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Biot-consistent elastic moduli of porous rocks: Low-frequency limit

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ABSTRACT

The semiphenomenological Biot-Gassmann (B-G) formulation of the low-frequency elastic moduli of porous rocks does contain two well-known predictions: (1) the shear modulus of an unsaturated rock (which is permeated by a compressible fluid, e.g., gas) is identical to that of the same rock saturated with liquid, and (2) the unsaturated bulk modulus differs from the saturated bulk modulus by a defined amount. These predictions are tested by ultrasonic data on a large number of sedimentary rocks and are approximately verified, despite the evident frequency discrepancy. The B-G theory makes only minimal assumptions about the microscopic geometry of the rock; therefore, any model theory which does make such assumptions (e.g., spherical pores) should be a special case of B-G theory. In particular, such model theories should also predict the two relations described above. Standard models for dilute concentrations of spherical pores and/or ellipsoidal cracks do predict these relationships. However, in general, the "Self-Consistent" (S-C) model (developed to deal with finite concentrations of heterogeneities) violates these predictions and hence is not consistent with the underlying Biot-Gassmann theory. [The special case of S-C theory, corresponding to pores only (no cracks), is consistent with the B-G model.] A new formulation of the model theory, for finite concentrations of heterogeneities of ideal shape, is developed so as to be explicitly consistent with B-G. This "Biot-consistent" (B-C) formalism is the first theory truly suitable for modeling most sedimentary rocks at seismic frequencies, in terms of porosity and pore shape.

INTRODUCTION

It is frequently of interest in exploration geophysics to be able to interpret seismic velocity measurements in terms of rock porosity. Although that goal remains elusive, considerable progress has been made on the corresponding direct problem: using assumed porosity and other lithologic information to predict rock elasticity. The present work constitutes a contribution to solving that problem.

Biot (1941) introduced a semiphenomenological formulation of the equations of elasticity for a porous aggregate. His chief contribution lay in recognizing the pressure and volume increment of the pore fluid as additional state varibles, along with stress and strain in the solid. Gassmann (1951a) expressed the phenomenological variables of the theory in terms of the separate properties of the pore fluid and the solid material. The theory is briefly reviewed here, highlighting two predictions of the otherwise phenomenological theory. Despite further development (e.g., Biot 1956a, b, c, 1962; Biot and Willis, 1957; Gassmann 1951b) and many applications (e.g., Geertsma, 1957; Geertsma and Smit, 1961; Gardner et al., 1974) notably to bright spot analysis (e.g., Domenico, 1974), the theory has not been adequately tested, either experimentally or theoretically, until recent times.

That situation has recently changed. On the theoretical side, Burridge and Keller (1981) have carefully derived the basic equations [equation (1)] which were first assumed by Biot, and have clarified the frequency regime in which they are valid. Experimentally, Plona (1980) observed the "Biot slow wave" predicted by the theory, thus suggesting the legitimacy of the other predictions of the theory. These other predictions, mentioned above, relate the elastic moduli of the "drained" state to those of the "undrained" state, or equivalently the unsaturated moduli to the saturated. They have been partially verified before, using a few rock samples; a much larger data set is presented here in partial support of these predictions.

Independently, a number of model theories have emerged, seeking to provide stronger predictions on the elasticity of aggregates through stronger assumptions about the microscopic geometry of the constituents. Since these model theories share the (minimal) assumptions of the Biot-Gassmann (B-G) theory about the structure of the pore space, they should also predict the relationships mentioned.

This requirement, although obvious once stated, has appar-

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ently not been applied to any of these theories. Gassmann (1951b) calculated unsaturated moduli using a model of spherical Hertzian grains, then subsequently used the B-G predictions to calculate the saturated moduli. Of course, the Hertzian model should be independently applicable to the saturated case; the difference from the unsaturated case should then be compared analytically to the B-G predictions. Gassmann didn't do this comparison; nor is this model so tested here. Similarly, Korringa and Thompson (1977) and Korringa et al. (1979) used a model calculation for the unsaturated moduli and then used the B-G predictions for the saturated moduli, without independently verifying the consistency of the model.

The requirement of Biot-consistency is applied later to the standard theory for dilute concentrations of interconnected spherical pores and/or thin cracks, where it is found that the theory does, in fact, contain the B-G predictions, in the limit of low porosity and crack density.

However, these limits (roughly, porosity less than 10 percent, crack density less than .1) are too low to permit application of the theory to most sedimentary rocks. Rocks of interest to the petroleum industry may commonly have porosity as high as 40 percent and/or crack density as high as .3 or more. To cope with this situation, the "Self-Consistent" (S-C) theory (Hershey, 1954; Hill, 1965; Budiansky, 1965) and the differential S-C theory (Bruner, 1976) have been proposed, with the suggestion that they adequately approximate the interactions between nearby inclusions. None of these model theories has been adequately tested against experimental data on aggregates of the assumed microscopic geometry. It will be shown later that, except for the special case of no cracks, the self-consistent theories are not consistent with B-G theory, in the above sense. Therefore, no model theories exist in the literature which are suitable for analyzing the majority of sedimentary rocks at seismic frequencies.

Finally, a new formalism is developed to extend the standard model theory (for dilute concentrations) of fluid heterogeneities to high concentrations of pores and cracks, while preserving explicitly the consistency with B-G theory. In the examples shown, the resulting saturated "Biot-Consistent moduli" differ numerically from the corresponding S-C moduli for the same system. In particular, the *difference* between saturated and unsaturated moduli (which is responsible for the bright-spot effect), is given correctly (i.e., according to Biot) by the present model, whereas it is substantially underestimated by S-C theory at high crack density.

BIOT-GASSMANN THEORY

Biot (1962) provided a relatively clear derivation of the equations of poroelasticity, unifying differences of notation and concept from his earlier papers. Following that discussion (with certain differences in notation) the constitutive equations for a porous, linearly elastic, isotropic aggregate at low frequency may be written

$$\bar{\tau} = G^*\bar{\gamma}, \tag{1a}$$

$$\bar{p} = -K^*\bar{\theta} + \alpha p_f, \tag{1b}$$

$$p_f = -\alpha M\bar{\theta} + M\Delta V_p/V. \tag{1c}$$

Here the bars refer to a volumetric average over an elemental volume V (fixed in the solid framework) which contains many grains and pores; $\bar{\tau}$ and $\bar{p}$ are shear stress and pressure, respectively, averaged over the entire volume V (both solid and fluid portions); $\bar{\gamma}$ and $\bar{\theta}$ are shear strain and dilatation, respectively, averaged over the entire volume V; p_f and ΔV_p are incremental fluid pressure and incremental pore volume; and G^*, K^*, α, and M are phenomenological elastic parameters to be discussed below.

Note that since p_f is not averaged, equations (1) assume that the pore space is interconnected and that the frequency is sufficiently low that the pore fluid pressure is uniform everywhere within V. Because of the possibility of fluid flow from V, the pore volume increment ΔV_p may differ from the specific pore fluid volume increment ΔV_f and may not in general be determined from p_f by using the fluid compressibility. Equations (1) implicitly assume that the solid parts of V are also homogeneous and isotropic on the small scale; Brown and Korringa (1975) discussed the relaxation of these assumptions.

Note that since equations (1) are simply linearized constitutive relations, all these state variables ($\bar{\tau}$, $\bar{p}$, $\bar{\gamma}$, $\bar{\theta}$, p_f, and ΔV_p) represent infinitesimal increments from an initial state of equilibrium. The elastic parameters (G^*, K^*, α, and M) are functions (possibly nonelastic and/or nonlinear) of the stresses and strains and fluid pressure (which all may be large) of the initial state, and possibly of the history leading to the initial state.

Biot and Willis (1957) gave an interpretation of the constants (G^*, K^*, α, and M) in terms of various compaction experiments. Gassmann (1951a) gave, without derivation, an interpretation of the constants in terms of more familiar properties of the solid and of the pore space separately. Because this extension of Biot's work is nontrivial, the results of this section are called the Biot-Gassmann (B-G) formulas. However, since Gassmann did not provide derivations, I comment briefly on these interpretations.

It is clear that if the pore fluid were permitted to drain out of the elemental volume V, upon application of incremental pressure $\bar{p}$ (so that $p_f = 0$), then equations (1a) and (1b) reduce to Hooke's equations with elastic moduli G^* and K^* characterizing the response of the drained, porous framework. This corresponds to the "drained" condition, which is not commonly encountered in a context of wave propagation. Nevertheless, the "frame moduli" G^* and K^* play an important role.

By considering separately the pore volume and the solid volume, Geertsma (1957) and Nur and Byerlee (1971) showed that, if the pore space is connected,

$$\alpha = 1 - \frac{K^*}{K_s}, \tag{2}$$

and

$$M = \frac{K_s}{\alpha - \phi}, \tag{3}$$

where K_s is the incompressibility of the solid grains and ϕ is the porosity. The four independent material parameters of the theory are then G^*, K^*, K_s, and ϕ.

So far the theory is completely phenomenological and descriptive, i.e., it has offered no predictions. I now consider the new effects which arise because of the explicit appearance of the fluid pressure in equations (1). The increment in fluid pres-

sure p_f clearly depends upon whether or not the fluid is permitted to drain from V during the application of $\bar{p}$. The drained case is encountered in situations of soil compaction, etc. For low-frequency wave propagation, neighboring elements are subject to virtually the same stress and strain as is V, so there is no place for the fluid to drain, which potentially affects the response to stress. However, it is evident immediately, since equation (1a) is not coupled to equation (1b), that the shear modulus is independent of p_f and hence of the conditions of drainage. Equation (1a) indicates that the shear modulus G of this "undrained" condition is

$$G = G^*, \tag{4}$$

i.e., that the undrained shear modulus is equal to the drained shear modulus. This constitutes an unambiguous prediction of the B-G theory which is tested below.

The equivalent prediction for the undrained bulk modulus requires specification of the pore pressure. Assuming that no fluid is lost from the elemental volume during application of $\bar{p}$,

$$\Delta V_p = \Delta V_f = -V_f p_f / K_f = -\phi V p_f / K_f, \tag{5}$$

where V_f and K_f are the specific volume and incompressibility of the fluid. Using equations (5), (1b), and (1c), it can easily be shown that the increment in fluid pressure is related to the dilatation by

$$p_f = -\alpha K_f \bar{\theta} \Big/ \left[\phi + \frac{K_f}{K_s}(\alpha - \phi) \right], \tag{6}$$

and hence that the drained incompressibility is (Gassmann, 1951a; Geertsma, 1957)

$$K \equiv -\frac{\bar{p}}{\bar{\theta}} = K^* - \alpha \frac{p_f}{\bar{\theta}} = K^* + K_f \alpha^2 \Big/ \left[\phi + \frac{K_f}{K_s}(\alpha - \phi) \right]. \tag{7}$$

This constitutes an unambiguous prediction of the B-G theory for the undrained bulk modulus K, analogous to equation (4) for G.

Before testing these predictions, notice that as K_f approaches 0 in equation (7), K approaches K^*, the frame incompressibility. This illustrates that, for a rock saturated with a highly compressible fluid (e.g., a gas), the drainage conditions are irrelevant and the observed bulk modulus is K^*, even if the experiment is undrained. It also shows that the predictions of equations (4) and (7) are equivalent to predictions concerning the functional form of the dependence of the moduli upon K_f, or [in view of equation (6)] upon p_f. Hence in the following, the moduli are often written with the explicit argument K_f; the asterisk indicates the special case $K_f = 0$. Equivalently, the argument is the fluid pressure increment p_f.

The equations of motion define elastic-wave velocities, formed in the usual way from the moduli and density ρ:

$$VP(K_f) = [(K(K_f) + \tfrac{4}{3}G)/\rho]^{1/2},$$

and

$$VS(K_f) = [G/\rho]^{1/2},$$

with the density given by

$$\rho = \phi\rho_f + (1 - \phi)\rho_s.$$

The equations of motion yield a third solution, the "Biot slow wave" (cf., Plona, 1980; Johnson, 1985) which does not enter the present discussion.

Strictly speaking, the predictions of equations (4) and (7) are valid only for low frequencies (lower than a critical frequency, which depends upon permeability and is usually in the sonic band). Nevertheless, they have usually been tested with ultrasonic data (e.g., Wyllie et al., 1962; Domenico, 1974; Gregory, 1976; Murphy, 1984) and usually with a relatively small set of samples. Figures 1 and 2 present similar tests (using ultrasonic data) of the predictions for G (K_f) and K (K_f), respectively. The merit of these figures is that they report results from many samples, both clastics and carbonates, taken with similar laboratory techniques (ultrasonic time-of-flight techniques on 1 inch cores; unsaturated by room drying; saturated with brine under pressure until density stabilizes).

For G, Figure 1 shows broad agreement between prediction and observation over more than an order of magnitude of variation in G, with significant scatter but no apparent bias. For K, Figure 2 shows, in addition to scatter, a significant bias: the predictions are too low. The two dashed lines indicate (for each of the two lithologies) the average locus of the naive prediction

$$K(K_f) = K^*.$$

The vertical difference between the dashed line and the solid line evidently gives the size of the correction term (second term) in equation (7). The points scatter roughly halfway between the dashed and solid lines. It is clear that the B-G correction for unsaturated-to-saturated bulk modulus is too small, by roughly a factor of two.

Winkler (1984) observed the same effect in a smaller data set and reached the conclusion that this observed bias is a high-frequency effect in the observations, not a fault in the (low-frequency) prediction. He specifically analyzed the bias in terms of local (non-Biot) flow mechanisms [arising from non-constant $p_f(\mathbf{x})$, within each volume V]. Presumably the scatter (in addition to the bias) is caused by similar high-frequency effects. It is clearly important that a similar test, using a large number of low-frequency data, should be done to examine the validity of the predictions more clearly.

Despite the imperfect nature of the test, the conclusion is drawn that the low-frequency Biot-Gassmann relations do have sufficient merit to justify their use as a theoretical constraint on model theories, as in the next section.

In the Biot-Gassmann theory, assumptions about the microscopic geometry of the rock are minimal. The primary assumption is that the pore space is entirely interconnected, ensuring that: (1) the fluid pressure p_f is constant everywhere in the mass element (not averaged, as with $\bar{p}$); and (2) the characterization of the primitive constants α and M, in terms of the more fundamental parameters K_s, and ϕ is exact, as given in equations (2) and (3).

Therefore, any theory which calculates K^* and K in terms of K_s, ϕ, etc., through an assumption about the microscopic geometry of the rock, should be consistent with Biot theory as long as it also assumes linear elastic behavior of all constituents and connected pore space. In other words, a theory which makes stronger assumptions than the Biot-Gassmann theory (e.g., assumptions about microscopic geometry) should reach stronger conclusions (e.g., calculating K^* itself), but should constitute a special case of the more general theory. In

particular, the model results for $G(K_f)$, and independently for G^*, should analytically verify the prediction of equation (4), and those for $K(K_f)$ and K^* should verify equation (7).

MODEL THEORY: NONINTERACTING

The standard theory for the elasticity of a solid, isotropic matrix permeated by a dilute concentration of spherical pores is due to Eshelby (1957) and his predecessors. The corresponding theory for a dilute concentration of thin, ellipsoidal cracks is due to Bristow (1960) and his predecessors. A thorough review was given in Watt et al. (1976).

Budiansky and O'Connell (1980) showed how to combine these results for the case where both spherical pores and thin cracks are present, with fluid pressure equalization between the two populations. Since the cracks are much more compliant than the pores and since fluid may flow from crack to pore, this fluid pressure equalization is an important consider-

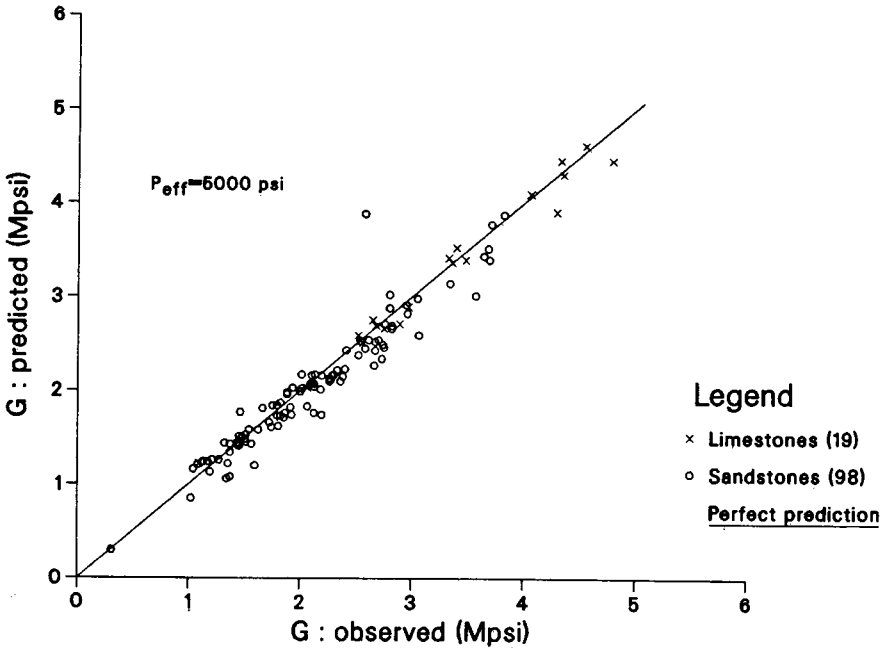

FIG. 1. Test of Biot: saturated shear modulus.

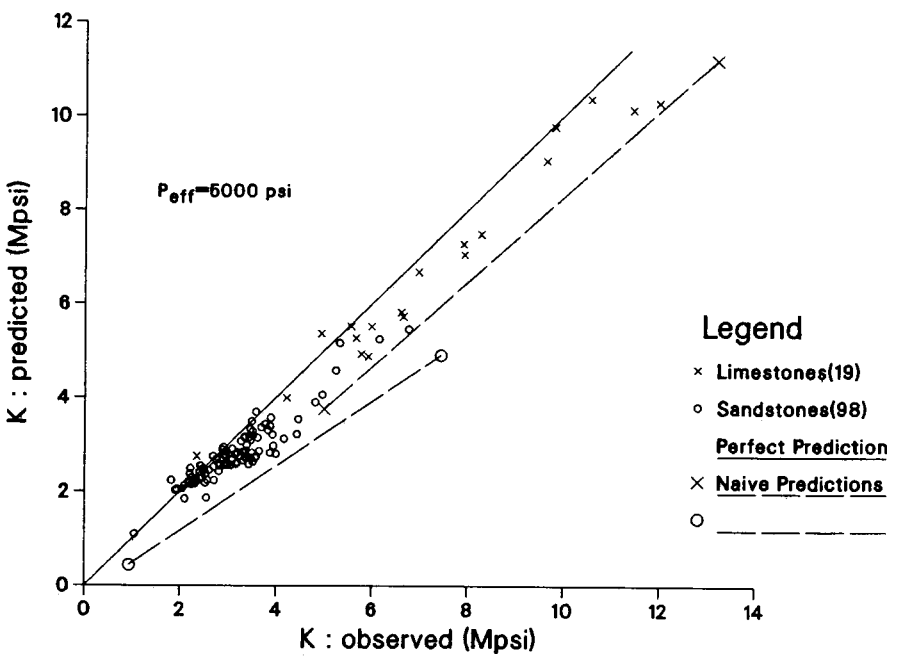

FIG. 2. Test of Biot: saturated bulk modulus.

ation. The issue of fluid pressure equalization for dilute concentrations of the pores is potentially embarrassing. It may be avoided conceptually by assuming a sparse network of very thin tubes which connect the pores and cracks. Such tubes would allow fluid flow and pressure equalization, but (by arguments reviewed in Watt et al., 1976) they would affect the porosity and the elasticity only negligibly. Since the theory for dilute concentrations will be shown to be consistent with Biot theory, the issue is moot.

In the model theory, the "canonical problem" of a single heterogeneity in an infinite medium is solved first. For dilute concentrations, the surrounding medium is taken to be identical to that of the solid grains. Since the pores and cracks are far apart, there is no elastic interaction between neighboring heterogeneities, and the effects on the moduli are simply additive, leading to formulas linear in porosity or in crack density.

Adapting the methods of Budiansky and O'Connell (1980) to the degenerate case of (noninteracting) dilute concentrations yields compact expressions for the elastic moduli:

$$\tilde{G}(K_f) = G_s\left[1 - \frac{\phi_p}{1 - b_s} - B_s\varepsilon\right], \tag{8}$$

and

$$\tilde{K}(K_f) = K_s \frac{1 - \left(1 - \frac{K_f}{K_s}\right)\left(\frac{\phi_p}{1 - a_s} + A_s\varepsilon\right)}{\left[1 + \frac{K_f}{K_s}\left(\frac{a_s}{1 - a_s}\frac{\phi_p}{\phi} + \frac{A_s\varepsilon}{\phi}\right)\right]}. \tag{9}$$

In these expressions, the tilde indicates a model prediction; G_s is the shear modulus of the solid grains; ϕ_p is the volume fraction of the pores, ϕ_c is that of the cracks, ϕ is the total porosity

$$\phi = \phi_p + \phi_c,$$

ε is the crack density, related to ϕ_c (for circular cracks) by

$$\varepsilon = \frac{3}{4\pi}\frac{\phi_c}{\lambda},$$

where the aspect ratio λ is the thickness/diameter ratio of the cracks. The model parameters in equations (8) and (9) are given by

$$b_s \equiv \frac{2}{15}\frac{4 - 5\nu_s}{1 - \nu_s}, \tag{10a}$$

and

$$a_s \equiv \frac{(1 + \nu_s)}{3(1 - \nu_s)}, \tag{10b}$$

for the pores, and

$$B_s \equiv \frac{32}{45}\frac{(1 - \nu_s)(5 - \nu_s)}{(2 - \nu_s)}, \tag{10c}$$

and

$$A_s \equiv \frac{16}{9}\frac{1 - \nu_s^2}{(1 - 2\nu_s)}, \tag{10d}$$

for the cracks. These are functions of only the Poisson's ratio of the solid:

$$\nu_s \equiv \frac{1 - 2G_s/3K_s}{2 + 2G_s/3K_s}. \tag{10e}$$

The theory may be generalized to a spectrum of crack shapes by replacing ε by $\langle\varepsilon\rangle$, the average over the spectrum, everywhere in equations (8) and (9). This leads to different formulas than were used in Hadley (1978) in a problem involving nonconnected cracks.

One can verify that the functional dependence of equations (8) and (9) on the fluid bulk modulus K_f is equivalent to that of the more general Biot-Gassmann relations of equations (4) and (7). For G, this is trivial. Since, according to equations (8) and (10a)–(10d) the shear modulus depends only on the properties of the solid and on ϕ_p and ε, there is no dependence on K_f and equation (4) is verified immediately.

For the bulk modulus some algebra is required, because the model parameters a_s, b_s, A_s, and B_s must be eliminated in favor of the model frame modulus $\tilde{K}^* = \tilde{K}(K_f = 0)$. A model α-parameter

$$\tilde{\alpha} = 1 - \tilde{K}^*/K_s$$

is defined analogously to equation (2). Then it is straightforward to show that equation (9) is equal to

$$\tilde{K}(K_f) = \tilde{K}^* + K_f\tilde{\alpha}^2\Big/\left[\phi + \frac{K_f}{K_s}(\tilde{\alpha} - \phi)\right], \tag{11}$$

i.e., equation (7) is verified for this model. The derivation of equation (11) from equation (9) is exact in the case of no cracks; with cracks, or both cracks and pores, it requires the explicit use of the dilute-concentration assumption. Since equations (8) and (9) already contain that assumption, it may be concluded that this model of microscopic geometry is, in fact, consistent with Biot-Gassmann theory. Since the equations were derived independently, this confirmation of consistency lends further credibility to both formulations.

Despite this success, however, the theory is not suitable for modeling most sedimentary rocks because it is limited to small values of porosity and crack density. It is worth noting at this point that one theory developed for dilute concentrations (Kuster and Toksöz, 1974) differs from the standard theory discussed here. It is easy to verify that this model is not consistent with the Biot-Gassmann theory; in particular, it predicts that shear modulus varies with fluid saturation or, equivalently, with K_f.

MODEL THEORY: "SELF-CONSISTENT"

Because the model discussed above is limited to small porosity ϕ_p, ϕ_c, and small crack density ε, it is important in upper crustal problems to consider extensions of the theory to finite concentrations of fluid inhomogeneities. An exact model would involve computation of interactions between neighboring inhomogeneities, at least on a statistical basis. This has not proven to be feasible; see Watt et al. (1976) for a review and discussion.

The "Self-Consistent" theory has been proposed as an adequate approximation for these interactions (cf., Budiansky and O'Connell 1976; O'Connell and Budiansky, 1974, 1977; Berryman, 1980). It assumes without proof that the effect of many spherical pores, for example, is given by the solution of the canonical problem of a single pore surrounded, not by the

solid, but by a uniform medium with the elastic properties of the fully porous aggregate itself. This is one justification (another is discussed later) of the label "self-consistent." Mathematically, it means replacing the model parameters (a_s, b_s, A_s, and B_s) in equation (10) with similar parameters ($\tilde{a}$, $\tilde{b}$, $\tilde{A}$, and $\tilde{B}$) defined in terms of the Poisson's ratio $\tilde{\nu}$ of the porous medium itself.

It is plausible to assume, for large porosity and/or crack density, that all voids are fully interconnected, allowing fluid pressure equalization during a low-frequency experiment. This is particularly important where more than one pore shape is present. The validity of Murphy's (1985) comparison of SC theory to data is diluted by his use of the inappropriate (unconnected) form of the theory (O'Connell and Budiansky, 1974) rather than the connected form (Budiansky and O'Connell, 1980). At higher frequencies, where fluid pressure equalization is imperfect, the concepts and methods of O'Connell and Budiansky (1977) are applicable.

In the case of connected pores and cracks at low frequency, the self-consistent expression is (Budiansky and O'Connell, 1980)

$$\tilde{K}(K_f) = K_s \frac{1 - \left(1 - \frac{K_f}{K_s}\right)\left(\frac{\phi_p}{1-\tilde{a}} + \tilde{A}\varepsilon\right)}{\left[1 + \frac{K_f}{\tilde{K}}\left(\frac{\tilde{a}}{1-\tilde{a}}\frac{\phi_p}{\phi} + \frac{\tilde{A}\varepsilon}{\phi}\right)\right]}. \tag{12}$$

This may be compared with equation (8); note that in one place on the right side of equation (12), K_s also has been changed to $\tilde{K}$.

If the inhomogeneities were solid, then the self-consistent theory for the shear modulus would be analogous to equation (12):

$$\tilde{G}(G_f, K_f) = G_s \frac{1 - \left(1 - \frac{G_f}{G_s}\right)\left(\frac{\phi_p}{1-\tilde{b}} + \tilde{B}\varepsilon\right)}{\left[1 + \frac{G_f}{\tilde{G}}\left(\frac{\tilde{b}}{1-\tilde{b}}\frac{\phi_p}{\phi} + \frac{\tilde{B}\varepsilon}{\phi}\right)\right]}.$$

However, O'Connell and Budiansky (1977) argued heuristically that for the present case of fluid inhomogeneities, "the cracked solid will respond in shear as if the cracks were empty." Hence they argued that the shear modulus of the saturated, undrained rock should be given by

$$\tilde{G}(K_f) = G_s\left(1 - \frac{\phi}{1-\tilde{b}^*} - \tilde{B}^*\varepsilon\right). \tag{13}$$

This differs from the above expression, not only by setting $G_f = 0$, but also by the appearance of the model parameters $\tilde{b}^*$, $\tilde{B}^*$ of the unsaturated, drained rock, despite the fact that the rock surrounding the pores is saturated and undrained.

The argument leading to equation (13) involved the differing orientations of the various cracks, with respect to the applied shear stress, and is clearly inapplicable to the present case which also contains spherical pores. The proposition that, in pure shear, "both cracks and pores behave as if they were empty," was in fact rigorously proven by Burridge and Keller (1981) in their derivation of Biot's equations (1). Since equation (1a) is independent of p_f, the shear modulus is independent of K_f and the cracks and pores are independent of the particularities of the pore fluid.

The argument (of Budiansky and O'Connell, 1977) is in fact a heuristic justification of an *assumption* (rather than a conclusion); it is an additional argument to those of the original S-C theory. It was de facto judged to be appropriate when the heterogeneities are fluid and interconnected. Since the response parameters $\tilde{b}^*$, $\tilde{B}^*$ in equation (13) do not reflect the properties of the surrounding medium (whose Poisson's ratio $\tilde{\nu}$ does depend upon K_f), the theory can no longer be described strictly as self-consistent in the sense justified earlier. I therefore call the results of equations (12) and (13) the "augmented self-consistent (AS-C) moduli." The original term has another justification: the final results are the same whether derived in terms of elastic moduli or compliances (Walpole, 1969). This symmetry is not disturbed by the augmenting assumption leading to equation (13).

Following the previous program, I wish to verify that the functional dependence of the AS-C moduli of equations (12) and (13) on K_f is that of the more general Biot-Gassmann theory. As previously, this is trival for the shear modulus $\tilde{G}$. By explicit assumption, the saturated $\tilde{G}$ [equation (13)] is given in terms of the same frame parameters $\tilde{b}^*$, $\tilde{B}^*$, and $\tilde{\nu}^*$ as is the frame modulus $\tilde{G}^*$. So the B-G prediction, equation (4), for the shear modulus is verified automatically.

For the bulk modulus, $\tilde{K}^*$ can be found by setting $K_f = 0$ in equation (12), both where it is explicit and where it is implicit in $\tilde{a}$, $\tilde{A}$, and $\tilde{\nu}$:

$$\tilde{K}(0) = K_s\left[1 - \frac{\phi_p}{1-\tilde{a}^*} - \tilde{A}^*\varepsilon\right] \equiv \tilde{K}^*. \tag{14}$$

It is apparent, since $\tilde{K}^*$ is a function of the unsaturated model parameters $\tilde{a}^*$, $\tilde{A}^*$, and $\tilde{1}6\nu^*$ (rather than the saturated model parameters $\tilde{a}$, $\tilde{A}$, and $\tilde{1}6\nu$) that $\tilde{a}$ and $\tilde{A}$ may not be eliminated from equation (12) in favor of K^*. Hence, for this rather general model, it is clear that the augmented self-consistent theory is not consistent with the more basic Biot-Gassmann theory. In other words, in general the AS-C equation (12) is not equivalent to the B-C equation (7).

However it may be shown (R. J. O'Connell, pers. comm., 1984), in the special case of no cracks (spherical pores only), that the two formulas are in fact algebraically equivalent. Hence for this special case, the augmented self-consistent moduli are also consistent with Biot-Gassmann.

In modeling sedimentary rocks, it is usually necessary to include some density of cracks in order to fit theory to data. The "cracks" thus inferred are not necessarily of brittle origin; they are simply that portion of the pore-shape distribution which is thin and flat. Similarly, the "spherical porosity" is simply a model representation of the more equant portion of the pore-shape distribution. Since, in general, a finite (nondilute) concentration of both types of porosity is required to model sedimentary rocks, then some generalization of the noninteracting theory described earlier is required. It is clear that the augmented self-consistent model is not, after all, appropriate for modeling such rocks since it is not consistent with the underlying Biot theory.

A variation of self-consistent theory has been proposed (Bruner, 1976); Clearly, 1978; Henyey and Pomphrey, 1982; Sheng and Calligari, 1984) in which the basic idea of self-consistency is applied in differential fashion, and the final re-

sults are expressed as integrals over crack density and/or porosity. These modifications have been criticized as physically unreasonable (O'Connell and Budiansky, 1976). However it is easy to generalize the results of this section to show that the differential self-consistent formalism (augmented or not), as applied to porous rocks, is also inconsistent with Biot theory. Hence it is not suitable for modeling most sedimentary rocks either. Further, the next section shows that this particular debate is moot.

MODEL THEORY: BIOT-CONSISTENT

It is clear from the foregoing that the character of the medium surrounding the heterogeneity in the canonical problem is not fixed in a necessary, logical way, but instead it offers a degree of freedom which may be used to approximate the elastic interactions between neighboring heterogeneities in the model rock itself. The self-consistent theory utilizes this degree of freedom in this way, assuming (except for the AS-C complication discussed above) that the surrounding canonical medium has the same bulk and shear moduli as the porous rock itself. In other words, the surrounding canonical medium is assumed to respond to external loads $\bar{p}$, $\bar{1}6\tau$ with dependent state variables (strains) $\bar{1}6\theta$, $\bar{1}6\gamma$ which are those of the rock itself.

However, Biot recognized that, in a porous aggregate, there are three dependent state variables rather than two. The third variable may be taken, as in equations (1), to be the fluid pressure p_f; in the undrained case this is equivalent to the increment in pore volume $\Delta V_p/V$. The corresponding modulus may be taken as the pore incompressibility, defined as

$$K_p \equiv -V_p \frac{\bar{p}}{\Delta V_p} = K_f \frac{\bar{p}}{p_f}. \tag{15}$$

Hence an aggregate obeying the Biot-Gassmann equations (a "Biot medium") has three characteristic moduli (G, K, and K_p), rather than two. These three moduli, specified at any particular value of K_f (e.g., any particular saturation) then define the elastic response at all K_f (e.g., all saturations), through equations (4) and (7).

Hence in the model theory for a porous rock, the surrounding medium of the canonical problem should also be a Biot medium and should possess three characteristic moduli rather than two. That is, the properties of the canonical medium should be chosen so that the predictions of equations (4) and (7) of Biot-Gassmann regarding the functional dependence of the moduli on K_f are satisfied identically. When this assumption is made, the resulting model elastic moduli will be consistent with the underlying Biot-Gassmann theory and may be called "Biot-Consistent."

In the derivation of explicit expressions for the Biot-consistent model moduli, a certain degree of arbitrariness arises. I wish to characterize the surrounding Biot medium of the canonical problem, at all values of K_f, by specifying three parameters only, recognizing that equations (4) and (7) provide the functional dependence upon K_f. Biot did not specify the values of K^* and G^*, but only the differences $\tilde{K}(K_f) - \tilde{K}^*$ [equation (7)] and $\tilde{G}(K_f) - \tilde{G}^*$ [null, equation (4)]. These differences depend upon the incremental fluid pressure p_f [equation (6)] and the corresponding fluid shear stress (null). Since the fluid stress is the same in both the fluid heterogeneity and the canonical Biot medium, this fixes, in effect, two of the three necessary parameters.

The third and final parameter must fix the overall stiffness of the Biot canonical medium. It is natural to assume that G_B^* of the canonical Biot medium should be set equal to $\tilde{G}^*$ of the model rock (following S-C or AS-C theory). Alternatively, it may also be natural to assume that K_B^* of the canonical Biot medium should be set equal to $\tilde{K}^*$ of the model rock. Here the former assumption ($G_B^* = \tilde{G}^*$) is made, on the grounds that this choice does not imply a preferred status of any particular saturation state (any particular K_f). However, it is recognized that some arbitrariness therefore remains in the present formulation.

Finally, assume that K_s and ϕ of the canonical Biot medium are those of the model rock, and that the fluid common to both has the incompressibility K_f. It is clear that, since all saturation states (all K_f) are coupled together, the final formulas and the associated iteration scheme will be more complicated than in the S-C or AS-C theories.

To implement these ideas, consider a Biot-Consistent model of a rock containing a substantial equant porosity and a substantial crack density, all interconnected. Following Budiansky and O'Connell (1980) and the discussion above, this model is solved with a combination of results from the associated canonical problems

$$\tilde{G}(p_f) = G_s\left[1 - \frac{\phi_p}{1 - b_B} - B_B\varepsilon\right], \tag{16a}$$

and

$$\tilde{K}(p_f) = K_s \frac{1 - \left(1 - \dfrac{K_f}{K_s}\right)\left(\dfrac{\phi_p}{1 - a_B} + A_B\varepsilon\right)}{\left[+\dfrac{K_f}{K_B}\left(\dfrac{a_B}{1 - a_B}\dfrac{\phi_p}{\phi} + \dfrac{A_B\varepsilon}{\phi}\right)\right]} \tag{16b}$$

Compare with equations (8) and (9) and with equations (12) and (13). The explicit argument here is taken as p_f, rather than the (equivalent) argument K_f emphasized earlier. The model parameters above are defined in terms of the Poisson's ratio of the Biot canonical medium (subscript B):

$$b_B \equiv \frac{2}{15}\frac{4 - 5\nu_B}{1 - \nu_B}, \tag{16c}$$

$$a_B \equiv \frac{(1 + \nu_B)}{3(1 - \nu_B)}, \tag{16d}$$

$$B_B \equiv \frac{32}{45}\frac{(1 - \nu_B)(5 - \nu_B)}{(2 - \nu_B)}, \tag{16e}$$

$$A_B \equiv \frac{16}{9}\frac{1 - \nu_B^2}{(1 - 2\nu_B)}, \tag{16f}$$

and

$$\nu_B \equiv \frac{1 - 2G_B/3K_B}{2 + 2G_B/3K_B}. \tag{16g}$$

It is obvious that these model parameters are dependent upon p_f or K_f, although the explicit display of the argument is omitted for brevity. Their values at $p_f = 0$ will be denoted with an asterisk, e.g., $\nu_B^* = \nu_B(p_f = 0)$, etc. The moduli of the canonical Biot medium, with their p_f dependences, is, from equations (4) and (7),

$$G_B(p_f) = G_B^*, \tag{17a}$$

and

$$K_B(p_f) = K_B^* - \alpha_B \frac{p_f}{\bar{1}6\theta_B}, \tag{17b}$$

where

$$\alpha_B \equiv 1 - K_B^*/K_s, \tag{17c}$$

and

$$\bar{1}6\theta_B = -\bar{p}/K_B(p_f). \tag{17d}$$

Combining equation (17b) and (17d), the canonical bulk modulus is

$$K_B(p_f) = K_B^* \Big/ \left[1 - \alpha_B \frac{p_f}{\bar{p}}\right]. \tag{17e}$$

As discussed, the Biot-consistent model moduli are defined to have the same value of $p_f/\bar{p}$ as the surrounding Biot medium.

In the derivation of the model shear modulus [equation (16a)], the quantity $\tilde{G}(p_f)$ was the ratio of pure shear stress $\bar{1}6\tau$ to pure shear strain $\bar{1}6\gamma$, with no change in volume, and hence no change in fluid pressure p_f. That is,

$$\tilde{G}(p_f) = \left.\frac{\partial\bar{\tau}}{\partial\bar{\gamma}}\right|_{p_f=0} = \tilde{G}(p_f = 0) = \tilde{G}^*. \tag{18}$$

Hence, the apparent p_f dependence in equation (16a) should be suppressed by indicating that $p_f = 0$ in the model parameters b_B, B_B:

$$\tilde{G}(p_f) = \tilde{G}(0) = G_s\left(1 - \frac{\phi}{1 - b_B^*} - B_B^*\varepsilon\right) = \tilde{G}^*. \tag{19}$$

This is substantially the AS-C argument leading to equation (13). However, here it appears not as an ad hoc additional assumption, but as a basic requirement of Biot-consistency.

Now consider the corresponding requirement on the model bulk modulus. The fluid pressure increment of the model theory is, from Budiansky and O'Connell (1980),

$$\frac{p_f}{\bar{p}} = \frac{K_f}{\tilde{K}} \frac{\dfrac{\phi_p}{1 - a_B} + A_B\varepsilon}{\phi + \dfrac{K_f}{K_B}\left(\dfrac{\phi_p}{1 - a_B} + A_B\varepsilon - \phi_p\right)}. \tag{20}$$

This may be combined with equation (16b) to yield

$$\frac{\tilde{K}}{K_s} = 1 - \left(1 - \frac{K_f}{K_s}\right)\frac{\tilde{K}}{K_f}\frac{p_f}{\bar{p}}\phi. \tag{21a}$$

This expression may also be found directly from a theorem by Hill (1963). Rearranging equation (21a),

$$\tilde{K}(K_f) = K_s \Big/ \left[1 + \phi\left(\frac{K_s}{K_f} - 1\right)\frac{p_f}{\bar{p}}\right]. \tag{21b}$$

The fluid pressure increment from the Biot theory is, from equation (6),

$$\frac{p_f}{\bar{p}} = \frac{\alpha_B K_f/K_B}{\phi + \dfrac{K_f}{K_s}(\alpha_B - \phi)}. \tag{22}$$

Using equation (17e) to eliminate K_B, this becomes

$$\frac{p_f}{\bar{p}} = \frac{K_f}{K_B^*} \frac{\alpha_B}{\phi + \dfrac{K_f}{K_B^*}(\alpha_B - \phi + \phi\alpha_B)}. \tag{23}$$

The two expressions (20) and (23) may be equated, fulfilling the second assumption proposed above for the Biot-consistent moduli, resulting in a closed expression involving only the unknown parameters K_B^*, ν_B. However, the result is algebraically cumbersome and may be avoided by the equivalent use of the Biot expression (23) for p_f in the model expression (21) for $\tilde{K}$, in place of the model p_f expression (20). This establishes the same Biot-consistency and is easier to implement.

Finally the third Biot-consistency assumption, as discussed above, is

$$\tilde{G} = G_B. \tag{24}$$

All of the necessary elements are now assembled. An iteration scheme, suitable for calculating the observable unknowns $\tilde{G}$, $\tilde{K}$ may be defined as follows.

(1) Choose modulus values (G_s, K_s) for the solid and (K_f) for the fluid, and porosity parameters (ϕ, ϕ_p, ε) for the aggregate.

(2) Assume a value of ν_B^*, the frame Poisson's ratio of the canonical medium. This initial value may be taken arbitrarily as that of the solid, or as the model value from the NI, S-C or AS-C theory.

(3) Calculate b_B^*, B_B^* using equations (16a)–(16c) as evaluated at $p_f = 0$.

(4) Calculate $\tilde{G} = \tilde{G}^*$ using equation (19).

(5) Using equation (24) for $G_B = G_B^*$, calculate K_B^* from equation (16g), as evaluated at $p_f = 0$.

(6) Calculate α_B using equation (17c) and the scaled fluid pressure increment $p_f/\bar{p}$ using equation (23) [*not* equation (20)].

(7) Calculate $\tilde{K}(p_f)$ using equation (21b), and $K_B(p_f)$ using equation (17e).

(8) Calculate $\nu_B(p_f)$ using G_B, $K_B(p_f)$ in equation (16g), then $a_B(p_f)$ and $A_B(p_f)$ using equations (16d–f).

(9) Calculate $\bar{7}mK(p_f)$ using equation (16b).

(10) Compare the values of $\tilde{K}(p_f)$, as calculated in steps (7) and (9); if they do not agree with sufficient accuracy, then adjust ν_B^* and iterate through steps (3)–(10) until convergence is achieved.

In this iteration scheme, the unsaturated model bulk modulus is completely decoupled. It may be found, after convergence, from

$$\tilde{K}^* = K_s\left[1 - \frac{\phi_p}{1 - a_B^*} - A_B^*\phi\right], \tag{25}$$

where a_B^* and A_B^* are found using equation (16g), as evaluated at $p_f = 0$.

I emphasize that this procedure is valid for modeling the elasticity of rocks at low (seismic) frequency. The extension of these concepts to include high-frequency effects, involving both nonlocal and local (non-Biot) flow of pore fluid will be the subject of another paper.

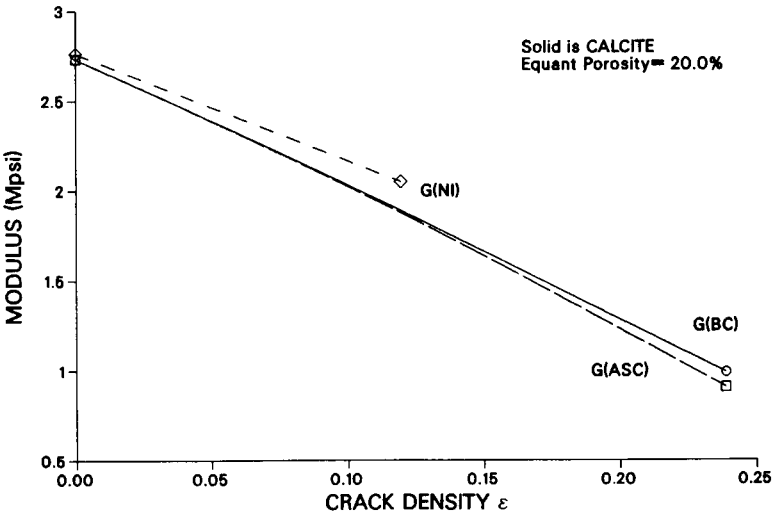

FIG. 3. Comparison of approximations: G.

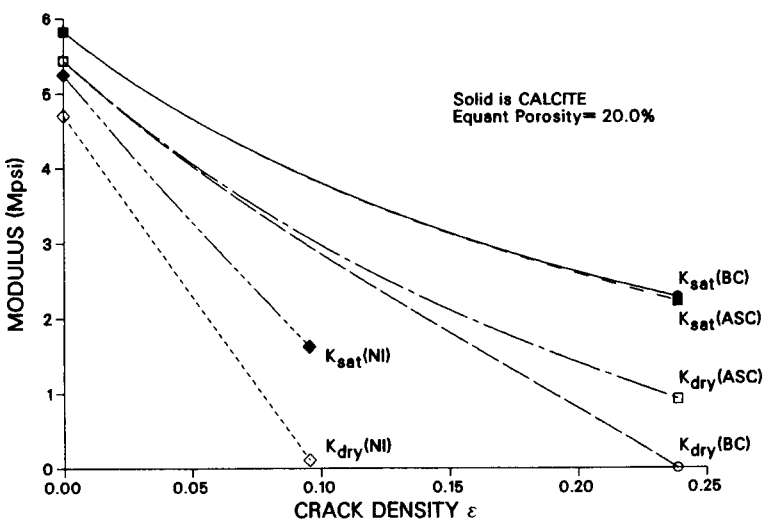

FIG. 4. Comparison of approximations: K.

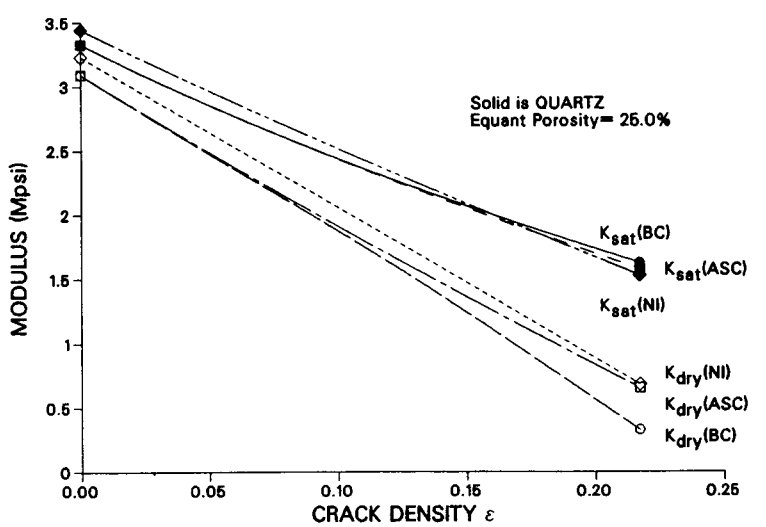

FIG. 5. Comparison of approximations: K.

NUMERICAL EXAMPLES

In any comparison of model theory with actual data, messy questions may arise concerning whether the actual material conforms to the model assumptions (e.g., are the pores really spherical? are the grains really isotropic?). Such questions are, of course, avoided in comparing theories containing identical assumptions. To concentrate on the internal consistency of implementation of these common assumptions, only the latter comparison is performed here.

Figure 3 shows the shear modulus for a model limestone containing 20 percent equant porosity and a large range of crack densities. The shear modulus $\tilde{G}$ is calculated with the three model theories (NI, AS-C, and Biot-Consistent); each predicts that $\tilde{G}$ should be independent of fluid modulus K_f. All three theories agree at small ϕ and small ε; their evident disagreement at $\varepsilon = 0$ is a result of the substantial background equant porosity (20 percent). (The NI theory is extended here past its range of strict validity.) The values of $\tilde{G}$ from the AS-C and the B-C theories are almost identical, as a result of the assumptions of the two formalisms discussed above.

Figure 4 shows the bulk modulus $\tilde{K}$ for the same model, with the same parameters, for the same three model theories. This time, there are two curves for each theory [one for gas saturation, ("dry"; $K_f = 0$), and one for brine saturation, ($K_f = .3$ Mpsi)]. With the B-C theory as formulated here, the saturated bulk modulus is almost identical to that of the AS-C theory, at all K_f and all ε. The differences between B-C theory and AS-C theory are, in effect, concentrated on $\tilde{K}^*$, the unsaturated bulk modulus. In this example, $\tilde{K}^*$(B-C) is everywhere less than $\tilde{K}^*$(AS-C), except at zero crack density where the two are analytically identical. The difference $\tilde{K}-\tilde{K}^*$ is substantially smaller in the AS-C model than in the B-C model—roughly 40 percent smaller at the larger crack densities. It is this difference which is critical for bright-spot analysis, and of course the B-C theory is correct on this point (cf., Figures 1, 2 and accompanying discussion for comparison to data). (Further, this difference is unaffected by the arbitrariness in formulation of the B-C moduli.)

The values of $K_B(p_f)$, the canonical Biot medium as the B-C theory, are not shown since they are not observable; by calculation they are somewhat larger than the corresponding observable B-C model quantities $\tilde{K}$, particularly in the unsaturated condition (*).

As another example, Figure 5 shows a similar calculation for a model sandstone. The comparison is shown for the bulk moduli only (as in the limestone example of Figure 3, the shear moduli from the three theories are similar). Similar remarks apply to the sandstone as to the limestone, although the numbers are different.

CONCLUSIONS

From the preceding discussion, the following conclusions emerge.

(1) The Biot-Gassmann theory of poroelasticity provides an adequate description of the dependence of elastic moduli on pore fluid compressibility, at low (seismic) frequencies. A systematic bias in k occurs when this low-frequency theory is applied at high (sonic and ultrasonic) frequencies.

(2) Any model of rock elasticity which shares the minimal B-G assumptions about the microstructure of the rock must also share the B-G predictions concerning the dependence of elastic moduli on pore fluid compressibility.

(3) The standard model for a rock containing dilute concentrations of pores and/or cracks is consistent with B-G in this sense. However, such a model is not valid for large porosity or crack density.

(4) The "Augmented Self-Consistent" model for a rock containing substantial concentrations of pores and/or cracks is inconsistent with B-G theory in general, although it is consistent for the special case of no cracks. Hence, it is not valid except in that special case.

(5) The "Biot-Consistent" formalism developed here is explicitly consistent with the Biot-Gassmann theory and not limited to small porosity or crack density. Hence, it may be used to model the elasticity of any porous rock at seismic frequencies.

(6) The error incurred in the use of non-B-C models is that the difference between saturated and unsaturated moduli (the bright-spot effect) is miscalculated. In the examples shown, the AS-C theory underestimates this difference by up to 40 percent.

REFERENCES

Berryman, J. G., 1980, Long-wavelength propagation in composite elastic media I. Spherical inclusions: J. Acoust. Soc. Am., **68**, 1809–1819.

Biot, M. A., 1941, General theory of three-dimensional consolidation, J. Appl. Phys., **12**, 155–164.

——— 1956a, General solutions of the equations of elasticity and consolidation for a porous material: J. Appl. Mech., **XX**, 92–96.

——— 1956b, Theory of propagation of elastic waves in a fluid-saturated porous solid, I. Low frequency range: J. Acoust. Soc. Am., **28**, 168–178.

——— 1956c, Theory of propagation of elastic waves in a fluid-saturated porous solid, II. Higher frequency range: J. Acoust. Soc. Am., **28**, 179–191.

——— 1962, Mechanics of deformation and acoustic propagation in porous media: J. Appl. Phys., **33**, 1482–1498.

Biot, M. A., and Willis, D. G., 1957, The elastic coefficients of the theory of consolidation, J. Appl. Mech., **24**, 594–601.

Bristow, J. R., 1960, Microcracks, and the static and dynamic constants of annealed and heavily cold-worked metals: Brit. J. Appl. Phys., **11**, 81–85.

Brown, R. J. S., and Korringa, J., 1975, On the dependence of the elastic properties of a porous rock on the compressibility of the pore fluid, Geophysics, **40**, 608–616.

Bruner, W. M., 1976, "Comment on Seismic velocities in dry and saturated cracked solids:" by R. J. O'Connell and B. Budiansky: J. Geophys. Res., **81**, 2573–2576.

Budiansky, B., 1965, On the elastic moduli of some heterogeneous materials: J. Mech. Phys. Solids, **13**, 223–227.

Budiansky, B., and O'Connell, R. J., 1976, Elastic moduli of a cracked solid: Int. J. Solids Structures, **12**, 81–97.

——— 1980, Bulk dissipation in heterogenous media: Solid Earth Geophys. and Geotech., **42**, 1–10.

Burridge, R., and Keller, J. B., 1981, Poroelasticity equations derived from microstructure: J. Acoust. Soc. Am., **70**, 1140–1146.

Cleary, M. P., 1978, Elastic and dynamic response regimes of fluid-impregnated solids with diverse microstructures: Int. J. Solids Structures, **14**, 795–819.

Domenico, S. N., 1974, Effect of water saturation on seismic reflectivity of sand reservoirs encased in shale: Geophysics, **39**, 759–769.

Eshelby, J. D., 1957, The determination of the elastic field of an ellipsoidal inclusion, and related problems: Proc. Roy. Soc. of London, **A241**, 376–396.

Gardner, G. H. F., Gardner, L. W., and Gregory, A. R., 1974, Formation velocity and density—The diagnostic basis for stratigraphic traps: Geophysics, **39**, 770–780.

Gassmann, F., 1951a, "Über die Elastizität poröser Medien, Mitteiluagen aus dem!: Inst. für Geophysik (Zu!rich), **17**, 1–23.

——— 1951b, Elastic waves through a packing of spheres: Geophysics, **16**, 673–685.
Geertsma, J., 1957, The effect of fluid pressure decline on volumetric changes of porous rock, Trans., Assoc. of Industr. and Mech. Eng., **210**, 331–340.
Geertsma, J., and Smit, D. C., 1961, Some aspects of elastic wave propagation in fluid-saturated porous solids: Geophysics, **26**, 169–181.
Gregory, A. R., 1976, Fluid saturation effects on dynamic elastic properties of sedimentary rocks: Geophysics, **41**, 895–921.
Hadley, K., 1978, Comparison of calculated and observed crack densities and seismic velocities in Westerly granite: J. Geophys. Res., **81**, 3484–3494.
Henyey, F. S., and Pomphrey, N., 1982, Self-consistent elastic moduli of a cracked solid, Geophys. Res. Lett., **9**, 903–906.
Hershey, A. V., 1954, The elasticity of an isotropic aggregate of anisotropic cubic crystals: J. Appl. Mech., **21**, 236–240.
Hill, R., 1963, Elastic properties of reinforced solids: Some theoretical principles: J. Mech. Phys. Solids, **11**, 357–372.
——— 1965, A sefl-consistent mechanics of composite materials: J. Mech. Phys. Solids, **13**, 213–222.
Johnson, D. L., 1985, Recent developments in the acoustic properties of porous media: Frontiers of Physical Acoustics, Proc. 1984 E. Fermi Intl. School Phys., North Holland Publ.
Korringa, J., Brown, R. J. S., Thompson, D. D., and Runge, R. J., 1979, Self-consistent imbedding and the ellipsoidal model for porous rocks: J. Geophys. Res., **84**, 5591–5598.
Korringa, J., and Thompson, D. D., 1977, Comment on the self-consistent imbedding approximation in the theory of elasticity of porous media: J. Geophys. Res., **82**, 933–934.
Kuster, G. T., and Toksöz, M. N. 1974, Velocity and attenuation of seismic waves in two-phase media; I. Theoretical formulations: Geophysics, **39**, 607–618.
Murphy, W. F., 1984, Acoustic measures of partial gas saturation in tight sandstones: J. Geophys. Res., **89**, 11549–11559.
——— 1985, Sonic and ultrasonic velocities: Theory vesus experiment: Geophy. Res. Lett., **12**, 85–88.
Nur, A., and Byerlee, J. D., 1971, An exact effective stress law for elastic deformation of rock with fluids: J. Geophys. Res., **76**, 6414–6419.
O'Connell, R. J., and Budiansky, B., 1974, Seismic velocities in dry and saturated cracked solids: J. Geophys. Res., **79**, 5412–5426.
——— 1976, Reply (to Bruner): J. Geophys. Res., **81**, 2577–2578.
——— 1977, Viscoelastic properties of fluid-saturated cracked solids: J. Geophys. Res., **82**, 5719–5735.
Plona, T. J., 1980, Observation of a second bulk compressional wave in a porous medium at ultrasonic frequencies: Appl. Phys. Lett., **36**, 259–261.
Sheng, P., and Callegari, A. J., 1984, Differential effective medium theory for sedimentary rocks: *in* Johnson, D. L., and Sen, P. N., Eds., Physics and chemistry of porous media: Am. Inst. Phys., 144–165.
Walpole, L. J., 1969, On the overall elastic moduli of composite materials: J. Mech. Phys. Solids, **17**, 235–251.
Watt, P., Davies, G. F., and O'Connell, R., 1976, The elastic properties of composite materials: Rev. Geophys. Space Phys., **14**, 541–563.
Winkler, K. W., 1985, Dispersion analysis of velocity and attenuation in Berea sandstone: Geophysics, **50**,
Wyllie, M. R. J., Gardner, G. H. F., and Gregory, A. R., 1962, Studies of Elastic wave attenuation in porous media: Geophysics, **27**, 569–589.