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Bipolar metric spaces and some fixed point theorems

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Abstract

In this paper we introduce the concept of bipolar metric space as a type of partial distance. We explore the link between metric spaces and bipolar metric spaces, especially in the context of completeness, and prove some extensions of known fixed point theorems.

Keywords: Fixed point, completeness, contraction, metric space, bipolar metric space.
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1. Introduction

Fréchet initiated the theory of metric spaces in 1906 [5]. Since then, there have been plenty of generalizations of metric spaces, by excluding or relaxing some axioms, modifying the metric function or abstracting the concept. In recent years, these structures have played a more central role in fixed point studies, and many valuable results have been obtained in this context [1, 2, 8, 9]. In this paper, we introduce a new structure, which generalizes metric spaces, by extending the possible domain of a metric to a product of two arbitrary nonempty sets.

Metric spaces and many generalizations of them allow us to consider the distances between points of a set, in classical or non-classical sense. However in some cases, distances arise between elements of two different sets, rather than between points of a unique set. In these cases, distances between same type of points are either undefined, or unknown because of absence of data. For example, under the assumption that each deliveryman must carry only one order at a time, it would be enough to know only the distances from locations of a pizza company to available delivery addresses, without handling the huge data of the distances between delivery addresses.

Examples of such types of distances are abundant in mathematics and science. Some basic examples are distance between lines and points in an Euclidean space, distance between points and sets in a metric

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space, suitability measurement of habitats to species, reaction rates of pairs from disjoint sets of chemical substances, affinity between a class of students and a set of activities, lifetime mean distances between people and places, inverse of visible luminosities of a set of stars from a set of planetary bodies and distances between sites and points forming a Voronoi diagram in a metric space [4].

Here we formalize these types of distances under the name bipolar metric, considering them only isometrically without exploring their topological structures in detail. We give some basic definitions and examples about bipolar metric spaces, then we describe maps and sequences, study completeness, discuss some related properties and finally give some fixed point theorems of contractive type on bipolar metric spaces.

2. Bipolar metric spaces

Definition 2.1. Let X and Y be nonempty sets and $d : X \times Y \to \mathbb{R}^+$ be a function, where $\mathbb{R}^+$ denotes the set of nonnegative real numbers. Consider the following properties:

(B0) If $d(x, y) = 0$ then $x = y$ for all $(x, y) \in X \times Y$.
(B1) If $x = y$, then $d(x, y) = 0$ for all $(x, y) \in X \times Y$.
(B2) $d(x, y) = d(y, x)$ for all $x, y \in X \cap Y$.
(B3) $d(x_1, y_2) \leq d(x_1, y_1) + d(x_2, y_1) + d(x_2, y_2)$ for all $x_1, x_2 \in X$ and $y_1, y_2 \in Y$.

Then,

(i) if (B1) and (B2) hold, then d is called a bipolar pseudo-semimetric on the pair (X, Y),
(ii) if d is a bipolar pseudo-semimetric satisfying (B3), it is called a bipolar pseudo-metric,
(iii) a bipolar pseudo-metric d satisfying (B0), is called a bipolar metric.

A bipolar (pseudo-(semi))metric space is a triple (X, Y, d), where d is a bipolar (pseudo-(semi))metric on (X, Y). In particular; if $X \cap Y = \varnothing$ the space is called disjoint, and otherwise it is called joint. The sets X and Y are respectively called the left pole and the right pole of (X, Y, d).

Example 2.2.

(i) Let (X, d) be a (pseudo-(semi))metric space. Then (X, X, d) is a bipolar (pseudo-(semi))metric space. On the other hand, if (X, Y, d) is a bipolar (pseudo-(semi))metric space such that $X = Y$, then (X, d) is a (pseudo-(semi))metric space.
(ii) Let (X, d) be a quasi-metric space. Define the set $Y = X \times \{X\}$. Then the sets X and Y are necessarily disjoint by the axiom of regularity so that (X, Y, d') is a disjoint bipolar metric space, where d' is defined as $d'(x_1, (x_2, X)) = d(x_1, x_2)$ for all $x_1, x_2 \in X$.
(iii) Let d be a pseudo-semimetric on a set X such that $X \cap \mathcal{P}(X) = \varnothing$. Then the point to set distance function $d' : X \times \mathcal{P}(X) \to \mathbb{R}^+$ is a bipolar pseudo-semimetric on the pair $(X, \mathcal{P}(X))$.
(iv) Let (X, d) be a dislocated metric space [6]. Define the sets X_0 and Y as $X_0 = \{x \in X : d(x, x) = 0\}$ and $Y = X_0 \cup (X_0^c \times \{X_0\} \times \{X_0^c\})$. Then (X, Y, d') is a bipolar metric space, where

$$d'(x, y) = \begin{cases} d(x, y), \text{ if } y \in X_0, \\ d(x, y'), \text{ if } y = (y', X_0, X_0^c) \text{ where } y' \in X_0^c, \end{cases}$$

for all $(x, y) \in X \times Y$.
(v) Let X and Y be disjoint sets. Then each relation $\beta \subseteq X \times Y$ defines a disjoint bipolar pseudo-semimetric d_β by $d_\beta(x, y) = 1 - \chi_\beta((x, y))$ for all $(x, y) \in X \times Y$. In particular if β satisfies the generalized Euclidean property

$$x_1 \beta y_1 \wedge x_2 \beta y_1 \wedge x_2 \beta y_2 \Longrightarrow x_1 \beta y_2$$

for all $x_1, x_2 \in X$, $y_1, y_2 \in Y$, then d_β becomes a bipolar pseudo-metric.
More generally, any fuzzy relation $\rho : X \times Y \to [0, 1]$, where X and Y are disjoint sets, defines a disjoint bipolar pseudo-semimetric by $d_\rho := 1 - \rho$.

(vi) Let X and Y be nonempty sets, (Z, ρ) be a pseudo-metric space and $f : X \cup Y \to Z$ be a function. Then $d : X \times Y \to \mathbb{R}^+$, $d(x, y) = \rho(f(x), f(y))$ defines a bipolar pseudo-metric on (X, Y). In particular, if X and Y are normed spaces, $d(x, y) = |\|x\| - \|y\||$ is a bipolar pseudo-metric on (X, Y).

(vii) Let C be the set of all functions and $f : \mathbb{R} \to [1, 3]$ and $d : C \times \mathbb{R} \to \mathbb{R}^+$ be defined as $d(f, x) = f(x)$. Then $(C, \mathbb{R}, d)$ is a disjoint bipolar metric space.

Here we give a series of definitions of some basic concepts related to bipolar metric spaces.

Definition 2.3.

(i) Let (X, Y, d) be a bipolar pseudo-semimetric space. Then the points of the sets X, Y and $X \cap Y$ are named as left, right and central points, respectively, and any sequence, that is consisted of only left (or right, or central) points is called a left (or right, or central) sequence in (X, Y, d).

(ii) Let (X, Y, d) be a bipolar pseudo-semimetric space and $A \subseteq X$, $B \subseteq Y$ be nonempty subsets. Then the triple $(A, B, d|_{A\times B})$, where $d|_{A\times B}$ is the restriction of d to $A \times B$, is also a bipolar pseudo-semimetric space and it is called a subspace of (X, Y, d). For simplicity, we generally write d, instead of $d|_{A\times B}$.

(iii) Let (X, Y, d) be a joint bipolar (pseudo-(semi))metric space. Then (pseudo-(semi))metric space $(X \cap Y, d)$, where d stands for $d|_{(X\cap Y)\times(X\cap Y)}$, is called the center of (X, Y, d).

(iv) The opposite of a bipolar pseudo-semimetric space (X, Y, d) is defined as the bipolar pseudo-semimetric space $\left(Y, X, \bar{d}\right)$, where the function $\bar{d} : Y \times X \to \mathbb{R}^+$ is defined as $\bar{d}(y, x) := d(x, y)$.

(v) Let (X_1, Y_1, d_1) and (X_2, Y_2, d_2) be bipolar pseudo-semimetric spaces and $f : X_1 \cup Y_1 \to X_2 \cup Y_2$ be a function. If $f(X_1) \subseteq X_2$ and $f(Y_1) \subseteq Y_2$, then f is called a covariant map, or a map from (X_1, Y_1, d_1) to (X_2, Y_2, d_2) and this is written as $f : (X_1, Y_1, d_1) \rightrightarrows (X_2, Y_2, d_2)$. If $f : (X_1, Y_1, d_1) \rightrightarrows \left(Y_2, X_2, \bar{d}_2\right)$ is a map, then f is called a contravariant map from (X_1, Y_1, d_1) to (X_2, Y_2, d_2) and this is denoted as $f : (X_1, Y_1, d_1) \rightleftarrows (X_2, Y_2, d_2)$.

In general, a bipolar pseudo-metric only deals with distances between points that are in different poles of the space. However, it also provides some information about the inner structure of these poles.

Definition 2.4. Let (X, Y, d) be a bipolar pseudo-metric space. Then the functions $d_X : X \times X \to \mathbb{R}^+$ and $d_Y : Y \times Y \to \mathbb{R}^+$ which are defined as $d_X(x_1, x_2) = \sup_{y\in Y} |d(x_1, y) - d(x_2, y)|$ for all $x_1, x_2 \in X$ and $d_Y(y_1, y_2) = \sup_{x\in X} |d(x, y_1) - d(x, y_2)|$ for all $y_1, y_2 \in Y$, respectively, are called inner pseudo-metrics generated by (X, Y, d).

Proposition 2.5. *Let (X, Y, d) be a bipolar pseudo-metric space. Then inner pseudo-metrics d_X and d_Y are actually pseudo-metrics on X and Y, respectively. Moreover, if the bipolar pseudo-metric space (X, Y, d) is joint, then d_X and d have same values on $X \times (X \cap Y)$ and similarly d_Y and d agree on $(X \cap Y) \times Y$.*

Proof. By (B3), $|d(x_1, y) - d(x_2, y)|$ is bounded above by $d(x_1, y_0) + d(x_2, y_0)$, where $y \in Y$ is any point. So the suprema in the above definition always exists. Also it is easy to show that d_X and d_Y are pseudo-metrics. To see d_X and d agree on $X \times (X \cap Y)$, let $(x, u) \in X \times (X \cap Y)$. Since $u \in Y$, the value $d(x, u) = |d(x, u) - d(u, u)|$ is counted in supremum so that $d(x, u) \le d_X(x, u)$. On the other hand, since $d(x, y) - d(u, y) \le d(x, u)$ and $d(u, y) - d(x, y) \le d(x, u)$,

$$d_X(x, u) = \sup_{y\in Y} |d(x, y) - d(u, y)| \le d(x, u).$$

□

Corollary 2.6. *Let (X, d) be a pseudo-metric space. Then $d_X = d$, where d_X is the inner pseudo-metric on X defined by the bipolar pseudo-metric space (X, X, d).*

Proposition 2.7. *Any bipolar (pseudo-)metric space (X, Y, d) is embeddable into a (pseudo-)metric space (Z, ρ), in the sense that (X, Y, d) is a subspace of the bipolar (pseudo-)metric space (Z, Z, ρ).*

Proof. Let (X,Y,d) be a bipolar pseudo-metric space and let $\rho : Z \times Z \to \mathbb{R}^+$ be defined as

$$\rho(z_1,z_2) = \begin{cases} d(z_1,z_2), & \text{if } z_1 \in X,\ z_2 \in Y, \\ d(z_2,z_1), & \text{if } z_1 \in Y,\ z_2 \in X, \\ d_X(z_1,z_2), & \text{if } z_1, z_2 \in X, \\ d_Y(z_1,z_2), & \text{if } z_1, z_2 \in Y, \end{cases}$$

where $Z = X \cup Y$. Then ρ is well-defined since $d_X|_{X\times(X\cap Y)} = d|_{X\times(X\cap Y)}$ and $d_Y|_{X\times(X\cap Y)} = d|_{X\times(X\cap Y)}$. It can be easily shown that ρ is a (pseudo-)metric on Z. In particular note that for the triangular inequality we consider multiple cases. As an example, for $x_1, x_2 \in X$ and $y_0 \in Y$, we have

$$\begin{aligned} \rho(y_0,x_2) = d(x_2,y_0) &= -[d(x_1,y_0) - d(x_2,y_0)] + d(x_1,y_0) \\ &\le |d(x_1,y_0) - d(x_2,y_0)| + d(x_1,y_0) \\ &\le \sup_{y\in Y} |d(x_1,y) - d(x_2,y)| + d(x_1,y_0) \\ &= d_X(x_1,x_2) + d(x_1,y_0) \\ &= \rho(y_0,x_1) + \rho(x_1,x_2). \end{aligned}$$

Other cases are similar. Moreover, (X,Y,d) is a subspace of (Z,Z,ρ) by the definition of ρ. □

An inner pseudo-metric generated by a bipolar metric space is not needed to be a metric, and an inner pseudo-metric generated by a bipolar pseudo-metric space, that is not bipolar metric space, yet may be a metric.

Example 2.8.

(i) Let X and Y be nonempty disjoint sets. Define $d : X \times Y \to \mathbb{R}^+$ such that $d(x,y) = 1$ for all $(x,y) \in X \times Y$. Then (X,Y,d) is a bipolar metric space, but

$$d_X(x_1,x_2) = \sup_{y\in Y} |d(x_1,y) - d(x_2,y)| = \sup_{y\in Y} |1-1| = 0$$

for all $x_1, x_2 \in X$, which gives a pseudo-metric space.

(ii) Let $Y = \{y_0\}$, where $y_0 \notin [0,1]$, and define $d : [0,1] \times Y \to \mathbb{R}^+$ as $d(x,y_0) = x$ for all $x \in [0,1]$. Then $([0,1], Y, d)$ is a bipolar pseudo-metric space, which is not bipolar metric space since $d(0,y_0) = 0$, while $d_{[0,1]}(x_1,x_2) = \sup_{y\in Y} |d(x_1,y) - d(x_2,y)| = |d(x_1,y_0) - d(x_2,y_0)| = |x_1 - x_2|$ is a metric on $[0,1]$.

Definition 2.9. Let (X,Y,d) be a bipolar metric space. If the inner pseudo-metric d_X is a metric on X, then we say that Y characterizes X, and if d_Y is a metric, we say that X characterizes Y. If X and Y characterize each other, then the space (X,Y,d) is called bicharacterized.

Example 2.10. Let $X = S^2$ be the unit sphere, Y be a nonempty subset of X, and $d(x,y)$ denotes the geodesic distance between the points $x, y \in S^2$. Then the bipolar metric space (X,Y,d) is bicharacterized if and only if Y has at least three points, which are pairwise non-antipodal.

3. Convergence and continuity

Definition 3.1. Let (X,Y,d) be a bipolar pseudo-semimetric space. A left sequence (x_n) converges to a right point y (symbolically $(x_n) \to y$ or $\lim_n x_n = y$) if and only if for every $\varepsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that $d(x_n,y) < \varepsilon$ for all $n \ge n_0$. Similarly, a right sequence (y_n) converges to a left point x (denoted as $(y_n) \to x$ or $\lim_n y_n = x$) if and only if, for every $\varepsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that, whenever $n \ge n_0$, $d(x,y_n) < \varepsilon$. When it is written $(u_n) \to v$ or $\lim_n u_n = v$ for a bipolar pseudo-semimetric space (X,Y,d), without exact information about the domain of the sequence, this means that either (u_n) is a left sequence and v is a right point, or (u_n) is a right sequence and v is a left point.

The above definition implies that, in a bipolar pseudo-metric space, a left sequence (x_n) converges to a right point y if and only if $(d(x_n, y)) \to 0$ on $\mathbb{R}^+$, and similarly a right sequence (y_n) converges to a left point x iff $(d(x, y_n)) \to 0$ on $\mathbb{R}^+$. Also note that, in a bipolar pseudo-semimetric space (X, Y, d), where $X = Y$, the convergence of any sequence (u_n) is equivalent to its convergence in the pseudo-semimetric space (X, d).

In contrast with the case of metric spaces, in a bipolar metric space a convergent sequence may has multiple limits.

Example 3.2. Let $X = (1, \infty)$ and $Y = [-1, 1]$. Define $d : X \times Y \to \mathbb{R}^+$ as $d(x, y) = \left|x^2 - y^2\right|$. Then (X, Y, d) is a bipolar metric space. Note that the left sequence $(1 + \frac{1}{n})$ converges to right points 1 and -1.

The following lemma gives two important conditions which force convergent sequences to have unique limit.

Lemma 3.3. *Let (X, Y, d) be a bipolar metric space.*

(i) *If (X, Y, d) is bicharacterized, then every convergent sequence has a unique limit.*
(ii) *If a central point is a limit of a sequence, then it is the unique limit of this sequence.*

Proof. We consider the left sequences, the proof for the right sequences is similar.

(i) Let (x_n) be a left sequence such that both $(x_n) \to y_1 \in Y$ and $(x_n) \to y_2 \in Y$. Then for each $x \in X$ we have $d(x, y_2) \leq d(x, y_1) + d(x_n, y_1) + d(x_n, y_2)$ and $d(x, y_1) \leq d(x, y_2) + d(x_n, y_2) + d(x_n, y_1)$. Since $\lim_n d(x_n, y_1) = \lim_n d(x_n, y_2) = 0$ on $\mathbb{R}^+$, $d(x, y_1) = d(x, y_2)$ for all $x \in X$. Hence $d_Y(y_1, y_2) = \sup_{x \in X} |d(x, y_1) - d(x, y_2)| = 0$ and since X characterizes Y, d_Y is a metric so that $y_1 = y_2$.
(ii) Let (x_n) be a left sequence and $(x_n) \to u \in X \cap Y$. If also $(x_n) \to y \in Y$, then $d(u, y) \leq d(u, u) + d(x_n, u) + d(x_n, y)$ and since $\lim_n d(x_n, u) = \lim_n d(x_n, y) = 0$, we have $d(u, y) = 0$ and thus $u = y$.

□

Now we define the continuity of covariant and contravariant maps between bipolar pseudo-semimetric spaces.

Definition 3.4. Let (X_1, Y_1, d_1) and (X_2, Y_2, d_2) be bipolar pseudo-semimetric spaces.

(i) A map $f : (X_1, Y_1, d_1) \rightrightarrows (X_2, Y_2, d_2)$ is said to be continuous at a point $x_0 \in X_1$, if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that whenever $y \in Y_1$ and $d_1(x_0, y) < \delta$, $d_2(f(x_0), f(y)) < \varepsilon$. It is continuous at a point $y_0 \in Y_1$, if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that whenever $x \in X_1$ and $d_1(x, y_0) < \delta$, $d_2(f(x), f(y_0)) < \varepsilon$. If f is continuous at each point $x \in X_1$ and $y \in Y_1$, then it is called continuous.
(ii) A contravariant map $f : (X_1, Y_1, d_1) \rightleftarrows (X_2, Y_2, d_2)$ is continuous if and only if it is continuous as a covariant map $f : (X_1, Y_1, d_1) \rightrightarrows (Y_2, X_2, \bar{d}_2)$.

This definition implies that a covariant or a contravariant map f from (X_1, Y_1, d_1) to (X_2, Y_2, d_2) is continuous, if and only if $(u_n) \to v$ on (X_1, Y_1, d_1) implies $(f(u_n)) \to f(v)$ on (X_2, Y_2, d_2).

Lemma 3.5. *Let (X, Y, d) be a bipolar pseudo-metric space. Then the mapping $d : X \times Y \to \mathbb{R}^+$ is continuous, in the sense that for each left sequence such that $(x_n) \to y$ and for each right sequence such that $(y_n) \to x$, we have $(d(x_n, y_n)) \to d(x, y)$ on $\mathbb{R}^+$.*

Proof. The inequality $|d(x_n, y_n) - d(x, y)| \leq d(x_n, y) + d(x, y_n)$ holds by (B3). By letting $n \to \infty$, we have $(d(x_n, y_n)) \to d(x, y)$. □

In the following we express the continuity of bipolar pseudo-metrics in another sense.

Lemma 3.6. *Let (X, Y, d) be a bipolar pseudo-metric space. The mapping $d : X \times Y \to \mathbb{R}^+$ is continuous with respect to the usual topology on $\mathbb{R}^+$ and the product topology on $X \times Y$, where X and Y has topologies induced by inner pseudo-metrics d_X and d_Y, respectively.*

Proof. The pseudo-metric defined as $d_{X\times Y}\left(\left(x_1, y_1\right), \left(x_2, y_2\right)\right) = \max\left\{d_X\left(x_1, x_2\right), d_Y\left(y_1, y_2\right)\right\}$ on $X \times Y$ generates the product topology. So d is continuous if and only if $(x_n, y_n) \to (x_0, y_0)$ on $(X \times Y, d_{X\times Y})$ implies $d\left(x_n, y_n\right) \to d\left(x_0, y_0\right)$ on $\mathbb{R}^+$.

Let $(x_n, y_n) \to (x_0, y_0)$ on $(X \times Y, d_{X\times Y})$. For every $\varepsilon > 0$, there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $d_{X\times Y}\left(\left(x_n, y_n\right), \left(x_0, y_0\right)\right) < \frac{\varepsilon}{2}$. Then

$$\begin{aligned}\left|d\left(x_n, y_n\right) - d\left(x_0, y_0\right)\right| &\leq \left|d\left(x_n, y_n\right) - d\left(x_0, y_n\right)\right| + \left|d\left(x_0, y_n\right) - d\left(x_0, y_0\right)\right| \\ &\leq d_X\left(x_n, x_0\right) + d_Y\left(y_n, y_0\right) \\ &\leq 2 \cdot d_{X\times Y}\left(\left(x_n, y_n\right), \left(x_0, y_0\right)\right) < \varepsilon.\end{aligned}$$

Therefore, d is continuous with respect to the product topology. □

Definition 3.7. Let (X_1, Y_1, d_1) and (X_2, Y_2, d_2) be bipolar metric spaces and $\lambda > 0$. A covariant map $f : (X_1, Y_1, d_1) \rightrightarrows (X_2, Y_2, d_2)$ such that $d(f(x), f(y)) \leq \lambda \cdot d(x, y)$ for all $x \in X_1$, $y \in Y_1$, or a contravariant map $f : (X_1, Y_1, d_1) \nearrow\!\!\!\swarrow (X_2, Y_2, d_2)$ such that $d(f(y), f(x)) \leq \lambda \cdot d(x, y)$ for all $x \in X_1$, $y \in Y_1$ is called Lipschitz continuous. If $\lambda = 1$, then this covariant or contravariant map is said to be non-expansive, and if $\lambda \in (0, 1)$, it is called a contraction.

Clearly, every contraction is non-expansive, hence Lipschitz continuous, which implies the continuity.

We define the category of bipolar metric spaces, **BMet**, with bipolar metric spaces as objects, non-expansive covariant maps as morphisms, and with usual composition and unit maps. It is readily seen that **Met** is a full subcategory of **BMet**, via the correspondence $(X, d) \longmapsto (X, X, d)$.

4. Complete bipolar metric spaces

Definition 4.1. Let (X, Y, d) be a bipolar pseudo-semimetric space.

(i) A sequence (x_n, y_n) on the set $X \times Y$ is called a bisequence on (X, Y, d).
(ii) If both (x_n) and (y_n) converge, then the bisequence (x_n, y_n) is said to be convergent. If (x_n) and (y_n) both converge to a same point $u \in X \cap Y$, then this bisequence is said to be biconvergent.
(iii) A bisequence (x_n, y_n) on (X, Y, d) is said to be a Cauchy bisequence, if for each $\varepsilon > 0$, there exists a number $n_0 \in \mathbb{N}$, such that for all positive integers $n, m \geq n_0$, $d\left(x_n, y_m\right) < \varepsilon$.

Proposition 4.2. *In a bipolar pseudo-metric space, every biconvergent bisequence is a Cauchy bisequence.*

Proof. Let (X, Y, d) be a bipolar pseudo-metric space and (x_n, y_n) be a bisequence biconvergent to a point $u \in X \cap Y$. Then for all positive integers m and n, we have $d\left(x_n, y_m\right) \leq d\left(x_n, u\right) + d\left(u, y_m\right)$, which implies that (x_n, y_n) is a Cauchy bisequence. □

Proposition 4.3. *In a bipolar metric space, every convergent Cauchy bisequence is biconvergent.*

Proof. Let (X, Y, d) be a bipolar metric space and (x_n, y_n) be a Cauchy bisequence, such that $(x_n) \to y \in Y$ and $(y_n) \to x \in X$. $d\left(x_n, y_n\right) \to 0$ implies $d\left(x, y\right) = 0$. Thus (x_n, y_n) biconverges to the point $x = y$. □

Definition 4.4. A bipolar metric space is called complete, if every Cauchy bisequence in this space is convergent.

In bipolar metric spaces, the notion of completeness is defined via bisequences, rather than sequences. Hence it is important to know, if it is equivalent to the notion of completeness in metric spaces, when a bipolar metric space (X, Y, d) represents a metric space, that is $X = Y$.

Proposition 4.5. *A metric space (X, d) is complete if and only if the corresponding bipolar metric space (X, X, d) is complete.*

Proof. Let (X, d) be a complete metric space and (x_n, x'_n) be a Cauchy bisequence on (X, X, d). Then $d\left(x_n, x_m\right) \leq d\left(x_n, x'_k\right) + d\left(x_m, x'_k\right)$ implies that for sufficiently large positive integers m and n; $d\left(x_n, x_m\right)$

can be made arbitrarily small, so that (x_n) is a Cauchy sequence on (X,d). By the completeness of (X,d), (x_n) converges, and similarly (x'_n) converges. Hence the bisequence (x_n, x'_n) converges on (X,X,d).

On the other hand, let (X,d) be a metric space, such that the corresponding bipolar metric space (X,X,d) is complete. Let (x_n) be a Cauchy sequence on (X,d). Then, since $d(x_n,x_m)$ can be made arbitrary small, (x_n,x_m) is a Cauchy bisequence on the complete bipolar metric space (X,X,d), so it is convergent. This implies that $(x_n) \to u$ for some $u \in X$, both in (X,X,d) and (X,d). □

For a bipolar metric space in the form (X,X,d), we have $d = d_X$. Therefore, completeness of (X,X,d) is equivalent to completeness of the inner pseudo-metric (in fact, metric) space (X,d_X). Now we explore a similar property for arbitrary bipolar metric spaces.

Definition 4.6. A bipolar metric space (X,Y,d) is called bicomplete, if both inner pseudo-metric spaces (X,d_X) and (Y,d_Y) are complete metric spaces.

Theorem 4.7. *Every bicomplete bipolar metric space is complete.*

Proof. Let (X,Y,d) be a bicomplete bipolar metric space and (x_n,y_m) be a Cauchy bisequence on (X,Y,d).

Let $\varepsilon > 0$. Then there exists $n_1 \in \mathbb{N}$ such that $d(x_n,y_m) < \frac{\varepsilon}{3}$ for all $m,n \geq n_1$. Then for all $k,m,n \geq n_1$ and $y \in Y$, we have $|d(x_n,y) - d(x_m,y)| \leq d(x_n,y_k) + d(x_m,y_k) < \frac{2\varepsilon}{3}$ by (B3), thus $d_X(x_n,x_m) < \frac{2\varepsilon}{3} < \varepsilon$. So (x_n) is a Cauchy sequence in the complete metric space (X,d_X).

Let $(x_n) \to x \in X$ in (X,d_X) and $\varepsilon > 0$. There exists an $n_2 \in \mathbb{N}$ such that $d_X(x_n,x) < \frac{\varepsilon}{3}$ for all $n \geq n_2$. Say $n_0 = \max\{n_1,n_2\}$. Then if $m,n \geq n_0$, $|d(x_n,y_m) - d(x,y_m)| \leq \sup_{y\in Y} |d(x_n,y) - d(x,y)| < \frac{\varepsilon}{3}$ implies that $d(x,y_m) < d(x_n,y_m) + \frac{\varepsilon}{3} < \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$, hence $(y_n) \to x$ in (X,Y,d). Similarly, (x_n) converges in (X,Y,d) to the point y, to which (y_n) converges in (Y,d_Y). Consequently, the bisequence (x_n,y_n) converges in (X,Y,d). □

For a bipolar metric space in the form (X,X,d), the notions of completeness and bicompleteness coincide. However in general, a complete bipolar metric space does not need to be bicomplete.

Example 4.8. Let $X = (0,1)$, $Y = (2,3)$. Consider the subspace (X,Y,d) of the space $(\mathbb{R},\mathbb{R},d)$ where d is the usual metric on $\mathbb{R}$. Since $d(x,y) > 1$ for any $(x,y) \in X \times Y$, there is no Cauchy bisequence in (X,Y,d). Thus, it is vacuously true that (X,Y,d) is complete. However, note that d_X is equal to the usual metric on $X = (0,1)$ and (X,d_X) is not complete, hence (X,Y,d) is not bicomplete.

5. Fixed point theorems

In this section, we first express and prove some different extensions and generalizations of Banach contraction principle [3] on bipolar metric spaces.

Theorem 5.1. *Let (X,Y,d) be a complete bipolar metric space and given a contraction $f : (X,Y,d) \rightrightarrows (X,Y,d)$. Then the function $f : X \cup Y \to X \cup Y$ has a unique fixed point.*

Proof. Since f is a contraction, there exists a $\lambda \in (0,1)$ such that $d(f(x),f(y)) \leq \lambda \cdot d(x,y)$ for all $(x,y) \in X \times Y$.

Let $x_0 \in X$ and $y_0 \in Y$. For each $n \in \mathbb{N}$, define $f(x_n) = x_{n+1}$ and $f(y_n) = y_{n+1}$. Then (x_n,y_n) is a bisequence on (X,Y,d).

Say $M := d(x_0,y_0) + d(x_0,y_1)$ and $K_n := \dfrac{\lambda^n M}{1-\lambda}$. Then, for each positive integer n and p, we have

$$\begin{aligned} d(x_n,y_n) &= d(f(x_{n-1}),f(y_{n-1})) \\ &\leq \lambda \cdot d(x_{n-1},y_{n-1}) \\ &\;\vdots \\ &\leq \lambda^n \cdot d(x_0,y_0), \end{aligned}$$

and also,

$$\begin{aligned} d(x_n, y_{n+1}) &= d(f(x_{n-1}), f(y_n)) \\ &\leq \lambda \cdot d(x_{n-1}, y_n) \\ &\vdots \\ &\leq \lambda^n \cdot d(x_0, y_1), \end{aligned}$$

$$\begin{aligned} d(x_{n+p}, y_n) &\leq d(x_{n+p}, y_{n+1}) + d(x_n, y_{n+1}) + d(x_n, y_n) \\ &\leq d(x_{n+p}, y_{n+1}) + \lambda^n M \\ &\leq d(x_{n+p}, y_{n+2}) + d(x_{n+1}, y_{n+2}) + d(x_{n+1}, y_{n+1}) + \lambda^n M \\ &\leq d(x_{n+p}, y_{n+2}) + \left(\lambda^{n+1} + \lambda^n\right) M \\ &\vdots \\ &\leq d(x_{n+p}, y_{n+p}) + \left(\lambda^{n+p-1} + \cdots + \lambda^{n+1} + \lambda^n\right) M \\ &\leq \left(\lambda^{n+p} + \cdots + \lambda^{n+1} + \lambda^n\right) M \leq \lambda^n M \sum_{k=0}^{\infty} \lambda^k = K_n, \end{aligned}$$

and similarly $d(x_n, y_{n+p}) \leq K_n$.

Let $\varepsilon > 0$. Since $\lambda \in (0,1)$, there exists an $n_0 \in \mathbb{N}$ such that $K_{n_0} = \dfrac{\lambda^{n_0} M}{1-\lambda} < \dfrac{\varepsilon}{3}$. Then $d(x_n, y_m) \leq d(x_n, y_{n_0}) + d(x_{n_0}, y_{n_0}) + d(x_{n_0}, y_m) \leq 3K_{n_0} < \varepsilon$ and (x_n, y_n) is a Cauchy bisequence.

Since (X, Y, d) is complete, (x_n, y_n) converges, and thus biconverges to a point $u \in X \cap Y$ and $(f(y_n)) = (y_{n+1}) \to u \in X \cap Y$ guarantees that $(f(y_n))$ has a unique limit. Since f is continuous $(f(y_n)) \to f(u)$, so $f(u) = u$. Hence u is a fixed point of f.

If v is any fixed point of f, then $f(v) = v$ implies that $v \in X \cap Y$ and we have $d(u, v) = d(f(u), f(v)) \leq \lambda \cdot d(u, v)$ where $0 < \lambda < 1$, which implies $d(u, v) = 0$, and so $u = v$. □

Below we prove a similar result for contravariant maps.

Theorem 5.2. *Let (X, Y, d) be a complete bipolar metric space and given a contravariant contraction $f : (X, Y, d) \rightleftarrows (X, Y, d)$. Then the function $f : X \cup Y \to X \cup Y$ has a unique fixed point.*

Proof. Since f is a contravariant contraction, there exists a $\lambda \in (0,1)$ such that $d(f(y), f(x)) \leq \lambda \cdot d(x, y)$ for all $(x, y) \in X \times Y$.

Let $x_0 \in X$. For each $n \in \mathbb{N}$ define $f(x_n) = y_n$ and $f(y_n) = x_{n+1}$. Then (x_n, y_n) is a bisequence on (X, Y, d).

Say $K_n := \dfrac{\lambda^{2n}}{1-\lambda} \cdot d(x_0, y_0)$. Then for all $n, p \in \mathbb{Z}^+$,

$$\begin{aligned} d(x_n, y_n) &= d(f(y_{n-1}), f(x_n)) \\ &\leq \lambda \cdot d(x_n, y_{n-1}) = \lambda \cdot d(f(y_{n-1}), f(x_{n-1})) \\ &\leq \lambda^2 \cdot d(x_{n-1}, y_{n-1}) \\ &\vdots \\ &\leq \lambda^{2n} \cdot d(x_0, y_0) = (1-\lambda) \cdot K_n \leq K_n, \end{aligned}$$

$$\begin{aligned} d(x_{n+1}, y_n) &= d(f(y_n), f(x_n)) \\ &\leq \lambda \cdot d(x_n, y_n) \\ &\leq \lambda^{2n+1} \cdot d(x_0, y_0), \end{aligned}$$

$$d(x_{n+p}, y_n) \leq d(x_{n+p}, y_{n+1}) + d(x_{n+1}, y_{n+1}) + d(x_{n+1}, y_n)$$

$$
\begin{aligned}
&\le d(x_{n+p}, y_{n+1}) + (\lambda^{2n+2} + \lambda^{2n+1}) \cdot d(x_0, y_0) \\
&\le d(x_{n+p}, y_{n+2}) + d(x_{n+2}, y_{n+2}) + d(x_{n+2}, y_{n+1}) \\
&\quad + \ (\lambda^{2n+2} + \lambda^{2n+1}) \cdot d(x_0, y_0) \\
&\le d(x_{n+p}, y_{n+2}) + d(x_{n+2}, y_{n+2}) + d(x_{n+2}, y_{n+1}) \\
&\quad + \ (\lambda^{2n+2} + \lambda^{2n+1}) \cdot d(x_0, y_0) \\
&\le d(x_{n+p}, y_{n+2}) + (\lambda^{2n+4} + \lambda^{2n+3} + \lambda^{2n+2} + \lambda^{2n+1}) \cdot d(x_0, y_0) \\
&\ \vdots \\
&\le d(x_{n+p}, y_{n+p-1}) + (\lambda^{2n+2p-2} + \cdots + \lambda^{2n+1}) \cdot d(x_0, y_0) \\
&\le (\lambda^{2n+2p-1} + \lambda^{2n+2p-2} + \lambda^{2n+2p-3} + \cdots + \lambda^{2n+1}) \cdot d(x_0, y_0) \\
&\le \lambda^{2n+1} \sum_{k=0}^{\infty} \lambda^k d(x_0, y_0) = \lambda K_n < K_n,
\end{aligned}
$$

$$
\begin{aligned}
d\left(x_n, y_{n+p}\right) &\le d(x_n, y_n) + d(x_{n+1}, y_n) + d(x_{n+1} + y_{n+p}) \\
&\le (\lambda^{2n} + \lambda^{2n+1}) \cdot d(x_0, y_0) + d(x_{n+1} + y_{n+p}) \\
&\le (\lambda^{2n} + \lambda^{2n+1}) \cdot d(x_0, y_0) + d(x_{n+1}, y_{n+1}) + d(x_{n+2}, y_{n+1}) \\
&\quad + \ d(x_{n+2}, y_{n+p}) \\
&\le (\lambda^{2n} + \lambda^{2n+1} + \lambda^{2n+2} + \lambda^{2n+3}) \cdot d(x_0, y_0) + d(x_{n+2}, y_{n+p}) \\
&\ \vdots \\
&\le (\lambda^{2n} + \lambda^{2n+1} + \cdots + \lambda^{2n+2p-1}) \cdot d(x_0, y_0) + d(x_{n+p}, y_{n+p}) \\
&\le (\lambda^{2n} + \lambda^{2n+1} + \cdots + \lambda^{2n+2p-1} + \lambda^{2n+2p}) \cdot d(x_0, y_0) \\
&\le \lambda^{2n} \sum_{k=0}^{\infty} \lambda^k d(x_0, y_0) = K_n.
\end{aligned}
$$

Now, since $0 < \lambda < 1$, for any $\varepsilon > 0$, we can find an integer n_0 such that $K_{n_0} = \dfrac{\lambda^{2n_0+1}}{1-\lambda} \cdot d(x_0, y_0) < \frac{\varepsilon}{3}$. Hence $d\left(x_n, y_m\right) \le d\left(x_n, y_{n_0}\right) + d(x_{n_0}, y_{n_0}) + d(x_{n_0}, y_m) \le 3K_{n_0} < \varepsilon$ and (x_n, y_n) is a Cauchy bisequence.

Since (X, Y, d) is complete bipolar metric space, (x_n, y_n) converges, and as a convergent Cauchy bisequence, in particular it biconverges. Let $(x_n) \to u$, $(y_n) \to u$, where $u \in X \cap Y$. Then (x_n) and (y_n) have a unique limit and since the contravariant map f is continuous $(x_n) \to u$ implies that $(y_n) = (f\left(x_n\right)) \to f\left(u\right)$ and combining this with $(y_n) \to u$ gives $f\left(u\right) = u$.

If also v is a fixed point of f, then $f\left(v\right) = v$ implies $v \in X \cap Y$ so that

$$d\left(u, v\right) = d\left(f\left(u\right), f\left(v\right)\right) \le \lambda \cdot d(u, v) \implies d\left(u, v\right) = 0 \implies u = v.$$

□

Definition 5.3. A map $f : (X_1, Y_1, d) \rightrightarrows (X_2, Y_2, d)$ between bipolar pseudo-semimetric spaces is said to be bisurjective, if $f(X_1) = X_2$ and $f\left(Y_1\right) = Y_2$. Similarly, a contravariant map $f : (X_1, Y_1, d) \nrightleftarrows (X_2, Y_2, d)$ is bisurjective, if $f(X_1) = Y_2$ and $f\left(Y_1\right) = X_2$.

The following two theorems illustrate that, it is also possible to obtain further fixed point results on bipolar metric spaces as direct applications of known results on metric spaces.

Theorem 5.4. *Let (X, Y, d) be a bicomplete bipolar metric space and $f : (X, Y, d) \rightrightarrows (X, Y, d)$ be a bisurjective map such that $d(f(x), f(y)) = \Phi(d(x, y))$ for all $(x, y) \in X \times Y$, where Φ is a contraction on $\mathbb{R}^+$. Then there exists a unique point $(x_0, y_0) \in X \times Y$ such that $(f\left(x_0\right), f\left(y_0\right)) = (x_0, y_0)$.*

Proof. By bisurjectivity $f(Y) = Y$, and since Φ is a contraction on $\mathbb{R}^+$, there exists a $\lambda \in (0,1)$ such that $|\Phi(\alpha) - \Phi(\beta)| \leq \lambda \cdot |\alpha - \beta|$ for all $\alpha, \beta \in \mathbb{R}^+$. Then we have

$$\begin{aligned} d_X(f(x_1), f(x_2)) &= \sup_{y\in Y} |d(f(x_1), y) - d(f(x_2), y)| \\ &= \sup_{y\in Y} |d(f(x_1), f(y)) - d(f(x_2), f(y))| \\ &= \sup_{y\in Y} |\Phi(d(x_1, y)) - \Phi(d(x_2, y))| \\ &\leq \sup_{y\in Y} \lambda \cdot |d(x_1, y) - d(x_2, y)| \\ &= \lambda \cdot d_X(x_1, x_2) \end{aligned}$$

for all $x_1, x_2 \in X$. Thus f (more precisely the restriction of f to X) is a contraction on (X, d_X). Since (X, Y, d) is bicomplete, (X, d_X) is a complete metric space. So f has a unique fixed point x on X. Similarly, there exists a unique point $y \in Y$, such that $f(y) = y$. □

Theorem 5.5. *Let (X, Y, d) be a bicomplete bipolar metric space and $f : (X, Y, d) \rightleftarrows (X, Y, d)$ be a bisurjective map such that $d(f(y), f(x)) = \Phi(d(x, y))$ for all $(x, y) \in X \times Y$, where Φ is a contraction on $\mathbb{R}^+$. Then there exists a unique point $(x_0, y_0) \in X \times Y$ such that $(f(y_0), f(x_0)) = (x_0, y_0)$.*

Proof. Since $f(X) = Y$ we have

$$\begin{aligned} d_X(f(y_1), f(y_2)) &= \sup_{y\in Y} |d(f(y_1), y) - d(f(y_2), y)| \\ &= \sup_{x\in X} |d(f(y_1), f(x)) - d(f(y_2), f(x))| \\ &= \sup_{x\in X} |\Phi(d(x, y_1)) - \Phi(d(x, y_2))| \\ &\leq \sup_{x\in X} \lambda \cdot |d(x, y_1) - d(x, y_2)| \\ &= \lambda \cdot d_Y(y_1, y_2) \end{aligned}$$

for all $y_1, y_2 \in Y$, and in a similar manner we have $d_Y(f(x_1), f(x_2)) \leq \lambda \cdot d_X(x_1, x_2)$ for all $x_1, x_2 \in X$. Then $d_X\left(f^2(x_1), f^2(x_2)\right) \leq \lambda^2 \cdot d_X(x_1, x_2)$, thus f^2 is a contraction on the complete metric space (X, d_X), hence there exists a unique point $x_0 \in X$ such that $f^2(x_0) = x_0$. Say $y_0 := f(x_0)$. Then $(f(y_0), f(x_0)) = (x_0, y_0)$.

If $(x_0', y_0') \in X \times Y$ is any point such that $(f(y_0'), f(x_0')) = (x_0', y_0')$, then we have $f^2(x_0') = x_0'$, which implies $x_0' = x_0$. Similarly $y_0' = y_0$. Hence, such a point (x_0, y_0) is unique. □

Finally, we express a theorem based of Kannan's fixed point result [7], which illustrates that many fixed point theorems of contraction type are likely to have some generalizations on bipolar metric spaces.

Theorem 5.6. *Let $T : (X, Y, d) \rightleftarrows (X, Y, d)$, where (X, Y, d) is a complete bipolar metric space and let $\alpha \in \left(0, \frac{1}{2}\right)$ such that the inequality $d(Ty, Tx) \leq \alpha\left(d(x, Tx) + d(Ty, y)\right)$ holds for all $(x, y) \in X \times Y$. Then the function $T : X \cup Y \to X \cup Y$ has a unique fixed point.*

Proof. Let $x_0 \in X$, for each nonnegative integer n, we define $y_n = Tx_n$ and $x_{n+1} = Ty_n$. Then we have

$$\begin{aligned} d(x_n, y_n) &= d(Ty_{n-1}, Tx_n) \\ &\leq \alpha \cdot (d(x_n, Tx_n) + d(Ty_{n-1}, y_{n-1})) \\ &= \alpha \cdot (d(x_n, y_n) + d(x_n, y_{n-1})) \end{aligned}$$

for all integers $n \geq 1$. Then,

$$d(x_n, y_n) \leq \frac{\alpha}{1-\alpha} \cdot d(x_n, y_{n-1}),$$

and

$$\begin{aligned} d\left(x_n, y_{n-1}\right) &= d\left(Ty_{n-1}, Tx_{n-1}\right) \\ &\leq \alpha \cdot \left(d\left(x_{n-1}, Tx_{n-1}\right) + d\left(Ty_{n-1}, y_{n-1}\right)\right) \\ &= \alpha \cdot \left(d\left(x_{n-1}, y_{n-1}\right) + d\left(x_n, y_{n-1}\right)\right), \end{aligned}$$

so that

$$d\left(x_n, y_{n-1}\right) \leq \frac{\alpha}{1-\alpha} \cdot d\left(x_{n-1}, y_{n-1}\right).$$

If we say $\lambda := \dfrac{\alpha}{1-\alpha}$, then we have $\lambda \in (0,1)$ since $\alpha \in (0, \frac{1}{2})$. Now

$$\begin{aligned} d\left(x_n, y_n\right) &\leq \lambda^{2n} \cdot d\left(x_0, y_0\right), \\ d\left(x_n, y_{n-1}\right) &\leq \lambda^{2n-1} \cdot d\left(x_0, y_0\right), \end{aligned}$$

and for all positive integers m and n

$$\begin{aligned} d\left(x_n, y_m\right) &\leq d\left(x_n, y_n\right) + d\left(x_{n+1}, y_n\right) + d\left(x_{n+1}, y_m\right) \\ &\leq \left(\lambda^{2n} + \lambda^{2n+1}\right) \cdot d\left(x_0, y_0\right) + d\left(x_{n+1}, y_m\right) \\ &\vdots \\ &\leq \left(\lambda^{2n} + \lambda^{2n+1} + \cdots + \lambda^{2m}\right) \cdot d\left(x_0, y_0\right), \end{aligned}$$

if $m > n$, and

$$\begin{aligned} d\left(x_n, y_m\right) &\leq d\left(x_{m+1}, y_m\right) + d\left(x_{m+1}, y_{m+1}\right) + d\left(x_n, y_{m+1}\right) \\ &\leq \left(\lambda^{2m+1} + \lambda^{2m+2}\right) \cdot d\left(x_0, y_0\right) + d\left(x_n, y_{m+1}\right) \\ &\vdots \\ &\leq \left(\lambda^{2m+1} + \lambda^{2m+2} + \cdots + \lambda^{2n}\right) \cdot d\left(x_0, y_0\right) + d\left(x_n, y_n\right) \\ &< \left(\lambda^{2m+1} + \lambda^{2m+2} + \cdots + \lambda^{2n+1}\right) \cdot d\left(x_0, y_0\right), \end{aligned}$$

if $m < n$.

Since $\lambda \in (0,1)$, these mean that $d\left(x_n, y_m\right)$ can be made arbitrarily small by m and n large enough, hence (x_n, y_m) is a Cauchy bisequence. Since (X, Y, d) is complete, so (x_n, y_m) converges, and in particular biconverges since it is a convergent Cauchy bisequence.

Let u be the point which (x_n, y_m) biconverges to it. Then $(x_n) \to u$, $(y_n) \to u$ and $u \in X \cap Y$. Since $(Tx_n) = (y_n) \to u$, $d(Tu, Tx_n) \to d(Tu, u)$. On the other hand,

$$d(Tu, Tx_n) \leq \alpha \cdot (d(x_n, Tx_n) + d(Tu, u)) = \alpha \cdot (d(x_n, y_n) + d(Tu, u)),$$

which in turn implies that $d(Tu, u) \leq \alpha \cdot d(Tu, u)$. Hence $Tu = u$ and T has a fixed point.

If v is any fixed point of T, then $Tv = v$ implies that v is in $X \cap Y$. Then

$$d\left(u, v\right) = d\left(Tu, Tv\right) \leq \alpha \cdot \left(d\left(u, Tu\right) + d\left(Tv, v\right)\right) = \alpha \cdot \left(d\left(u, u\right) + d\left(v, v\right)\right) = 0.$$

Consequently $u = v$. □

Example 5.7. Let X be the class of all singleton subsets of $\mathbb{R}$ and Y be the class of all nonempty compact subsets of $\mathbb{R}$. We define $d : X \times Y \to \mathbb{R}^+$ as $d(x, A) = |x - \inf(A)| + |x - \sup(A)|$. Then the triple (X, Y, d) is a complete bipolar metric space. The contravariant map $T : (X, Y, d) \rightleftarrows (X, Y, d)$, defined as $TA = \left\{\frac{\inf(A)+\sup(A)+6}{8}\right\} \subset \mathbb{R}$ for all $A \in X \cup Y$, satisfies the inequality $d(Ty, Tx) \leq \alpha \left(d\left(x, Tx\right) + d\left(Ty, y\right)\right)$ for $\alpha = \frac{1}{3}$. Hence T must have a unique fixed point, which in fact is the set $\{1\} \in X \cap Y$.

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