

BINET'S FORMULA FOR THE TRIBONACCI SEQUENCE

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1. Introduction

The terms of a recursive sequence are usually defined by a recurrence procedure; that is, any term is the sum of preceding terms. Such a definition might not be entirely satisfactory, because the computation of any term could require the computation of all of its predecessors. An alternative definition gives any term of a recursive sequence as a function of the index of the term. For the simplest nontrivial recursive sequence, the Fibonacci sequence, Binet's formula [1]

$$u_n = (1/\sqrt{5})(\alpha^{n+1} - \beta^{n+1})$$

defines any Fibonacci number as a function of its index and the constants

$$\alpha = \frac{1}{2}(1 + \sqrt{5}) \quad \text{and} \quad \beta = \frac{1}{2}(1 - \sqrt{5}).$$

In this paper, an analog of Binet's formula for the Tribonacci sequence

$$1, 1, 2, 4, 7, \dots, u_{n+1} = u_n + u_{n-1} + u_{n-2}, \dots$$

(see [2]), is derived. Binet's formula defines any term of the Tribonacci sequence as a function of the index of the term and three constants, ρ , σ , and τ .

2. Binet's Formula for the Tribonacci Sequence

Binet's formula is derived by determining the generating function for the difference equation

$$u_0 = u_1 = 1, u_2 = 2$$

$$u_{n+1} = u_n + u_{n-1} + u_{n-2} \quad n \geq 2.$$

Let $f(x) = u_0 + u_1x + u_2x^2 + \dots + u_nx^n + \dots = \sum_{i=0}^{\infty} u_i x^i$ be the generating function; then

$$(1 - x - x^2 - x^3)f(x) = 1,$$

so

$$f(x) = \frac{1}{1 - x - x^2 - x^3} = \frac{1}{(1 - \rho x)(1 - \sigma x)(1 - \tau x)} = \frac{1}{p(x)}.$$

The roots of $p(x) = 0$ are $1/\rho$, $1/\sigma$, and $1/\tau$, where ρ , σ , and τ are the roots of

$$p\left(\frac{1}{x}\right) = x^3 - x^2 - x - 1 = 0.$$

Applying Cardan's formulas to $p\left(\frac{1}{x}\right) = 0$ yields

$$\rho = \frac{1}{3}\left(\sqrt[3]{19 + 3\sqrt{33}} + \sqrt[3]{19 - 3\sqrt{33}} + 1\right),$$

$$\sigma = \frac{1}{6}\left([2 - \sqrt[3]{19 + 3\sqrt{33}} - \sqrt[3]{19 - 3\sqrt{33}} + \sqrt{3}i \sqrt[3]{19 + 3\sqrt{33}} - \sqrt[3]{19 - 3\sqrt{33}}]\right),$$

and

$$\tau = \bar{\sigma}, \text{ the complex conjugate of } \sigma.$$

Approximate numerical values for ρ , σ , and $\bar{\sigma}$ are:

$$\rho = 1.8393, \sigma = -0.4196 + 0.6063i, \bar{\sigma} = -0.4196 - 0.6063i.$$

Since the roots of $p(x) = 0$ are distinct, by partial fractions

$$f(x) = \frac{1}{(1 - \rho x)(1 - \sigma x)(1 - \bar{\sigma} x)} = \frac{A}{1 - \rho x} + \frac{B}{1 - \sigma x} + \frac{C}{1 - \bar{\sigma} x}.$$

Here

$$A = \frac{1}{\left(1 - \frac{\sigma}{\rho}\right)\left(1 - \frac{\bar{\sigma}}{\rho}\right)} = \frac{\rho^2}{(\rho - \sigma)(\rho - \bar{\sigma})},$$

$$B = \frac{1}{\left(1 - \frac{\rho}{\sigma}\right)\left(1 - \frac{\bar{\sigma}}{\sigma}\right)} = \frac{\sigma^2}{(\sigma - \rho)(\sigma - \bar{\sigma})},$$

and

$$C = \frac{1}{\left(1 - \frac{\rho}{\bar{\sigma}}\right)\left(1 - \frac{\sigma}{\bar{\sigma}}\right)} = \frac{\bar{\sigma}^2}{(\bar{\sigma} - \rho)(\bar{\sigma} - \sigma)}.$$

Consequently,

$$\begin{aligned} f(x) &= \frac{\rho^2}{(\rho - \sigma)(\rho - \bar{\sigma})} \sum_{i=0}^{\infty} \rho^i x^i + \frac{\sigma^2}{(\sigma - \rho)(\sigma - \bar{\sigma})} \sum_{i=0}^{\infty} \sigma^i x^i + \frac{\bar{\sigma}^2}{(\bar{\sigma} - \rho)(\bar{\sigma} - \sigma)} \sum_{i=0}^{\infty} \bar{\sigma}^i x^i \\ &= \sum_{i=0}^{\infty} \left(\frac{\rho^{i+2}}{(\rho - \sigma)(\rho - \bar{\sigma})} + \frac{\sigma^{i+2}}{(\sigma - \rho)(\sigma - \bar{\sigma})} + \frac{\bar{\sigma}^{i+2}}{(\bar{\sigma} - \rho)(\bar{\sigma} - \sigma)} \right) x^i. \end{aligned}$$

Thus, Binet's formula for the Tribonacci sequence is

$$u_n = \frac{\rho^{n+2}}{(\rho - \sigma)(\rho - \bar{\sigma})} + \frac{\sigma^{n+2}}{(\sigma - \rho)(\sigma - \bar{\sigma})} + \frac{\bar{\sigma}^{n+2}}{(\bar{\sigma} - \rho)(\bar{\sigma} - \sigma)}.$$

Multiplying the numerators and denominators of the last two terms by $(\rho - \bar{\sigma})$ and $(\rho - \sigma)$, respectively, yields

$$u_n = \frac{\rho^{n+2}}{|\rho - \sigma|^2} + \frac{(\rho - \bar{\sigma})\sigma^{n+2}}{-2iI(\sigma)|\rho - \sigma|^2} + \frac{(\rho - \sigma)\bar{\sigma}^{n+2}}{2iI(\sigma)|\rho - \sigma|^2}.$$

Using the relations $\sigma = r(\cos \theta + i \sin \theta)$,

$$\sigma^n = r^n(\cos n \theta + i \sin n \theta), \quad \theta = \tan^{-1}(I(\sigma)/R(\sigma))$$

and combining terms:

$$u_n = \frac{\rho^2}{|\rho - \sigma|^2} \rho^n + \frac{r(r - 2\rho \cos \theta)}{|\rho - \sigma|^2} r^n \cos n \theta \\ + \frac{r^2 \cos \theta - \rho r(1 - 2 \sin^2 \theta)}{\sin \theta |\rho - \sigma|^2} r^n \sin n \theta.$$

Denoting the coefficients of ρ^n , $r^n \cos n \theta$, and $r^n \sin n \theta$ by α , β , and γ , respectively, yields

$$u_n = \alpha \rho^n + r^n(\beta \cos n \theta + \gamma \sin n \theta).$$

Approximate values for the constants are:

$$\begin{aligned} \rho &= 1.8393, & \theta &= 124.69^\circ, & r &= 0.7374, \\ \alpha &= 0.6184, & \beta &= 0.3816, & \gamma &= 0.0374. \end{aligned}$$

3. An Application

Since $|r| = .7374 < 1$, the n th Tribonacci number is the integer nearest $\alpha \rho^n$ when

$$|r^n(\beta \cos n \theta + \gamma \sin n \theta)| < \frac{1}{2}.$$

Using calculus, the value of $|\beta \cos n \theta + \gamma \sin n \theta|$ is at a maximum when

$$n\theta = 5.60^\circ + k\pi, \quad \text{for } k \text{ an integer.}$$

Consequently,

$$|r^n(\beta \cos n \theta + \gamma \sin n \theta)| < \frac{1}{2} \text{ for } n \geq 1.$$

Since $[\alpha + .5] = 1$ (where $[]$ is the greatest integer function), a short form of the formula that is suitable for calculating the terms of the Tribonacci sequence is

$$u_n = [\alpha \rho^n + .5] \text{ for } n \geq 0.$$

References

1. Vorob'ev, N. *The Fibonacci Numbers*. Boston: Heath, 1963, pp. 12-15.
2. Feinberg, Mark. "Fibonacci-Tribonacci." *The Fibonacci Quarterly* 1, no. 1 (1963):71-74.
