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Biorthogonal Wavelets on Local Fields of Positive Characteristic

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Abstract

We generalize the concept of biorthogonal wavelets to a local field K of positive characteristic. We show that if the translates of the scaling functions of two multiresolution analyses are biorthogonal, then the associated wavelet families are also biorthogonal. Under mild assumptions on the scaling functions and the wavelets, we also show that the wavelets generate Riesz bases for $L^2(K)$.

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1 Introduction

The concept of wavelet is defined and studied extensively in the Euclidean spaces $\mathbb{R}^n$, see [10, 14, 21, 31, 32, 37, 38] and references therein. Subsequently, it has been extended to many different setups. Dahlke [13] introduced this concept on locally compact abelian groups. This was generalized to abstract Hilbert spaces by Han, Larson, Papadakis and

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Stavropoulos [20, 35]. Lemarie [28] extended this concept to stratified Lie groups. Recently, R. L. Benedetto and J. J. Benedetto [6, 7] developed a wavelet theory for local fields and related groups.

A field K equipped with a topology is called a local field if both the additive and multiplicative groups of K are locally compact abelian groups. The local fields are essentially of two types (excluding the connected local fields $\mathbb{R}$ and $\mathbb{C}$). The local fields of characteristic zero include the p-adic field $\mathbb{Q}_p$. Examples of local fields of positive characteristic are the Cantor dyadic group and the Vilenkin p-groups. Even though the structures and metrics of local fields of zero and positive characteristics are similar, their wavelet and MRA (multiresolution analysis) theory are quite different.

Khrennikov, Shelkovich and Skopina [23] constructed a number of scaling functions generating an MRA of $L^2(\mathbb{Q}_p)$. But later on in [3], Albeverio, Evdokimov and Skopina proved that all these scaling functions lead to the same Haar MRA and that there exist no other orthogonal test scaling functions generating an MRA of $L^2(\mathbb{Q}_p)$ except those described in [23]. Some wavelet bases for $L^2(\mathbb{Q}_p)$ different from the Haar system were constructed in [15] and [2]. These wavelet bases were obtained by relaxing the basis condition in the definition of an MRA. Moreover, these systems form Riesz bases without any dual wavelet systems.

Haar type wavelets can also be constructed on certain metric-measure spaces without any algebraic structures, namely on a space (X,d,μ) of homogeneous type. A space of homogeneous type is a quasi-metric space X with quasi-metric d such that the d-balls are open sets, and μ is a regular measure defined on the σ-algebra containing the d-balls that satisfies the “doubling condition”, i. e., there is a constant A such that the measure of a ball of radius $2r$ is at most A times the measure of the ball of radius r with the same centre. We refer to [1] for the details of this construction. Novikov and Skopina have observed that this can also be done in the absence of a metric. In [33] they showed the existence of Haar MRA on a measure space (Ω,Σ,μ) equipped with a topology such that Σ contains all the open sets and satisfies some other conditions.

On the other hand, Lang [25, 26, 27] constructed several examples of compactly supported wavelets for the Cantor dyadic group. Farkov [16, 17] has constructed many examples of wavelets for the Vilenkin p-groups. Several examples of biorthogonal wavelets on the Vilenkin groups were constructed by Farkov in [18] and by Farkov and Rodionov in [19]. By choosing the parameters appearing in these constructions suitably, we can see that these wavelets are not orthogonal. Also, in [19], the authors have provided an algorithm to construct biorthogonal wavelets on such groups.

For related works on zero-dimensional groups, we refer to [30] and references cited there.

Jiang, Li and Jin [22] gave the definition of an MRA on a local field K of positive characteristic and constructed the wavelets from an MRA. In [4], among other results, we characterized the scaling functions of MRAs of local fields of positive characteristic, and in [5], we constructed the wavelet packets and wavelet frame packets associated with such MRAs.

The concept of biorthogonal wavelets plays an important role in applications. We refer to [11, 12, 24] for various aspects of this theory on $\mathbb{R}$. For the higher dimensional situation on $\mathbb{R}^n$, we refer to the articles [8, 9, 29].

In this article we generalize the concept of biorthogonal wavelets to a local field K of positive characteristic. We show that if φ and $\tilde{\varphi}$ are the scaling functions of two multiresolution analyses (MRAs) such that their translates are biorthogonal, then the associated families of wavelets are also biorthogonal. Under mild decay conditions on the scaling functions and the wavelets, we also show that the wavelets generate Riesz bases for $L^2(K)$.

The article is organized as follows. In section 2, we give a brief introduction to local fields and Fourier analysis on such a field. In section 3, we find necessary and sufficient conditions for the translates of a function to form a Riesz basis for its closed linear span. We give the definition of an MRA in section 4, where we also define the projection operators associated with the MRAs and show that they are uniformly bounded on $L^2(K)$. In the last section, we prove that the wavelets associated with dual MRAs are biorthogonal and generate Riesz bases for $L^2(K)$.

2 Preliminaries on local fields

Let K be a field and a topological space. Then K is called a *locally compact field* or a *local field* if both K^+ and K^* are locally compact abelian groups, where K^+ and K^* denote the additive and multiplicative groups of K respectively.

If K is any field and is endowed with the discrete topology, then K is a local field. Further, if K is connected, then K is either $\mathbb{R}$ or $\mathbb{C}$. If K is not connected, then it is totally disconnected. So by a local field, we mean a field K which is locally compact, nondiscrete and totally disconnected.

We use the notation of the book by Taibleson [36]. Proofs of all the results stated in this section can be found in the books [36] and [34].

Let K be a local field. Since K^+ is a locally compact abelian group, we choose a Haar measure dx for K^+. If $\alpha \neq 0, \alpha \in K$, then $d(\alpha x)$ is also a Haar measure. Let $d(\alpha x) = |\alpha| dx$. We call $|\alpha|$ the *absolute value* or *valuation* of α. We also let $|0| = 0$.

The map $x \to |x|$ has the following properties:

(a) $|x| = 0$ if and only if $x = 0$;

(b) $|xy| = |x||y|$ for all $x, y \in K$;

(c) $|x + y| \leq \max\{|x|, |y|\}$ for all $x, y \in K$.

Property (c) is called the *ultrametric inequality*. It follows that

$$|x + y| = \max\{|x|, |y|\} \text{ if } |x| \neq |y|.$$

The set $\mathfrak{D} = \{x \in K : |x| \leq 1\}$ is called the *ring of integers* in K. It is the unique maximal compact subring of K. Define $\mathfrak{P} = \{x \in K : |x| < 1\}$. The set $\mathfrak{P}$ is called the *prime ideal* in K. The prime ideal in K is the unique maximal ideal in $\mathfrak{D}$. It is principal and prime.

Since K is totally disconnected, the set of values $|x|$ as x varies over K is a discrete set of the form $\{s^k : k \in \mathbb{Z}\} \cup \{0\}$ for some $s > 0$. Hence, there is an element of $\mathfrak{P}$ of maximal absolute value. Let $\mathfrak{p}$ be a fixed element of maximum absolute value in $\mathfrak{P}$. Such an element is called a *prime element* of K. Note that as an ideal in $\mathfrak{D}, \mathfrak{P} = \langle \mathfrak{p} \rangle = \mathfrak{p}\mathfrak{D}$.

It can be proved that $\mathfrak{D}$ is compact and open. Hence, $\mathfrak{P}$ is compact and open. Therefore, the residue space $\mathfrak{D}/\mathfrak{P}$ is isomorphic to a finite field $GF(q)$, where $q = p^c$ for some prime p and $c \in \mathbb{N}$. For a proof of this fact we refer to [36].

For a measurable subset E of K, let $|E| = \int_K \chi_E(x)dx$, where χ_E is the characteristic function of E and dx is the Haar measure of K normalized so that $|\mathfrak{D}| = 1$. Then, it is easy to see that $|\mathfrak{P}| = q^{-1}$ and $|\mathfrak{p}| = q^{-1}$ (see [36]). It follows that if $x \neq 0$, and $x \in K$, then $|x| = q^k$ for some $k \in \mathbb{Z}$.

Let $\mathfrak{D}^* = \mathfrak{D} \setminus \mathfrak{P} = \{x \in K : |x| = 1\}$. $\mathfrak{D}^*$ is the group of units in K^*. If $x \neq 0$, we can write $x = \mathfrak{p}^k x'$, with $x' \in \mathfrak{D}^*$.

Recall that $\mathfrak{D}/\mathfrak{P} \cong GF(q)$. Let $\mathcal{U} = \{a_i : i = 0, 1, \ldots, q-1\}$ be any fixed full set of coset representatives of $\mathfrak{P}$ in $\mathfrak{D}$. Let $\mathfrak{P}^k = \mathfrak{p}^k\mathfrak{D} = \{x \in K : |x| \leq q^{-k}\}, k \in \mathbb{Z}$. These are called *fractional ideals*. Each $\mathfrak{P}^k$ is compact and open and is a subgroup of K^+ (see [34]).

If K is a local field, then there is a nontrivial, unitary, continuous character χ on K^+. It can be proved that K^+ is self dual (see [36]).

Let χ be a fixed character on K^+ that is trivial on $\mathfrak{D}$ but is nontrivial on $\mathfrak{P}^{-1}$. We can find such a character by starting with any nontrivial character and rescaling. We will define such a character for a local field of positive characteristic. For $y \in K$, we define $\chi_y(x) = \chi(yx)$, $x \in K$.

Definition 2.1. If $f \in L^1(K)$, then the Fourier transform of f is the function $\hat{f}$ defined by

$$\hat{f}(\xi) = \int_K f(x)\overline{\chi_\xi(x)}\, dx.$$

Note that

$$\hat{f}(\xi) = \int_K f(x)\overline{\chi(\xi x)}\, dx = \int_K f(x)\chi(-\xi x)\, dx.$$

Similar to the standard Fourier analysis on the real line, one can prove the following results.

(a) The map $f \to \hat{f}$ is a bounded linear transformation of $L^1(K)$ into $L^\infty(K)$, and $\|\hat{f}\|_\infty \leq \|f\|_1$.

(b) If $f \in L^1(K)$, then $\hat{f}$ is uniformly continuous.

(c) If $f \in L^1(K) \cap L^2(K)$, then $\|\hat{f}\|_2 = \|f\|_2$.

To define the Fourier transform of function in $L^2(K)$, we introduce the functions Φ_k. For $k \in \mathbb{Z}$, let Φ_k be the characteristic function of $\mathfrak{P}^k$.

Definition 2.2. For $f \in L^2(K)$, let $f_k = f\Phi_{-k}$ and

$$\hat{f}(\xi) = \lim_{k\to\infty} \hat{f}_k(\xi) = \lim_{k\to\infty} \int_{|x|\leq q^k} f(x)\overline{\chi_\xi(x)}\, d\xi,$$

where the limit is taken in $L^2(K)$.

We have the following theorem (see Theorem 2.3 in [36]).

Theorem 2.3. *The Fourier transform is unitary on $L^2(K)$.*

A set of the form $h+\mathfrak{P}^k$ will be called a *sphere* with centre h and radius q^{-k}. It follows from the ultrametric inequality that if S and T are two spheres in K, then either S and T are disjoint or one sphere contains the other. Also, note that the characteristic function of the sphere $h+\mathfrak{P}^k$ is $\Phi_k(\cdot - h)$ and that $\Phi_k(\cdot - h)$ is constant on cosets of $\mathfrak{P}^k$.

Definition 2.4. The set $\mathcal{S}$ is the space of all finite linear combinations of functions of the form $\Phi_k(\cdot - h)$, $h \in K$, $k \in \mathbb{Z}$.

This class of functions can also be described in the following way. A function $g \in \mathcal{S}$ if and only if there exist integers k, l such that g is constant on cosets of $\mathfrak{P}^k$ and is supported on $\mathfrak{P}^l$.

It follows that $\mathcal{S}$ is closed under Fourier transform and is an algebra of continuous functions with compact support, which is dense in $C_0(K)$ as well as $L^p(K), 1 \le p < \infty$. We have the following theorem (see [36]).

Theorem 2.5. *If $g \in \mathcal{S}$ is constant on cosets of $\mathfrak{P}^k$ and is supported on $\mathfrak{P}^l$, then $\hat{g} \in \mathcal{S}$ is constant on cosets of $\mathfrak{P}^{-l}$ and is supported on $\mathfrak{P}^{-k}$.*

Let χ_u be any character on K^+. Since $\mathfrak{D}$ is a subgroup of K^+, the restriction $\chi_u|_{\mathfrak{D}}$ is a character on $\mathfrak{D}$. Also, as characters on $\mathfrak{D}, \chi_u = \chi_v$ if and only if $u - v \in \mathfrak{D}$. That is, $\chi_u = \chi_v$ if $u+\mathfrak{D} = v+\mathfrak{D}$ and $\chi_u \ne \chi_v$ if $(u+\mathfrak{D}) \cap (v+\mathfrak{D}) = \phi$. Hence, if $\{u(n)\}_{n=0}^{\infty}$ is a complete list of distinct coset representative of $\mathfrak{D}$ in K^+, then $\{\chi_{u(n)}\}_{n=0}^{\infty}$ is a list of distinct characters on $\mathfrak{D}$. It is proved in [36] that this list is complete. That is, we have the following proposition.

Proposition 2.6. *Let $\{u(n)\}_{n=0}^{\infty}$ be a complete list of (distinct) coset representatives of $\mathfrak{D}$ in K^+. Then $\{\chi_{u(n)}\}_{n=0}^{\infty}$ is a complete list of (distinct) characters on $\mathfrak{D}$. Moreover, it is a complete orthonormal system on $\mathfrak{D}$.*

Given such a list of characters $\{\chi_{u(n)}\}_{n=0}^{\infty}$, we define the Fourier coefficients of $f \in L^1(\mathfrak{D})$ as

$$\hat{f}(u(n)) = \int_{\mathfrak{D}} f(x)\overline{\chi_{u(n)}(x)}dx.$$

The series $\sum_{n=0}^{\infty} \hat{f}(u(n))\chi_{u(n)}(x)$ is called the Fourier series of f. From the standard L^2-theory for compact abelian groups we conclude that the Fourier series of f converges to f in $L^2(\mathfrak{D})$ and Parseval's identity holds:

$$\int_{\mathfrak{D}} |f(x)|^2 dx = \sum_{n=0}^{\infty} |\hat{f}(u(n))|^2.$$

Also, if $f \in L^1(\mathfrak{D})$ and $\hat{f}(u(n)) = 0$ for all $n = 0, 1, 2, \ldots$, then $f = 0$ a. e.

These results hold irrespective of the ordering of the characters. We now proceed to impose a natural order on the sequence $\{u(n)\}_{n=0}^{\infty}$. Note that $\Gamma = \mathfrak{D}/\mathfrak{P}$ is isomorphic to the finite field $GF(q)$ and $GF(q)$ is a c-dimensional vector space over the field $GF(p)$. We choose a set $\{1 = \epsilon_0, \epsilon_1, \epsilon_2, \cdots, \epsilon_{c-1}\} \subset \mathfrak{D}^*$ such that $\mathrm{span}\{\epsilon_j\}_{j=0}^{c-1} \cong GF(q)$. Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $n \in \mathbb{N}_0$ such that $0 \le n < q$, we have

$$n = a_0 + a_1 p + \cdots + a_{c-1} p^{c-1}, \quad 0 \le a_k < p, k = 0, 1, \cdots, c-1.$$

Define

$$u(n) = (a_0 + a_1\epsilon_1 + \cdots + a_{c-1}\epsilon_{c-1})\mathfrak{p}^{-1}. \tag{2.1}$$

Note that $\{u(n) : n = 0,1,\ldots,q-1\}$ is a complete set of coset representatives of $\mathfrak{D}$ in $\mathfrak{P}^{-1}$. Now, for $n \ge 0$, write

$$n = b_0 + b_1q + b_2q^2 + \cdots + b_sq^s, \quad 0 \le b_k < q, k = 0,1,2,\cdots,s,$$

and define

$$u(n) = u(b_0) + u(b_1)\mathfrak{p}^{-1} + \cdots + u(b_s)\mathfrak{p}^{-s}. \tag{2.2}$$

This defines $u(n)$ for all $n \in \mathbb{N}_0$. In general, it is not true that $u(m+n) = u(m)+u(n)$. But it follows that

$$u(rq^k + s) = u(r)\mathfrak{p}^{-k} + u(s) \quad \text{if } r \ge 0, k \ge 0 \text{ and } 0 \le s < q^k.$$

In the following proposition we list some properties of $\{u(n)\}$ which will be used later. For a proof, we refer to [5].

Proposition 2.7. *For $n \in \mathbb{N}_0$, let $u(n)$ be defined as in* (2.1) *and* (2.2). *Then*

(a) $u(n) = 0$ *if and only if* $n = 0$. *If* $k \ge 1$, *then* $|u(n)| = q^k$ *if and only if* $q^{k-1} \le n < q^k$*;*

(b) $\{u(k) : k \in \mathbb{N}_0\} = \{-u(k) : k \in \mathbb{N}_0\}$*;*

(c) for a fixed $l \in \mathbb{N}_0$, *we have* $\{u(l)+u(k) : k \in \mathbb{N}_0\} = \{u(k) : k \in \mathbb{N}_0\}$.

For brevity, we will write $\chi_n = \chi_{u(n)}$ for $n \in \mathbb{N}_0$. As mentioned before, $\{\chi_n : n \in \mathbb{N}_0\}$ is a complete set of characters on $\mathfrak{D}$.

Let K be a local field of characteristic $p > 0$ and $\epsilon_0, \epsilon_1, \ldots, \epsilon_{c-1}$ be as above. We define a character χ on K as follows (see [39]):

$$\chi(\epsilon_\mu \mathfrak{p}^{-j}) = \begin{cases} \exp(2\pi i/p), & \mu = 0 \text{ and } j = 1, \\ 1, & \mu = 1,\cdots,c-1 \text{ or } j \ne 1. \end{cases}$$

Note that χ is trivial on $\mathfrak{D}$ but nontrivial on $\mathfrak{P}^{-1}$.

In order to be able to define the concepts of multiresolution analysis and wavelet on local fields, we need analogous notions of translation and dilation. Since $\bigcup_{j\in\mathbb{Z}} \mathfrak{p}^{-j}\mathfrak{D} = K$, we can regard $\mathfrak{p}^{-1}$ as the dilation (note that $|\mathfrak{p}^{-1}| = q$) and since $\{u(n) : n \in \mathbb{N}_0\}$ is a complete list of distinct coset representatives of $\mathfrak{D}$ in K, the set $\{u(n) : n \in \mathbb{N}_0\}$ can be treated as the translation set. Note that it follows from Proposition 2.7 that the translation set $\{u(n) : n \in \mathbb{N}_0\}$ is a subgroup of K^+.

Since the dilation is induced by $\mathfrak{p}^{-1}$ and $\left|\mathfrak{p}^{-1}\right| = q$, as in the case of $\mathbb{R}^n$, we expect the existence of $q-1$ number of functions $\{\psi_1, \psi_2, \cdots, \psi_{q-1}\}$ to form a set of basic wavelets.

For $f \in L^2(K)$, $j \in \mathbb{Z}$, and $k \in \mathbb{N}_0$, we define the dilation operator δ_j and the translation operator τ_k as follows:

$$\delta_j f(x) = q^{j/2} f(\mathfrak{p}^{-1}x) \quad \text{and} \quad \tau_k f(x) = f(x - u(k)).$$

Let $f_{j,k} = \delta_j \tau_k f$. Then

$$f_{j,k}(x) = q^{j/2} f(\mathfrak{p}^{-j} x - u(k)), \quad j \in \mathbb{Z}, k \in \mathbb{N}_0.$$

It is easy to see that

$$\|f_{j,k}\|_2 = \|f\|_2, \langle \delta_j f, \delta_j g\rangle = \langle f, g\rangle, \langle f, \delta_j g\rangle = \langle \delta_{-j} f, g\rangle.$$

and

$$(f_{j,k})^\wedge(\xi) = q^{-j/2}\overline{\chi_k(\mathfrak{p}^j \xi)}\hat{f}(\mathfrak{p}^j \xi).$$

A function f on K will be called *integral-periodic* if

$$f(x + u(k)) = f(x) \text{ for all } k \in \mathbb{N}_0.$$

3 Riesz bases of translates

In this section we consider translates of a single function and find necessary and sufficient conditions when they form Riesz bases for their closed linear span.

Definition 3.1. Let $\{\psi_n : n \in \mathbb{N}_0\}$ and $\{\tilde{\psi}_n : n \in \mathbb{N}_0\}$ be two collections of functions in $L^2(K)$. We say that they are biorthogonal if

$$\langle \psi_n, \tilde{\psi}_m\rangle = \delta_{n,m} \quad \text{for every } m, n \in \mathbb{N}_0.$$

A collection $\{\psi_n : n \in \mathbb{N}_0\}$ of functions in $L^2(K)$ is said to be linearly independent if for any ℓ^2-sequence $\{a_n : n \in \mathbb{N}_0\}$ of coefficients with $\sum_{n \in \mathbb{N}_0} a_n \psi_n = 0$ in $L^2(K)$, we have $a_n = 0$ for all $n \in \mathbb{N}_0$. It is easy to see that biorthogonal sets are linearly independent.

Lemma 3.2. *Let $\{\psi_n : n \in \mathbb{N}_0\}$ be a collection of functions in $L^2(K)$. Suppose that there is a collection $\{\tilde{\psi}_n : n \in \mathbb{N}_0\}$ in $L^2(K)$ which is biorthogonal to $\{\psi_n : n \in \mathbb{N}_0\}$. Then $\{\psi_n : n \in \mathbb{N}_0\}$ is linearly independent.*

Proof. Let $\{a_n : n \in \mathbb{N}_0\}$ be an ℓ^2-sequence satisfying $\sum_{n \in \mathbb{N}_0} a_n \psi_n = 0$ in $L^2(K)$. Then for each $m \in \mathbb{N}_0$, we have

$$0 = \langle 0, \tilde{\psi}_m\rangle = \Big\langle \sum_{n \in \mathbb{N}_0} a_n \psi_n, \tilde{\psi}_m \Big\rangle = \sum_{n \in \mathbb{N}_0} a_n \langle \psi_n, \tilde{\psi}_m\rangle = a_m.$$

Hence, $\{\psi_n : n \in \mathbb{N}_0\}$ is linearly independent. □

Definition 3.3. Let $\{x_n : n \in \mathbb{N}_0\}$ be a subset of a Hilbert space H. Then $\{x_n : n \in \mathbb{N}_0\}$ is said to form a Riesz basis for H if

(a) $\{x_n : n \in \mathbb{N}_0\}$ is linearly independent, and

(b) there exist constants A and B with $0 < A \le B < \infty$ such that

$$A\|x\|_2^2 \le \sum_{n \in \mathbb{N}_0} |\langle x, x_n\rangle|^2 \le B\|x\|_2^2 \quad \text{for every } x \in H.$$

Note. The condition in (b) is known as the "frame condition".

Remark 3.4. The above definition is equivalent to the following definition. A subset $\{x_n : n \in \mathbb{N}_0\}$ of a Hilbert space H forms a Riesz basis for H if

(a) $\overline{\text{span}}\{x_n : n \in \mathbb{N}_0\} = H$, and

(b) there exist constants A and B with $0 < A \le B < \infty$ such that

$$A \sum_{n\in\mathbb{N}_0} |c_n|^2 \le \Big\| \sum_{n\in\mathbb{N}_0} c_n x_n \Big\|^2 \le B \sum_{n\in\mathbb{N}_0} |c_n|^2 \quad \text{for every } \{c_n\} \in \ell^2(\mathbb{N}_0).$$

In the following lemma, we provide a necessary and sufficient condition for the translates of two functions to be biorthogonal.

Lemma 3.5. *Let $\varphi, \tilde{\varphi} \in L^2(K)$ be given. Then $\{\varphi(\cdot - u(n)) : n \in \mathbb{N}_0\}$ is biorthogonal to $\{\tilde{\varphi}(\cdot - u(n)) : n \in \mathbb{N}_0\}$ if and only if*

$$\sum_{n\in\mathbb{N}_0} \hat{\varphi}(\xi + u(n))\overline{\hat{\tilde{\varphi}}(\xi + u(n))} = 1 \quad a.e.$$

Proof. For a fixed $l \in \mathbb{N}_0$, we have $\{u(l) + u(k) : k \in \mathbb{N}_0\} = \{u(k) : k \in \mathbb{N}_0\}$ (see Proposition 2(c)). Hence, it follows that $\langle \varphi(\cdot - u(n)), \tilde{\varphi}(\cdot - u(m))\rangle = \delta_{n,m}$ if and only if $\langle \varphi, \tilde{\varphi}(\cdot - u(m))\rangle = \delta_{0,m}$. Since

$$\begin{aligned} \langle \varphi, \tilde{\varphi}(\cdot - u(m))\rangle &= \int_K \hat{\varphi}(\xi)\overline{\hat{\tilde{\varphi}}(\xi)}\chi_m(\xi)d\xi \\ &= \int_{\mathfrak{D}} \sum_{l\in\mathbb{N}_0} \hat{\varphi}(\xi + u(l))\overline{\hat{\tilde{\varphi}}(\xi + u(l))}\chi_m(\xi)d\xi, \end{aligned}$$

the result follows from the uniqueness of the Fourier series and the fact that $\{\chi_m : m \in \mathbb{N}_0\}$ is an orthonormal basis for $L^2(\mathfrak{D})$. □

The following lemma provides a sufficient condition for the translates of a function to be linearly independent.

Lemma 3.6. *Let $\varphi \in L^2(K)$. Assume that there exist constants $c_1, c_2 > 0$ such that*

$$c_1 \le \sum_{k\in\mathbb{N}_0} |\hat{\varphi}(\xi + u(k))|^2 \le c_2 \quad \text{for a.e } \xi \in K. \tag{3.1}$$

Then $\{\varphi(\cdot - u(n)) : n \in \mathbb{N}_0\}$ is linearly independent.

Proof. By Lemma 3.2, it suffices to find a function $\tilde{\varphi}$ whose translates are biorthogonal to the translates of φ. We define $\tilde{\varphi}$ by

$$\widehat{\tilde{\varphi}}(\xi) = \frac{\hat{\varphi}(\xi)}{\sum_{k\in\mathbb{N}_0} |\hat{\varphi}(\xi + u(k))|^2}.$$

By (3.1), this function is well-defined. Now

$$\begin{aligned}\sum_{m\in\mathbb{N}_0}\hat{\varphi}(\xi+u(m))\overline{\hat{\tilde{\varphi}}(\xi+u(m))} &= \sum_{m\in\mathbb{N}_0}\hat{\varphi}(\xi+u(m))\frac{\overline{\hat{\varphi}(\xi+u(m))}}{\sum_{k\in\mathbb{N}_0}|\hat{\varphi}(\xi+u(k)+u(m))|^2}\\ &= \frac{\sum\limits_{m\in\mathbb{N}_0}|\hat{\varphi}(\xi+u(m))|^2}{\sum\limits_{l\in\mathbb{N}_0}|\hat{\varphi}(\xi+u(l))|^2}=1.\end{aligned}$$

By Lemma 3.5, $\{\varphi(\cdot-u(n)) : n\in\mathbb{N}_0\}$ is biorthogonal to $\{\tilde{\varphi}(\cdot-u(n)) : n\in\mathbb{N}_0\}$. □

Lemma 3.7. *Suppose that φ satisfies* (3.1). *Any f in* $\operatorname{span}\{\varphi(\cdot-u(n)) : n\in\mathbb{N}_0\}$ *is of the form $f=\sum\limits_{n\in\mathbb{N}_0}a_n\varphi(\cdot-u(n))$, where $\{a_n\}$ is a finite sequence. Let $\hat{a}$ be its Fourier transform, that is, $\hat{a}(\xi)=\sum\limits_{n\in\mathbb{N}_0}a_n\overline{\chi_n}(\xi)$. Then*

$$c_1\int_{\mathfrak{D}}|\hat{a}(\xi)|^2\,d\xi\le\|f\|_2^2\le c_2\int_{\mathfrak{D}}|\hat{a}(\xi)|^2\,d\xi.$$

Proof. By Placherel's theorem, we have

$$\begin{aligned}\int_K|f(x)|^2\,dx &= \int_K\Big|\sum_{n\in\mathbb{N}_0}a_n\varphi(x-u(n))\Big|^2\,dx\\ &= \int_K\Big|\sum_{n\in\mathbb{N}_0}a_n\hat{\varphi}(\xi)\overline{\chi_n}(\xi)\Big|^2\,d\xi\\ &= \int_K|\hat{\varphi}(\xi)|^2\Big|\sum_{n\in\mathbb{N}_0}a_n\overline{\chi_n}(\xi)\Big|^2\,d\xi\\ &= \int_K|\hat{\varphi}(\xi)|^2|\hat{a}(\xi)|^2\,d\xi\\ &= \int_{\mathfrak{D}}\sum_{k\in\mathbb{N}_0}|\hat{\varphi}(\xi+u(k))^2\hat{a}(\xi)|^2\,d\xi.\end{aligned}$$

The result follows by (3.1). □

Remark 3.8. In particular, for a finite sequence $\{a_n\}$, we have

$$\Big\|\sum_{n\in\mathbb{N}_0}a_n\varphi(\cdot-u(n))\Big\|_2^2\le c_2\sum_{n\in\mathbb{N}_0}|a_n|^2.$$

Theorem 3.9. *Let $\{\varphi(\cdot-u(n)) : n\in\mathbb{N}_0\}$ be a Riesz basis for its closed linear span. Suppose that there exists a function $\tilde{\varphi}$ such that $\{\tilde{\varphi}(\cdot-u(n)) : n\in\mathbb{N}_0\}$ is biorthogonal to $\{\varphi(\cdot-u(n)) : n\in\mathbb{N}_0\}$. Then*

(a) *for every $f\in\overline{\operatorname{span}}\{\varphi(\cdot-u(n)) : n\in\mathbb{N}_0\}$, we have*

$$f=\sum_{n\in\mathbb{N}_0}\langle f,\tilde{\varphi}(\cdot-u(n))\rangle\varphi(\cdot-u(n));$$

(b) *there exist constants* $A, B > 0$ *such that for every* $f \in \overline{\text{span}}\{\varphi(\cdot - u(n)) : n \in \mathbb{N}_0\}$

$$A\|f\|_2^2 \leq \sum_{n=1}^{\infty} |\langle f, \tilde{\varphi}(\cdot - u(n))\rangle|^2 \leq B\|f\|_2^2.$$

Proof. Since $\{\varphi(\cdot - u(n)) : n \in \mathbb{N}_0\}$ forms a Riesz basis for its closed linear span, there exist constants c_1 and c_2 such that (3.1) holds (see Lemma 3.4 in [5]). We will first prove (a) and (b) for $f \in \text{span}\{\varphi(\cdot - u(n)) : n \in \mathbb{N}_0\}$ and then generalize the results to $\overline{\text{span}}\{\varphi(\cdot - u(n)) : n \in \mathbb{N}_0\}$.

(a) Let $f \in \text{span}\{\varphi(\cdot - u(n)) : n \in \mathbb{N}_0\}$, then there exist a finite sequence $\{a_n\}$ such that $f = \sum_{n \in \mathbb{N}_0} a_n \varphi(\cdot - u(n))$. Using biorthogonality, we have

$$\begin{aligned}
\langle f, \tilde{\varphi}(\cdot - u(k))\rangle &= \Big\langle \sum_{n \in \mathbb{N}_0} a_n \varphi(\cdot - u(n)), \tilde{\varphi}(\cdot - u(k))\Big\rangle \\
&= \sum_{n \in \mathbb{N}_0} a_n \langle \varphi(\cdot - u(n)), \tilde{\varphi}(\cdot - u(k))\rangle \\
&= a_k.
\end{aligned}$$

(b) Since (3.1) is satisfied, by Lemma 3.7, for every $f \in \text{span}\{\varphi(\cdot - u(n)) : n \in \mathbb{N}_0\}$, we have

$$c_2^{-1}\|f\|_2^2 \leq \int_{\mathfrak{D}} |\hat{a}(\xi)|^2 d\xi \leq c_1^{-1}\|f\|_2^2.$$

By Plancherel formula for Fourier series and the fact that $a_n = \langle f, \tilde{\varphi}(\cdot - u(n))\rangle$,

$$\int_{\mathfrak{D}} |\hat{a}(\xi)|^2 d\xi = \sum_{n \in \mathbb{N}_0} |a_n|^2 = \sum_{n \in \mathbb{N}_0} |\langle f, \tilde{\varphi}(\cdot - u(n))\rangle|^2.$$

So (b) is proved.

We now generalize the results to $\overline{\text{span}}\{\varphi(\cdot - u(n)) : n \in \mathbb{N}_0\}$. First we will prove (b). For $f \in \overline{\text{span}}\{\varphi(\cdot - u(n)) : n \in \mathbb{N}_0\}$, there exists a sequence $\{f_m : m \in \mathbb{N}_0\}$ in $\text{span}\{\varphi(\cdot - u(n)) : n \in \mathbb{N}_0\}$ such that $\|f_m - f\|_2 \to 0$ as $m \to \infty$. Hence, for each $n \in \mathbb{N}_0$,

$$\langle f_m, \tilde{\varphi}(\cdot - u(n))\rangle \to \langle f, \tilde{\varphi}(\cdot - u(n))\rangle \text{ as } m \to \infty.$$

The result holds for each f_m. Hence,

$$\begin{aligned}
\sum_{n=0}^{N} |\langle f, \tilde{\varphi}(\cdot - u(n))\rangle|^2 &= \sum_{n=0}^{N} \lim_{m \to \infty} |\langle f_m, \tilde{\varphi}(\cdot - u(n))\rangle|^2 \\
&= \lim_{m \to \infty} \sum_{n=0}^{N} |\langle f_m, \tilde{\varphi}(\cdot - u(n))\rangle|^2 \\
&\leq B \lim_{m \to \infty} \|f_m\|_2^2 \\
&= B\|f\|_2^2.
\end{aligned}$$

Letting $N \to \infty$ in the above expression, we get

$$\sum_{n\in\mathbb{N}_0} |\langle f, \tilde{\varphi}(\cdot - u(n))\rangle|^2 \le B\|f\|_2^2.$$

Hence, the upper bound in (3.9) holds. Now

$$\begin{aligned}
&\Big(\sum_{n\in\mathbb{N}_0} |\langle f_m, \tilde{\varphi}(\cdot - u(n))\rangle|^2\Big)^{\frac{1}{2}} \\
&\le \Big(\sum_{n\in\mathbb{N}_0} |\langle f_m - f, \tilde{\varphi}(\cdot - u(n))\rangle|^2\Big)^{\frac{1}{2}} + \Big(\sum_{n\in\mathbb{N}_0} |\langle f, \tilde{\varphi}(\cdot - u(n))\rangle|^2\Big)^{\frac{1}{2}}.
\end{aligned}$$

Since the upper bound in (3.9) holds for each $f_m - f$ and the lower bound holds for each f_m, we have

$$A^{\frac{1}{2}}\|f_m\|_2 \le B^{\frac{1}{2}}\|f_m - f\|_2 + \Big(\sum_{n\in\mathbb{N}_0} |\langle f, \tilde{\varphi}(\cdot - u(n))\rangle|^2\Big)^{\frac{1}{2}}.$$

Taking limit as $m \to \infty$, we get

$$A\|f\|_2^2 \le \sum_{n\in\mathbb{N}_0} |\langle f, \tilde{\varphi}(\cdot - u(n))\rangle|^2.$$

Now, we will prove (a) for $f \in \overline{\text{span}}\{\varphi(\cdot - u(n)) : n \in \mathbb{N}_0\}$. Let $\epsilon > 0$ and $g \in \text{span}\{\varphi(\cdot - u(n)) : n \in \mathbb{N}_0\}$ such that $\|f - g\|_2 < \epsilon$. Since (a) holds for g, for large enough $N \in \mathbb{N}_0$, we have

$$\begin{aligned}
&f - \sum_{n=0}^{N} \langle f, \tilde{\varphi}(\cdot - u(n))\rangle \varphi(\cdot - u(n)) \\
&= f - g + \sum_{n=0}^{N} \langle g, \tilde{\varphi}(\cdot - u(n))\rangle \varphi(\cdot - u(n)) - \sum_{n=0}^{N} \langle f, \tilde{\varphi}(\cdot - u(n))\rangle \varphi(\cdot - u(n)) \\
&= f - g + \sum_{n=0}^{N} \langle g - f, \tilde{\varphi}(\cdot - u(n))\rangle \varphi(\cdot - u(n)).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\Big\| f - \sum_{n=0}^{N} \langle f, \tilde{\varphi}(\cdot - u(n))\rangle \varphi(\cdot - u(n))\Big\|_2 \\
&\le \|f - g\|_2 + \Big\| \sum_{n=0}^{N} \langle g - f, \tilde{\varphi}(\cdot - u(n))\rangle \varphi(\cdot - u(n))\Big\|_2 \\
&\le \|f - g\|_2 + \sqrt{c_2}\Big(\sum_{n=0}^{N} |\langle g - f, \tilde{\varphi}(\cdot - u(n))\rangle|^2\Big)^{\frac{1}{2}} \text{(by Remark 3.8)} \\
&\le \|f - g\|_2 + \sqrt{c_2}\sqrt{B}\|f - g\|_2 < (1 + \sqrt{c_2 B})\epsilon.
\end{aligned}$$

Since ϵ is arbitrary, the result follows. □

4 Multiresolution analysis on a local field

Definition 4.1. Let K be a local field of characteristic $p > 0$, $\mathfrak{p}$ be a prime element of K and $u(n), n \in \mathbb{N}_0$, be as defined in (2.1) and (2.2). A multiresolution analysis (MRA) of $L^2(K)$ is a sequence $\{V_j : j \in \mathbb{Z}\}$ of closed subspaces of $L^2(K)$ satisfying the following properties:

(a) $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$;

(b) $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(K)$;

(c) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$;

(d) $f \in V_j$ if and only if $f(\mathfrak{p}^{-1}\cdot) \in V_{j+1}$ for all $j \in \mathbb{Z}$;

(e) there is a function $\varphi \in V_0$, called the *scaling function*, such that $\{\varphi(\cdot - u(k)) : k \in \mathbb{N}_0\}$ forms a Riesz basis for V_0.

In the usual definition of an MRA, it is required that $\{\varphi(\cdot - u(k)) : k \in \mathbb{N}_0\}$ forms an orthonormal basis for V_0. In [5], we proved that if $\{\varphi(\cdot - u(k)) : k \in \mathbb{N}_0\}$ forms a Riesz basis for V_0, then we can find another function $\varphi_1 \in V_0$ such that $\{\varphi_1(\cdot - u(k)) : k \in \mathbb{N}_0\}$ forms an orthonormal basis for V_0. In the same paper, we also proved the following result.

Lemma 4.2. *Let $\varphi \in L^2(K)$ be such that $\{\varphi(\cdot - u(k)) : k \in \mathbb{N}_0\}$ forms a Riesz basis of its closed linear span. Then, there exist C_1 and C_2 such that for a.e. $\xi \in \mathfrak{D}$,*

$$C_1 \le \sum_{k\in\mathbb{N}_0} |\hat{\varphi}(\xi + u(k))|^2 \le C_2.$$

We can use the condition (e) in the definition of an MRA to get Riesz bases for V_j.

Lemma 4.3. *Let φ be the scaling function for an MRA $\{V_j : j \in \mathbb{Z}\}$. Then, for each $j \in \mathbb{Z}$, $\{\varphi_{j,k} : k \in \mathbb{N}_0\}$ is a Riesz basis for V_j.*

Proof. If we define $\tilde{\varphi}$ by

$$\hat{\tilde{\varphi}}(\xi) = \frac{\hat{\varphi}(\xi)}{\sum_{k\in\mathbb{N}_0} |\hat{\varphi}(\xi + u(k))|^2},$$

then $\{\tilde{\varphi}(\cdot - u(k)) : k \in \mathbb{N}_0\}$ is biorthogonal to $\{\varphi(\cdot - u(k)) : k \in \mathbb{N}_0\}$ (see the proof of Lemma 3.6). Hence,

$$\langle \varphi_{j,n}, \tilde{\varphi}_{j,m}\rangle = \langle \delta_j \varphi(\cdot - u(n)), \delta_j \tilde{\varphi}(\cdot - u(m))\rangle = \langle \varphi(\cdot - u(n)), \tilde{\varphi}(\cdot - u(m))\rangle = \delta_{n,m}.$$

That is, $\{\tilde{\varphi}_{j,k} : k \in \mathbb{N}_0\}$ is biorthogonal to $\{\varphi_{j,k} : k \in \mathbb{N}_0\}$ for every $j \in \mathbb{Z}$. Hence, by Lemma 3.2, $\{\varphi_{j,k} : k \in \mathbb{N}_0\}$ is linearly independent.

We need to show that $\{\varphi_{j,k} : k \in \mathbb{N}_0\}$ satisfies the frame condition. For any $f \in V_j$, we have

$$\sum_{k\in\mathbb{N}_0} |\langle f, \varphi_{j,k}\rangle|^2 = \sum_{k\in\mathbb{N}_0} |\langle f, \delta_j \varphi(\cdot - u(k))\rangle|^2 = \sum_{k\in\mathbb{N}_0} |\langle \delta_{-j} f, \varphi(\cdot - u(k))\rangle|^2.$$

Since $\{\varphi(\cdot - u(k)) : k \in \mathbb{N}_0\}$ is a Riesz basis for V_0 and $\delta_{-j} f \in V_0$, there are constants $A, B > 0$ such that for every $f \in V_j$,

$$A\|\delta_{-j} f\|_2^2 \le \sum_{k\in\mathbb{N}_0} |\langle \delta_{-j} f, \varphi(\cdot - u(k))\rangle|^2 \le B\|\delta_{-j} f\|_2^2.$$

This is equivalent to

$$A\|f\|_2^2 \le \sum_{k\in\mathbb{N}_0} |\langle f, \delta_j \varphi(\cdot - u(k))\rangle|^2 \le B\|f\|_2^2.$$

Hence, $\{\varphi_{j,k} : k \in \mathbb{N}_0\}$ satisfies the frame condition. □

Lemma 4.4. *Suppose that $\{V_j : j \in \mathbb{Z}\}$ is an MRA with scaling function φ. Then there exists a sequence $\{h_n : n \in \mathbb{N}_0\}$ in $l^2(\mathbb{N}_0)$ such that*

$$\varphi(x) = \sum_{n\in\mathbb{N}_0} h_n q^{1/2} \varphi(\mathfrak{p}^{-1} x - u(n))$$

and an integral periodic function m_0 such that

$$\hat{\varphi}(\xi) = m_0(\mathfrak{p}\xi)\hat{\varphi}(\mathfrak{p}\xi).$$

Proof. Since $q^{-1}\varphi(\mathfrak{p}\cdot) \in V_{-1} \subset V_0$, by Theorem 3.9(a), we have

$$q^{-1}\varphi(\mathfrak{p}x) = \sum_{n\in\mathbb{N}_0} \langle f, \tilde{\varphi}(\cdot - u(n))\rangle \varphi(x - u(n)) = \sum_{n\in\mathbb{N}_0} h_n \varphi(x - u(n)).$$

By Theorem 3.9(b), $\{h_n\} \in \ell^2(\mathbb{N}_0)$. Taking Fourier transform, we get

$$\hat{\varphi}(\mathfrak{p}^{-1}\xi) = \sum_{n\in\mathbb{N}_0} h_n \overline{\chi_n(\xi)} \hat{\varphi}(\xi) = m_0(\xi)\hat{\varphi}(\xi).$$

This is equivalent to

$$\hat{\varphi}(\xi) = m_0(\mathfrak{p}\xi)\hat{\varphi}(\mathfrak{p}\xi).$$

As in Proposition 3 in [22], we can show that m_0 is integral-periodic. □

Definition 4.5. A pair of MRAs $\{V_j : j \in \mathbb{Z}\}$ and $\{\tilde{V}_j : j \in \mathbb{Z}\}$ with scaling functions φ and $\tilde{\varphi}$ respectively are said to be dual to each other if $\{\varphi(\cdot - u(k)) : k \in \mathbb{N}_0\}$ and $\{\tilde{\varphi}(\cdot - u(k)) : k \in \mathbb{N}_0\}$ are biorthogonal.

Definition 4.6. Let φ and $\tilde{\varphi}$ be scaling functions for dual MRAs. For each $j \in \mathbb{Z}$, define the operator P_j, $\tilde{P}_j$ on $L^2(K)$ by

$$\begin{aligned} P_j f &= \sum_{k\in\mathbb{N}_0} \langle f, \tilde{\varphi}_{j,k}\rangle \varphi_{j,k}, \\ \tilde{P}_j f &= \sum_{k\in\mathbb{N}_0} \langle f, \varphi_{j,k}\rangle \tilde{\varphi}_{j,k}. \end{aligned}$$

We first note that the series defining these operators are convergent in $L^2(K)$ and that these operators are uniformly bounded on $L^2(K)$.

Lemma 4.7. *The opeartors P_j and $\tilde{P}_j$ are uniformly bounded.*

Proof. Since the translates of φ and $\tilde{\varphi}$ form Riesz bases for their closed linear spans, by Lemma 4.2, there exist constants C_1 and C_2 such that

$$C_1 \leq \sum_{k\in\mathbb{N}_0} |\hat{\varphi}(\xi+u(k))|^2 \leq C_2 \quad \text{and} \quad C_1 \leq \sum_{k\in\mathbb{N}_0} |\hat{\tilde{\varphi}}(\xi+u(k))|^2 \leq C_2.$$

Now, let $\{c_k : k \in \mathbb{N}_0\} \in \ell^2(\mathbb{N}_0)$. Then, by Remark 3.4, there exists $B > 0$ such that

$$\Big\|\sum_{k\in\mathbb{N}_0} c_k \varphi_{0,k}\Big\|_2^2 \leq B \sum_{k\in\mathbb{N}_0} |c_k|^2.$$

Now, for $f \in L^2(K)$, we have

$$\begin{aligned}
\sum_{k\in\mathbb{N}_0} |\langle f, \varphi_{0,k}\rangle|^2 &= \sum_{k\in\mathbb{N}_0} \Big|\int_K \hat{f}(\xi)\overline{\hat{\varphi}(\xi)}\chi_k(\xi)\, d\xi\Big|^2 \\
&= \sum_{k\in\mathbb{N}_0} \Big|\int_{\mathfrak{D}} \sum_{l\in\mathbb{N}_0} \hat{f}(\xi+u(l))\overline{\hat{\varphi}(\xi+u(l))}\chi_k(\xi)\, d\xi\Big|^2 \\
&= \sum_{k\in\mathbb{N}_0} \Big|\int_{\mathfrak{D}} F(\xi)\chi_k(\xi)\, d\xi\Big|^2 = \sum_{k\in\mathbb{N}_0} |\hat{F}(u(k))|^2 = \|F\|^2_{L^2(\mathfrak{D})} \\
&= \int_{\mathfrak{D}} \Big|\sum_{l\in\mathbb{N}_0} \hat{f}(\xi+u(l))\overline{\hat{\varphi}(\xi+u(l))}\Big|^2\, d\xi \\
&\leq \int_{\mathfrak{D}} \Big(\sum_{l\in\mathbb{N}_0} |\hat{f}(\xi+u(l))|^2\Big)\Big(\sum_{l\in\mathbb{N}_0} |\hat{\varphi}(\xi+u(l))|^2\Big)\, d\xi \\
&\leq C_2 \int_{\mathfrak{D}} \Big(\sum_{l\in\mathbb{N}_0} |\hat{f}(\xi+u(l))|^2\Big)\, d\xi \\
&= C_2 \int_K |\hat{f}(\xi)|^2\, d\xi = C_2\|f\|_2^2.
\end{aligned}$$

Similar estimates hold for $\tilde{\varphi}$. Hence, for $f \in L^2(K)$, we have

$$\|P_0 f\|_2^2 = \Big\|\sum_{k\in\mathbb{N}_0} \langle f, \tilde{\varphi}_{0,k}\rangle \varphi_{0,k}\Big\|_2^2 \leq B \sum_{k\in\mathbb{N}_0} |\langle f, \tilde{\varphi}_{0,k}\rangle|^2 \leq BC_2\|f\|_2^2.$$

Thus, P_0 is a bounded operator on $L^2(K)$ with norm at most $\sqrt{BC_2} = C$, say. Now, since the dilation operators are unitary and since

$$P_j f = \sum_{k\in\mathbb{N}_0} \langle f, \tilde{\varphi}_{j,k}\rangle \varphi_{j,k} = \sum_{k\in\mathbb{N}_0} \langle \delta_{-j} f, \tilde{\varphi}_{0,k}\rangle \delta_{-j}\varphi_{0,k},$$

we conclude that the operator norm of P_j is at most C. Similar arguments work for $\tilde{P}_j$. This finishes the proof of the lemma. □

In the following lemma, we prove some useful properties of the operators P_j and $\tilde{P}_j$.

Lemma 4.8. *The operators P_j and $\tilde{P}_j$ satisfy the following properties.*

(a) $P_j f = f$ *if and only if* $f \in V_j$ *and* $\tilde{P}_j f = f$ *if and only if* $f \in \tilde{V}_j$*;*

(b) $\lim_{j\to\infty} \|P_j f - f\|_2 = 0$ *and* $\lim_{j\to-\infty} \|P_j f\|_2 = 0$ *for every* $f \in L^2(K)$*.*

Proof. (a) $P_j f = f$ if and only if $f = \sum_{n\in\mathbb{N}_0} \langle f, \tilde{\varphi}_{j,n}\rangle \varphi_{j,n}$. Since $\{\varphi_{j,n} : n \in \mathbb{N}_0\}$ is a Riesz basis for V_j and $\{\tilde{\varphi}_{j,n}\}$ is biorthogonal to $\{\varphi_{j,n}\}$, the result follows from Theorem 3.9. Similar argument works for $\tilde{P}_j f$.

(b) Let $f \in L^2(K)$ and $\epsilon > 0$. Since $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(K)$, there exists $J \in \mathbb{Z}$ and $g \in V_J$ such that $\|f - g\|_2 < \frac{\epsilon}{1+C}$, where C is as in Lemma 4.7. If $g \in V_j$, then $P_j g = g$ for every $j \ge J$. Thus for $j \ge J$,

$$\begin{aligned} \|f - P_j f\|_2 &\le \|f-g\|_2 + \|P_j(f-g)\|_2 \\ &\le (1 + \|P_j\|)\|f-g\|_2 \\ &< (1+C)\|f-g\|_2 < \epsilon. \end{aligned}$$

This shows that

$$\lim_{j\to\infty} \|P_j f - f\|_2 = 0.$$

Now consider $h \in \mathcal{S}$ (see Definition 2.4). Then

$$\|P_j h\|_2^2 = \Big\| \sum_{k\in\mathbb{N}_0} \langle h, \tilde{\varphi}_{j,k}\rangle \varphi_{j,k} \Big\|_2^2 \le B \sum_{k\in\mathbb{N}_0} |\langle h, \tilde{\varphi}_{j,k}\rangle|^2.$$

In Theorem 4.1 in [5], we proved that if $h \in \mathcal{S}$, then $\sum_{k\in\mathbb{N}_0} |\langle h, \tilde{\varphi}_{j,k}\rangle|^2 \to 0$ as $j \to -\infty$. Hence, $\|P_j h\|_2 \to 0$ as $j \to -\infty$. Since $\mathcal{S}$ is dense in $L^2(K)$, given $\epsilon > 0$, there exists $h \in \mathcal{S}$ such that $\|f - h\|_2 < \epsilon$. Hence,

$$\|P_j f\|_2 \le \|P_j(f-h)\|_2 + \|P_j h\|_2 \le C\|f-h\|_2 + \|P_j h\|_2.$$

Therefore, $\|P_j f\|_2 \to 0$ as $j \to -\infty$. □

5 Biorthogonality of the wavelets

Let $\{V_j : j \in \mathbb{Z}\}$ and $\{\tilde{V}_j : j \in \mathbb{Z}\}$ be biorthogonal MRAs with scaling function φ and $\tilde{\varphi}$ respectively. By Lemma 4.4, there exist integral periodic functions m_0 and $\tilde{m}_0$ such that $\hat{\varphi}(\xi) = m_0(\mathfrak{p}\xi)\hat{\varphi}(\mathfrak{p}\xi)$ and $\widehat{\tilde{\varphi}}(\xi) = \tilde{m}_0(\mathfrak{p}\xi)\hat{\tilde{\varphi}}(\mathfrak{p}\xi)$. Assume that there exist integral periodic functions m_l and $\tilde{m}_l$, $1 \le l \le q-1$, such that

$$M(\xi)\tilde{M}^*(\xi) = I, \tag{5.1}$$

where $M(\xi) = \left(m_l(\mathfrak{p}\xi + \mathfrak{p}u(k))\right)_{l,k=0}^{q-1}$ and $\tilde{M}(\xi) = \left(\tilde{m}_l(\mathfrak{p}\xi + \mathfrak{p}u(k))\right)_{l,k=0}^{q-1}$. Now for $1 \le l \le q-1$, we define the associated wavelets ψ_l and $\tilde{\psi}_l$ as follows:

$$\hat{\psi}_l(\xi) = m_l(\mathfrak{p}\xi)\hat{\varphi}(\mathfrak{p}\xi) \quad \text{and} \quad \hat{\tilde{\psi}}_l(\xi) = \tilde{m}_l(\mathfrak{p}\xi)\hat{\tilde{\varphi}}(\mathfrak{p}\xi).$$

We have the following lemma.

Lemma 5.1. *Let φ and $\tilde{\varphi}$ be the scaling functions for dual MRAs and $\psi_l, \tilde{\psi}_l$, $1 \le l \le q-1$, be the associated wavelets satisfying the matrix condition* (5.1)*. Then the following hold.*

(a) $\{\psi_{l,0,n} : n \in \mathbb{N}_0\}$ *is biorthogonal to* $\{\tilde{\psi}_{l,0,n} : n \in \mathbb{N}_0\}$*;*

(b) $\langle \psi_{l,0,n}, \tilde{\varphi}_{0,m} \rangle = \langle \tilde{\psi}_{l,0,n}, \varphi_{0,m} \rangle = 0$ *for all* $m, n \in \mathbb{N}_0$.

Proof. (a) We have

$$\begin{aligned}
&\sum_{n\in\mathbb{N}_0} \hat{\psi}_l(\xi+u(n))\overline{\hat{\tilde{\psi}}_l(\xi+u(n))} \\
&= \sum_{n\in\mathbb{N}_0} m_l(\mathfrak{p}\xi+\mathfrak{p}u(n))\hat{\varphi}(\mathfrak{p}\xi+\mathfrak{p}u(n))\overline{\tilde{m}_l(\mathfrak{p}\xi+\mathfrak{p}u(n))}\overline{\hat{\tilde{\varphi}}(\mathfrak{p}\xi+\mathfrak{p}u(n))} \\
&= \sum_{s=0}^{q-1}\sum_{k\in\mathbb{N}_0} m_l(\mathfrak{p}\xi+\mathfrak{p}u(qk+s))\hat{\varphi}(\mathfrak{p}\xi+\mathfrak{p}u(qk+s)) \\
&\quad \times \overline{\tilde{m}_l(\mathfrak{p}\xi+\mathfrak{p}u(qk+s))}\overline{\hat{\tilde{\varphi}}(\mathfrak{p}\xi+\mathfrak{p}u(qk+s))} \\
&= \sum_{s=0}^{q-1}\sum_{k\in\mathbb{N}_0} m_l(\mathfrak{p}\xi+\mathfrak{p}u(s))\hat{\varphi}(\mathfrak{p}\xi+\mathfrak{p}u(s)+u(k)) \\
&\quad \times \overline{\tilde{m}_l(\mathfrak{p}\xi+\mathfrak{p}u(s))}\overline{\hat{\tilde{\varphi}}(\mathfrak{p}\xi+\mathfrak{p}u(s)+u(k))} \\
&= \sum_{s=0}^{q-1} m_l(\mathfrak{p}\xi+\mathfrak{p}u(s))\overline{\tilde{m}_l(\mathfrak{p}\xi+\mathfrak{p}u(s))} \\
&= 1.
\end{aligned}$$

Hence, by Lemma 3.5, $\{\psi_{l,0,n} : n \in \mathbb{N}_0\}$ is biorthogonal to $\{\tilde{\psi}_{l,0,n} : n \in \mathbb{N}_0\}$.

(b) For $m, n \in \mathbb{N}_0$, we have

$$\begin{aligned}
&\langle \psi_{l,0,n}, \tilde{\varphi}_{0,m} \rangle \\
&= \langle \psi_l(\cdot - u(n)), \tilde{\varphi}(\cdot - u(m)) \rangle \\
&= \langle \hat{\psi}_l \overline{\chi_n}, \hat{\tilde{\varphi}} \overline{\chi_m} \rangle \\
&= \int_K m_l(\mathfrak{p}\xi)\hat{\varphi}(\mathfrak{p}\xi)\overline{\chi_n}(\xi)\overline{\tilde{m}_0(\mathfrak{p}\xi)}\overline{\hat{\tilde{\varphi}}(\mathfrak{p}\xi)}\chi_m(\xi)\,d\xi \\
&= \int_{\mathfrak{D}} \sum_{k\in\mathbb{N}_0} m_l(\mathfrak{p}\xi+\mathfrak{p}u(k))\hat{\varphi}(\mathfrak{p}\xi+\mathfrak{p}u(k))\overline{\chi_n}(\xi) \\
&\qquad \overline{\tilde{m}_0(\mathfrak{p}\xi+\mathfrak{p}u(k))}\overline{\hat{\tilde{\varphi}}(\mathfrak{p}\xi+\mathfrak{p}u(k))}\chi_m(\xi)\,d\xi \\
&= \int_{\mathfrak{D}} \sum_{s=0}^{q-1}\sum_{k\in\mathbb{N}_0} m_l(\mathfrak{p}\xi+\mathfrak{p}u(s)+u(k))\hat{\varphi}(\mathfrak{p}\xi+\mathfrak{p}u(s)+u(k))\overline{\chi_n}(\xi) \\
&\qquad \overline{\tilde{m}_0(\mathfrak{p}\xi+\mathfrak{p}u(s)+u(k))}\overline{\hat{\tilde{\varphi}}(\mathfrak{p}\xi+\mathfrak{p}u(s)+u(k))}\chi_m(\xi)\,d\xi \\
&= \int_{\mathfrak{D}} \Big\{\sum_{s=0}^{q-1} m_l(\mathfrak{p}\xi+\mathfrak{p}u(s))\overline{\tilde{m}_0(\mathfrak{p}\xi+\mathfrak{p}u(s))}\Big\}\overline{\chi_n}(\xi)\chi_m(\xi)\,d\xi \\
&= 0.
\end{aligned}$$

Similarly, we can show that $\langle \tilde{\psi}_{l,0,n}, \varphi_{0,m} \rangle = 0$. □

Our aim is to show that the wavelets associated with dual MRAs are biorthogonal and they form Riesz bases for $L^2(K)$. The following proposition is crucial for the proof of the main result.

Proposition 5.2. *Let* $\varphi, \tilde{\varphi}$ *and* $\psi_l, \tilde{\psi}_l$ *for* $1 \le l \le q-1$ *be as in Lemma* 5.1. *Denote* $\psi_0 = \varphi$ *and* $\tilde{\psi}_0 = \tilde{\varphi}$. *Then for every* $f \in L^2(K)$, *we have*

$$P_1 f = P_0 f + \sum_{l=1}^{q-1} \sum_{k \in \mathbb{N}_0} \langle f, \tilde{\psi}_{l,0,k} \rangle \psi_{l,0,k} \tag{5.2}$$

and

$$\tilde{P}_1 f = \tilde{P}_0 f + \sum_{l=1}^{q-1} \sum_{k \in \mathbb{N}_0} \langle f, \psi_{l,0,k} \rangle \tilde{\psi}_{l,0,k}, \tag{5.3}$$

where the series converge in $L^2(K)$.

Proof. It is enough to prove (5.2) as the proof of (5.3) is similar. Moreover, it is enough to prove (5.2) in the weak sense, that is, for all $f, g \in L^2(K)$

$$\begin{aligned} \langle P_1 f, g \rangle &= \langle P_0 f, g \rangle + \sum_{l=1}^{q-1} \sum_{k \in \mathbb{N}_0} \langle f, \tilde{\psi}_{l,0,k} \rangle \overline{\langle g, \psi_{l,0,k} \rangle} \\ &= \sum_{l=0}^{q-1} \sum_{k \in \mathbb{N}_0} \langle f, \tilde{\psi}_{l,0,k} \rangle \overline{\langle g, \psi_{l,0,k} \rangle}. \end{aligned}$$

We have

$$\begin{aligned} &\sum_{l=0}^{q-1} \sum_{k \in \mathbb{N}_0} \langle f, \tilde{\psi}_{l,0,k} \rangle \overline{\langle g, \psi_{l,0,k} \rangle} \\ &= \sum_{l=0}^{q-1} \sum_{k \in \mathbb{N}_0} \Big(\int_K \hat{f}(\xi) \overline{\hat{\tilde{\psi}}_l(\xi)} \chi_k(\xi) d\xi \Big) \Big(\int_K \overline{\hat{g}(\xi)} \hat{\psi}_l(\xi) \overline{\chi_k(\xi)} d\xi \Big) \\ &= \sum_{l=0}^{q-1} \sum_{k \in \mathbb{N}_0} \Big(\int_{\mathfrak{D}} \sum_{\alpha \in \mathbb{N}_0} \hat{f}(\xi + u(\alpha)) \overline{\hat{\tilde{\psi}}_l(\xi + u(\alpha))} \chi_k(\xi) d\xi \Big) \\ &\qquad \times \Big(\int_{\mathfrak{D}} \sum_{\beta \in \mathbb{N}_0} \overline{\hat{g}(\xi + u(\beta))} \hat{\psi}_l(\xi + u(\beta)) \overline{\chi_k(\xi)} d\xi \Big) \\ &= \sum_{l=0}^{q-1} \int_{\mathfrak{D}} \Big(\sum_{\alpha \in \mathbb{N}_0} \hat{f}(\xi + u(\alpha)) \overline{\hat{\tilde{\psi}}_l(\xi + u(\alpha))} \Big) \Big(\sum_{\beta \in \mathbb{N}_0} \overline{\hat{g}(\xi + u(\beta))} \hat{\psi}_l(\xi + u(\beta)) \Big) d\xi \\ &= \int_{\mathfrak{D}} \sum_{l=0}^{q-1} \Big(\sum_{\alpha \in \mathbb{N}_0} \hat{f}(\xi + u(\alpha)) \overline{\tilde{m}_l(\mathfrak{p}\xi + \mathfrak{p}u(\alpha))} \overline{\hat{\tilde{\varphi}}(\mathfrak{p}\xi + \mathfrak{p}u(\alpha))} \end{aligned}$$

$$
\begin{aligned}
&\times \sum_{\beta\in\mathbb{N}_0} \overline{\hat{g}(\xi+u(\beta))} m_l(\mathfrak{p}\xi+\mathfrak{p}u(\beta))\hat{\varphi}(\mathfrak{p}\xi+\mathfrak{p}u(\beta))\Big)d\xi \\
= \;& \int_{\mathfrak{D}} \sum_{l=0}^{q-1}\Big(\sum_{\nu=0}^{q-1}\sum_{\alpha'\in\mathbb{N}_0} \hat{f}(\xi+u(q\alpha')+u(\nu))\overline{\tilde{m}_l(\mathfrak{p}\xi+u(\alpha')+\mathfrak{p}u(\nu))} \\
&\times\overline{\hat{\tilde{\varphi}}(\mathfrak{p}\xi+u(\alpha')+\mathfrak{p}u(\nu))} \\
&\times\sum_{\nu'=0}^{q-1}\sum_{\beta'\in\mathbb{N}_0} \overline{\hat{g}(\xi+u(q\beta')+u(\nu'))} m_l(\mathfrak{p}\xi+u(\beta')+\mathfrak{p}u(\nu')) \\
&\times\hat{\varphi}(\mathfrak{p}\xi+u(\beta')+\mathfrak{p}u(\nu'))\Big)d\xi \\
= \;& \int_{\mathfrak{D}} \sum_{\alpha'}\sum_{\beta'}\sum_{\nu}\sum_{\nu'}\Big(\sum_{l} \overline{\tilde{m}_l(\mathfrak{p}\xi+\mathfrak{p}u(\nu))} m_l(\mathfrak{p}\xi+\mathfrak{p}u(\nu')) \\
&\times\hat{f}(\xi+u(q\alpha')+u(\nu))\overline{\hat{\tilde{\varphi}}(\mathfrak{p}\xi+u(\alpha')+\mathfrak{p}u(\nu))} \\
&\times\overline{\hat{g}(\xi+u(q\beta')+u(\nu'))}\hat{\varphi}(\mathfrak{p}\xi+u(\beta')+\mathfrak{p}u(\nu'))\Big)d\xi \\
= \;& \int_{\mathfrak{D}} \sum_{\nu}\sum_{\alpha'}\sum_{\beta'}\Big(\hat{f}(\xi+u(q\alpha')+u(\nu))\overline{\hat{\tilde{\varphi}}(\mathfrak{p}\xi+u(\alpha')+\mathfrak{p}u(\nu))} \\
&\times\overline{\hat{g}(\xi+u(q\beta')+u(\nu))}\hat{\varphi}(\mathfrak{p}\xi+u(\beta')+\mathfrak{p}u(\nu))\Big)d\xi \\
= \;& \sum_{\nu}\int_{\mathfrak{D}+u(\nu)}\sum_{\alpha'}\sum_{\beta'} \hat{f}(\xi+u(q\alpha'))\overline{\hat{\tilde{\varphi}}(\mathfrak{p}\xi+u(\alpha'))}\,\overline{\hat{g}(\xi+u(q\beta'))}\hat{\varphi}(\mathfrak{p}\xi+u(\beta'))d\xi. \qquad (5.4)
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
&\sum_{k\in\mathbb{N}_0}\langle f,\tilde{\varphi}_{1,k}\rangle\overline{\langle g,\varphi_{1,k}\rangle} \\
&= \sum_{k\in\mathbb{N}_0}\Big(\int_K \hat{f}(\xi)\overline{\hat{\tilde{\varphi}}(\mathfrak{p}\xi)}\chi_k(\mathfrak{p}\xi)d\xi\Big)\Big(\int_K \overline{\hat{g}(\xi)}\hat{\varphi}(\mathfrak{p}\xi)\overline{\chi_k(\mathfrak{p}\xi)}d\xi\Big) \\
&= \sum_{k\in\mathbb{N}_0}\Big(\int_{\mathfrak{P}^{-1}}\sum_{\alpha} \hat{f}(\xi+\mathfrak{p}^{-1}u(\alpha))\overline{\hat{\tilde{\varphi}}(\mathfrak{p}\xi+u(\alpha))}\chi_k(\mathfrak{p}\xi)d\xi\Big) \\
&\qquad\times\Big(\int_{\mathfrak{P}^{-g}}\sum_{\beta} \overline{\hat{g}(\xi+\mathfrak{p}^{-1}u(\beta))}\hat{\varphi}(\mathfrak{p}\xi+u(\beta))\overline{\chi_k(\mathfrak{p}\xi)}d\xi\Big) \\
&= \int_{\mathfrak{P}^{-1}}\sum_{\alpha}\sum_{\beta} \hat{f}(\xi+u(q\alpha))\overline{\hat{\tilde{\varphi}}(\mathfrak{p}\xi+u(\alpha))}\,\overline{\hat{g}(\xi+u(q\beta))}\hat{\varphi}(\mathfrak{p}\xi+u(\beta))d\xi. \qquad (5.5)
\end{aligned}
$$

Since the right sides of (5.4) and (5.5) are same, the proof is finished. □

Combining Lemma 4.8 and Proposition 5.2, we have the following proposition.

Proposition 5.3. *Let $\varphi,\tilde{\varphi}$ and $\psi_l,\tilde{\psi}_l$ for $1\le l\le q-1$ be as above. Then for every $f\in L^2(K)$, we have*

$$
f=\sum_{l=1}^{q-1}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{N}_0}\langle f,\tilde{\psi}_{l,j,k}\rangle\psi_{l,j,k}=\sum_{l=1}^{q-1}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{N}_0}\langle f,\psi_{l,j,k}\rangle\tilde{\psi}_{l,j,k}, \qquad (5.6)
$$

where the series converge in $L^2(K)$.

We now prove the main results of the article.

Theorem 5.4. *Let φ and $\tilde{\varphi}$ be the scaling functions for dual MRAs and $\psi_l, \tilde{\psi}_l$, $1 \le l \le q-1$, be the associated wavelets satisfying the matrix condition* (5.1). *Then the collections $\{\psi_{l,j,k} : 1 \le l \le q-1, j \in \mathbb{Z}, k \in N_0\}$ and $\{\tilde{\psi}_{l,j,k} : 1 \le l \le q-1, j \in \mathbb{Z}, k \in N_0\}$ are biorthogonal. In addition, if*

$$|\hat{\varphi}(\xi)| \le C(1+|\xi|)^{-\frac{1}{2}-\epsilon}, |\hat{\tilde{\varphi}}(\xi)| \le C(1+|\xi|)^{-\frac{1}{2}-\epsilon}, |\hat{\psi}_l(\xi)| \le C|\xi|, \text{ and } |\hat{\tilde{\psi}}_l(\xi)| \le C|\xi|, \tag{5.7}$$

for some constant $C > 0$, $\epsilon > 0$ and for a. e. $\xi \in K$, then $\{\psi_{l,j,k} : 1 \le l \le q-1, j \in \mathbb{Z}, k \in N_0\}$ and $\{\tilde{\psi}_{l,j,k} : 1 \le l \le q-1, j \in \mathbb{Z}, k \in N_0\}$ form Riesz bases for $L^2(K)$.

Proof. We begin by proving that $\{\psi_{l,j,k} : 1 \le l \le q-1, j \in \mathbb{Z}, k \in \mathbb{N}_0\}$ and $\{\tilde{\psi}_{l,j,k} : 1 \le l \le q-1, j \in \mathbb{Z}, k \in \mathbb{N}_0\}$ are biorthogonal to each other. First we will show that, for $l = 1, 2 \ldots, q-1$ and $j \in \mathbb{Z}$,

$$\langle \psi_{l,j,k}, \tilde{\psi}_{l,j,k'} \rangle = \delta_{k,k'}.$$

We have already proved it for $j = 0$ in Lemma 5.1(a). If $j \neq 0$, then

$$\begin{aligned} \langle \psi_{l,j,k}, \tilde{\psi}_{l,j,k'} \rangle &= \langle \delta_{-j}\psi_{l,0,k}, \delta_{-j}\tilde{\psi}_{l,0,k} \rangle \\ &= \langle \psi_{l,0,k}, \tilde{\psi}_{l,0,k} \rangle \\ &= \delta_{k,k'}. \end{aligned}$$

Let $k, k' \in \mathbb{N}_0$ be fixed and let $j, j' \in \mathbb{Z}$. Assume that $j < j'$. We will show that

$$\langle \psi_{l,j,k}, \tilde{\psi}_{l',j',k'} \rangle = 0.$$

It can be shown that $\psi_{l,0,k} \in V_1$. Hence, $\psi_{l,j,k} = \delta_{-j}\psi_{l,0,k} \in V_{j+1} \subseteq V_{j'}$. Therefore, it will be enough to show that $\tilde{\psi}_{l',j',k'}$ is orthogonal to every element of $V_{j'}$. Let $f \in V_{j'}$. By Lemma 4.3, $\{\varphi_{j',k} : k \in \mathbb{N}_0\}$ is a Riesz basis for $V_{j'}$. Hence, there exists an l^2-sequence $\{c_k\}$ such that $f = \sum_{k\in\mathbb{N}_0} c_k \varphi_{j',k}$ in $L^2(K)$. By Lemma 5.1(b),

$$\langle \tilde{\psi}_{l',j',k'}, \varphi_{j',k} \rangle = \langle \delta_{-j'}\tilde{\psi}_{l',0,k'}, \delta_{-j'}\varphi_{0,k} \rangle = \langle \tilde{\psi}_{l',0,k'}, \varphi_{0,k} \rangle = 0.$$

Hence,

$$\langle \tilde{\psi}_{l',j',k'}, f \rangle = \left\langle \tilde{\psi}_{l',j',k'}, \sum_{k\in\mathbb{N}_0} c_k \varphi_{j',k} \right\rangle = \sum_{k\in\mathbb{N}_0} \overline{c_k} \langle \tilde{\psi}_{l',j',k'}, \varphi_{j',k} \rangle = 0.$$

In order to show that these two collections form Riesz bases for $L^2(K)$, we must verify that they are linearly independent and satisfy the frame condition. Since they are biorthogonal to each other, both the collections are linearly independent by Lemma 3.2.

To show the frame condition, we must show that there exist constants $A, B, \tilde{A}$, and $\tilde{B} > 0$ such that for every $f \in L^2(K)$,

$$A\|f\|_2^2 \le \sum_{l=1}^{q-1} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{N}_0} |\langle f, \psi_{l,j,k} \rangle|^2 \le B\|f\|_2^2, \tag{5.8}$$

and

$$\tilde{A}\|f\|_2^2 \le \sum_{l=1}^{q-1}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{N}_0}|\langle f,\tilde{\psi}_{l,j,k}\rangle|^2 \le \tilde{B}\|f\|_2^2. \tag{5.9}$$

We first show the existence of upper bounds in (5.8) and (5.9). We have

$$\begin{aligned}
&\sum_{k\in\mathbb{N}_0}|\langle f,\psi_{l,j,k}\rangle|^2\\
&= \sum_{k\in\mathbb{N}_0}\Big|\int_K \hat{f}(\xi)q^{-j/2}\overline{\hat{\psi}_l(\mathfrak{p}^j\xi)}\chi_k(\mathfrak{p}^j\xi)\,d\xi\Big|^2\\
&= q^{-j}\sum_{k\in\mathbb{N}_0}\Big|\int_{\mathfrak{P}^{-j}}\sum_{m\in\mathbb{N}_0}\hat{f}(\xi+\mathfrak{p}^{-j}u(m))\overline{\hat{\psi}_l(\mathfrak{p}^j\xi+u(m))}\chi_k(\mathfrak{p}^j\xi)\,d\xi\Big|^2\\
&= \int_{\mathfrak{P}^{-j}}\Big|\sum_{m\in\mathbb{N}_0}\hat{f}(\xi+\mathfrak{p}^{-j}u(m))\overline{\hat{\psi}_l(\mathfrak{p}^j\xi+u(m))}\Big|^2\,d\xi\\
&\le \int_{\mathfrak{P}^{-j}}\Big(\sum_{m\in\mathbb{N}_0}|\hat{f}(\xi+\mathfrak{p}^{-j}u(m))|^2|\hat{\psi}_l(\mathfrak{p}^j\xi+u(m))|^{2\delta}\Big)\Big(\sum_{n\in\mathbb{N}_0}|\hat{\psi}_l(\mathfrak{p}^j\xi+u(n))|^{2(1-\delta)}\Big)\,d\xi\\
&= \int_K|\hat{f}(\xi)|^2|\hat{\psi}_l(\mathfrak{p}^j\xi)|^{2\delta}\sum_{n\in\mathbb{N}_0}|\hat{\psi}_l(\mathfrak{p}^j\xi+u(n))|^{2(1-\delta)}\,d\xi.
\end{aligned}$$

We have assumed that $|\hat{\varphi}(\xi)| \le C(1+|\xi|)^{-\frac{1}{2}-\epsilon}$, hence we have, $|\hat{\psi}_l(\xi)| \le C(1+|\mathfrak{p}\xi|)^{-\frac{1}{2}-\epsilon}$. So $\sum_{n\in\mathbb{N}_0}|\hat{\psi}_l(\mathfrak{p}^j\xi+u(n))|^{2(1-\delta)}$ is uniformly bounded if $\delta < 2\epsilon(1+2\epsilon)^{-1}$. Hence, there exists $C>0$ such that

$$\begin{aligned}
&\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{N}_0}|\langle f,\psi_{l,j,k}\rangle|^2\\
&\le C\int_K|\hat{f}(\xi)|^2\sum_{j\in\mathbb{Z}}|\hat{\psi}_l(\mathfrak{p}^j\xi)|^{2\delta}\,d\xi\\
&\le C\sup\Big\{\sum_{j\in\mathbb{Z}}|\hat{\psi}_l(\mathfrak{p}^j\xi)|^{2\delta} : \xi\in\mathfrak{P}^{-1}\setminus\mathfrak{D}\Big\}\|f\|_2^2.
\end{aligned}$$

The last step follows because K is a disjoint union of $\mathfrak{P}^j$, $j\in\mathbb{Z}$, and the function $F(\xi) = \sum_{j\in\mathbb{Z}}|\hat{\psi}_l(\mathfrak{p}^j\xi)|^{2\delta}$ has the property that $F(\xi) = F(\mathfrak{p}\xi)$. Note that $\mathfrak{D} = \mathfrak{P}^0$. For $\xi\in\mathfrak{P}^{-1}\setminus\mathfrak{D}$, we have $1 < |\xi| \le q$. Hence,

$$\begin{aligned}
\sum_{j=-\infty}^{0}|\hat{\psi}_l(\mathfrak{p}^j\xi)|^{2\delta} &\le \sum_{j=0}^{\infty}\frac{C^{2\delta}}{(1+|\mathfrak{p}^{-j+1}\xi|)^{\delta(1+2\epsilon)}}\\
&\le \sum_{j=0}^{\infty}\frac{C^{2\delta}}{(1+q^{j-1})^{\delta(1+2\epsilon)}}\\
&\le \sum_{j=0}^{\infty}\frac{C^{2\delta}}{q^{(j-1)\delta(1+2\epsilon)}} = C^{2\delta}\frac{q^{\delta(1+2\epsilon)}}{1-q^{-\delta(1+2\epsilon)}}.
\end{aligned}$$

Also,

$$\begin{aligned}\sum_{j=1}^{\infty}|\hat{\psi}_l(\mathfrak{p}^j\xi)|^{2\delta} &\leq \sum_{j=1}^{\infty}(Cq^{-j}|\xi|)^{2\delta}\\ &\leq C^{2\delta}\sum_{j=1}^{\infty}q^{(-j+1)2\delta}=C^{2\delta}\frac{1}{1-q^{-2\delta}}.\end{aligned}$$

These two estimates show that $\sup\{\sum_{j\in\mathbb{Z}}|\hat{\psi}_l(\mathfrak{p}^j\xi)|^{2\delta}:\xi\in\mathfrak{P}^{-1}\setminus\mathfrak{D}\}$ is finite. Hence, there exists $B>0$ such that the second inequality in (5.8) holds. Similarly, we can show that the upper bound in (5.9) holds.

Using the existence of the upper bounds, we now show that the lower bounds in (5.8) and (5.9) also exist. It follows from Proposition 5.3 that, if $f\in L^2(K)$, then we have

$$f=\sum_{l=1}^{q-1}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{N}_0}\langle f,\tilde{\psi}_{l,j,k}\rangle\psi_{l,j,k}=\sum_{l=1}^{q-1}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{N}_0}\langle f,\psi_{l,j,k}\rangle\tilde{\psi}_{l,j,k}.$$

Therefore,

$$\begin{aligned}\|f\|_2^2 &= \langle f,f\rangle\\ &= \Big\langle\sum_l\sum_j\sum_k\langle f,\tilde{\psi}_{l,j,k}\rangle\psi_{l,j,k},f\Big\rangle\\ &= \sum_l\sum_j\sum_k\langle f,\tilde{\psi}_{l,j,k}\rangle\langle\psi_{l,j,k},f\rangle\\ &\leq \Big(\sum_l\sum_j\sum_k|\langle f,\tilde{\psi}_{l,j,k}\rangle|^2\Big)^{\frac{1}{2}}\Big(\sum_l\sum_j\sum_k|\langle f,\psi_{l,j,k}\rangle|^2\Big)^{\frac{1}{2}}\\ &\leq (\tilde{B})^{\frac{1}{2}}\|f\|_2\Big(\sum_l\sum_j\sum_k|\langle f,\psi_{l,j,k}\rangle|^2\Big)^{\frac{1}{2}}.\end{aligned}$$

Hence,

$$\frac{1}{\tilde{B}}\|f\|_2^2\leq\sum_l\sum_j\sum_k|\langle f,\psi_{l,j,k}\rangle|^2.$$

Similarly, we can show that

$$\frac{1}{B}\|f\|_2^2\leq\sum_l\sum_j\sum_k|\langle f,\tilde{\psi}_{l,j,k}\rangle|^2.$$

This completes the proof of the theorem. □

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