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BIORTHOGONAL SYSTEMS AND REFLEXIVITY OF BANACH SPACES

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The author gives characterizations of reflexivity by means of properties of biorthogonal systems and by means of the geometrical structure of the space considered.

In the present paper we intend to examine some simple geometrical properties of biorthogonal systems in a Banach space E and their connection with the reflexivity of E. It turns out that the behaviour of biorthogonal systems is different in reflexive and nonreflexive spaces and may be used as a simple criterion of reflexivity. At the same time the results obtained yield a description of the geometrical structure of a nonreflexive Banach space, so that we are able to complete some earlier results of R. C. JAMES [1]. The proofs are based on a method developed in an earlier paper of the authors [4].

Let E be a Banach space. A system consisting of a sequence $e_1, e_2, \ldots$ of points of E and a sequence $f_1, f_2, \ldots$ of points of E' is said to constitute a *biorthogonal system in E* if $\langle e_i, f_i \rangle = 1$ for $i = 1, 2, \ldots$ and $\langle e_i, f_j \rangle = 0$ for $i \neq j$. A biorthogonal system is said to be *bounded* if there exists a number μ such that $|e_i| \leqq \mu$, $|f_j| \leqq \mu$ for every i and j. If $S = (e_i, f_j)$ is a biorthogonal system in E, we shall denote by $A(S)$ the closed subspace of E spanned by the sequence $e_1, e_2 \ldots$. Further, let $B_1(S)$ be the norm-closed subspace of E' spanned by the sequence $f_1, f_2, \ldots$ We shall denote by $B(S)$ the $\sigma(E', E)$ closure of $B_1(S)$ in E'.

We intend to prove the following three theorems:

Theorem 1. *A Banach space E is reflexive if and only if, for every bounded biorthogonal system (e_i, f_j), the sequence $e_1 + \ldots + e_n$ is unbounded.*

Theorem 2. *A Banach space E is reflexive if and only if, for every bounded biorthogonal system (e_i, f_j), the sequence $f_1 + \ldots + f_n$ is unbounded.*

Theorem 3. *Let E be a Banach space. Then the following three conditions are equivalent*:

1° *the space E is non-reflexive;*

2° *there exists a bounded biorthogonal system (e_i, f_j) and a number $\omega > 0$ such that, for every monotonic sequence $\alpha_1, \alpha_2, \ldots$ tending to zero, the sum $x = \sum_1^\infty \alpha_i e_i$ exists and fulfills $|x| \leqq \omega|\alpha_1|$;*

3° *there exists a bounded biorthogonal system $S = (e_i, f_j)$ with the following property: if S is considered as a biorthogonal system in the space $E/B(S)^0$ (with norm $\|x\|$), there exists a number $\eta > 0$ such that $\|\sum_1^\infty \lambda_i e_i\| \geqq \eta \sum_1^\infty \lambda_i$ for every sequence $\lambda_1, \lambda_2, \ldots$ of nonnegative numbers such that $\sum_1^\infty \lambda_i < \infty$.*

The main idea of the proof being nearly the same for all three theorems, we intend to prove them simultaneously.

Proof. Let E be a non-reflexive Banach space..

1. Choose first an element $r \in E''$, $|r| = 1$ such that r non $\in E$. Let σ be an arbitrary positive number. We are going to show that, for every finite sequence $f_1, \ldots, f_n \in E'$, there exists a $b \in E$, $|b| \leqq 1 + \sigma$ such that $\langle b, f_j\rangle = \langle r, f_j\rangle$ for $j = 1, 2, \ldots, n$. To see that, consider, in the n-dimensional Euclidean space, the set W consisting of all vectors of the form $(\langle x, f_1\rangle, \ldots, \langle x, f_n\rangle)$, where $|x| \leqq 1$. Clearly we may restrict ourselves to the case where the f_j are linearly independent. It follows that W is a neighbourhood of zero in E_n. Consider now the vector $z = (\zeta_1, \ldots, \zeta_n)$, where $\zeta_j = \langle r, f_j\rangle$ and suppose that z does not belong to the closure of W. Then there exist real numbers $\lambda_1, \ldots, \lambda_n$ such that

$$\lambda_1\zeta_1 + \ldots + \lambda_n\zeta_n > 1 \quad \text{while} \quad \lambda_1\xi_1 + \ldots + \lambda_n\xi_n \leqq 1$$

for every $(\xi_1, \ldots, \xi_n) \in W$. It follows that the norm of the functional $f = \lambda_1 f_1 + \ldots + \lambda_n f_n$ is at most one; at the same time $\langle r, f\rangle = \lambda_1\zeta_1 + \ldots + \lambda_n\zeta_n > 1$, which is a contradiction. The inclusion $z \in \overline{W}$ is thus established. The set W being a neighbourhood of zero in E_n, we have $\overline{W} \subset (1 + \sigma) W$. Hence $z \in (1 + \sigma) W$ which proves our assertion.

2. Consider now an arbitrary finite-dimensional subspace $H \subset E$. We assert now that the number

$$\varrho = \sup_{y \in U^\circ \cap H^\circ} \langle r, y\rangle$$

is positive. Indeed, if this were not the case, we should have $\langle r, y\rangle = 0$ for ever y for which $\langle H, y\rangle = 0$. A simple algebraic argument shows, however, that this would imply the inclusion $r \in H$ which is impossible.

3. Let us turn now to the construction of our biorthogonal systems. Since $|r| = 1$, there exists a point $y_1 \in E'$, $|y_1| = 1$ such that $\beta_1 = \langle r, y_1\rangle > \frac{1}{2}$; there exists a point $b_1 \in E$ such that $|b_1| \leqq 1 + \sigma$ and $\langle b_1, y_1\rangle = \beta_1$. Let us denote by E_1 the subspace of E generated by b_1.

According to the preceding remark, the number $\varrho_1 = \sup_{y \in U^0 \cap E_1^0} \langle r, y\rangle$ is positive.

It follows that there exists a point $y_2 \in E_1^0$. $|y_2| = 1$ such that $\beta_2 = \langle r, y_2\rangle > > \frac{1}{2}\varrho_1$; further, a point $b_2 \in E$ may be found such that $|b_2| \leqq 1 + \sigma$ and $\langle b_2, y_1\rangle = \beta_1$, $\langle b_2, y_2\rangle = \beta_2$. We shall denote by E_2 the subspace of E generated by b_1 and b_2. The number $\varrho_2 = \sup_{y \in U^0 \cap E_2^0} \langle r, y\rangle$ is positive.

The construction proceeds by a simple induction. Suppose we have already defined points $b_1, \ldots, b_n \in E$ and functionals $y_1, \ldots, y_n \in E'$ with the following properties:

$$|b_i| \leqq 1 + \sigma\,, \quad |y_j| = 1 \quad \text{for} \quad 1 \leqq i, j \leqq n\,,$$
$$\langle b_i, y_j\rangle = \beta_j \quad \text{for} \quad 1 \leqq j \leqq i \leqq n\,,$$
$$\langle b_i, y_j\rangle = 0 \quad \text{for} \quad 1 \leqq i < j \leqq n\,,$$
$$\beta_j = \langle r, y_j\rangle > \tfrac{1}{2}\varrho_j \quad \text{for} \quad 1 \leqq j \leqq n\,,$$

where $\varrho_j = \sup_{y \in U^0 \cap E^0_{j-1}} \langle r, y\rangle$ and E_p is the subspace of E spanned by $b_1, \ldots, b_p$.

The number $\varrho_{n+1} = \sup_{y \in U^0 \cap E^0_n} \langle r, y\rangle$ being positive, there exists a point $y_{n+1} \in E_n^0$, $|y_{n+1}| = 1$ such that $\beta_{n+1} = \langle r, y_{n+1}\rangle > \frac{1}{2}\varrho_{n+1}$. Further a point $b_{n+1} \in E$ may be found such that $|b_{n+1}| \leqq 1 + \sigma$ and $\langle b_{n+1}, y_j\rangle = \beta_j$ for $j = 1, 2, \ldots, n + 1$. The induction is thus complete.

In this manner, we have defined a sequence $b_i \in E$ and a sequence $y_j \in E'$ with the following properties:

1° $|b_i| \leqq 1 + \sigma\,, \quad |y_j| = 1\,;$

2° the matrix $\langle b_i, y_j\rangle$ has the subdiagonal form

$$\begin{matrix} \beta_1, & 0, & 0, & 0, & \ldots, \\ \beta_1, & \beta_2, & 0, & 0, & \ldots, \\ \beta_1, & \beta_2, & \beta_3, & 0, & \ldots, \\ \beta_1, & \beta_2, & \beta_3, & \beta_4, & \ldots; \end{matrix}$$

3° $\beta_j = \langle r, y_j\rangle > \frac{1}{2}\varrho_j = \frac{1}{2} \sup_{y \in U^0 \cap E^0_{j-1}} \langle r, y\rangle$

where E_p is the subspace of E spanned by $b_1, \ldots, b_p$.

Clearly we have $\varrho_1 \geqq \varrho_2 \geqq \varrho_3 \geqq \ldots$ We intend to show now that $\inf \varrho_j > 0$. Suppose not. Since $\varrho_j > 0$ for every j, we have then $\lim \varrho_j = 0$. At the same time, we have $0 < \frac{1}{2}\varrho_j < \beta_j \leqq \varrho_j$ for every j.

Let ε be an arbitrary positive number. Let n be a natural number such that $\varrho_n < \frac{1}{3}\varepsilon$. Let $y \in U^0$ and let $\tau = \max_{1 \leqq i \leqq n} |\langle b_i, y\rangle|$.

Let

$$z = \sum_{j=1}^{n} \frac{1}{\beta_j} (\langle b_j, y\rangle - \langle b_{j-1}, y\rangle)\, y_j$$

where $b_0 = 0$. We have, clearly, the estimate $|z| \leqq \frac{2}{\varrho_n} \cdot 2\tau \cdot n$. It is easy to see that $\langle b_i, y - z \rangle = 0$ for $i = 1, 2, \ldots, n$ so that $|\langle r, y - z \rangle| \leqq |y - z| \, \varrho_n \leqq \leqq \left(1 + \frac{4n\tau}{\varrho_n}\right) \varrho_n = \varrho_n + 4n\tau$. Further,

$$\langle r, y \rangle = \langle r, z \rangle + \langle r, y - z \rangle = \langle b_n, y \rangle + \langle r, y - z \rangle$$

whence

$$|\langle r, y \rangle| \leqq \tau + \varrho_n + 4n\tau < \tfrac{1}{3}\varepsilon + (4n + 1)\,\tau\,.$$

It follows that $|\langle r, y \rangle| < \varepsilon$ whenever $y \in U^0$ and $\tau \leqq \frac{\varepsilon}{8n + 2}$. We have thus shown that r is weakly continuous on U^0. This is a contradiction since r does not belong to E. There exists consequently a positive number β such that $\varrho_j \geqq 2\beta$ for every j: hence $\beta_j \geqq \beta > 0$ for every j.

4. For every natural number i, let $e_i = b_i - b_{i-1}$. Clearly $|e_i| \leqq 2(1 + \sigma)$ for every i. For every natural j, let $f_j = \frac{1}{\beta_j} y_j$. According to what has been shown above, there is a common bound $\frac{1}{\beta}$ for the norms $|f_j|$. It is easy to see that $S = (e_i, f_j)$ is a biorthogonal system.

5. Let $\alpha_1 \geqq \alpha_2 \geqq \ldots$ be a sequence tending to 0. Then the sum $s(\alpha) = \alpha_1 e_1 + + \alpha_2 e_2 + \ldots$ exists and we have $|s(\alpha)| \leqq 2(1 + \sigma)\, |\alpha_1|$. Indeed, we may write

$$\alpha_1 e_1 + \ldots + \alpha_n e_n =$$
$$= \alpha_1 b_1 + \alpha_2(b_2 - b_1) + \alpha_3(b_3 - b_2) + \ldots + \alpha_{n-1}(b_{n-1} - b_{n-2}) + \alpha_n(b_n - b_{n-1}) =$$
$$= (\alpha_1 - \alpha_2)\, b_1 + (\alpha_2 - \alpha_3)\, b_2 + \ldots + (\alpha_{n-1} - \alpha_n)\, b_{n-1} + \alpha_n b_n\,.$$

The last expression has a limit for n tending to infinity; in fact, the term $\alpha_n b_n$ converges to zero on account of the equiboundedness of b_n, the numbers $\alpha_i - \alpha_{i+1}$ are nonnegative and the series of general term $\alpha_i - \alpha_{i+1}$ converges with sum α_1. We may remark here that there is also an estimate of $|s(\alpha)|$ from below; indeed, we have

$$|s(\alpha)| \geqq \langle s(\alpha), y_1 \rangle = \langle \alpha_1 e_1, y_1 \rangle = \alpha_1\,.$$

6. On the other hand, there exists a monotonic sequence such that the partial sums of the series $\alpha_1 e_1 + \alpha_2 e_2 + \ldots$ are bounded and the series itself is not even weakly convergent.

To see that, consider the series $e_1 + e_2 + e_3 + \ldots$ We have

$$e_1 + \ldots + e_n = b_1 + (b_2 - b_1) + \ldots + (b_n - b_{n-1}) = b_n\,.$$

Suppose that there exists a point v_0 such that b_n converges weakly to v_0. Clearly $v_0 \in A(S)$. Take a fixed f_j. Since $\langle b_n, f_j \rangle = 1$ for $n \geqq j$, we must have $\langle v_0, f_j \rangle = 1$. Hence $\langle v_0, f_j \rangle = 1$ for every j which is impossible since $\lim \langle x, f_j \rangle = 0$ for every $x \in A(S)$.

7. For every natural number i, put $g_i = b_i$, $h_i = \frac{1}{\beta_i} y_i - \frac{1}{\beta_{i+1}} y_{i+1}$.

We have $|g_i| \leqq 1 + \sigma$, $|h_i| \leqq \frac{2}{\beta}$. It is easy to see that (g_i, h_j) is a biorthogonal system.

8. Consider the sum $h_1 + h_2 + \ldots + h_n$. We have

$$h_1 + \ldots + h_n = \frac{1}{\beta_1} y_1 - \frac{1}{\beta_2} y_2 + \frac{1}{\beta_2} y_2 - \frac{1}{\beta_3} y_3 + \ldots + \frac{1}{\beta_n} y_n - \\ - \frac{1}{\beta_{n+1}} y_{n+1} = \frac{1}{\beta_1} y_1 - \frac{1}{\beta_{n+1}} y_{n+1}.$$

It follows that there is a common bound $\frac{2}{\beta}$ for the sums $h_1 + \ldots + h_n$.

9. Let $\lambda_1, \lambda_2, \ldots$ be a sequence of nonnegative numbers such that $\sum_1^\infty \lambda_i < \infty$. Let z belong to the annihilator $B(S)^0$ of the sequence h_j. We have then

$$\left|\sum_1^n \lambda_i g_i + z\right| \geqq \frac{\beta}{2} \left\langle \sum_1^n \lambda_i g_i \,, \quad h_1 + \ldots + h_n \right\rangle = \frac{\beta}{2} \sum_1^n \lambda_i \,.$$

Consider the space $E/B(S)^0$, the norm of which will be denoted by $\|x\|$. If the g_i are considered as elements of $E/B(S)^0$, we have clearly

$$(1 + \sigma) \sum_1^\infty \lambda_i \geqq \left\| \sum_1^\infty \lambda_i g_i \right\| \geqq \frac{\beta}{2} \sum_1^\infty \lambda_i \,.$$

10. We have seen in section 6 that every nonreflexive Banach space possesses a bounded biorthogonal system (e_i, f_j) such the sums $e_1 + \ldots + e_n$ are bounded. Consider now an arbitrary Banach space E and a bounded biorthogonal system $S = (e_i, f_j)$ in E. We intend to show that $\lim \langle x, f_j \rangle = 0$ for every $x \in A(S)$. To see that, take an arbitrary positive ε. There exists a natural q and real numbers $\alpha_1, \ldots, \alpha_q$ such that $|x - \sum_{i=1}^q \alpha_i e_i| < \varepsilon$. If $n > q$, we have

$$\langle x, f_n \rangle = \langle \sum_{i=1}^q \alpha_i e_i, f_n \rangle + \Theta \varepsilon |f_n| = \Theta \varepsilon |f_n| \,,$$

where $|\Theta| < 1$.

Since there is a common bound for the $|f_n|$, our assertion is proved.

Suppose further that E is reflexive and that the sums $s_n = e_1 + \ldots + e_n$ are bounded. It follows that the sequence s_n has a weak limit point $s \in A(S)$. Let j be a fixed natural number. We have $\langle s_n, f_j \rangle = 1$ for $n \geqq j$ whence $\langle s, f_j \rangle = 1$. This is a contradiction with $\lim \langle s, f_j \rangle = 0$. The proof of the first theorem is thus complete.

11. Let us turn now to theorem 2. Consider first an arbitrary Banach space E and a bounded biorthogonal system $S = (e_i, f_j)$ in E. Let us denote by

$B_1(S)$ the strongly closed subspace of E' generated by the sequence f_j. We are going to show that $\lim \langle e_n, f \rangle = 0$ for every $f \in B_1(S)$. Take an arbitrary $\varepsilon > 0$. There exists a linear combination $h = \sum_{j=1}^{p} \alpha_j f_j$ such that $|f - h| < \varepsilon$. If $n > p$, we have $\langle e_n, h \rangle = 0$ whence $|\langle e_n, f \rangle| \leqq \varepsilon |e_n|$. Suppose now that E is reflexive and that the sequence $f_1 + \ldots + f_n$ is bounded. It follows that the sequence $f_1 + \ldots + f_n$ has a weak limit point $s \in B(S)$. The space E being reflexive, we have $B(S) = B_1(S)$, whence $s \in B_1(S)$ so that $\lim \langle e_n, s \rangle = 0$. This is, however, a contradiction, since $\langle e_n, f_1 + \ldots + f_r \rangle = 1$ for $r \geqq n$, whence $\langle e_n, s \rangle = 1$ for every n.

On the other hand, if E is non-reflexive, the existence of a biorthogonal system with the required properties follows from 8. The second theorem is thus established.

12. Consider now the third theorem. First of all, let E be a non-reflexive Banach space. It follows from 5 that there exists a bounded biorthogonal system (e_i, f_j) with the properties described in 2°. We may take $\omega = 2(1 + \sigma)$. Hence 1° implies 2°. Next, consider a Banach space E with a bounded biorthogonal system (e_i, f_j) fulfilling 2°. We intend to show that the space $A(S)$ is nonreflexive. Every functional on E may be considered also as an element of $A(S)'$. Let us denote by $\|f\|$ the norm of an $f \in A(S)'$. Let $\lambda_1, \ldots, \lambda_n$ be an arbitrary sequence of nonnegative numbers. Since $|e_1 + \ldots + e_n| \leqq \omega$, we have

$$\frac{1}{\omega} \sum_{i=1}^{n} \lambda_i = \frac{1}{\omega} \langle e_1 + \ldots + e_n, \lambda_1 f_1 + \ldots + \lambda_n f_n \rangle \leqq \left\| \sum_{i=1}^{n} \lambda_j f_j \right\| .$$

According to a well-known result [2], [3], the last inequality is sufficient for the existence of an element $r \in A(S)''$ such that $\langle r, f_j \rangle \geqq \frac{1}{\omega}$ for every j. We have seen, however, that $\lim \langle x, f_j \rangle = 0$ for every $x \in A(S)$. (See section 10.) It follows that $A(S)$ is not reflexive so that E cannot be reflexive, either.

Further, we have seen in section 9 that every non reflexive Banach space possesses a biorthogonal system with the properties described in 3° of theorem 3. On the other hand, let E be a Banach space with a bounded biorthogonal system $S = (e_i, f_j)$ fulfilling 3°. It is easy to see that we have $|\sum \lambda_i e_i + z| \geqq \eta \sum \lambda_i$ for every finite sequence of nonnegative numbers and every $z \in B(S)^0$. This inequality ensures the existence of a functional $f \in E'$ such that $\langle B(S)^0, f \rangle = 0$ and $\langle e_i, f \rangle \geqq \eta$ for every i. Since $\langle B(S)^0, f \rangle = 0$ we have $f \in B(S)^{00}$ whence $f \in B(S)$. We have seen already that $\lim \langle e_n, h \rangle = 0$ for every $h \in B_1(S)$. We see thus that f does not belong to $B_1(S)$ so that $B_1(S)$ is properly contained in $B(S)$. The space E cannot be reflexive. The proof is now complete.

To complete the picture of the structure of a non-reflexive Banach space given by theorem 3 we intend to give an example of a reflexive space E and a bounded biorthogonal system (e_i, f_j) in E with the following property: there exist two positive numbers $\alpha \geqq \beta > 0$ such that

$$\alpha \sum \lambda_i \geqq |\sum \lambda_i e_i| \geqq \beta \sum \lambda_i$$

for every finite sequence of nonnegative numbers λ_i.

Let E be the real Hilbert space corresponding to the interval $\langle -1, +1\rangle$. Let $s_0 = 0$, $s_i = s_{i-1} + \frac{1}{2^i}$ for $i = 1, 2, \ldots$ and let us denote by H_i the interval $\langle s_{i-1}, s_i\rangle$. Let a_0 be the characteristic function of the interval $\langle -1, 0\rangle$, let a_i be the characteristic function of H_i. For $i = 1, 2, \ldots$ let h_i be defined as follows: $h_i(t) = 1$ on the first half of H_i, $h_i(t) = -1$ on the second half of H_i and $h_i(t) = 0$ outside H_i. For $i = 1, 2, \ldots$ put

$$e_i = -\frac{1}{v^i} a_0 + v^i h_i + \sum_{\substack{j=1 \\ j \neq i}}^{\infty} a_j ,$$

$$f_i = a_0 + (v^i + 1) h_i + \sum_{\substack{j=1 \\ j \neq i}}^{\infty} h_j$$

where $v = \sqrt{2}$.

It is easy to see that $|e_i| \leqq \sqrt{3}$ and $|f_i| \leqq \sqrt{6}$ for every $i = 1, 2, \ldots$. At the same time it may be verified that (e_i, f_j) is a biorthogonal system in E. Now let $i \neq j$. We have then

$$\int e_i(t)\, e_j(t) = \frac{1}{v^{i+j}} + 1 - \frac{1}{2^i} - \frac{1}{2^j} \geqq \frac{1}{4}.$$

If $\lambda_1, \ldots, \lambda_n$ are arbitrary nonnegative numbers, we have

$$\begin{aligned} &\int (\lambda_1 e_1(t) + \ldots + \lambda_n e_n(t))^2 \, \mathrm{d}t = \\ &= \lambda_1^2 + \ldots + \lambda_n^2 + \sum_{i \neq j} \lambda_i \lambda_j \int e_i(t)\, e_j(t)\, \mathrm{d}t \geqq \\ &\geqq \lambda_1^2 + \ldots + \lambda_n^2 + \tfrac{1}{4} \sum_{i \neq j} \lambda_i \lambda_j \geqq \tfrac{1}{4}(\lambda_1 + \ldots + \lambda_n)^2 \end{aligned}$$

so that $|\lambda_1 e_1 + \ldots + \lambda_n e_n| \geqq \frac{1}{2}(\lambda_1 + \ldots + \lambda_n)$. The above inequality is thus satisfied with $\alpha = 1$ and $\beta = \frac{1}{2}$.

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Резюме

БИОРТОГОНАЛЬНЫЕ СИСТЕМЫ И РЕФЛЕКСИВНОСТЬ БАНАХОВЫХ ПРОСТРАНСТВ

ВЛАСТИМИЛ ПТАК (Vlastimil Pták), Прага

(Поступило в редакцию 7/VII 1958 г.)

В предлагаемой работе исследуются простые геометрические свойства биортогональных систем в пространстве Банаха E и их связь с рефлексивностью пространства E. Полученные результаты позволяют также описать геометрическую структуру нерефлексивных пространств Банаха и дополняют таким образом прежние результаты R. C. JAMES'а [1]. В основу исследований положен метод, разработанный в более ранней работе автора [4]. Доказываются следующие три теоремы:

Теорема 1. *Банахово пространство E является рефлексивным тогда и только тогда, если для каждой ограниченной биортогональной системы (e_i, f_j) последовательность $e_1 + e_2 + \ldots + e_n$ неограничена.*

Теорема 2. *Банахово пространство E является рефлексивным тогда и только тогда, если для каждой ограниченной биортогональной системы (e_i, f_j) последовательность $f_1 + f_2 + \ldots + f_n$ неограничена.*

Теорема 3. *Пусть E — пространство Банаха. Тогда следующие три условия являются эквивалентными:*

1° *пространство E нерефлексивно:*

2° *существует ограниченная биортогональная система (e_i, f_j) и число $\omega > 0$ так, что для каждой монотонной последовательности $\alpha_1, \alpha_2, \ldots$, сходящейся к нулю, сумма $x = \sum_1^\infty \alpha_i e_i$ существует и выполняет неравенство $|x| \leqq \omega |\alpha_1|$;*

3° *существует ограниченная биортогональная система $S = (e_i, f_j)$ со свойством: если принять S за биортогональную систему в пространстве $E/B(S)^\circ$ (с нормой $\|x\|$), то существует число $\eta > 0$ так, что*

$$\|\sum_1^\infty \lambda_i e_i\| \geqq \eta \sum_1^\infty \lambda_i$$

для любой последовательности $\lambda_1, \lambda_2, \ldots$ неотрицательных чисел, для которой $\sum_1^\infty \lambda_i < \infty$.