Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Filomat **24:1** (2010), 41–46 DOI: 10.2298/FIL1001041C

BINOMIAL SYMMETRIES INSPIRED BY BRUCKMAN'S PROBLEM

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Abstract

The partial fraction decomposition method is employed to establish two general algebraic identities, which contain consequently several binomial identities and their q–analogues as special cases.

1 Introduction and Motivation

Recently, Gould and Quaintance [3] published a short paper devoted to a problem posed by Bruckman, who asked to show the binomial identity

$$\sum_{k=1}^{n}(-1)^k\binom{n}{k}\binom{n+k}{k}\frac{k}{(1-2k)(k+m)}=\frac{1+2n}{1+2m},$$

where both m and n are natural numbers subject to $m \leq n$.

By utilizing telescoping technique and partial fraction expansion of Lagrange interpolation, Gould and Quaintance [3] not only proved Bruckman's identity, but also extended it in two different manners.

After having read carefully their paper, we find that partial fraction method can further be employed to derive two general symmetric relations. In the 2nd section, we shall prove the first relation (cf. Theorem 1) which is symmetric with respect to two indeterminate x and y. Then the second relation (cf. Theorem 4) will be established in the 3rd section, which is symmetric with respect to two sets of free parameters $\{\beta_i\}_{i=0}^m$ and $\{\gamma_j\}_{j=0}^n$. Both symmetric relations contain not only the formulae of Gould and Quaintance [3] as special cases, but also several q-binomial symmetries.

2010 *Mathematics Subject Classifications.* Primary 05A10; Secondary 05A30.
Key words and Phrases. Partial fraction; Binomial coefficient; Gaussian Binomial coefficient.
Received: February 20, 2010
Communicated by Dragan Stevanović

2 Symmetry with respect to x and y

For the rational function defined by

$$f(x)=\frac{1}{x+\gamma_0}\prod_{i=1}^{n}\frac{x+\alpha_i}{x+\gamma_i},$$

it can be decomposed into the following partial fractions

$$f(x)=\sum_{k=0}^{n}\frac{A_k}{x+\gamma_k},$$

where the coefficients $\{A_k\}$ are determined by

$$A_k=\lim_{x\to-\gamma_k}(x+\gamma_k)f(x)=\frac{\prod_{i=1}^{n}(\alpha_i-\gamma_k)}{\prod_{\substack{j=0\\ j\neq k}}^{n}(\gamma_j-\gamma_k)}.$$

This leads us to the following identity

$$\frac{1}{x+\gamma_0}\prod_{i=1}^{n}\frac{x+\alpha_i}{x+\gamma_i}=\frac{1}{x+\gamma_0}\frac{\prod_{i=1}^{n}(\alpha_i-\gamma_0)}{\prod_{j=1}^{n}(\gamma_j-\gamma_0)}+\sum_{k=1}^{n}\frac{1}{x+\gamma_k}\frac{\prod_{i=1}^{n}(\alpha_i-\gamma_k)}{\prod_{\substack{j=0\\ j\neq k}}^{n}(\gamma_j-\gamma_k)},$$

where the initial term corresponding to $k=0$ has been singled out. Multiplying across the last equation by $x+\gamma_0$ and then replacing γ_0 by $-y$, we can easily check that the resulting equality is equivalent to the following symmetric relation with respect to x and y.

Theorem 1. *Let x and y be two indeterminate and $\{\alpha_k,\gamma_k\}_{k=1}^{n}$ complex numbers with $\{\gamma_k\}_{k=1}^{n}$ being distinct. Then there holds the algebraic identity*

$$\sum_{k=1}^{n}\frac{(\alpha_k-\gamma_k)(x-y)}{(x+\gamma_k)(y+\gamma_k)}\prod_{\substack{i=1\\ i\neq k}}^{n}\frac{\alpha_i-\gamma_k}{\gamma_i-\gamma_k}=\prod_{i=1}^{n}\frac{y+\alpha_i}{y+\gamma_i}-\prod_{i=1}^{n}\frac{x+\alpha_i}{x+\gamma_i}.$$

We remark that for $\gamma_k=k$ with $k=1,2,\cdots,n$, this identity reduces equivalently to the binomial symmetry due to Gould [2, Equation Z.14]. Instead, specializing the parameters by $\alpha_k=-k$ and $\gamma_k=k$ in Theorem 1 and observing that

$$\prod_{\substack{i=1\\ i\neq k}}^{n}\frac{\alpha_i-\gamma_k}{\gamma_i-\gamma_k}\Longrightarrow\prod_{\substack{i=1\\ i\neq k}}^{n}\frac{k+i}{k-i}=\frac{(-1)^{n-k}}{2}\binom{n}{k}\binom{n+k}{k},$$

we recover the following binomial identity, which corrects a sign error appeared in the paper by Gould and Quaintance.

Corollary 2 (Gould and Quaintance [3, Equation 3.3])**.**

$$\sum_{k=1}^{n}(-1)^k\binom{n}{k}\binom{n+k}{k}\frac{k(x-y)}{(x+k)(y+k)}=\frac{\binom{n-x}{n}}{\binom{n+x}{n}}-\frac{\binom{n-y}{n}}{\binom{n+y}{n}}.$$

When $x = m$ and $y = -1/2$ with $m = 1, 2, \cdots, n$, the first binomial fraction on the right hand side vanishes. In this case, the last identity reduces to Bruckman's binomial formula [3, Equation 1.4] anticipated at the beginning of this paper.

Furthermore, the q–analogue of Corollary 2 can be derived from Theorem 1. For this sake, recall the shifted factorial of x based on q

$$(x;q)_0 = 1 \quad \text{and} \quad (x;q)_n = (1-x)(1-qx)\cdots(1-q^{n-1}x) \quad \text{for} \quad n = 1, 2, \cdots$$

and the Gaussian binomial coefficient

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}.$$

Then letting $\alpha_k = -q^k$ and $\gamma_k = -q^{-k}$ in Theorem 1 and then keeping in mind

$$\prod_{\substack{i=1 \\ i \neq k}}^{n} \frac{\alpha_i - \gamma_k}{\gamma_i - \gamma_k} \Longrightarrow \prod_{\substack{i=1 \\ i \neq k}}^{n} \frac{1 - q^{k+i}}{1 - q^{k-i}} = \frac{(-1)^{n-k}}{1+q^k} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} n+k \\ k \end{bmatrix} q^{\binom{n-k+1}{2}},$$

we find, under some routine simplification, the following q–analogue of Corollary 2.

Corollary 3.

$$\sum_{k=1}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} n+k \\ k \end{bmatrix} \frac{(1-q^k)(x-y)}{(1-q^k x)(1-q^k y)} q^{\binom{n-k}{2}} = q^{\binom{n}{2}} \frac{(q/y;q)_n}{(qy;q)_n} y^n - q^{\binom{n}{2}} \frac{(q/x;q)_n}{(qx;q)_n} x^n.$$

In particular for $y = q^{-1/2}$ and $x = q^m$ with $m = 1, 2, \cdots, n$, this corollary provides the following q-analogue of Bruckman's binomial formula

$$\sum_{k=1}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} n+k \\ k \end{bmatrix} \frac{(1-q^k)(1-q^{1/2})}{(1-q^{m+k})(1-q^{\frac{1}{2}-k})} q^{\binom{k-n}{2}} = \frac{1-q^{n+1/2}}{1-q^{m+1/2}} q^{n^2/2}.$$

3 Symmetry with respect to β_i and γ_j

For the polynomial $P(x)$ of degree $\leq m$ in x, define another rational function

$$g(x) = \frac{P(x)}{\prod_{j=1}^{m}(x-\beta_j)} \frac{\prod_{i=1}^{n}(x-\alpha_i)}{\prod_{j=0}^{n}(x-\gamma_j)}.$$

Decomposing it into the partial fractions

$$g(x) = \sum_{k=1}^{m} \frac{A_k}{x-\beta_k} + \sum_{k=0}^{n} \frac{B_k}{x-\gamma_k},$$

and evaluating the coefficients A_k and B_k respectively by

$$\begin{aligned} A_k &= \lim_{x \to \beta_k} (x-\beta_k) g(x) = \frac{P(\beta_k)}{\prod_{\substack{j=1 \\ j \neq k}}^{m}(\beta_k - \beta_j)} \frac{\prod_{i=1}^{n}(\beta_k - \alpha_i)}{\prod_{j=0}^{n}(\beta_k - \gamma_j)}, \\ B_k &= \lim_{x \to \gamma_k} (x-\gamma_k) g(x) = \frac{P(\gamma_k)}{\prod_{j=1}^{m}(\gamma_k - \beta_j)} \frac{\prod_{i=1}^{n}(\gamma_k - \alpha_i)}{\prod_{\substack{j=0 \\ j \neq k}}^{n}(\gamma_k - \gamma_j)}; \end{aligned}$$

we get the following identity

$$\frac{P(x)}{\prod_{j=1}^m(x-\beta_j)}\frac{\prod_{i=1}^n(x-\alpha_i)}{\prod_{j=0}^n(x-\gamma_j)}=\sum_{k=1}^m\frac{P(\beta_k)}{(x-\beta_k)\prod_{\substack{j=1\\j\neq k}}^m(\beta_k-\beta_j)}\frac{\prod_{i=1}^n(\beta_k-\alpha_i)}{\prod_{j=0}^n(\beta_k-\gamma_j)}$$
$$+\sum_{k=0}^n\frac{P(\gamma_k)}{(x-\gamma_k)\prod_{j=1}^m(\gamma_k-\beta_j)}\frac{\prod_{i=1}^n(\gamma_k-\alpha_i)}{\prod_{\substack{j=0\\j\neq k}}^n(\gamma_k-\gamma_j)}.$$

Replacing x by β_0 and then absorbing the left member in the first sum on the right hand side, we get another symmetric relation with respect to β_i and γ_j.

Theorem 4. *Let $P(x)$ be a polynomial of degree $\leq m$ in x and $\{\beta_i\}_{i=0}^m\cup\{\gamma_j\}_{j=0}^n$ distinct complex numbers. There holds the equality*

$$\sum_{k=0}^m\frac{P(\beta_k)}{\prod_{\substack{j=0\\j\neq k}}^m(\beta_k-\beta_j)}\frac{\prod_{i=1}^n(\beta_k-\alpha_i)}{\prod_{j=0}^n(\beta_k-\gamma_j)}+\sum_{k=0}^n\frac{P(\gamma_k)}{\prod_{j=0}^m(\gamma_k-\beta_j)}\frac{\prod_{i=1}^n(\gamma_k-\alpha_i)}{\prod_{\substack{j=0\\j\neq k}}^n(\gamma_k-\gamma_j)}=0.$$

Similarly, Theorem 4 contains several interesting consequences.

Firstly for $\alpha_k=-k$ and $\gamma_k=k$, it becomes the following identity which is equivalent to the generalized Bruckman series due to Gould and Quaintance [3, Equation 3.7].

Corollary 5. *Let $P(x)$ be a polynomial of degree $\leq m$ and $\{\beta_i\}_{i=0}^m$ distinct complex numbers. There holds the identity*

$$\sum_{k=0}^n(-1)^k\binom{n}{k}\binom{n+k}{k}\frac{P(k)}{\prod_{j=0}^m(\beta_j-k)}=\sum_{k=0}^m\frac{\binom{n+\beta_k}{n}}{\binom{n-\beta_k}{n}}\frac{P(\beta_k)}{\beta_k\prod_{\substack{j=0\\j\neq k}}^m(\beta_j-\beta_k)}.$$

Then for $\beta_k=\beta+k$ and $\gamma_k=\gamma+k$, it reduces to another binomial symmetry.

Corollary 6. *Let $P(x)$ be a polynomial of degree $\leq m$. There holds the identity*

$$\sum_{k=0}^m\frac{(-1)^k}{\beta-\gamma+k}\binom{m}{k}\frac{P(\beta+k)}{\binom{\gamma-\beta+n-k}{n}}\prod_{i=1}^n(\beta+k-\alpha_i)$$
$$=\sum_{k=0}^n\frac{(-1)^k}{\beta-\gamma-k}\binom{n}{k}\frac{P(\gamma+k)}{\binom{\beta-\gamma+m-k}{m}}\prod_{i=1}^n(\gamma+k-\alpha_i).$$

A slightly different setting with $\beta_k=\beta+k$ and $\gamma_k=\gamma-k$ will lead Theorem 4 to the following binomial symmetry.

Corollary 7. *Let $P(x)$ be a polynomial of degree $\leq m$. There holds the identity*

$$\sum_{k=0}^m\frac{(-1)^k}{\beta-\gamma+k}\binom{m}{k}\frac{P(\beta+k)}{\binom{\beta-\gamma+n+k}{n}}\prod_{i=1}^n(\beta+k-\alpha_i)$$
$$=\sum_{k=0}^n\frac{(-1)^k}{\beta-\gamma+k}\binom{n}{k}\frac{P(\gamma-k)}{\binom{\beta-\gamma+m+k}{m}}\prod_{i=1}^n(\gamma-k-\alpha_i).$$

In particular, one can check the limiting case $\beta \to \gamma$ of this corollary is equivalent to Gould [2, Equation Z.16].

Finally, the q-analogues of these three symmetric relations are produced as follows.

Corollary 8 ($\alpha_k = q^{-k}$ and $\gamma_k = q^k$ in Theorem 4)**.** *Let $P(x)$ be a polynomial of degree $\leq m$. There holds the identity*

$$\sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} n+k \\ k \end{bmatrix} \frac{q^{\binom{n-k}{2}} P(q^k)}{\prod_{j=0}^{m}(q^k - \beta_j)} = \sum_{k=0}^{m} \frac{(q\beta_k;q)_n}{(q/\beta_k;q)_n} \frac{q^{\binom{n}{2}} P(\beta_k)}{(1-\beta_k)\beta_k^n \prod_{\substack{j=0 \\ j\neq k}}^{m}(\beta_k - \beta_j)}.$$

Corollary 9 ($\beta_k = q^k\beta$ and $\gamma_k = q^k\gamma$ in Theorem 4)**.** *Let $P(x)$ be a polynomial of degree $\leq m$. There holds the identity*

$$\begin{aligned} &\sum_{k=0}^{m} \frac{(-1)^k q^{\binom{m+n-k}{2}}}{\beta^{m+n}(q;q)_m} \begin{bmatrix} m \\ k \end{bmatrix} \frac{P(q^k\beta)}{q^k\beta - \gamma} \frac{\prod_{i=1}^{n}(\alpha_i - q^k\beta)}{(q^{1-k}\gamma/\beta;q)_n} \\ = &\sum_{k=0}^{n} \frac{(-1)^k q^{\binom{m+n-k}{2}}}{\gamma^{m+n}(q;q)_n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{P(q^k\gamma)}{\beta - q^k\gamma} \frac{\prod_{i=1}^{n}(\alpha_i - q^k\gamma)}{(q^{1-k}\beta/\gamma;q)_m}. \end{aligned}$$

Corollary 10 ($\beta_k = q^k\beta$ and $\gamma_k = q^{-k}\gamma$ in Theorem 4)**.** *Let $P(x)$ be a polynomial of degree $\leq m$. There holds the identity*

$$\begin{aligned} &\sum_{k=0}^{m} \frac{(-1)^k q^{\binom{m-k}{2}}}{\beta^{m}(q;q)_m} \begin{bmatrix} m \\ k \end{bmatrix} \frac{P(q^k\beta)}{q^k\beta - \gamma} \frac{\prod_{i=1}^{n}(\alpha_i - q^k\beta)}{(q^{1+k}\gamma/\beta;q)_n} \\ = &\sum_{k=0}^{n} \frac{(-1)^k q^{\binom{m+k}{2}}}{\gamma^{m}(q;q)_n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{P(q^{-k}\gamma)}{\beta - q^{-k}\gamma} \frac{\prod_{i=1}^{n}(\alpha_i - q^{-k}\gamma)}{(q^{1+k}\beta/\gamma;q)_m}. \end{aligned}$$

From the formulae displayed in this paper, one sees that partial fraction method is indeed powerful tool to treat binomial identities. For the application to trigonometric identities, the interested reader can consult a recent paper by Chu [1].

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