{\phi}}\otimes\hat{\boldsymbol{\phi}})$ and $(\hat{\boldsymbol{\theta}}\otimes\hat{\boldsymbol{\phi}} + \hat{\boldsymbol{\phi}}\otimes\hat{\boldsymbol{\theta}})$ are two TT-basis tensors which describe the two physical polarization states, h_+^{TT} and $h_\times^{TT}$, respectively, of the gravitational field. The pure-spin harmonics with vanishing coefficients (see equations 58–76) are not given.