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Birational Canonical Transformations and Classical Solutions of the Sixth Painlevé Equation

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Abstract. Two topics on the sixth Painlevé equation are treated in this paper. In Section 1, a simple construction of a group of birational canonical transformations of the sixth equation isomorphic to the affine Weyl group of D_4 root system is given by exploiting an affine Weyl group symmetry of the Hamiltonian structure of the sixth equation defined on the defining variety of the equation. In the rest of this paper (Sections 2-4), based on Umemura's theory on algebraic differential equations, all one-parameter families of classical solutions of the sixth equation are determined, and the irreducibility of the sixth equation is proved. The latter is a rigorous proof of what Painlevé asserted in C. R. Acad. Sci. Paris **143** (1906), 1111-1117.

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0. – Introduction

In this paper we give a simple construction of a group of birational canonical transformations of the sixth Painlevé equation isomorphic to the affine Weyl group of D_4 root system, and determine all one-parameter families of classical solutions of the sixth Painlevé equation. Especially, the latter includes the proof of the irreducibility of the generic solutions of the equation, which is a rigorous proof of what Painlevé asserted in [5].

In general, one understands by the sixth Painlevé equation the following one:

$$\tag{1} \begin{aligned} \frac{d^2q}{dt^2} = \frac{1}{2}\left(\frac{1}{q}+\frac{1}{q-1}+\frac{1}{q-t}\right)\left(\frac{dq}{dt}\right)^2 - \left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{q-t}\right)\frac{dq}{dt} \\ + \frac{q(q-1)(q-t)}{t^2(t-1)^2}\left[\alpha+\beta\frac{t}{q^2}+\gamma\frac{t-1}{(q-1)^2}+\delta\frac{t(t-1)}{(q-t)^2}\right]. \end{aligned}$$

This equation (1) does not, however, seem to be suitable for our purpose because it is hard to explain geometric and/or algebraic properties of the sixth

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Painlevé equation by means of this form (1). Okamoto gave a Hamiltonian system with a polynomial Hamiltonian equivalent to (1) in connection with the isomonodromic deformation of a second order Fuchsian differential equation ([3]), and developed a theory of birational canonical transformations and tau functions of the sixth Painlevé equation from the view point of the Hamiltonian structure ([4]). According to his theory, various properties of the sixth Painlevé equation are written by means of such a Hamiltonian system more systematically than by means of only the original single equation (1). Moreover, Okamoto [2] constructed a seven-dimensional phase space bundle E (see Section 1) over the space of time and parameters of the sixth Painlevé equation on which the Hamiltonian foliation of the sixth equation is naturally defined. The space E, which we would like to call the defining variety, inherits a lot of properties of the Hamiltonian structure of the sixth Painlevé equation. For example, starting from the space E, one can recover the Hamiltonian structure of the sixth Painlevé equation on E under a natural assumption ([6]). In this paper, therefore, we understand by the sixth Painlevé equation the pair of the variety E and the Hamiltonian structure of the sixth Painlevé equation defined on E (see Section 1).

The content of this paper is as follows. In Section 1, after reviewing the defining variety and the definition of the sixth equation defined on E, we explain how to construct the group $\mathbf{B}$ of birational canonical transformations of the sixth equation isomorphic to the affine Weyl group of the D_4 root system. In [4] Okamoto constructed such transformations by exploiting the symmetry of a second order differential equation which is satisfied by a Hamiltonian function of the sixth Painlevé equation. His calculation in construction seems to be complicated, and it is not explicitly specified where such birational canonical transformations are considered. In this paper we take the position that birational canonical transformations of the sixth Painlevé equation should be considered on the defining variety E. This easily leads us to a simpler construction of them. In fact, we explicitly write out birational canonical transformations of E corresponding to generators of the affine Weyl group of D_4 root system W_a, which acts on the space of parameters of the sixth Painlevé equation, by taking into consideration an affine Weyl group symmetry of the Hamiltonian structure on E (Theorem 1.1). By an elementary property of the Coxeter system, we also see that the group $\mathbf{B}$ generated by such birational canonical transformations is isomorphic to the affine Weyl group W_a (Corollary 1.2). Our method releases us from complicated calculations, and clarifies the geometric meaning of birational canonical transformations of the sixth Painlevé equation. Moreover, there seem to be applications of our method to the other Painlevé equations and the Garnier systems, etc. (e.g. [13], [14]).

Sections 2-4 deal with the determination of classical solutions and the irreducibility of the sixth Painlevé equation. In Section 2 we first prepare some materials relating to the parameter space $\mathbb{C}^4$ of the sixth Painlevé equation: hyperplanes invariant under the action of the group W_a and fundamental regions for the group W_a. Next we state the main theorem concerning the determination of one-parameter families of classical solutions and the irreducibility of the equation (Theorem 2.1). Since the notion of classical function in the sense of Umemura (e.g. [8]) is invariant under birational transformations, thanks to the action of the group $\mathbf{B}$ on the parameter space of the sixth equation, we can reduce the main theorem to Theorem 2.2 where all one-parameter families within a fundamental region Γ of the parameter space for W_a are determined. We see that all the classical solutions come from the hypergeometric differential equation of Gauss (Remark 2.2).

Sections 3 and 4 are devoted to the proof of Theorem 2.2. Our proof here as well as those for the other Painlevé equations ([10], [11], [12]) is based on Umemura's theory on algebraic differential equations ([7], [8], [9]). In his theory Umemura introduced an algebraic criterion for the irreducibility of a given ordinary differential equation, which is called the condition (J) (see Section 3). In Proposition 3.1, we establish a necessary condition of the parameters of the sixth equation under which the condition (J) fails. This condition is crucial because all results concerning the irreducibility and the determination of classical solutions follow from Proposition 3.1. In the proof of Proposition 3.1, as was done in the other Painlevé equations, we investigate the equation $X(\boldsymbol{a})F = GF$ in detail, where $X(\boldsymbol{a})$ is the Hamiltonian vector field corresponding to the sixth equation, and F and G are polynomials in the canonical variables q and p with coefficients in a differential overfield K of $\mathbb{C}(t)$ the field of rational functions in one variable. We introduce two gradings to the polynomial ring $K[q,p]$, and decompose F in two ways with respect to such gradings: $F = F_m + \cdots + F_0 = F'_n + \cdots + F'_{n'}$, where m is a non-negative integer, and n and n' are integers such that $n \geq n'$. We determine the explicit forms of the polynomials F_m, F_{m-1}, F'_n and G by Lemmas 3.2-3.6, and obtain the desired condition by equating two equivalent expressions of a coefficient ρ in G. As a corollary of Proposition 3.1 (Corollary 3.7), we prove that, for all vectors in the fundamental region Γ but not on the hyperplanes invariant under the action of the group W_a, there exists no one-parameter family of classical solutions of the sixth equation. In Section 4 we determine all $X(\boldsymbol{a})$-invariant principal ideals of $K[q,p]$ for all vectors $\boldsymbol{a}$ on the intersection of the region Γ and the invariant hyperplanes (Propositions 4.1-4.8). Thus the results in Sections 3 and 4 complete the proof of Theorem 2.2.

1. – Defining variety and birational canonical transformations

We review the defining variety of the sixth Painlevé equation constructed by Okamoto [2], and define the sixth Painlevé equation on it. Let $\mathbb{P}^1$ denote the complex projective line, and let t and s be inhomogeneous coordinates of $\mathbb{P}^1$ with the condition

$$ts = 1. \tag{1}$$

We set $B = \mathbb{P}^1 - \{t = 0, 1, \infty\}$, and regard t or s as coordinates of B. Let U_0, U_1, U_2, U_3, U_4, U_∞ be copies of the product space $\mathbb{C}^4 \times B \times \mathbb{C}^2$ with corresponding coordinate systems $(a_1, a_2, a_3, a_4; t; q_i, p_i)$ $(i = 0, 1, 2, 3, 4, \infty)$. With a brief notation $\boldsymbol{a} = (a_1, a_2, a_3, a_4) \in \mathbb{C}^4$, we also write $(\boldsymbol{a}; t; q_i, p_i) = (a_1, a_2, a_3, a_4; t; q_i, p_i)$. The rule of patching the six spaces is as follows:

(i) on $U_0 \cap U_\infty$,

$$q_0 q_\infty = 1, \tag{2}$$

$$q_0 p_0 + q_\infty p_\infty = a_3 - a_2; \tag{3}$$

(ii) on $U_0 \cap U_1$,

$$q_0 p_0 + q_1 p_1 = a_3 + a_4, \tag{4}$$

$$p_0 p_1 = 1; \tag{5}$$

(iii) on $U_0 \cap U_2$,

$$(q_0 - 1) p_0 + q_2 p_2 = a_3 - a_4, \tag{6}$$

$$p_0 p_2 = 1; \tag{7}$$

(iv) on $U_0 \cap U_3$,

$$(q_0 - t) p_0 + q_3 p_3 = 1 - a_1 - a_2, \tag{8}$$

$$p_0 p_3 = 1; \tag{9}$$

(v) on $U_\infty \cap U_4$,

$$q_\infty p_\infty + q_4 p_4 = a_1 - a_2, \tag{10}$$

$$p_\infty p_4 = 1. \tag{11}$$

We denote by E the seven-dimensional open variety thus obtained. The variety E has been first constructed by Okamoto [2], p.37-49. The above conditions (i)-(v) are taken from Shioda and Takano [6]. Obviously, the variety E has a natural fibration $\pi : E \to \mathbb{C}^4 \times B$.

We next define the sixth Painlevé equation on E. To this end, we give polynomial functions H_i on open sets U_i $(i = 0, 1, 2, 3, 4, \infty)$ by the following equations (see [4]):

$$\begin{aligned} t(t-1)H_0 = {} & q_0(q_0 - 1)(q_0 - t)p_0^2 - (a_3 + a_4)(q_0 - 1)(q_0 - t)p_0 \\ & - (a_3 - a_4)q_0(q_0 - t)p_0 \\ & + (a_1 + a_2)q_0(q_0 - 1)p_0 + (a_3 - a_2)(a_3 - a_1)(q_0 - t); \end{aligned}$$

$$\begin{aligned} s(s-1)H_\infty = {} & q_\infty(q_\infty - 1)(q_\infty - s)p_\infty^2 - (a_1 - a_2)(q_\infty - 1)(q_\infty - s)p_\infty \\ & - (a_3 - a_4)q_\infty(q_\infty - s)p_\infty + (a_1 + a_2)q_\infty(q_\infty - 1)p_\infty \\ & + (a_2 - a_3)(a_2 + a_4)(q_\infty - s); \end{aligned}$$

$$\begin{aligned}t(t-1)H_1 = &-(q_1p_1-a_4-a_1)(q_1p_1-a_4-a_2)(q_1p_1-a_4-a_3)p_1\\ &-t(q_1p_1-a_4)^2-(q_1p_1-a_1-a_4)(q_1p_1-a_2-a_3)-tq_1;\end{aligned}$$

$$\begin{aligned}t(t-1)H_2 = &-(q_2p_2+a_4-a_1)(q_2p_2+a_4-a_2)(q_2p_2+a_4-a_3)p_2\\ &+(1-t)(q_2p_2+a_4)^2+(q_2p_2-a_1+a_4)(q_2p_2-a_2-a_3)\\ &-(1-t)q_2;\end{aligned}$$

$$\begin{aligned}t(t-1)H_3 = &-(q_3p_3+a_1+a_2-1)(q_3p_3+a_2+a_3-1)(q_3p_3+a_3+a_1-1)p_3\\ &+t(q_3p_3+a_1+a_2-1)(q_3p_3+a_3-a_4-1)\\ &+(t-1)(q_3p_3+a_1+a_2-1)(q_3p_3+a_3+a_4-1)-t(t-1)q_3;\end{aligned}$$

$$\begin{aligned}s(s-1)H_4 = &-(q_4p_4-a_4-a_1)(q_4p_4-a_1+a_3)(q_4p_4-a_1+a_2)p_4\\ &-(q_4p_4-a_1)^2-s(q_4p_4-a_1-a_4)(q_4p_4+a_2+a_3)-sq_4.\end{aligned}$$

Then we have the following relations:
(i) on $U_0 \cap U_\infty$,

$$H_\infty = -t^2H_0 + (a_3-a_2)(a_1+a_2)t; \tag{12}$$

(ii) on $U_0 \cap U_1$,

$$H_0 = H_1 + \frac{a_1a_3-a_1a_4+a_3a_4}{t-1} - \frac{(a_1-a_3)(a_2-a_4)}{t}; \tag{13}$$

(iii) on $U_0 \cap U_2$,

$$H_0 = H_2 + \frac{a_1a_3-a_3a_4+a_1a_4}{t} + \frac{(a_3-a_1)(a_2+a_4)}{t-1}; \tag{14}$$

(iv) on $U_0 \cap U_3$,

$$H_0 = H_3 + \frac{1}{p_3}; \tag{15}$$

(v) on $U_\infty \cap U_4$,

$$H_\infty = H_4 - \frac{a_2^2}{s} - \frac{a_1a_2+a_2a_4+a_1a_4}{s-1}. \tag{16}$$

Let d be the exterior differential with respect to the variables t, q_i, p_i ($i = 0,1,2,3,4,\infty$). Thus we have $da_i = 0$ ($i = 1,2,3,4$). From (1)-(16), we have the following:
(i) on $U_0 \cap U_\infty$,

$$dp_0 \wedge dq_0 - dH_0 \wedge dt = dp_\infty \wedge dq_\infty - dH_\infty \wedge ds; \tag{17}$$

(ii) on $U_0 \cap U_1$,

$$dp_0 \wedge dq_0 - dH_0 \wedge dt = dp_1 \wedge dq_1 - dH_1 \wedge dt; \tag{18}$$

(iii) on $U_0 \cap U_2$,

$$dp_0 \wedge dq_0 - dH_0 \wedge dt = dp_2 \wedge dq_2 - dH_2 \wedge dt; \tag{19}$$

(iv) on $U_0 \cap U_3$,

$$dp_0 \wedge dq_0 - dH_0 \wedge dt = dp_3 \wedge dq_3 - dH_3 \wedge dt; \tag{20}$$

(v) on $U_\infty \cap U_4$,

$$dp_\infty \wedge dq_\infty - dH_\infty \wedge ds = dp_4 \wedge dq_4 - dH_4 \wedge ds. \tag{21}$$

These conditions (17)-(21) mean that the foliations $\mathcal{F}_i$ in U_i $(i = 0, 1, 2, 3, 4, \infty)$ defined by the Hamiltonian systems

$$S_i \begin{cases} \dfrac{dq_i}{dt_i} = \dfrac{\partial H_i}{\partial p_i}, \\ \dfrac{dp_i}{dt_i} = -\dfrac{\partial H_i}{\partial q_i}, \end{cases}$$

are mutually compatible. Here $t_i = t$ if $i = 0, 1, 2, 3$, and $t_i = s$ if $i = 4, \infty$. Therefore the six foliations determine a unique foliation $\mathcal{F}$ of codimension 2 in E such that the restriction of $\mathcal{F}$ to each U_i coincides with the foliation $\mathcal{F}_i$. In [2] the following properties are shown concerning the variety E and the foliation $\mathcal{F}$:

(i) the foliation $\mathcal{F}$ in E has no singularity except those of the first class (for the definition, see [2]);

(ii) the variety E is maximal with respect to inclusion among seven-dimensional complex smooth varieties possessing the property (i).

We call the system S_i the sixth Painlevé equation defined on U_i. By the sixth Painlevé equation defined on E we mean the collection of all the sixth Painlevé equations defined on trivial affine open subbundles of E. Thus we can identify the foliation $\mathcal{F}$ with the set of solutions of the sixth Painlevé equation defined on E, and a leaf of $\mathcal{F}$ with a solution of the sixth Painlevé equation defined on E. We call E the defining variety of the sixth Painlevé equation, or the Painlevé-Okamoto variety of sixth kind. In fact, Shioda and Takano [6] show that the foliation $\mathcal{F}$ is a unique holomorphic Hamiltonian foliation in E. Let π be the natural projection $E \to \mathbb{C}^4 \times B$. Following Okamoto, we call each fibre $\pi^{-1}(\boldsymbol{a}; t)$ $((\boldsymbol{a}; t) \in \mathbb{C}^4 \times B)$ the space of initial conditions of the sixth Painlevé equation for $(\boldsymbol{a}; t) \in \mathbb{C}^4 \times B$.

Let V_1, V_3, V_4 be again copies of $\mathbb{C}^4 \times B \times \mathbb{C}^2$ with corresponding coordinate systems $(\boldsymbol{a}; t; Q_i, P_i)$ $(i = 1, 3, 4)$. The rule of patching U_0, V_1, V_3, V_4 is as follows (see [4]):

(i) on $U_0 \cap V_1$,

$$(Q_1 - t)(q_0 - t) = t(t-1), \tag{22}$$

$$(Q_1 - t)P_1 + (q_0 - t)p_0 = a_3 - a_2; \tag{23}$$

(ii) on $U_0 \cap V_3$,

$$Q_3 q_0 = t, \tag{24}$$

$$Q_3 P_3 + q_0 p_0 = a_3 - a_2; \tag{25}$$

(iii) on $U_0 \cap V_4$,

$$(Q_4 - 1)(q_0 - 1) = 1 - t, \tag{26}$$

$$(Q_4 - 1)P_4 + (q_0 - 1)p_0 = a_3 - a_2. \tag{27}$$

The variety thus obtained is obviously identified with the union $U_0 \cup U_\infty$ in E. In the following we identify V_1, V_3, V_4 with affine open subbundles of E. We represent the sixth Painlevé equation on each open set V_i. Let K_i $(i = 1, 3, 4)$ be polynomial functions on V_i given by the following equations:

$$\begin{aligned} t(t-1)K_1 = {} & Q_1(Q_1-1)(Q_1-t)P_1^2 - (a_3-a_4)(Q_1-1)(Q_1-t)P_1 \\ & - (a_3+a_4)Q_1(Q_1-t)P_1 + (1-a_1+a_2)Q_1(Q_1-1)P_1 \\ & + (a_3-a_2)(a_3+a_1-1)(Q_1-t); \end{aligned}$$

$$\begin{aligned} t(t-1)K_3 = {} & Q_3(Q_3-1)(Q_3-t)P_3^2 - (a_1-a_2)(Q_3-1)(Q_3-t)P_3 \\ & - (1-a_1-a_2)Q_3(Q_3-t)P_3 + (1+a_4-a_3)Q_3(Q_3-1)P_3 \\ & - (a_3-a_2)(a_2+a_4)(Q_3-t); \end{aligned}$$

$$\begin{aligned} t(t-1)K_4 = {} & Q_4(Q_4-1)(Q_4-t)P_4^2 - (1-a_1-a_2)(Q_4-1)(Q_4-t)P_4 \\ & - (a_1-a_2)Q_4(Q_4-t)P_4 + (1-a_4-a_3)Q_4(Q_4-1)P_4 \\ & + (a_3-a_2)(a_4-a_2)(Q_4-t). \end{aligned}$$

Then we have the following relations:
(i) on $U_0 \cap V_1$,

$$\begin{aligned} K_1 = {} & H_0 - \frac{1}{t(t-1)} q_0(q_0-1)p_0 + \frac{a_3-a_2}{t(t-1)} q_0 \\ & + \frac{(a_3-a_2)(a_4-a_1+1)}{t} - \frac{(a_3-a_2)(a_1+a_4)}{t-1}; \end{aligned} \tag{28}$$

(ii) on $U_0 \cap V_3$,

$$K_3 = H_0 - \frac{1}{t} q_0 p_0 + (a_3 - a_2)(a_4 - a_1 + 1)\frac{1}{t}; \tag{29}$$

(iii) on $U_0 \cap V_4$,

$$K_4 = H_0 - \frac{1}{t(t-1)}(q_0 - 1)p_0 + (a_3 - a_2)(1 - a_1 - a_4)\frac{1}{t-1}. \tag{30}$$

Moreover we have:
(i) on $U_0 \cap V_1$,

$$dp_0 \wedge dq_0 - dH_0 \wedge dt = dP_1 \wedge dQ_1 - dK_1 \wedge dt; \tag{31}$$

(ii) on $U_0 \cap V_3$,

$$dp_0 \wedge dq_0 - dH_0 \wedge dt = dP_3 \wedge dQ_3 - dK_3 \wedge dt; \tag{32}$$

(iii) on $U_0 \cap V_4$,

$$dp_0 \wedge dq_0 - dH_0 \wedge dt = dP_4 \wedge dQ_4 - dK_4 \wedge dt. \tag{33}$$

Thus we see that the sixth Painlevé equation defined on V_i is also represented as the Hamiltonian system

$$T_i \begin{cases} \dfrac{dQ_i}{dt} = \dfrac{\partial K_i}{\partial P_i}, \\ \dfrac{dP_i}{dt} = -\dfrac{\partial K_i}{\partial Q_i}. \end{cases}$$

Now we construct a group of birational canonical transformations of E isomorphic to the affine Weyl group of D_4 root system. To this end, let s_i ($i = 1, 2, 3, 4, 5$) be affine transformations of $\mathbb{C}^4$ defined by $s_1(\boldsymbol{a}) = (a_2, a_1, a_3, a_4)$, $s_2(\boldsymbol{a}) = (a_1, a_3, a_2, a_4)$, $s_3(\boldsymbol{a}) = (a_1, a_2, a_4, a_3)$, $s_4(\boldsymbol{a}) = (a_1, a_2, -a_4, -a_3)$ and $s_5(\boldsymbol{a}) = (-a_2 + 1, -a_1 + 1, a_3, a_4)$ for $\boldsymbol{a} = (a_1, a_2, a_3, a_4) \in \mathbb{C}^4$. We have $s_i^2 = \mathbf{1}$ ($i = 1, 2, 3, 4, 5$), $s_i s_j = s_j s_i$ ($i, j \neq 2$) and $(s_2 s_i)^3 = (s_i s_2)^3 = \mathbf{1}$ ($i \neq 2$), where $\mathbf{1}$ denotes the identity transformation of $\mathbb{C}^4$. Let W_a be the subgroup generated by these five transformations in the group of all affine transformations. We see that W_a is isomorphic to the affine Weyl group of D_4 root system, and the pair $(W_a, \{s_1, s_2, s_3, s_4, s_5\})$ is a Coxeter system (cf. [15], Corollary 2.4). We prove the following theorem concerning the construction of birational canonical transformations of the sixth Painlevé equation (cf. [4], Theorem 1.).

THEOREM 1.1. *If $g \in W_a$, then there exists a birational canonical transformation γ of E such that the following diagram* (34) *is commutative:*

$$
\begin{tikzcd}
E \arrow[r, "\gamma"] \arrow[d, "\pi_1"'] & E \arrow[d, "\pi_1"] \\
\mathbb{C}^4 \arrow[r, "g"'] & \mathbb{C}^4,
\end{tikzcd}
\tag{34}
$$

where π_1 represents the natural projection $E \to \mathbb{C}^4$.

PROOF. Since the group W_a is generated by s_i $(i = 1, 2, 3, 4, 5)$, it is sufficient to construct birational canonical transformations σ_i $(i = 1, 2, 3, 4, 5)$ of E such that the diagram (34) is commutative for $\gamma = \sigma_i$ and $g = s_i$.

(i) Construction of σ_1. Since $s_1(\boldsymbol{a}) = (a_2, a_1, a_3, a_4)$, the Hamiltonian H_0 is invariant under s_1. Thus we have the locally trivial birational canonical transformation σ_1 of U_0:

$$\sigma_1(\boldsymbol{a}; t; q_0, p_0; H_0) = (s_1(\boldsymbol{a}); t; q_0, p_0; H_0).$$

If extending the domain of definition of σ_1 to U_∞ by (2), (3) and (12), we have on U_∞

$$\sigma_1(\boldsymbol{a}; s; q_\infty, p_\infty; H_\infty) = \left(s_1(\boldsymbol{a}); s; q_\infty, p_\infty - \frac{a_1 - a_2}{q_\infty}; H_\infty - \frac{(a_1 - a_2)(a_1 + a_2)}{s}\right).$$

In a way, the birational canonical transformation σ_1 on $U_0 \cup U_\infty$ is extended to the whole space E. We omit the detail.

(ii) Construction of σ_3. Since $s_3(\boldsymbol{a}) = (a_1, a_2, a_4, a_3)$, the Hamiltonian K_4 is invariant under s_3. Thus we have the locally trivial birational canonical transformation σ_3 of V_4:

$$\sigma_3(\boldsymbol{a}; t; Q_4, P_4; K_4) = (s_3(\boldsymbol{a}); t; Q_4, P_4; K_4).$$

If extending the domain of definition of σ_3 to U_0 by (26), (27) and (30), we have on U_0

$$\sigma_3(\boldsymbol{a}; t; q_0, p_0; H_0) = \left(s_3(\boldsymbol{a}); t; q_0, p_0 - \frac{a_3 - a_4}{q_0 - 1}; H_0 - \frac{(a_3 - a_4)(a_1 + a_2)}{t - 1}\right).$$

By (2), (3) and (12), we have on U_∞

$$\sigma_3(\boldsymbol{a}; s; q_\infty, p_\infty; H_\infty) = \left(s_3(\boldsymbol{a}); s; q_\infty, p_\infty - \frac{a_3 - a_4}{q_\infty - 1}; H_\infty - \frac{(a_3 - a_4)(a_1 + a_2)}{s - 1}\right).$$

By the similar way the birational canonical transformation σ_3 of $U_0 \cup U_\infty \cup V_4 = U_0 \cup U_\infty$ is extended to the whole space E. We omit the detail.

(iii) Construction of σ_4. Since $s_4(\boldsymbol{a}) = (a_1, a_2, -a_4, -a_3)$, the Hamiltonian H_∞ is invariant under s_4. Thus we have the locally trivial birational canonical transformation σ_4 of U_∞:

$$\sigma_4(\boldsymbol{a}; s; q_\infty, p_\infty; H_\infty) = (s_4(\boldsymbol{a}); s; q_\infty, p_\infty; H_\infty).$$

If extending the domain of definition of σ_4 to U_0 by (2), (3) and (12), we have on U_0

$$\sigma_4(\boldsymbol{a}; t; q_0, p_0; H_0) = \left(s_4(\boldsymbol{a}); t; q_0, p_0 - \frac{a_3 + a_4}{q_0}; H_0 - \frac{(a_3 + a_4)(a_1 + a_2)}{t} \right).$$

By the similar way, the birational canonical transformation σ_4 of $U_0 \cup U_\infty$ is extended to the whole space E. We omit the detail.

(iv) Construction of σ_5. Since $s_5(\boldsymbol{a}) = (1 - a_2, 1 - a_1, a_3, a_4)$, the Hamiltonian K_1 is invariant under s_5. Thus we have the locally trivial birational canonical transformation σ_5 of V_1:

$$\sigma_5(\boldsymbol{a}; t; Q_1, P_1; K_1) = (s_5(\boldsymbol{a}); t; Q_1, P_1; K_1).$$

If extending the domain of definition of σ_5 to U_0 by (22), (23) and (28), we have on U_0

$$\begin{aligned} \sigma_5(\boldsymbol{a}; t; q_0, p_0; H_0) = \Bigg(& s_5(\boldsymbol{a}); t; q_0, p_0 - \frac{1 - a_1 - a_2}{q_0 - t}; \\ & H_0 - \frac{1 - a_1 - a_2}{q_0 - t} + (a_1 + a_2 - 1)\left(\frac{1 - a_3 - a_4}{t} + \frac{1 - a_3 + a_4}{t - 1} \right) \Bigg). \end{aligned}$$

By (2), (3) and (12), we have on U_∞

$$\begin{aligned} \sigma_5(\boldsymbol{a}; s; q_\infty, p_\infty; H_\infty) = \Bigg(& s_5(\boldsymbol{a}); s; q_\infty, p_\infty - \frac{1 - a_1 - a_2}{q_\infty - s}; \\ & H_\infty - \frac{1 - a_1 - a_2}{q_\infty - s} + (a_1 + a_2 - 1)\left(\frac{1 - a_1 + a_2}{s} + \frac{1 - a_3 + a_4}{s - 1} \right) \Bigg). \end{aligned}$$

By the similar way, the birational canonical transformation σ_5 of $U_0 \cup U_\infty \cup V_1 = U_0 \cup U_\infty$ is extended to the whole space E. We omit the detail.

(v) Construction of σ_2. Since $s_2(\boldsymbol{a}) = (a_1, a_3, a_2, a_4)$, the Hamiltonian H_1 is invariant under s_2. Thus we have the locally trivial birational canonical transformation σ_2 of U_1:

$$\sigma_2(\boldsymbol{a}; t; q_1, p_1; H_1) = (s_2(\boldsymbol{a}); t; q_1, p_1; H_1).$$

If extending the domain of definition of σ_2 to U_0 by (4), (5) and (13), we have on U_0

$$\sigma_2(\boldsymbol{a}; t; q_0, p_0; H_0) = \Bigg(s_2(\boldsymbol{a}); t; q_0 - \frac{a_3 - a_2}{p_0}, p_0; \\ H_0 + \frac{(a_1 + a_4)(a_2 - a_3)}{t - 1} + \frac{(a_1 - a_4)(a_2 - a_3)}{t}\Bigg).$$

By (2), (3) and (12), we have on U_∞

$$\sigma_2(\boldsymbol{a}; s; q_\infty, p_\infty; H_\infty) = \Bigg(s_2(\boldsymbol{a}); s; q_\infty - \frac{a_3 - a_2}{p_\infty}, p_\infty; \\ H_\infty + \frac{(a_2 - a_3)(a_2 + a_3)}{s} + \frac{(a_2 - a_3)(a_1 + a_4)}{s - 1}\Bigg).$$

By the similar way the birational canonical transformation σ_2 of $U_0 \cup U_\infty \cup U_1$ is extended to the whole space E. We omit the detail. Theorem 1.1 is thus proved.

Corollary 1.2 (cf. [4], Remark 3.2). *Let* **B** *be the subgroup of birational canonical transformations of* E *generated by the five transformations* σ_i *(*$i = 1, 2, 3, 4, 5$*) in the proof of the theorem. Then* **B** *is isomorphic to the group* W_a*, and therefore to the affine Weyl group of* D_4 *root system.*

Proof. Obviously, we have a homomorphism $\mathbf{B} \ni \gamma \to g \in W_a$ by the diagram (34). Conversely, we have $\sigma_i^2 = \mathbf{1}$ $(i = 1, 2, 3, 4, 5)$, $\sigma_i\sigma_j = \sigma_j\sigma_i$ $(i, j \neq 2)$ and $(\sigma_i\sigma_2)^3 = (\sigma_2\sigma_i)^3 = \mathbf{1}$ $(i \neq 2)$, where $\mathbf{1}$ denotes the identity transformation of E. Since the pair $(W_a, \{s_1, s_2, s_3, s_4, s_5\})$ is a Coxeter system, by the universality (cf. [1], Chap. IV, Section 1), we have another homomorphism $W_a \ni g \to \gamma \in \mathbf{B}$. Obviously, these homomorphisms are mutually reciprocal.

Remark 1.1. We find various notations for the parameters of the sixth Painlevé equation in the literature. For example Okamoto [4] denotes a parameter of the equation (or, the Hamiltonian) by $\boldsymbol{b} = (b_1, b_2, b_3, b_4)$. The following shows the relation between our notation $\boldsymbol{a} = (a_1, a_2, a_3, a_4)$ and Okamoto's $\boldsymbol{b}$:

$$a_1 = -b_4, \;\; a_2 = -b_3, \;\; a_3 = b_1, \;\; a_4 = b_2.$$

2. – Classical solutions and irreducibility

First, we consider the parameter space of the sixth Painlevé equation $\mathbb{C}^4$, and review some results about it (for the details, see [15]). We fix the usual hermitian inner product $(\boldsymbol{a}|\boldsymbol{b}) = a_1\bar{b}_1 + a_2\bar{b}_2 + a_3\bar{b}_3 + a_4\bar{b}_4$ for two vectors $\boldsymbol{a} = (a_1, a_2, a_3, a_4)$ and $\boldsymbol{b} = (b_1, b_2, b_3, b_4)$ in $\mathbb{C}^4$, where $\bar{b}$ denotes the complex conjugate of a complex number b. Let R be the collection of the following 24 vectors: $(\pm1, \pm1, 0, 0)$, $(\pm1, 0, \pm1, 0)$, $(\pm1, 0, 0, \pm1)$, $(0, \pm1, \pm1, 0)$, $(0, \pm1, 0, \pm1)$, $(0, 0, \pm1, \pm1)$; the set R is the root system of type D_4. For $\alpha \in R$ and $k \in \mathbb{Z}$ (the set of integers), let $H_{\alpha,k}$ be the complex hyperplane of $\mathbb{C}^4$ defined by $H_{\alpha,k} = \{\boldsymbol{a} \in \mathbb{C}^4 \mid (\boldsymbol{a}|\alpha) = k\}$. We define ten subsets M, P, P_1, P_2, L, L_1, L_2, D, D_1 and D_2 of $\mathbb{C}^4$ by the following:

M: union of all $H_{\alpha,k}$'s for $\alpha \in R$ and $k \in \mathbb{Z}$;

P: union of all intersections $H_{\alpha,k} \cap H_{\beta,l}$ for $\alpha, \beta \in R$ which are linearly independent (or equivalently, $(\alpha|\beta) \neq \pm2$), and for $k, l \in \mathbb{Z}$;

P_1: union of all intersections $H_{\alpha,k} \cap H_{\beta,l}$ for $\alpha, \beta \in R$ such that $(\alpha|\beta) = 0$, and for $k, l \in \mathbb{Z}$;

P_2: union of all intersections $H_{\alpha,k} \cap H_{\beta,l}$ for $\alpha, \beta \in R$ such that $(\alpha|\beta) = -1$, and for $k, l \in \mathbb{Z}$;

L: union of all intersections $H_{\alpha,k} \cap H_{\beta,l} \cap H_{\gamma,m}$ for $\alpha, \beta, \gamma \in R$ which are linearly independent, and for $k, l, m \in \mathbb{Z}$;

L_1: union of all intersections $H_{\alpha,k} \cap H_{\beta,l} \cap H_{\gamma,m}$ for $\alpha, \beta, \gamma \in R$ such that $(\alpha|\beta) = (\beta|\gamma) = (\gamma|\alpha) = 0$, and for $k, l, m \in \mathbb{Z}$;

L_2: union of all intersections $H_{\alpha,k} \cap H_{\beta,l} \cap H_{\gamma,m}$ for $\alpha, \beta, \gamma \in R$ such that $(\alpha|\beta) = 0$ and $(\beta|\gamma) = (\gamma|\alpha) = -1$, and for $k, l, m \in \mathbb{Z}$;

D: union of all intersections $H_{\alpha,k} \cap H_{\beta,l} \cap H_{\gamma,m} \cap H_{\delta,n}$ for $\alpha, \beta, \gamma, \delta \in R$ which are linearly independent, and for $k, l, m, n \in \mathbb{Z}$;

D_1: union of all intersections $H_{\alpha,k} \cap H_{\beta,l} \cap H_{\gamma,m} \cap H_{\delta,n}$ for $\alpha, \beta, \gamma, \delta \in R$ such that $(\alpha|\beta) = (\beta|\gamma) = (\gamma|\alpha) = (\alpha|\delta) = (\beta|\delta) = (\gamma|\delta) = 0$, and for $k, l, m, n \in \mathbb{Z}$ such that $k + l + m + n \equiv 1 \mod 2$;

D_2: union of all intersections $H_{\alpha,k} \cap H_{\beta,l} \cap H_{\gamma,m} \cap H_{\delta,n}$ for $\alpha, \beta, \gamma, \delta \in R$ such that $(\alpha|\beta) = (\beta|\gamma) = (\gamma|\alpha) = (\alpha|\delta) = (\beta|\delta) = (\gamma|\delta) = 0$, and for $k, l, m, n \in \mathbb{Z}$ such that $k + l + m + n \equiv 0 \mod 2$.

Obviously, we have $M \supset P \supset L \supset D$, $P \supset P_1 \cup P_2$, $L \supset L_1 \cup L_2$, $D \supset D_1 \cup D_2$ and $D_1 \cap D_2 = \emptyset$. Moreover, by [15], Proposition 1.2, we have (i) $P_1 \not\supset P_2$ and $P_1 \not\subset P_2$; (ii) $P = P_1 \cup P_2$; (iii) $L_1 \not\supset L_2$ and $L_1 \not\subset L_2$; (iv) $L = L_1 \cup L_2$; (v) $D = D_1 \cup D_2$. By [15], Proposition 1.3, we see that these ten subsets are invariant under the action of the group W_a.

Let $\pi_1 : E \to \mathbb{C}^4$ be the natural projection, and let $\mathcal{F}(\boldsymbol{a})$ be the restriction of the foliation $\mathcal{F}$ to the fibre $\pi_1^{-1}(\boldsymbol{a})$ ($\boldsymbol{a} \in \mathbb{C}^4$). By a one-parameter family of solutions of $\mathcal{F}(\boldsymbol{a})$ we mean the collection of leaves of $\mathcal{F}(\boldsymbol{a})$ which are defined on each open set $\pi_1^{-1}(\boldsymbol{a}) \cap U_i$ ($i = 0, 1, 2, 3, 4, \infty$) in $\pi_1^{-1}(\boldsymbol{a})$ by a common non-trivial single algebraic equation in at least one variable q_i or p_i. By a classical solution of $\mathcal{F}(\boldsymbol{a})$, we mean a leaf of $\mathcal{F}(\boldsymbol{a})$ which are parametrized by a classical solution (in the sense of Umemura, for example see [8]) of the sixth Painlevé equation defined on each open set $U_i \cap \pi_1^{-1}(\boldsymbol{a})$ in $\pi_1^{-1}(\boldsymbol{a})$. The rest of this paper is devoted to the proof of the following:

THEOREM 2.1. (i) *For $\boldsymbol{a} \in M$ but $\boldsymbol{a} \notin P$, there exists a one-parameter family of classical solutions of $\mathcal{F}(\boldsymbol{a})$.*
(ii) *For $\boldsymbol{a} \in P$ but $\boldsymbol{a} \notin L$, there exist two one-parameter families of classical solutions of $\mathcal{F}(\boldsymbol{a})$.*
(iii) *For $\boldsymbol{a} \in L$ but $\boldsymbol{a} \notin D$, there exist three one-parameter families of classical solutions of $\mathcal{F}(\boldsymbol{a})$.*
(iv) *For $\boldsymbol{a} \in D$, there exist four one-parameter families of classical solutions of $\mathcal{F}(\boldsymbol{a})$.*
(v) *Let α be a leaf of $\mathcal{F}(\boldsymbol{a})$ ($\boldsymbol{a} \in \mathbb{C}^4$) defined by a transcendental solution of the sixth Painlevé equation different from those in* (i)-(iv). *Then α does not belong to any one-parameter family of $\mathcal{F}(\boldsymbol{a})$, and does not define any classical solution of the sixth Painlevé equation.*

REMARK 2.1. Assertion (v) implies the irreducibility of the sixth Painlevé equation (cf. [5]).

We consider reduction of the proof of the theorem. First, observing the forms of the systems S_i ($i = 1, 2, 3, 4$), we see that there exists no one-parameter family of solutions on E defined by $p_1 = 0$ or $p_2 = 0$ or $p_3 = 0$ or $p_4 = 0$. Therefore it is sufficient to prove the theorem on the open subset $U_0 \cup U_\infty$. We write the systems S_0 and S_∞ explicitly:

$$S_0 \begin{cases} t(t-1)\dfrac{dq}{dt} = t(t-1)\dfrac{\partial H_0}{\partial p} \\ \qquad = 2q(q-1)(q-t)p + (a_1 + a_2 - 2a_3)q^2 \\ \qquad\quad + (2a_3 t - a_1 - a_2 + a_3 + a_4)q - (a_3 + a_4)t, \\ t(t-1)\dfrac{dp}{dt} = -t(t-1)\dfrac{\partial H_0}{\partial q} \\ \qquad = -3q^2p^2 + 2(1+t)qp^2 - tp^2 - 2(a_1 + a_2 - 2a_3)qp \\ \qquad\quad - (2a_3 t - a_1 - a_2 + a_3 + a_4)p - (a_3 - a_2)(a_3 - a_1), \end{cases}$$

$$S_\infty \begin{cases} s(s-1)\dfrac{dQ}{ds} = s(s-1)\dfrac{\partial H_\infty}{\partial P} \\ \qquad = 2Q(Q-1)(Q-s)P + (2a_2 - a_3 + a_4)Q^2 \\ \qquad\quad + [(a_1 - a_2 + a_3 - a_4)s - 2a_2]Q - (a_1 - a_2)s, \\ s(s-1)\dfrac{dP}{ds} = -s(s-1)\dfrac{\partial H_\infty}{\partial Q} \\ \qquad = -3Q^2P^2 + 2(1+s)QP^2 - sP^2 - 2(2a_2 - a_3 + a_4)QP \\ \qquad\quad - [(a_1 - a_2 + a_3 - a_4)s - 2a_2]P - (a_2 - a_3)(a_2 + a_4). \end{cases}$$

Here we set

$$q = q_0, \quad p = p_0, \quad Q = q_\infty, \quad P = p_\infty,$$

and always adopt this notation in the following. There seems to be no possibility of confusion of the variable P with the subset P of $\mathbb{C}^4$. When we emphasize the dependence on the vector $\boldsymbol{a} \in \mathbb{C}^4$ of the systems S_0 and S_∞, we also write $S_0(\boldsymbol{a})$ and $S_\infty(\boldsymbol{a})$. By the argument of the proof of Theorem 1.1, we can regard the group $\mathbf{B}$ in Corollary 1.2 as a group of birational canonical transformations of the subbundle $U_0 \cup U_\infty$ of E. Since, by definition (e.g. [8]), the notion of classical function is invariant under birational transformations, and since the group $\mathbf{B}$ acts naturally on $\mathbb{C}^4$ the parameter space of the systems S_0 and S_∞, we may restrict the parameter $\boldsymbol{a}$ of the systems S_0 and S_∞ to a fundamental region of $\mathbb{C}^4$ for the group W_a isomorphic to $\mathbf{B}$. We adopt as a fundamental region the collection, denoted Γ, of all vectors $\boldsymbol{a} = (a_1, a_2, a_3, a_4) \in \mathbb{C}^4$ subject to the following conditions (see [15], Corollary 2.5):

(i) $\Re(a_1 - a_2) \geq 0$;

(ii) $\Re(a_2 - a_3) \geq 0$;

(iii) $\Re(a_3 - a_4) \geq 0$;

(iv) $\Re(a_3 + a_4) \geq 0$;

(v) $\Re(a_1 + a_2) \leq 1$;

(vi) $\Im(a_1 - a_2) \geq 0$ if $\Re(a_1 - a_2) = 0$;

(vii) $\Im(a_2 - a_3) \geq 0$ if $\Re(a_2 - a_3) = 0$;

(viii) $\Im(a_3 - a_4) \geq 0$ if $\Re(a_3 - a_4) = 0$;

(ix) $\Im(a_3 + a_4) \geq 0$ if $\Re(a_3 + a_4) = 0$;

(x) $\Im(a_1 + a_2) \leq 0$ if $\Re(a_1 + a_2) = 1$.

Here we denote by $\Re(a)$ and $\Im(a)$ the real and imaginary parts of a complex number a, respectively. Therefore, to prove Theorem 2.1, it is sufficient to prove the following:

THEOREM 2.2. (i) *For* $\boldsymbol{a}_1 = (a_1, a_2, a_3, a_4) \in \mathbb{C}^4$ *such that* $a_1 = a_2$, *there exists a one-parameter family of classical solutions of* $S_\infty(\boldsymbol{a}_1)$. *It consists of the solutions of the form* $(0, P)$ *where* P *satisfies the Riccati equation*

$$(1) \qquad s(s-1)\frac{dP}{ds} = -sP^2 - [(a_3 - a_4)s - 2a_2]P - (a_2 - a_3)(a_2 + a_4).$$

(ii) *For* $\boldsymbol{a}_2 = (a_1, a_2, a_3, a_4) \in \mathbb{C}^4$ *such that* $a_2 = a_3$, *there exists a one-parameter family of classical solutions of* $S_0(\boldsymbol{a}_2)$. *It consists of the solutions of the form* $(q, 0)$ *where* q *satisfies the Riccati equation*

$$(2) \qquad t(t-1)\frac{dq}{dt} = (a_1 - a_3)q^2 + (2a_3 t - a_1 + a_4)q - (a_3 + a_4)t.$$

(iii) *For* $\boldsymbol{a}_3 = (a_1, a_2, a_3, a_4) \in \mathbb{C}^4$ *such that* $a_3 = a_4$, *there exists a one-parameter family of classical solutions of* $S_0(\boldsymbol{a}_3)$. *It consists of the solutions of the form* $(1, p)$ *where* p *satisfies the Riccati equation*

$$(3)\quad t(t-1)\frac{dp}{dt} = (t-1)p^2 + (-2a_3t - a_1 - a_2 + 2a_3)p - (a_3 - a_2)(a_3 - a_1).$$

(iv) *For* $\boldsymbol{a}_4 = (a_1, a_2, a_3, a_4) \in \mathbb{C}^4$ *such that* $a_3 = -a_4$, *there exists a one-parameter family of classical solutions of* $S_0(\boldsymbol{a}_4)$. *It consists of the solutions of the form* $(0, p)$ *where* p *satisfies the Riccati equation*

$$(4)\qquad t(t-1)\frac{dp}{dt} = -tp^2 - (2a_3t - a_1 - a_2)p - (a_3 - a_2)(a_3 - a_1).$$

(v) *For* $\boldsymbol{a}_5 = (a_1, a_2, a_3, a_4) \in \mathbb{C}^4$ *such that* $a_1 + a_2 = 1$, *there exists a one-parameter family of classical solutions of* $S_0(\boldsymbol{a}_5)$. *It consists of the solutions of the form* (t, p) *where* p *satisfies the Riccati equation*

$$(5)\quad t(t-1)\frac{dp}{dt} = -t(t-1)p^2 + ((2a_3-2)t + 1 - a_3 - a_4)p - (a_3 - a_2)(a_3 + a_2 - 1).$$

(vi) *For* $\boldsymbol{a} \in \Gamma$, *let* (q, p) *(or* (Q, P)*) be a transcendental solution of the system* $S_0(\boldsymbol{a})$ *(or* $S_\infty(\boldsymbol{a})$*) different from those in* (i)-(v). *Then none of the functions* q, p, Q, P *is classical, and the transcendence degree of* $\mathbb{C}(t, q, p)$ $(\mathbb{C}(s, Q, P))$ *over* $\mathbb{C}(t) = \mathbb{C}(s)$ *equals two.*

The assertions (i)-(v) are obvious. The proof of the assertion (vi) consists of the following two parts:

(a) to determine the condition of the parameter $\boldsymbol{a} \in \mathbb{C}^4$ where the Hamiltonian vector field $X(\boldsymbol{a})$ (which will be defined in Section 3) does not satisfy the algebraic condition (J) of Umemura (Proposition 3.1 and Corollary 3.7);

(b) to determine $X(\boldsymbol{a})$-invariant principal ideals for $\boldsymbol{a} \in \Gamma$ (Propositions 4.1-4.8).

Once these steps have been established, Theorem 2.2 follows immediately from Umemura's theory summarized in [10], Section 1 (see also [7], [8], [9]).

REMARK 2.2. The Riccati equations (1)-(5) come from the hypergeometric differential equation of Gauss. For example, in (5), if setting $p = (du/dt)/u$, then we have the equation for u

$$t(t-1)\frac{d^2u}{dt^2} - [(2a_3 - 2)t + 1 - a_3 - a_4]\frac{du}{dt} + (a_3 - a_2)(a_3 + a_2 - 1)u = 0.$$

Similarly, the equations (1), (3) and (4) are also transformed to the hypergeometric equation. In (2), we introduce a new variable $\bar{q}$ by $\bar{q} = (a_1 - a_3)q$. Then $\bar{q}$ satisfies the following Riccati equation

$$t(t-1)\frac{d\bar{q}}{dt} = \bar{q}^2 + (2a_3t - a_1 + a_4)\bar{q} - (a_3 + a_4)(a_1 - a_3)t,$$

which is also transformed to the hypergeometric equation.

3. – Necessary condition for the existence of invariant ideals

Let K be an ordinary differential overfield of $\mathbb{C}(t)$, and let $K[q, p]$ be the polynomial ring over K in two variables q and p. We consider the following derivation on $K[q, p]$:

$$\begin{aligned}X(\boldsymbol{a}) = t(t-1)\frac{\partial}{\partial t} &+ [2q^3p - 2(1+t)q^2p + 2tqp + (a_1 + a_2 - 2a_3)q^2 \\ &+ (2a_3t - a_1 - a_2 + a_3 + a_4)q - (a_3 + a_4)t]\frac{\partial}{\partial q} \\ &+ [-3q^2p^2 + 2(1+t)qp^2 - tp^2 - 2(a_1 + a_2 - 2a_3)qp \\ &- (2a_3t - a_1 - a_2 + a_3 + a_4)p - (a_2 - a_3)(a_1 - a_3)]\frac{\partial}{\partial p},\end{aligned}$$

where $\boldsymbol{a} = (a_1, a_2, a_3, a_4) \in \mathbb{C}^4$. We call this the Hamiltonian vector field of the system S_0 (Section 2). From now on we fix the vector $\boldsymbol{a} \in \mathbb{C}^4$. In [10], Section 1, Umemura introduced the condition (J) for $X(\boldsymbol{a})$:

(J) For any ordinary differential field extension $K/\mathbb{C}(t)$, there exists no principal ideal I of $K[q, p]$ such that $0 \subsetneq I \subsetneq K[q, p]$ and $X(\boldsymbol{a})I \subset I$.

Here we consider the following cases for the parameter $\boldsymbol{a} = (a_1, a_2, a_3, a_4) \in \mathbb{C}^4$:

Case 1.

$$\boldsymbol{a} = \left(\frac{1}{2}, \frac{1}{2}, 0, 0\right); \tag{1}$$

Case 2.

$$a_1 - a_2 \neq 0 \tag{2}$$

and

$$a_2 - a_3 = 0; \tag{3}$$

Case 3.

$$(a_1 - a_2)(a_1 - a_3)(a_2 - a_3) \neq 0. \tag{4}$$

All cases are exhausted in these three. In fact, assume $\boldsymbol{a} \neq (\frac{1}{2}, \frac{1}{2}, 0, 0)$. Then we have $a_1 - a_2 \neq 0$ or $a_3 - a_4 \neq 0$ or $a_3 + a_4 \neq 0$ or $a_1 + a_2 \neq 1$. By considering the sixth Painlevé equation on an appropriate open set among the four U_0, V_1, V_3 and V_4, we may assume $a_1 - a_2 \neq 0$. Since the system S_0 has a symmetry with respect to the transformation s_1 (Section 1), if $(a_2 - a_3)(a_1 - a_3) = 0$ and $a_1 - a_2 \neq 0$, we may assume $a_1 - a_3 \neq 0$. The Cases 1 and 2 will be treated in the next section. Here we assume the Case 3. We show the following crucial result:

PROPOSITION 3.1. *Assume the Case 3. If the derivation $X(\boldsymbol{a})$ does not satisfy the condition (J) for a vector $\boldsymbol{a} = (a_1, a_2, a_3, a_4) \in \mathbb{C}^4$ with the condition (4), then there exist non-negative integers a, b, i, j and k such that*

$$i + j + k \geq 1 \tag{5}$$

and

$$a(a_2 - a_3) + b(a_1 - a_3) + i(a_3 + a_4) + j(a_3 - a_4) + k(1 - a_1 - a_2) = 0. \tag{6}$$

PROOF. The proof is accomplished in seven steps.

Step 1. By hypothesis there exists an $X(\boldsymbol{a})$-invariant principal ideal I properly between the zero-ideal and $K[q, p]$. Let $F \in K[q, p]$ be a generator of I: $I = (F)$. Then we have

$$F \notin K \tag{7}$$

and

$$X(\boldsymbol{a})F = GF \tag{8}$$

with some $G \in K[q, p]$.

To investigate the equality (8), we introduce the following two gradings in the polynomial ring $K[q, p]$.

In the first grading we define the weight of a monomial $\gamma q^i p^j$ $(0 \neq \gamma \in K)$ in $K[q, p]$ as j. Let R_d be the K-linear subspace of $K[q, p]$ generated over K by all the monomials of weight d. We have $R_d = K[q] \cdot p^d$ for every non-negative integer d. Thus $K[q, p]$ becomes a graded ring: $K[q, p] = \oplus_{d \geq 0} R_d$, $R_d \cdot R_{d'} \subseteq R_{d+d'}$. We define three derivations X_i's $(i = -1, 0, 1)$ by

$$\begin{aligned}
X_1 &= 2q(q-1)(q-t)p\frac{\partial}{\partial q} + [-3q^2 + 2(1+t)q - t]p^2\frac{\partial}{\partial p}, \\
X_0 &= t(t-1)\frac{\partial}{\partial t} \\
&\quad + [(a_1 + a_2 - 2a_3)q^2 + (2a_3 t - a_1 - a_2 + a_3 + a_4)q - (a_3 + a_4)t]\frac{\partial}{\partial q} \\
&\quad - [2(a_1 + a_2 - 2a_3)q + 2a_3 t - a_1 - a_2 + a_3 + a_4]p\frac{\partial}{\partial p}, \\
X_{-1} &= -(a_2 - a_3)(a_1 - a_3)\frac{\partial}{\partial p}.
\end{aligned}$$

Then we see that $X(\boldsymbol{a}) = X_1 + X_0 + X_{-1}$ and that each X_i maps R_d to R_{d+i}.

In the second grading we define the weight of a monomial $\gamma q^i p^j$ $(0 \neq \gamma \in K)$ in $K[q, p]$ as $i - j$. Let R'_d be the K-linear subspace of $K[q, p]$ generated over K by all the monomials of weight d. We have $R'_d = K[qp] \cdot q^d$ and

$R'_d = K[qp] \cdot p^d$ for every non-negative integer d. Thus $K[q, p]$ has another grading structure: $K[q, p] = \oplus_{-\infty<d<\infty} R'_d$, $R'_d \cdot R'_{d'} \subseteq R'_{d+d'}$. We define three derivations X'_i's $(i = -1, 0, 1)$ by

$$\begin{aligned}
X'_1 &= [2qp + a_1 + a_2 - 2a_3]q^2 \frac{\partial}{\partial q} \\
&\quad - [3q^2p^2 + 2(a_1 + a_2 - 2a_3)qp + (a_2 - a_3)(a_1 - a_3)]\frac{\partial}{\partial p}, \\
X'_0 &= t(t-1)\frac{\partial}{\partial t} + [-2(1+t)qp + 2a_3t - a_1 - a_2 + a_3 + a_4]q\frac{\partial}{\partial q} \\
&\quad + [2(1+t)qp - (2a_3t - a_1 - a_2 + a_3 + a_4)]p\frac{\partial}{\partial p}, \\
X'_{-1} &= [2tqp - (a_3 + a_4)t]\frac{\partial}{\partial q} - tp^2\frac{\partial}{\partial p}.
\end{aligned}$$

Then we see that $X(\boldsymbol{a}) = X'_1 + X'_0 + X'_{-1}$ and that each X'_i maps R'_d to R'_{d+i}.

We determine the form of the polynomial G in (8). Since the highest part X_1 of $X(\boldsymbol{a})$ is of weight one with respect to the first grading, the polynomial G belongs to the direct sum $R_0 \oplus R_1$. Namely we have $G = g_1 p + g_0$ with some $g_1, g_0 \in K[q]$. Moreover, since the highest part X'_1 of $X(\boldsymbol{a})$ is also of weight one with respect to the second grading, the polynomial G belongs to the direct sum $R'_{-1} \oplus R'_0 \oplus R'_1$. Therefore we see that the polynomial g_1 is of degree at most two in q and the polynomial g_0 is of degree at most one in q. Namely we have

$$G = (\lambda q^2 + \mu q + \nu)p + \rho q + \sigma \tag{9}$$

with some $\lambda, \mu, \nu, \rho, \sigma \in K$.

We decompose the polynomial F with respect to the first grading of $K[q, p]$. Then there exist a non-negative integer m and a unique collection of $m + 1$ homogeneous polynomials $F_d \in R_d$ $(0 \le d \le m)$ such that

$$F = F_m + \cdots + F_0 \tag{10}$$

and

$$F_m \notin K. \tag{11}$$

We can surely assume the condition (11) because of (7). Substituting (9) and (10) into (8), we have

$$(X_1 + X_0 + X_{-1})(F_m + \cdots + F_0) = \{(\lambda q^2 + \mu q + \nu)p + \rho q + \sigma\}(F_m + \cdots + F_0).$$

Comparing the homogeneous parts of both sides of the equality, we have a system of $m + 3$ equations equivalent to (8):

$$(12)_d \qquad X_1 F_d = (\lambda q^2 + \mu q + \nu)pF_d + (\rho q + \sigma)F_{d+1} - X_0 F_{d+1} - X_{-1} F_{d+2}$$

where d is an integer such that $-2 \leq d \leq m$, and $F_{m+2} = F_{m+1} = F_{-1} = F_{-2} = 0$.

On the other hand, we decompose F with respect to the second grading of $K[q, p]$. Then there exist two integers n and n' with $n \geq n'$, and a unique collection of $n - n' + 1$ homogeneous polynomials $F'_d \in R'_d$ ($n' \leq d \leq n$) such that

$$F = F'_n + \cdots + F'_{n'} \tag{13}$$

and

$$F'_n F'_{n'} \neq 0. \tag{14}$$

Substituting (9) and (13) into (8), we have

$$(X'_1 + X'_0 + X'_{-1})(F'_n + \cdots + F'_{n'}) = \{(\lambda qp + \rho)q + (\mu qp + \sigma) + \nu p\}(F'_n + \cdots + F'_{n'}).$$

Comparing the homogeneous parts of both sides of the equality, we have a system of $n - n' + 3$ equations equivalent to (8):

$$(15)_d \quad X'_1 F'_d = (\lambda qp + \rho) q F'_d + (\mu qp + \sigma) F'_{d+1} + \nu p F'_{d+2} - X'_0 F'_{d+1} - X'_{-1} F'_{d+2}$$

where d is an integer such that $n' - 2 \leq d \leq n$, and $F'_{n+2} = F'_{n+1} = F'_{n'-1} = F'_{n'-2} = 0$.

Thus we have obtained the two systems $(12)_d$ and $(15)_d$ equivalent to (8). In the following we investigate them exclusively.

REMARK 3.1. The gradings above come from the Newton polygon of the derivation $X(\boldsymbol{a})$, which is described in the following figure:

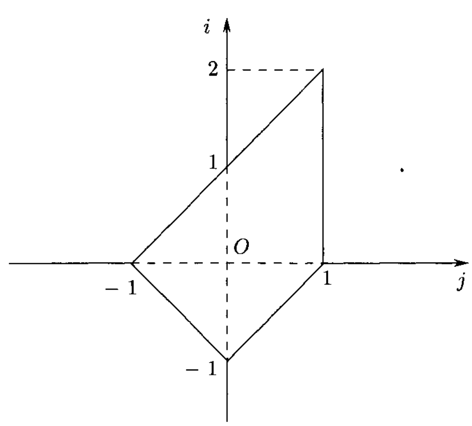

Fig. 1.

Here an integral point $(j,i) \neq (0,0)$ in $\mathbb{R}^2$ represents the derivation in $X(\boldsymbol{a})$ of the form $uq^{i+1}p^j(\partial/\partial q) + vq^ip^{j+1}(\partial/\partial p)$ $(u, v \in K)$; the point $(0,0)$ represents that of the form $t(t-1)(\partial/\partial t) + uq(\partial/\partial q) + vp(\partial/\partial p)$ $(u, v \in K)$ (cf. [10], [11], [12]).

Step 2. To investigate the systems $(12)_d$ and $(15)_d$ we need the lemmas below (Lemmas 3.2-3.6).

LEMMA 3.2. *Let d be a non-negative integer and e be a positive integer. Let A be a polynomial in R_d, and let λ', μ' and ν' be elements of K. If $\nu' + (d-2l+2)t \neq 0$ for every integer l such that $1 \leq l \leq e$ and if A satisfies a congruence*

$$X_1A \equiv (\lambda' q^2 + \mu' q + \nu')pA \mod q^e, \tag{16}$$

then $A \equiv 0 \mod q^e$.

LEMMA 3.3. *Let d, e, A, λ', μ', and ν' be as above. If $\lambda' + \mu' + \nu' + (d-2l+2)(1-t) \neq 0$ for every integer l such that $1 \leq l \leq e$, and if A satisfies a congruence*

$$X_1A \equiv (\lambda' q^2 + \mu' q + \nu')pA \mod (q-1)^e, \tag{17}$$

then $A \equiv 0 \mod (q-1)^e$.

LEMMA 3.4. *Let d, e, A, λ', μ', and ν' be as above. If $\lambda' t^2 + \mu' t + \nu' + (d-2l+2)t(t-1) \neq 0$ for every integer l such that $1 \leq l \leq e$, and if A satisfies a congruence*

$$X_1A \equiv (\lambda' q^2 + \mu' q + \nu')pA \mod (q-t)^e, \tag{18}$$

then $A \equiv 0 \mod (q-t)^e$.

PROOF OF LEMMA 3.2. We denote by $K[T]$ the polynomial ring in one variable T over K. Let φ_0 be the K-algebra morphism of $K[q,p]$ onto $K[T]$ defined by $\varphi_0(q) = 0$ and $\varphi_0(p) = (1-t)T$. Then the following diagram is commutative:

$$\begin{array}{ccc} K[q,p] & \xrightarrow{\varphi_0} & K[T] \\ {\scriptstyle X_1}\big\downarrow & & \big\downarrow{\scriptstyle t(t-1)T^2\frac{d}{dT}} \\ K[q,p] & \xrightarrow[\varphi_0]{} & K[T]. \end{array} \tag{19}$$

The kernel of the morphism φ_0 is a principal ideal generated by q. Since $X_1(q) = 2q(q-1)(q-t)p$, it is an X_1-invariant ideal. Now we show $A \equiv 0 \mod q^l$ by induction on l $(1 \leq l \leq e)$. We set $A = Bp^d$ with some $B \in R_0$. If we apply φ_0 to both sides of (16), we have

$$\varphi_0(X_1A) = \varphi_0(\lambda' q^2 + \mu' q + \nu')\varphi_0(pA).$$

This is equivalent to

$$t(t-1)T^2\frac{d}{dT}\varphi_0(A) = \nu'(1-t)T\varphi_0(A)$$

by the diagram (19). Since $\varphi_0(A) = \varphi_0(B)(1-t)^dT^d$, it follows that

$$-dt(1-t)^{d+1}\varphi_0(B)T^{d+1} = \nu'(1-t)^{d+1}\varphi_0(B)T^{d+1},$$

that is,

$$(\nu' + dt)\varphi_0(B) = 0.$$

Since $\nu' + dt \neq 0$ by hypothesis, we have $\varphi_0(B) = 0$ and hence $A \equiv 0 \mod q$. This proves the case $l = 1$. Assume that $A \equiv 0 \mod q^{l-1}$ for $l \geq 2$. We show $A \equiv 0 \mod q^l$. We set $A = Cq^{l-1}p^d$ with some $C \in R_0$. If we substitute this expression into (16) and divide both sides of the resulting congruence by q^{l-1}, then we obtain

$$X_1(Cp^d) \equiv [(\lambda'-2l+2)q^2 + \{\mu'+2(l-1)(1+t)\}q + \nu' - 2(l-1)t]pCp^d \mod q^{e-l+1}.$$

If we apply φ_0 to this congruence, we have

$$\{\nu' + dt - 2(l-1)t\}\varphi_0(C) = 0.$$

Since $\nu' + dt - 2(l-1)t \neq 0$ by hypothesis, we have $\varphi_0(C) = 0$ and hence $A \equiv 0 \mod q^l$. Thus Lemma 3.2 is proved.

PROOF OF LEMMA 3.3. Let φ_1 be the K-algebra morphism of $K[q,p]$ onto $K[T]$ defined by $\varphi_1(q) = 1$ and $\varphi_1(p) = tT$. Then the following diagram is commutative:

$$(20)\qquad \begin{array}{ccc} K[q,p] & \xrightarrow{\varphi_1} & K[T] \\ {\scriptstyle X_1}\downarrow & & \downarrow{\scriptstyle t(t-1)T^2\frac{d}{dT}} \\ K[q,p] & \xrightarrow[\varphi_1]{} & K[T]. \end{array}$$

The kernel of the morphism φ_1 is a principal ideal generated by $q-1$. Since $X_1(q-1) = 2q(q-1)(q-t)p$, it is an X_1-invariant ideal. By the same argument as in the proof of Lemma 3.2, we can show $A \equiv 0 \mod (q-1)^l$ by induction on l $(1 \leq l \leq e)$. So we omit the rest of the proof of Lemma 3.3.

PROOF OF LEMMA 3.4. Let φ_t be the K-algebra morphism of $K[q,p]$ onto $K[T]$ defined by $\varphi_t(q) = t$ and $\varphi_t(p) = -T$. Then the following diagram is commutative:

$$(21)\qquad \begin{array}{ccc} K[q,p] & \xrightarrow{\varphi_t} & K[T] \\ {\scriptstyle X_1}\downarrow & & \downarrow{\scriptstyle t(t-1)T^2\frac{d}{dT}} \\ K[q,p] & \xrightarrow[\varphi_t]{} & K[T]. \end{array}$$

The kernel of the morphism φ_t is a principal ideal generated by $q - t$. Since $X_1(q-t) = 2q(q-1)(q-t)p$, it is an X_1-invariant ideal. By the same argument as in the proof of Lemma 3.2, we can show $A \equiv 0 \mod (q-t)^l$ by induction on l $(1 \leq l \leq e)$. So we omit the rest of the proof of this lemma.

REMARK 3.2. The commutative diagrams (19), (20) and (21) are obtained in the following way (cf. [10]). Assume that there exists a homogeneous K-algebra morphism Φ such that the following diagram is commutative:

$$\begin{tikzcd} K[q,p] \arrow[r, "\Phi"] \arrow[d, "X_1"'] & K[T] \arrow[d, "T^2\frac{d}{dT}"] \\ K[q,p] \arrow[r, "\Phi"'] & K[T]. \end{tikzcd}$$

Here the polynomial ring $K[q,p]$ is regarded as the graded ring $K[q,p] = \oplus_{d\geq 0} R_d$, and $K[T]$ as the usual graded ring $K[T] = \oplus_{d \geq 0} K \cdot T^d$. Then we can set $\Phi(q) = \alpha$ and $\Phi(p) = \beta T$ with $\alpha, \beta \in K$ $(\beta \neq 0)$. By the commutative diagram, we have a system of algebraic equations:

$$\begin{cases} \alpha(\alpha-1)(\alpha-t)\beta = 0, \\ \{-3\alpha^2 + 2(1+t)\alpha - t\}\beta = 1. \end{cases}$$

All the solutions of this system are $(\alpha, \beta) = (0, -\frac{1}{t})$, $(1, \frac{1}{t-1})$ and $(t, -\frac{1}{t(t-1)})$, which define the expected morphisms φ_0, φ_1 and φ_t, respectively.

LEMMA 3.5. *Let d be an integer and e be a positive integer. Let A be a polynomial in R_d, and let λ' and ρ' be elements of K. Furthermore, let the derivation X_1', depending on the vector $\mathbf{a} = (a_1, a_2, a_3, a_4)$, be considered under the condition $(a_1 - a_2)(a_3 - a_2) \neq 0$. If $(a_3 - a_2)\lambda' + \rho' + (l - 1 - d)(a_1 - a_2) \neq 0$ for every integer l such that $1 \leq l \leq e$ and if A satisfies a congruence*

$$X_1'A \equiv (\lambda' qp + \rho')qA \mod (qp - a_3 + a_2)^e, \tag{22}$$

then $A \equiv 0 \mod (qp - a_3 + a_2)^e$.

LEMMA 3.6. *Let d, e, A, λ' and ρ' be as above, and let the derivation X_1' be considered under the condition $(a_2 - a_1)(a_3 - a_1) \neq 0$. If $(a_3 - a_1)\lambda' + \rho' + (l - 1 - d)(a_2 - a_1) \neq 0$ for every integer l such that $1 \leq l \leq e$ and if A satisfies a congruence*

$$X_1'A \equiv (\lambda' qp + \rho')qA \mod (qp - a_3 + a_1)^e, \tag{23}$$

then $A \equiv 0 \mod (qp - a_3 + a_1)^e$.

PROOF OF LEMMA 3.5. We denote by $K[T, T^{-1}]$ the Laurent polynomial ring in one variable T over K. Let ψ_+ be the K-algebra morphism of $K[q, p]$ onto $K[T, T^{-1}]$ defined by $\psi_+(q) = (a_1-a_2)^{-1}T$ and $\psi_+(p) = (a_3-a_2)(a_1-a_2)T^{-1}$. The definition is well-defined by the hypothesis $(a_1 - a_2)(a_3 - a_2) \neq 0$. Then the following diagram is commutative:

$$\begin{array}{ccc} K[q,p] & \xrightarrow{\psi_+} & K[T,T^{-1}] \\ {\scriptstyle X_1'}\downarrow & & \downarrow{\scriptstyle T^2\frac{d}{dT}} \\ K[q,p] & \xrightarrow[\psi_+]{} & K[T,T^{-1}]. \end{array} \tag{24}$$

The kernel of the morphism ψ_+ is a principal ideal generated by $qp - a_3 + a_2$. Since $X_1'(qp - a_3 + a_2) = -(qp - a_3 + a_2)(qp - a_3 + a_1)q$, it is X_1'-invariant ideal. We show $A \equiv 0 \mod (qp - a_3 + a_2)^l$ by induction on l $(1 \leq l \leq e)$. Here we need to consider two cases: (a) $d \geq 0$; (b) $d \leq 0$. Assume the case (a). Then we set $A = Bq^d$ with some $B \in R_0'$. If we apply ψ_+ to both sides of (22), we have

$$\psi_+(X_1'A) = \psi_+(\lambda' qp + \rho')\psi_+(qA).$$

This is equivalent to

$$T^2\frac{d}{dT}\psi_+(A) = \psi_+(\lambda' qp + \rho')\psi_+(qA)$$

by the commutative diagram (24). Since $\psi_+(A) = \psi_+(B)(a_1 - a_2)^{-d}T^d$, it follows that

$$[(a_3 - a_2)\lambda' + \rho' - d(a_1 - a_2)]\psi_+(B) = 0.$$

Since $(a_3 - a_2)\lambda' + \rho' - d(a_1 - a_2) \neq 0$ by hypothesis, we have $\psi_+(B) = 0$ and hence $A \equiv 0 \mod (qp - a_3 + a_2)$. This proves the case $l = 1$. Assume that $A \equiv 0 \mod (qp - a_3 + a_2)^{l-1}$ for $l \geq 2$. We show $A \equiv 0 \mod (qp - a_3 + a_2)^l$. We set $A = C(qp - a_3 + a_2)^{l-1}q^d$ with some $C \in R_0'$. If we substitute this expression into (22) and divide both sides of the resulting congruence by $(qp - a_3 + a_2)^{l-1}$, then we obtain

$$X_1'(Cq^d) \equiv [(\lambda' + l - 1)qp + \rho' - (a_3 - a_1)(l-1)]qCq^d \mod (qp - a_3 + a_2)^{e-l+1}.$$

If we apply ψ_+ to this congruence, then we have

$$[(a_3 - a_2)\lambda' + \rho' + (l - 1 - d)(a_1 - a_2)]\psi_+(C) = 0.$$

Since $(a_3 - a_2)\lambda' + \rho' + (l - 1 - d)(a_1 - a_2) \neq 0$ by hypothesis, we have $\psi_+(C) = 0$ and hence $A \equiv 0 \mod (qp - a_3 + a_2)^l$. In the case (b) (i.e. $d \leq 0$), the discussion is similar to the case (a) by setting $A = Bp^{-d}$ with $B \in R_0'$. So we omit the detail. Lemma 3.5 is proved.

Proof of Lemma 3.6. Let ψ_- be the K-algebra morphism of $K[q,p]$ onto $K[T,T^{-1}]$ defined by $\psi_-(q)=(a_1-a_2)^{-1}T$ and $\psi_-(p)=(a_3-a_1)(a_1-a_2)T^{-1}$. The definition is well-defined by the hypothesis $(a_2-a_1)(a_3-a_1)\neq 0$. Then the following diagram is commutative:

$$\begin{array}{ccc} K[q,p] & \xrightarrow{\psi_-} & K[T,T^{-1}] \\ {\scriptstyle X_1'}\big\downarrow & & \big\downarrow{\scriptstyle -T^2\frac{d}{dT}} \\ K[q,p] & \xrightarrow[\psi_-]{} & K[T,T^{-1}]. \end{array} \tag{25}$$

The kernel of the morphism ψ_- is a principal ideal generated by $qp-a_3+a_1$. Since $X_1'(qp-a_3+a_1)=-(qp-a_3+a_2)(qp-a_3+a_1)q$, it is X_1'-invariant ideal. The rest of the proof is similar to the corresponding part of the proof of Lemma 3.5. In fact, we can show $A\equiv 0 \mod (qp-a_3+a_1)^l$ by induction on l $(1\leq l\leq e)$. So we omit the detail.

Remark 3.3. The commutative diagrams (24) and (25) are also obtained by the same argument as in Remark 3.2. In fact we can determine all homogeneous K-algebra morphisms Ψ such that the following diagram is commutative:

$$\begin{array}{ccc} K[q,p] & \xrightarrow{\Psi} & K[T,T^{-1}] \\ {\scriptstyle X_1'}\big\downarrow & & \big\downarrow{\scriptstyle T^2\frac{d}{dT}} \\ K[q,p] & \xrightarrow[\Psi]{} & K[T,T^{-1}]. \end{array}$$

Here the polynomial ring $K[q,p]$ is regarded as the graded ring $K[q,p]=\oplus_{d\in\mathbb{Z}}R_d'$, and the Laurent polynomial ring $K[T,T^{-1}]$ as the usual graded ring $K[T,T^{-1}]=\oplus_{d\in\mathbb{Z}}K\cdot T^d$.

Step 3. We return to the proof of the proposition. Consider the equation

$$X_1F_m=(\lambda q^2+\mu q+\nu)pF_m. \tag*{$(12)_m$}$$

Because of (11), it follows from Lemmas 3.2-3.4 that the three elements of K

$$i=\frac{1}{2}(\nu t^{-1}+m),$$

$$j=\frac{1}{2}\{(\lambda+\mu+\nu)(1-t)^{-1}+m\}$$

and

$$k=\frac{1}{2}\{(\lambda t^2+\mu t+\nu)t^{-1}(t-1)^{-1}+m\}$$

are non-negative integers. From these we have

$$\lambda = 2i + 2j + 2k - 3m, \tag{26}$$

$$\mu = 2m - 2i - 2k - t(2i + 2j - 2m), \tag{27}$$

and

$$\nu = (2i - m)t. \tag{28}$$

Using Lemmas 3.2-3.4 again, we see that there exists a non-zero element $c \in R_0 = K[q]$ such that

$$F_m = cq^i(q-1)^j(q-t)^k p^m. \tag{29}$$

Substituting (29) into $(12)_m$, we have $X_1 c = 0$, and therefore $c \in K$. By (11) we also see

$$i + j + k + m \geq 1. \tag{30}$$

Step 4. We first show that the integer n in (13) is a non-negative integer. In fact, otherwise, since $n' \leq n \leq -1$, the polynomial $F = F'_n + \cdots F'_{n'}$ is divisible by the monomial p^{-n}. Namely there exists a polynomial F' such that $p \nmid F'$ and $F = F'p^{-n}$. Substituting $F = F'p^{-n}$ into the equality (8), we have

$$(X(\boldsymbol{a})F')p^{-n} - nF'p^{-n-1}X(\boldsymbol{a})p = GF'p^{-n}.$$

Since $p \nmid F'$, it follows that the polynomial $X(\boldsymbol{a})p$ is divisible by p, and so $(a_2 - a_3)(a_1 - a_3) = 0$. This contradicts the hypothesis (4), and therefore we have $n \geq 0$.

Consider the equation

$$X'_1 F'_n = (\lambda qp + \rho)qF'_n. \tag{$15)_n$}$$

Because of (11), it follows from Lemmas 3.5 and 3.6 that the elements

$$a = -(a_1 - a_2)^{-1}(a_3 - a_2)\lambda - (a_1 - a_2)^{-1}\rho + n$$

and

$$b = (a_1 - a_2)^{-1}(a_3 - a_1)\lambda + (a_1 - a_2)^{-1}\rho + n$$

are non-negative integers. From these we have

$$\lambda = 2n - a - b \tag{31}$$

and

$$\rho = (a - n)(a_3 - a_1) + (b - n)(a_3 - a_2). \tag{32}$$

Using Lemmas 3.5 and 3.6 again, we see that there exists a non-zero element $c' \in R'_0 = K[qp]$ such that

$$F'_n = c'(qp - a_3 + a_2)^a (qp - a_3 + a_1)^b q^n. \tag{33}$$

Substituting (33) into $(15)_n$, we have $X'_1 c' = 0$. Since c' is a polynomial in qp over K and $X'_1(qp) \neq 0$, we have $c' \in K$.

Step 5. By the same argument as in [10], Subsection 2.5, we find the generic figure of the Newton polygon of the polynomial F:

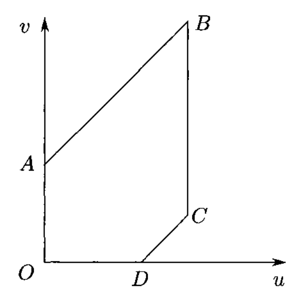

Fig. 2.

Here an integral point (u, v) in $\mathbb{R}^2$ represents a monomial $\gamma q^v p^u$ ($\gamma \in K$). In the figure the Cartesian coordinates of the vertices O, A, B, C, D are $(0, 0)$, $(0, n)$, $(a+b, a+b+n) = (m, i+j+k)$, (m, i), $(m-i, 0)$, respectively. The coefficient of each monomial out of the pentagon $OABCD$ is equal to zero. The side AB represents the polynomial F'_n. The side BC represents the polynomial F_m. It is also easy to see that the side CD represents the polynomial $c''(qp - a_3 - a_4)^i p^{m-i}$ ($c'' \in K$). Since the monomials $cq^{i+j+k}p^m$ in F_m and $c'q^{a+b+n}p^{a+b}$ in F'_n represent the same vertex B, we have the equalities

$$i + j + k = a + b + n, \tag{34}$$

$$m = a + b, \tag{35}$$

$$c = c' \ (\neq 0). \tag{36}$$

Similarly we have the equality at the vertex C:

$$c'' = (-1)^j (-t)^k c. \tag{37}$$

In particular we see that the two expressions of λ, (26) and (31), are equal to each other via (34) and (35). A polynomial $c^{-1}F$ is $X(\boldsymbol{a})$-invariant and generates the ideal $I = (F)$ introduced at the beginning of Step 1. And so we may assume $c = 1$. It follows from (29), (33) and (36) that

$$F_m = q^i(q-1)^j(q-t)^k p^m, \tag{38}$$

and

$$F'_n = (qp - a_3 + a_2)^a(qp - a_3 + a_1)^b q^n. \tag{39}$$

From (30), (34), and (35), we have

$$i + j + k = a + b + n \geq 1. \tag{40}$$

This is the desired relation (5).

Step 6. Consider the equation

$$X_1 F_{m-1} = (\lambda q^2 + \mu q + \nu)pF_{m-1} + (\rho q + \sigma)F_m - X_0 F_m. \tag*{$(12)_{m-1}$}$$

Substituting (38) into $(12)_{m-1}$, we have

$$\begin{aligned} X_1 F_{m-1} =&(\lambda q^2 + \mu q + \nu)pF_{m-1} \\ &+ (\rho q + \sigma)q^i(q-1)^j(q-t)^k p^m \\ &+ i(a_3 + a_4)tq^{i-1}(q-1)^j(q-t)^k p^m \\ &+ j(a_3 - a_4)(1-t)q^i(q-1)^{j-1}(q-t)^k p^m \\ &+ k(1 - a_1 - a_2)t(t-1)q^i(q-1)^j(q-t)^{k-1}p^m \\ &+ \{(i + j + k - 2m)(2a_3 - a_1 - a_2)q \\ &+ 2(m - i - j)a_3 t - k(a_1 + a_2)t \\ &+ (i + k - m)(a_1 + a_2 - a_3 - a_4) \\ &+ j(a_3 - a_4)\}q^i(q-1)^j(q-t)^k p^m, \end{aligned} \tag{41}$$

where λ, μ, ν and ρ are given by (26), (27), (28), (31) and (32). We assume $m \geq 1$ in this step, and treat the remaining case $m = 0$ in Step 7. Since X_1 is a derivation, we have

$$\begin{aligned} X_1(q(q-1)(q-t)F_{m-1}) = 2[3q^2 - 2(1+t)q + t]q(q-1)(q-t)pF_{m-1} \\ + q(q-1)(q-t)X_1 F_{m-1}. \end{aligned} \tag{42}$$

Eliminating X_1F_{m-1} from (41) and (42), we have

$$
\begin{aligned}
&X_1(q(q-1)(q-t)F_{m-1})\\
=&[(\lambda+6)q^2+(\mu-4-4t)q+\nu+2t]p\cdot q(q-1)(q-t)F_{m-1}\\
&+(\rho q+\sigma)q^{i+1}(q-1)^{j+1}(q-t)^{k+1}p^m\\
&+i(a_3+a_4)tq^i(q-1)^{j+1}(q-t)^{k+1}p^m\\
&+j(a_3-a_4)(1-t)q^{i+1}(q-1)^j(q-t)^{k+1}p^m\\
&+k(1-a_1-a_2)t(t-1)q^{i+1}(q-1)^{j+1}(q-t)^kp^m\\
&+\{(i+j+k-2m)(2a_3-a_1-a_2)q\\
&+2(m-i-j)a_3t-k(a_1+a_2)t+(i+k-m)(a_1+a_2-a_3-a_4)\\
&+j(a_3-a_4)\}q^{i+1}(q-1)^{j+1}(q-t)^{k+1}p^m.
\end{aligned}
\tag{43}
$$

Applying Lemmas 3.2-3.4 to the homogeneous polynomial $A=q(q-1)(q-t)F_{m-1}$, we see that there exists an element $B\in R_0=K[q]$ such that

$$
q(q-1)(q-t)F_{m-1}=Bq^i(q-1)^j(q-t)^kp^{m-1}. \tag{44}
$$

Substituting (44) into (43) and dividing the resulting equation by $q^i(q-1)^j(q-t)^kp^m$, we have an equation for B:

$$
\begin{aligned}
L(B)=&(\rho q+\sigma)q(q-1)(q-t)+i(a_3+a_4)t(q-1)(q-t)\\
&+j(a_3-a_4)(1-t)q(q-t)+k(1-a_1-a_2)t(t-1)q(q-1)\\
&+\{(i+j+k-2m)(2a_3-a_1-a_2)q+2(m-i-j)a_3t-k(a_1+a_2)t\\
&+(i+k-m)(a_1+a_2-a_3-a_4)+j(a_3-a_4)\}q(q-1)(q-t).
\end{aligned}
\tag{45}
$$

Here we set $L(B)=p^{-1}X_1(B)-[3q^2-2(1+t)q+t]B=2q(q-1)(q-t)(\partial B/\partial q)-[3q^2-2(1+t)q+t]B$. The operator L defines a K-linear endomorphism of the K-linear space R_0. We have the following formulae:

$$
L(1)=-3q^2+2(1+t)q-t, \tag{46}
$$

$$
L(q)=-q^3+tq, \tag{47}
$$

$$
L(q^2)=q^4-2(1+t)q^3+3tq^2. \tag{48}
$$

Moreover, if A is a polynomial in R_0 of degree $d\geq 3$, then $L(A)$ is a polynomial in R_0 of degree $d+2$. Let V_0 be the three-dimensional K-linear subspace of R_0 generated by the three monomials 1, q, q^2, i.e., the K-linear subspace of R_0 consisting of all the polynomials in q of degree at most two. Let V_1 be the three-dimensional K-linear subspace of R_0 generated by the three polynomials $-3q^2+2(1+t)q-t$, $-q^3+tq$, $q^4-2(1+t)q^3+3tq^2$. By (46), (47) and (48), we see that the restriction of L to V_0 induces a K-linear isomorphism of V_0 onto V_1. Thus, if there exists a solution B of the equation (45), then B must be a polynomial in q of degree at most two and the right hand side of (45) must belong to the K-linear space V_1. If we set

$$B = x + yq + zq^2 \ (x, y, z \in K), \tag{49}$$

then we have

$$L(B) = zq^4 - \{2(1+t)z + y\}q^3 + 3(tz - x)q^2 + \{ty + 2(1+t)x\}q - tx. \tag{50}$$

Substituting (50) into (45), we have the following:

$$\begin{aligned}
zq^4 &- \{2(1+t)z + y\}q^3 + 3(tz - x)q^2 + \{ty + 2(1+t)x\}q - tx \\
&= \{\rho + (i + j + k - 2m)(2a_3 - a_1 - a_2)\}q^4 \\
&\quad + [-(1+t)\rho + \sigma + \{-(i+j)(4a_3 - a_1 - a_2) \\
&\quad + 2m(3a_3 - a_1 - a_2) - 2ka_3\}t + (i+k)(2a_1 + 2a_2 - 3a_3 - a_4) \\
&\quad + j(a_1 + a_2 - a_3 - a_4) + m(-3a_1 - 3a_2 + 5a_3 + a_4)]q^3 \\
&\quad + [\rho t - (1+t)\sigma + \{k + 2(i + j - m)a_3\}t^2 \\
&\quad + \{2i(-a_1 - a_2 + 3a_3 + a_4) \\
&\quad + j(-a_1 - a_2 + 2a_3 + 2a_4) + k(3a_3 + a_4 - 1) \\
&\quad + m(3a_1 + 3a_2 - 7a_3 - a_4)\}t \\
&\quad + (m - i - k)(a_1 + a_2 - a_3 - a_4)]q^2 \\
&\quad + [\sigma t + \{-i(3a_3 + a_4) - j(a_3 + a_4) - k + 2ma_3\}t^2 \\
&\quad + \{i(a_1 + a_2 - 2a_3 - 2a_4) + k(1 - a_3 - a_4) \\
&\quad - m(a_1 + a_2 - a_3 - a_4)\}t]q + i(a_3 + a_4)t^2.
\end{aligned} \tag{51}$$

Comparing the coefficients of powers of q in both sides of (51), we have the following system of equations (52)-(56) for x, y, z, ρ, σ:

$$z - \rho = (i + j + k - 2m)(2a_3 - a_1 - a_2), \tag{52}$$

$$\begin{aligned}
-2(1+t)z - y + (1+t)\rho - \sigma \\
&= \{-(i+j)(4a_3 - a_1 - a_2) + 2m(3a_3 - a_1 - a_2) - 2ka_3\}t \\
&\quad + (i+k)(2a_1 + 2a_2 - 3a_3 - a_4) + j(a_1 + a_2 - a_3 - a_4) \\
&\quad + m(-3a_1 - 3a_2 + 5a_3 + a_4),
\end{aligned} \tag{53}$$

$$\begin{aligned}
3tz - 3x - \rho t + (1+t)\sigma \\
&= \{k + 2(i + j - m)a_3\}t^2 + \{2i(-a_1 - a_2 + 3a_3 + a_4) \\
&\quad + j(-a_1 - a_2 + 2a_3 + 2a_4) + k(3a_3 + a_4 - 1) \\
&\quad + m(3a_1 + 3a_2 - 7a_3 - a_4)\}t \\
&\quad + (m - i - k)(a_1 + a_2 - a_3 - a_4),
\end{aligned} \tag{54}$$

$$\begin{aligned}
ty + 2(1+t)x - \sigma t \\
&= \{-i(3a_3 + a_4) - j(a_3 + a_4) - k + 2ma_3\}t^2 \\
&\quad + \{i(a_1 + a_2 - 2a_3 - 2a_4) + k(1 - a_3 - a_4) \\
&\quad - m(a_1 + a_2 - a_3 - a_4)\}t,
\end{aligned} \tag{55}$$

$$(56)\qquad -tx = i(a_3+a_4)t^2.$$

From them, we have

$$(57)\qquad x = -i(a_3+a_4)t,$$

$$(58)\qquad y = \{i(a_3+a_4)+j(a_3-a_4)\}t + i(a_3+a_4)+k(1-a_1-a_2),$$

$$(59)\qquad z = -i(a_3+a_4)-j(a_3-a_4)-k(1-a_1-a_2),$$

$$(60)\qquad \begin{aligned}\rho &= i(a_1+a_2-3a_3-a_4)+j(a_1+a_2-3a_3+a_4)\\ &\quad + k(2a_1+2a_2-2a_3-1)-2m(a_1+a_2-2a_3),\end{aligned}$$

$$(61)\qquad \begin{aligned}\sigma &= (2ia_3+2ja_3+k-2ma_3)t - i(a_1+a_2-a_3-a_4)\\ &\quad - k(a_1+a_2-a_3-a_4)+m(a_1+a_2-a_3-a_4).\end{aligned}$$

Then the two expressions of ρ, (32) and (60), must coincide with each other:

$$\begin{aligned}\rho &= (a-n)(a_3-a_1)+(b-n)(a_3-a_2)\\ &= i(a_1+a_2-3a_3-a_4)+j(a_1+a_2-3a_3+a_4)\\ &\quad + k(2a_1+2a_2-2a_3-1)-2m(a_1+a_2-2a_3).\end{aligned}$$

By substituting (34) and (35) into this equality, we have

$$a(a_2-a_3)+b(a_1-a_3)+i(a_3+a_4)+j(a_3-a_4)+k(1-a_1-a_2)=0.$$

Thus we have the desired condition (6). Finally, we determine the form of F_{m-1}. Substituting (57), (58) and (59) into (49), we have

$$(62)\qquad \begin{aligned}B &= -i(a_3+a_4)t\\ &\quad + [\{i(a_3+a_4)+j(a_3-a_4)\}t + i(a_3+a_4)+k(1-a_1-a_2)]q\\ &\quad - \{i(a_3+a_4)+j(a_3-a_4)+k(1-a_1-a_2)\}q^2.\end{aligned}$$

From (44) and (62), we have

$$(63)\qquad \begin{aligned}F_{m-1} &= -i(a_3+a_4)tq^{i-1}(q-1)^{j-1}(q-t)^{k-1}p^{m-1}\\ &\quad + [\{i(a_3+a_4)+j(a_3-a_4)\}t\\ &\quad + i(a_3+a_4)+k(1-a_1-a_2)]q^i(q-1)^{j-1}(q-t)^{k-1}p^{m-1}\\ &\quad - \{i(a_3+a_4)+j(a_3-a_4)\\ &\quad + k(1-a_1-a_2)\}q^{i+1}(q-1)^{j-1}(q-t)^{k-1}p^{m-1}.\end{aligned}$$

Step 7. Here we consider the case $m=0$. From (35), (34) and (32), we have $a=b=0$, $i+j+k=n$ and $\rho=(i+j+k)(a_1+a_2-2a_3)$. Thus the equality (41) is turned to

$$\begin{aligned}0 &= \{\sigma - 2(i+j)a_3t - k(a_1+a_2)t + (i+k)(a_1+a_2-a_3-a_4)\\ &\quad + j(a_3-a_4)\}q(q-1)(q-t)\\ &\quad + i(a_3+a_4)t(q-1)(q-t)+j(a_3-a_4)(1-t)q(q-t)\\ &\quad + k(1-a_1-a_2)t(t-1)q(q-1).\end{aligned}$$

Then we have

$$i(a_3 + a_4) = j(a_3 - a_4) = k(1 - a_1 - a_2) = 0 \tag{64}$$

and

$$\sigma = 2(i + j)a_3 t + k(a_1 + a_2)t - (i + k)(a_1 + a_2 - a_3 - a_4) - j(a_3 - a_4). \tag{65}$$

The relation (64) with $a = b = 0$ is a special case of the relation (6), and (65) is a special case of (61). Thus Proposition 3.1 is completely proved.

COROLLARY 3.7. *Let the notation be as in Sections* 1-2. *The vector* $\boldsymbol{a}$ *in Proposition* 3.1 *does not belong to the set* $(\Gamma \cup s_1(\Gamma)) - M$.

PROOF. It is sufficient to prove that, for arbitrary non-negative integers a, b, i, j and k such that $a + b + i + j + k \geq 1$, the complex hyperplane

$$a(a_2 - a_3) + b(a_1 - a_3) + i(a_3 + a_4) + j(a_3 - a_4) + k(1 - a_1 - a_2) = 0 \tag{66}$$

does not intersect $(\Gamma \cup s_1(\Gamma)) - M$. Assume the contrary. Then there exist non-negative integers a, b, i, j, k and a vector $\boldsymbol{a} = (a_1, a_2, a_3, a_4) \in (\Gamma \cup s_1(\Gamma)) - M$ such that $a + b + i + j + k \geq 1$ and the relation (66) holds. Since the relation (66) has a symmetry with respect to the transformation s_1 (Section 1), we may assume that $\boldsymbol{a} \in \Gamma - M$. The relation (66) is equivalent to the following two relations:

$$a\Re(a_2 - a_3) + b\Re(a_1 - a_3) + i\Re(a_3 + a_4) + j\Re(a_3 - a_4) + k\Re(1 - a_1 - a_2) = 0 \tag{67}$$

and

$$a\Im(a_2 - a_3) + b\Im(a_1 - a_3) + i\Im(a_3 + a_4) + j\Im(a_3 - a_4) + k\Im(-a_1 - a_2) = 0. \tag{68}$$

The rest of the proof is an analogy of the proof of [12], Corollary 2.6. So we omit the detail.

4. – Determination of invariant ideals

Corollary 3.7 leads us to determine all the non-trivial $X(\boldsymbol{a})$-invariant principal ideals of $K[q, p]$ for $\boldsymbol{a} \in \Gamma \cap M$. This includes the consideration for the Cases 1 and 2 at the beginning of Section 3. We first prove:

PROPOSITION 4.1. (i) *Let $\boldsymbol{a}_1$ be a vector in $\Gamma \cap \{\boldsymbol{a} \in \mathbb{C}^4 \mid a_3 + a_4 = 0\}$ and not in P (Section 2). For every positive integer i, a principal ideal (q^i) is $X(\boldsymbol{a}_1)$-invariant. Conversely, if I is an $X(\boldsymbol{a}_1)$-invariant principal ideal properly between the zero-ideal and $K[q, p]$, then there exists a positive integer i such that $I = (q^i)$.*

(ii) *Let $\boldsymbol{a}_2$ be a vector in $\Gamma \cap \{\boldsymbol{a} \in \mathbb{C}^4 \mid a_3 - a_4 = 0\}$ and not in P. For every positive integer j, a principal ideal $((q-1)^j)$ is $X(\boldsymbol{a}_2)$-invariant. Conversely, if I is an $X(\boldsymbol{a}_2)$-invariant principal ideal properly between the zero-ideal and $K[q, p]$, then there exists a positive integer j such that $I = ((q-1)^j)$.*

(iii) *Let $\boldsymbol{a}_3$ be a vector in $\Gamma \cap \{\boldsymbol{a} \in \mathbb{C}^4 \mid a_1 + a_2 = 1\}$ and not in P. For every positive integer k, a principal ideal $((q-t)^k)$ is $X(\boldsymbol{a}_3)$-invariant. Conversely, if I is an $X(\boldsymbol{a}_3)$-invariant principal ideal properly between the zero-ideal and $K[q, p]$, then there exists a positive integer k such that $I = ((q-t)^k)$.*

PROOF. Since the three assertions are proved in the same way, we prove only the assertion (i) and omit the proofs of the others. The first half of (i) is obvious. We show the second half. To this end, the notation being as in Proposition 3.1, it is sufficient to prove that the $X(\boldsymbol{a}_1)$-invariant polynomial F is equal to q^i with some positive integer i. We set $\boldsymbol{a}_1 = (a_1, a_2, a_3, a_4)$. By hypothesis the vector $\boldsymbol{a}_1$ satisfies the condition (4) in Section 3. Since $a_3 + a_4 = 0$, it follows from (6) in Proposition 3.1 that

$$a(a_2 - a_3) + b(a_1 - a_3) + j(a_3 - a_4) + k(1 - a_1 - a_2) = 0,$$

which is equivalent to the following pair of equalities:

$$\begin{cases} a\Re(a_2 - a_3) + b\Re(a_1 - a_3) + j\Re(a_3 - a_4) + k\Re(1 - a_1 - a_2) = 0, \\ a\Im(a_2 - a_3) + b\Im(a_1 - a_3) + j\Im(a_3 - a_4) + k\Im(-a_1 - a_2) = 0. \end{cases}$$

Since $\boldsymbol{a}_1 \in \Gamma \cap \{\boldsymbol{a} \in \mathbb{C}^4 \mid a_3 + a_4 = 0\}$ and $\boldsymbol{a}_1 \notin P$, we have $a = b = j = k = 0$ by the same argument as in the proof of Corollary 3.7. Thus we have $i \geq 1$ from (5) in Section 3, and $m = 0$ from (35) in Section 3. Therefore it follows from (10) and (38) in Section 3 that $F = F_0 = q^i$.

REMARK 4.1. For $\bar{\boldsymbol{a}} \in \Gamma \cap \{\boldsymbol{a} \in \mathbb{C}^4 \mid a_1 - a_2 = 0\}$ and $\bar{\boldsymbol{a}} \notin P$, there exists no non-trivial $X(\bar{\boldsymbol{a}})$-invariant principal ideal. However, taking $\bar{X}(\bar{\boldsymbol{a}})$ for the Hamiltonian vector field of the sixth Painlevé equation defined on V_3 (Section 2), we find a $\bar{X}(\bar{\boldsymbol{a}})$-invariant principal ideal (Q_3^N) with a positive integer N of $K[Q_3, P_3]$. This implies that the system $S_0(\bar{\boldsymbol{a}})$ has a unique one-parameter family of classical solutions defined by $q = q_0 = \infty$.

PROPOSITION 4.2. (i) *Let $\boldsymbol{a}_4$ be a vector in $\Gamma \cap \{\boldsymbol{a} \in \mathbb{C}^4 \mid a_3 - a_4 = 1 - a_1 - a_2 = 0\}$ and not in L (Section 2). For arbitrary non-negative integers j and k such that $j + k \geq 1$, a principal ideal $((q-1)^j(q-t)^k)$ is $X(\boldsymbol{a}_4)$-invariant. Conversely, if I is an $X(\boldsymbol{a}_4)$-invariant principal ideal properly between the zero-ideal and $K[q, p]$, then there exist non-negative integers j and k such that $j + k \geq 1$ and $I = ((q-1)^j(q-t)^k)$.*

(ii) *Let $\boldsymbol{a}_5$ be a vector in $\Gamma \cap \{\boldsymbol{a} \in \mathbb{C}^4 \mid a_3 + a_4 = 1 - a_1 - a_2 = 0\}$ and not in L. For arbitrary non-negative integers i and k such that $i + k \geq 1$, a principal ideal $(q^i(q-t)^k)$ is $X(\boldsymbol{a}_5)$-invariant. Conversely, if I is an $X(\boldsymbol{a}_5)$-invariant principal ideal properly between the zero-ideal and $K[q,p]$, then there exist non-negative integers i and k such that $i + k \geq 1$ and $I = (q^i(q-t)^k)$.*

(iii) *Let $\boldsymbol{a}_6$ be a vector in $\Gamma \cap \{\boldsymbol{a} \in \mathbb{C}^4 \mid a_3 + a_4 = a_3 - a_4 = 0\}$ and not in L. For arbitrary non-negative integers i and j such that $i + j \geq 1$, a principal ideal $(q^i(q-1)^j)$ is $X(\boldsymbol{a}_6)$-invariant. Conversely, if I is an $X(\boldsymbol{a}_6)$-invariant principal ideal properly between the zero-ideal and $K[q,p]$, then there exist non-negative integers i and j such that $i + j \geq 1$ and $I = (q^i(q-1)^j)$.*

PROOF. Since the three assertions are proved in the same way, we prove only the assertion (i) and omit the proofs of the others. The first half of (i) is obvious. We show the second half. To this end, the notation being as in Proposition 3.1, it is sufficient to prove that the $X(\boldsymbol{a}_4)$-invariant polynomial F is equal to $(q-1)^j(q-t)^k$ with some non-negative integers j and k such that $j + k \geq 1$. We set $\boldsymbol{a}_4 = (a_1, a_2, a_3, a_4)$. By hypothesis the vector $\boldsymbol{a}_4$ satisfies the condition (4) in Section 3. Since $a_3 - a_4 = 1 - a_1 - a_2 = 0$, it follows from (6) in Proposition 3.1 that

$$a(a_2 - a_3) + b(a_1 - a_3) + i(a_3 + a_4) = 0,$$

which is equivalent to the following pair of equalities:

$$\begin{cases} a\Re(a_2 - a_3) + b\Re(a_1 - a_3) + i\Re(a_3 + a_4) = 0, \\ a\Im(a_2 - a_3) + b\Im(a_1 - a_3) + i\Im(a_3 + a_4) = 0. \end{cases}$$

Since $\boldsymbol{a}_4 \in \Gamma \cap \{\boldsymbol{a} \in \mathbb{C}^4 \mid a_3 - a_4 = 1 - a_1 - a_2 = 0\}$ and $\boldsymbol{a}_4 \notin L$, we have $a = b = i = 0$ by the same argument as in the proof of Corollary 3.7. Thus we have $j + k \geq 1$ from (5) in Section 3, and $m = 0$ from (35) in Section 3. Therefore it follows from (10) and (38) in Section 3 that $F = F_0 = (q-1)^j(q-t)^k$.

REMARK 4.2. For $\bar{\boldsymbol{a}} \in \Gamma \cap \{\boldsymbol{a} \in \mathbb{C}^4 \mid a_1 - a_2 = 1 - a_1 - a_2 = 0\}$, we find a $\bar{X}(\bar{\boldsymbol{a}})$-invariant principal ideal $(Q_3^M(Q_3 - 1)^N)$with some non-negative integers M and N such that $M + N \geq 1$ of $K[Q_3, P_3]$. By the same argument we can find all the invariant ideals on U_0 or V_1 or V_3 or V_4 for $\boldsymbol{a} \in \Gamma \cap P_1$ but $\boldsymbol{a} \notin L$.

PROPOSITION 4.3. *Let $\boldsymbol{a}_7$ be a vector in $\Gamma \cap \{\boldsymbol{a} \in \mathbb{C}^4 \mid a_3 + a_4 = a_3 - a_4 = 1 - a_1 - a_2 = 0\}$ and not in D (Section 2). For arbitrary non-negative integers i, j and k such that $i + j + k \geq 1$, a principal ideal $(q^i(q-1)^j(q-t)^k)$ is $X(\boldsymbol{a}_7)$-invariant. Conversely, if I is an $X(\boldsymbol{a}_7)$-invariant principal ideal properly between the zero-ideal and $K[q,p]$, then there exist non-negative integers i, j and k such that $i + j + k \geq 1$ and $I = (q^i(q-1)^j(q-t)^k)$.*

PROOF. The first half is obvious. We show the second half. To this end, the notation being as in Proposition 3.1, it is sufficient to prove that the $X(\boldsymbol{a}_7)$-invariant polynomial F is equal to $q^i(q-1)^j(q-t)^k$ with some non-negative integers i, j and k such that $i+j+k \geq 1$. We set $\boldsymbol{a}_7 = (a_1, a_2, a_3, a_4)$. By hypothesis the vector $\boldsymbol{a}_7$ satisfies the condition (4) in Section 3. Since $a_3 + a_4 = a_3 - a_4 = 1 - a_1 - a_2 = 0$, it follows from (6) in Proposition 3.1 that

$$a(a_2 - a_3) + b(a_1 - a_3) = 0,$$

which is equivalent to the following pair of equalities:

$$\begin{cases} a\Re(a_2 - a_3) + b\Re(a_1 - a_3) = 0, \\ a\Im(a_2 - a_3) + b\Im(a_1 - a_3) = 0. \end{cases}$$

Since $\boldsymbol{a}_7 \in \Gamma \cap \{\boldsymbol{a} \in \mathbb{C}^4 \mid a_3 + a_4 = a_3 - a_4 = 1 - a_1 - a_2 = 0\}$ and $\boldsymbol{a}_7 \notin D$, we have $a = b = 0$ by the same argument as in the proof of Corollary 3.7. Thus we have $m = 0$ from (35) in Section 3. Therefore it follows from (10) and (38) in Section 3 that $F = F_0 = q^i(q-1)^j(q-t)^k$ with (5) in Section 3.

REMARK 4.3. By the same argument as in Remark 4.2, we can determine all the invariant ideals on U_0 or V_1 or V_3 or V_4 for $\boldsymbol{a} \in \Gamma \cap L_1$ but $\boldsymbol{a} \notin D$.

The rest of this section concerns the consideration of the remaining Cases 1 and 2 at the beginning of Section 3. We first prove:

PROPOSITION 4.4. *Let $\boldsymbol{a}_8$ be a vector in $\Gamma \cap \{\boldsymbol{a} \in \mathbb{C}^4 \mid a_2 - a_3 = 0\}$ and not in P. For every positive integer m, a principal ideal (p^m) is $X(\boldsymbol{a}_8)$-invariant. Conversely, if I is an $X(\boldsymbol{a}_8)$-invariant principal ideal properly between the zero-ideal and $K[q, p]$, then there exists a positive integer m such that $I = (p^m)$.*

PROOF. The first half is obvious. We show the second half. To this end, the notation being as in Proposition 3.1, it is sufficient to prove that the $X(\boldsymbol{a}_8)$-invariant polynomial F is equal to p^m with some positive integer m. We set $\boldsymbol{a}_8 = (a_1, a_2, a_3, a_4)$. Although $a_2 = a_3$, we can use the results obtained in Steps 3 and 6 of the proof of Proposition 3.1 without assuming (31) and (32) in Section 3. From Step 3, we have

$$F_m = q^i(q-1)^j(q-t)^k p^m \tag{1}$$

with

$$i + j + k \geq 1. \tag{2}$$

(Since $c \in K$ in (29) in Section 3, we may assume $c = 1$.) Here we claim that $m \geq 1$. Otherwise, we have $m = 0$ and $i + j + k \geq 1$ from (2). Since $F_{m-1} = F_{-1} = 0$, the equation (41) in Section 3 with $a_2 = a_3$ is turned to

$$\begin{aligned} 0 = {} & (\rho q + \sigma)q^i(q-1)^j(q-t)^k + i(a_3 + a_4)tq^{i-1}(q-1)^j(q-t)^k \\ & + j(a_3 - a_4)(1-t)q^i(q-1)^{j-1}(q-t)^k \\ & + k(1 - a_1 - a_2)t(t-1)q^i(q-1)^j(q-t)^{k-1} \\ & + \{(i+j+k)(a_3 - a_1)q - 2(i+j)a_3 t - k(a_1 + a_2)t + (i+k)(a_1 - a_4) \\ & + j(a_3 - a_4)\}q^i(q-1)^j(q-t)^k, \end{aligned}$$

from which it follows immediately that

$$i(a_3+a_4) = j(a_3-a_4) = k(1-a_1-a_2) = 0.$$

Since $\boldsymbol{a}_8 \notin P$, we have $(a_3+a_4)(a_3-a_4)(1-a_1-a_2) \neq 0$. Thus we have $i=j=k=0$, which contradicts $i+j+k \geq 1$.

We determine the form of the polynomial F'_n ($n \in \mathbb{Z}$) in (13) in Section 3. We need the following:

SUBLEMMA 1. *Let e be a positive integer, and A be a polynomial in R'_0, and let λ' be an element of K. Let X'_1 be the derivation introduced in the proof of Proposition* 3.1, *and assume that $a_2 - a_3 = 0$ and $a_1 - a_3 \neq 0$. If $\lambda' + l - 1 \neq 0$ for every integer l such that $1 \leq l \leq e$ and if A satisfies a congruence*

$$X'_1 A \equiv \lambda' q^2 p A \mod (qp - a_3 + a_1)^e, \tag{3}$$

then $A \equiv 0 \mod (qp - a_3 + a_1)^e$.

PROOF OF SUBLEMMA 1. We denote by $K[T, T^{-1}]$ the Laurent polynomial ring in one variable T over K. By hypothesis, the derivation X'_1 is given by

$$X'_1 = (2qp + a_1 - a_3)q^2\frac{\partial}{\partial q} - (3qp + 2a_1 - 2a_3)qp\frac{\partial}{\partial p}.$$

Let ψ be the K-algebra morphism of $K[q,p]$ onto $K[T, T^{-1}]$ defined by $\psi(q) = (a_3 - a_1)^{-1}T$ and $\psi(p) = (a_3 - a_1)^2 T^{-1}$. Then the following diagram is commutative:

$$\begin{tikzcd} K[q,p] \arrow[r, "\psi"] \arrow[d, "X'_1"'] & K[T,T^{-1}] \arrow[d, "T^2\frac{d}{dT}"] \\ K[q,p] \arrow[r, "\psi"'] & K[T,T^{-1}]. \end{tikzcd}$$

The kernel of the morphism ψ is a principal ideal generated by $qp + a_1 - a_3$. Since $X'_1(qp + a_1 - a_3) = -(qp + a_1 - a_3)q^2 p$, it is X'_1-invariant. By the same argument as in the proof of Lemma 3.2, we can show $A \equiv 0 \mod (qp+a_1-a_3)^l$ by induction on l ($1 \leq l \leq e$). So we omit the rest of the proof.

Now we consider the equation $(15)_n$ in Section 3

$$X'_1 F'_n = (\lambda qp + \rho)qF'_n. \tag{4}$$

We set

$$F'_n = Aq^u p^v \tag{5}$$

with non-negative integers u and v and $A \in R'_0$ such that $qp \nmid A$. Then we have

$$n = u - v. \tag{6}$$

Substituting (5) into (4) and dividing the resulting equation by $q^u p^v$, we have

$$X_1' A = \{(\lambda - 2u + 3v)qp + \rho + (2v - u)(a_1 - a_3)\}qA. \tag{7}$$

Since $qp | X_1' A$ and $qp \nmid A$, we have

$$\rho = (u - 2v)(a_1 - a_3), \tag{8}$$

and the equation (7) is turned to

$$X_1' A = (\lambda - 2u + 3v)q^2 pA. \tag{9}$$

Here we claim that $b = -(\lambda - 2u + 3v)$ is a non-negative integer. In fact, otherwise, we would have $(\lambda - 2u + 3v) + l - 1 \neq 0$ for every integer $l \geq 1$. It would follow from Sublemma 1 that $A \equiv 0 \mod (qp + a_1 - a_3)^e$ for every integer $e \geq 1$ and therefore $F_n' = 0$. This contradicts (14) in Section 3. Therefore b is a non-negative integer. Thus we have

$$\lambda = -b + 2u - 3v. \tag{10}$$

If $b \geq 1$, we have $(\lambda - 2u + 3v) + l - 1 \neq 0$ for every integer l such that $1 \leq l \leq b$. It follows from Sublemma 1 that $A \equiv 0 \mod (qp + a_1 - a_3)^b$. Therefore there exists a non-zero element $c \in R_0'$ such that

$$A = c(qp + a_1 - a_3)^b. \tag{11}$$

Substituting (11) into (9), we have $X_1'(c) = 0$, and therefore $c \in K$. From (11) and (5), we have

$$F_n' = c(qp + a_1 - a_3)^b q^u p^v. \tag{12}$$

Since the Newton polygon of F is given in Step 4 in the proof of Proposotion 3.1, by comparing (1) and (12), we have

$$c = 1, \tag{13}$$

$$b + u = i + j + k, \tag{14}$$

$$b + v = m. \tag{15}$$

Here, the two expressions of λ, (26) in Section 3 and (10), are equal to each other via (14) and (15). On the other hand, we have two expressions of ρ. Namely, from (60) in Section 3, using $a_2 - a_3 = 0$, we have

$$\rho = i(a_1 - 2a_3 - a_4) + j(a_1 - 2a_3 + a_4) + k(2a_1 - 1) - 2m(a_1 - a_3). \tag{16}$$

Equating (8) and (16), and eliminating u and v by (14) and (15), we have

$$i(a_3 + a_4) + j(a_3 - a_4) + k(1 - a_1 - a_3) + b(a_1 - a_3) = 0.$$

Since $\boldsymbol{a}_8 \in \Gamma \cap \{\boldsymbol{a} \in \mathbb{C}^4 \mid a_2 - a_3 = 0\}$ and $\boldsymbol{a}_8 \notin P$, by the same argument as in the proof of Corollary 3.7, we have $i = j = k = b = 0$, and therefore $u = 0$ and $m = v = -n$ from (6), (14) and (15). Consequently it follows from (1), (8), (10), (12) that

$$F_m = p^m, \tag{17}$$

$$\rho = -2m(a_1 - a_3), \tag{18}$$

$$\lambda = -3m, \tag{19}$$

$$F'_{-m} = p^m. \tag{20}$$

Moreover, from (27), (28), (61), (63) in Section 3, we have

$$\mu = 2m(1 + t), \tag{21}$$

$$\nu = -mt, \tag{22}$$

$$\sigma = -2ma_3t + m(a_1 - a_4), \tag{23}$$

$$F_{m-1} = 0. \tag{24}$$

Since $X_{-1} = 0$, and since $(12)_d$ in Section 3 is turned to

$$X_1F_d = (\lambda q^2 + \mu q + \nu)pF_d + (\rho q + \sigma)F_{d+1} - X_0F_{d+1},$$

the proposition follows immediately from

SUBLEMMA 2. *Let d be an integer such that $0 \leq d < m$, and A be a polynomial in R_d. If A satisfies an equation*

$$X_1A = (\lambda q^2 + \mu q + \nu)pA$$

with (19), (21) *and* (22), *then $A = 0$.*

In fact, since $\nu + (d - 2l + 2)t = (-m + d - 2l + 2)t \neq 0$ for every $l \geq 1$, it follows from Lemma 3.2 that $A \equiv 0 \mod q^e$ for every integer $e \geq 1$. Thus we have $A = 0$. Proposition 4.4 is proved.

REMARK 4.4. In order to prove Sublemma 2, we may use Lemmas 3.3 or 3.4 instead of Lemma 3.2.

PROPOSITION 4.5. (i) *Let* $\boldsymbol{a}_9$ *be a vector in* $\Gamma \cap \{\boldsymbol{a} \in \mathbb{C}^4 \mid a_2 - a_3 = a_3 + a_4 = 0\}$ *and not in* L. *For arbitrary non-negative integers* i *and* m *such that* $i + m \geq 1$, *a principal ideal* $(q^i p^m)$ *is* $X(\boldsymbol{a}_9)$*-invariant. Conversely, if* I *is an* $X(\boldsymbol{a}_9)$*-invariant principal ideal properly between the zero-ideal and* $K[q, p]$, *then there exist non-negative integers* i *and* m *such that* $i + m \geq 1$ *and* $I = (q^i p^m)$.

(ii) *Let* $\boldsymbol{a}_{10}$ *be a vector in* $\Gamma \cap \{\boldsymbol{a} \in \mathbb{C}^4 \mid a_2 - a_3 = a_3 - a_4 = 0\}$ *and not in* L. *For arbitrary non-negative integers* j *and* m *such that* $j + m \geq 1$, *a principal ideal* $((q-1)^j p^m)$ *is* $X(\boldsymbol{a}_{10})$*-invariant. Conversely, if* I *is an* $X(\boldsymbol{a}_{10})$*-invariant principal ideal properly between the zero-ideal and* $K[q, p]$, *then there exist non-negative integers* j *and* m *such that* $j + m \geq 1$ *and* $I = ((q-1)^j p^m)$.

(iii) *Let* $\boldsymbol{a}_{11}$ *be a vector in* $\Gamma \cap \{\boldsymbol{a} \in \mathbb{C}^4 \mid a_2 - a_3 = 1 - a_1 - a_2 = 0\}$ *and not in* L. *For arbitrary non-negative integers* k *and* m *such that* $k + m \geq 1$, *a principal ideal* $((q-t)^k p^m)$ *is* $X(\boldsymbol{a}_{11})$*-invariant. Conversely, if* I *is an* $X(\boldsymbol{a}_{11})$*-invariant principal ideal properly between the zero-ideal and* $K[q, p]$, *then there exist non-negative integers* k *and* m *such that* $k + m \geq 1$ *and* $I = ((q-t)^k p^m)$.

PROOF. Since the three assertions are proved in the same way, we prove only the assertion (i) and omit the proofs of the others. The first half of (i) is obvious. We show the second half. To this end, the notation being as in Proposition 3.1, it is sufficient to prove that the $X(\boldsymbol{a}_9)$-invariant polynomial F is equal to $q^i p^m$ with some non-negative integers i and m such that $i+m \geq 1$. We set $\boldsymbol{a}_9 = (a_1, a_2, a_3, a_4)$. By the same reason as in the proof of Proposition 4.4, we have (1) and (2). Let us show $i + m \geq 1$. Otherwise, we have $i = m = 0$ and $j + k \geq 1$ from (2). Since $F_{m-1} = F_{-1} = 0$, the equation (41) in Section 3 with $a_2 - a_3 = a_3 + a_4 = 0$ is turned to

$$\begin{aligned}0 = {} & (\rho q + \sigma)(q-1)^j (q-t)^k + j(a_3 - a_4)(1-t)(q-1)^{j-1}(q-t)^k \\ & + k(1 - a_1 - a_2)t(t-1)(q-1)^j (q-t)^{k-1} \\ & + \{(j+k)(a_3 - a_1)q - 2ja_3 t - k(a_1 + a_2)t + k(a_1 - a_4) \\ & + j(a_3 - a_4)\}(q-1)^j (q-t)^k,\end{aligned}$$

from which it follows immediately that

$$j(a_3 - a_4) = k(1 - a_1 - a_2) = 0.$$

Since $\boldsymbol{a}_9 \notin L$, we have $(a_3 - a_4)(1 - a_1 - a_2) \neq 0$. Thus we have $j = k = 0$, which is a contradiction.

Since $a_2 - a_3 = 0$ and $a_1 - a_3 \neq 0$, we can develop an argument similar to that in the proof of Proposition 4.4. Therefore we may use (6), (8), (10), (12)-(15). Moreover we have two expressions of ρ. Namely, from (60) in Section 3, using $a_2 - a_3 = a_3 + a_4 = 0$, we have

$$\rho = i(a_1 - a_3) + j(a_1 - 2a_3 + a_4) + k(2a_1 - 1) - 2m(a_1 - a_3). \tag{25}$$

Equating (8) and (25), and eliminating u and v by (14) and (15), we have

$$j(a_3 - a_4) + k(1 - a_1 - a_3) + b(a_1 - a_3) = 0.$$

Since $\boldsymbol{a}_9 \in \Gamma \cap \{\boldsymbol{a} \in \mathbb{C}^4 \mid a_2 - a_3 = a_3 + a_4 = 0\}$ and $\boldsymbol{a}_9 \notin L$, by the same argument as in the proof of Corollary 3.7, we have $j = k = b = 0$, and therefore $u = i$, $v = m$ and $n = i - m$ from (6), (14) and (15). Consequently, it follows from (1), (8), (10), (12) that

$$F_m = q^i p^m, \tag{26}$$

$$\rho = (i - 2m)(a_1 - a_3), \tag{27}$$

$$\lambda = 2i - 3m, \tag{28}$$

$$F'_{i-m} = q^i p^m. \tag{29}$$

Moreover, from (27), (28), (61), (63) in Section 3, we have

$$\mu = (2m - 2i)(1 + t), \tag{30}$$

$$\nu = (2i - m)t, \tag{31}$$

$$\sigma = (2ia_3 - 2ma_3)t - i(a_1 - a_4) + m(a_1 - a_4), \tag{32}$$

$$F_{m-1} = 0. \tag{33}$$

Thus the proposition follows immediately from

SUBLEMMA. *Let d be an integer such that $0 \leq d < m$, and let A be a polynomial in R_d. If A satisfies an equation*

$$X_1 A = (\lambda q^2 + \mu q + \nu) p A$$

with (28), (30) *and* (31), *then $A = 0$.*

In fact, since $\lambda + \mu + \nu + (d - 2l + 2)(1 - t) = (m - d + 2l - 2)(t - 1) \neq 0$ for every $l \geq 1$, it follows from Lemma 3.3 that $A \equiv 0 \mod (q - 1)^e$ for every $e \geq 1$. Thus we have $A = 0$. Proposition 4.5 is proved.

REMARK 4.5. By the same argument as in Remark 4.1, for $\bar{\boldsymbol{a}} \in \Gamma \cap \{\boldsymbol{a} \in \mathbb{C}^4 \mid a_2 - a_3 = a_1 - a_2 = 0\}$ but $\bar{\boldsymbol{a}} \notin L$, we find an $\bar{X}(\bar{\boldsymbol{a}})$-invariant principal ideal $(Q_3^M P_3^N)$ with $M + N \geq 1$ of $K[Q_3, P_3]$.

REMARK 4.6. In order to prove Sublemma, we may use Lemma 3.4 instead of Lemma 3.3. However, we cannot use Lemma 3.2.

PROPOSITION 4.6. (i) *Let $\boldsymbol{a}_{12}$ be a vector in $\Gamma \cap \{\boldsymbol{a} \in \mathbb{C}^4 \mid a_2 - a_3 = a_3 - a_4 = 1 - a_1 - a_2 = 0\}$ and not in D. For arbitrary non-negative integers j, k and m such that $j + k + m \geq 1$, a principal ideal $((q-1)^j (q-t)^k p^m)$ is $X(\boldsymbol{a}_{12})$-invariant. Conversely, if I is an $X(\boldsymbol{a}_{12})$-invariant principal ideal properly between the zero-ideal and $K[q, p]$, then there exist non-negative integers j, k and m such that $j + k + m \geq 1$ and $I = ((q-1)^j (q-t)^k p^m)$.*

(ii) *Let $\boldsymbol{a}_{13}$ be a vector in $\Gamma \cap \{\boldsymbol{a} \in \mathbb{C}^4 \mid a_2 - a_3 = a_3 + a_4 = 1 - a_1 - a_2 = 0\}$ and not in D. For arbitrary non-negative integers i, k and m such that $i + k + m \geq 1$, a principal ideal $(q^i (q-t)^k p^m)$ is $X(\boldsymbol{a}_{13})$-invariant. Conversely, if I is an $X(\boldsymbol{a}_{13})$-invariant principal ideal properly between the zero-ideal and $K[q, p]$, then there exist non-negative integers i, k and m such that $i+k+m \geq 1$ and $I = (q^i (q-t)^k p^m)$.*

(iii) *Let $\boldsymbol{a}_{14}$ be a vector in $\Gamma \cap \{\boldsymbol{a} \in \mathbb{C}^4 \mid a_2 - a_3 = a_3 + a_4 = a_3 - a_4 = 0\}$ and not in D. For arbitrary non-negative integers i, j and m such that $i + j + m \geq 1$, a principal ideal $(q^i (q-1)^j p^m)$ is $X(\boldsymbol{a}_{14})$-invariant. Conversely, if I is an $X(\boldsymbol{a}_{14})$-invariant principal ideal properly between the zero-ideal and $K[q, p]$, then there exist non-negative integers i, j and m such that $i + j + m \geq 1$ and $I = (q^i (q-1)^j p^m)$.*

PROOF. Since the three assertions are proved in the same way, we prove only the assertion (iii), and omit the proofs of the others. The first half of (iii) is obvious. We show the second half. To this end, the notation being as in Proposition 3.1, it is sufficient to prove that the $X(\boldsymbol{a}_{14})$-invariant polynomial F is equal to $q^i (q-1)^j p^m$ with some non-negative integers i, j and m such that $i + j + m \geq 1$. We set $\boldsymbol{a}_{14} = (a_1, a_2, a_3, a_4)$. By the same reason as in the proof of Proposition 4.4, we have (1) and (2). Let us show $i + j + m \geq 1$. Otherwise, we have $i = j = m = 0$ and $k \geq 1$ from (2). Since $F_{m-1} = F_{-1} = 0$, the equation (41) in Section 3 with $a_2 - a_3 = a_3 + a_4 = a_3 - a_4 = 0$ is turned to

$$\begin{aligned} 0 =& (\rho q + \sigma)(q-t)^k + k(1 - a_1 - a_2) t (t-1)(q-t)^{k-1} \\ &+ \{k(a_3 - a_1) q - k(a_1 + a_2) t + k(a_1 - a_4)\}(q-t)^k, \end{aligned}$$

from which it follows immediately that

$$k(1 - a_1 - a_2) = 0.$$

Since $\boldsymbol{a}_{14} \notin D$, we have $1 - a_1 - a_2 \neq 0$, and therefore $k = 0$. This is a contradiction.

Since $a_2 - a_3 = 0$ and $a_1 - a_3 \neq 0$, we can develop an argument similar to that in the proof of Proposition 4.4. Therefore we may use (6), (8), (10), (12)-(15). Moreover we have two expressions of ρ. Namely, from (60) in Section 3, using $a_2 - a_3 = a_3 + a_4 = a_3 - a_4 = 0$, we have

$$\rho = i(a_1 - a_3) + j(a_1 - a_3) + k(2a_1 - 1) - 2m(a_1 - a_3). \tag{34}$$

Equating (8) and (34), and eliminating u and v by (14) and (15), we have

$$k(1 - a_1 - a_3) + b(a_1 - a_3) = 0.$$

Since $a_{14} \in \Gamma \cap \{a \in \mathbb{C}^4 \mid a_2 - a_3 = a_3 + a_4 = a_3 - a_4 = 0\}$ and $a_{14} \notin D$, by the same argument as in the proof of Corollary 3.7, we have $k = b = 0$, and therefore $u = i + j$, $v = m$ and $n = i + j - m$ from (6), (14) and (15). Consequently, it follows from (1), (8), (10), (12) that

$$F_m = q^i (q-1)^j p^m, \tag{35}$$

$$\rho = (i + j - 2m)(a_1 - a_3), \tag{36}$$

$$\lambda = 2(i + j) - 3m, \tag{37}$$

$$F'_{i+j-m} = q^{i+j} p^m. \tag{38}$$

Moreover, from (27), (28), (61), (63) in Section 3, we have

$$\mu = 2m - 2i - t(2i + 2j - 2m), \tag{39}$$

$$\nu = (2i - m)t, \tag{40}$$

$$\sigma = (m - i)a_1, \tag{41}$$

$$F_{m-1} = 0. \tag{42}$$

Thus the proposition follows immediately from

SUBLEMMA. *Let d be an integer such that $0 \le d < m$, and let A be a polynomial in R_d. If A satisfies an equation*

$$X_1 A = (\lambda q^2 + \mu q + \nu) p A$$

with (37), (39) *and* (40), *then* $A = 0$.

In fact, since $\lambda t^2 + \mu t + \nu + (d - 2l + 2)t(t-1) = (-m + d - 2l + 2)(t^2 - t) \neq 0$ for every $l \ge 1$, it follows from Lemma 3.4 that $A \equiv 0 \mod (q-t)^e$ for every $e \ge 1$. Thus we have $A = 0$. Proposition 4.6 is proved.

REMARK 4.7. We can determine all the invariant ideals on U_0 or V_1 or V_3 or V_4 for $a \in \Gamma \cap L_2$ but $a \notin D$. See Remarks 4.2 and 4.3.

REMARK 4.8. We can use neither Lemma 3.2 nor Lemma 3.3 instead of Lemma 3.4 to prove Sublemma.

PROPOSITION 4.7. *For arbitrary non-negative integers i, j, k and m such that $i+j+k+m \ge 1$, a principal ideal $(q^i(q-1)^j(q-t)^k p^m)$ is $X(1,0,0,0)$-invariant. Conversely, if I is an $X(1,0,0,0)$-invariant principal ideal properly between the zero-ideal and $K[q,p]$, then there exist non-negative integers i, j, k and m such that $i+j+k+m \ge 1$ and $I = (q^i(q-1)^j(q-t)^k p^m)$.*

PROOF. The first half is obvious. We show the second half. To this end, the notation being as in Proposition 3.1, it is sufficient to prove that the $X(1,0,0,0)$-invariant polynomial F is equal to $q^i(q-1)^j(q-t)^k p^m$ with some non-negative integers i, j, k and m such that $i+j+k+m \geq 1$. By the same reason as in the proof of Proposition 4.4, we have (1) and (2). Substituting $(a_1, a_2, a_3, a_4) = (1,0,0,0)$ into (60), (61), (63) in Section 3, we have

$$\rho = i + j + k - 2m, \tag{43}$$

$$\sigma = kt - i - k + m, \tag{44}$$

$$F_{m-1} = 0. \tag{45}$$

On the other hand, since $a_2 - a_3 = 0$ and $a_1 - a_3 = 1 \neq 0$, we can develop an argument similar to that in the proof of Proposition 4.4. Therefore we may use (6), (8), (10), (12)-(15). Equating (8) and (43), and eliminating u and v by (14) and (15), we have $b = 0$, and therefore

$$F_n' = q^{i+j+k} p^m \tag{46}$$

from (12) and (13), where

$$n = i + j + k - m. \tag{47}$$

Let h be an integer such that $0 \leq h \leq j + k$. We first show

$$F_{n-h}' = \left(\sum_{0\leq d\leq j, 0\leq e\leq k, d+e=h} \binom{j}{d}\binom{k}{e}(-1)^d(-t)^e \right) q^{i+j+k-h} p^m. \tag*{$(48)_h$}$$

We have already proved the case $h = 0$ by (46). Assume that $h \geq 1$, and that F_{n-h+2}' and F_{n-h+1}' are given by $(48)_{h-2}$ and $(48)_{h-1}$. The polynomial F_{n-h}' satisfies the equation $(15)_{n-h}$ in Section 3:

$$\begin{aligned} X_1' F_{n-h}' = {} & (\lambda qp + \rho) q F_{n-h}' + (\mu qp + \sigma) F_{n-h+1}' \\ & + \nu p F_{n-h+2}' - X_0' F_{n-h+1}' - X_{-1}' F_{n-h+2}'. \end{aligned} \tag{49}$$

Here λ, μ and ν are given by (26), (27), (28) in Section 3, and ρ and σ are given by (43) and (44). Moreover X_1', X_0' and X_{-1}' are given by

$$\begin{aligned} X_1' &= (2qp+1)q^2 \frac{\partial}{\partial q} - (3qp+2)qp \frac{\partial}{\partial p}, \\ X_0' &= t(t-1)\frac{\partial}{\partial t} - [2(1+t)qp+1] q \frac{\partial}{\partial q} + [2(1+t)qp+1] p \frac{\partial}{\partial p}, \\ X_{-1}' &= 2tqp \frac{\partial}{\partial q} - tp^2 \frac{\partial}{\partial p}. \end{aligned}$$

Thus the equation (49) is turned to

$$\begin{aligned} X_1' F_{n-h}' &= (\lambda qp + \rho) q F_{n-h}' \\ &\quad - h(2qp+1) \sum_{0\le d\le j, 0\le e\le k, d+e=h} \binom{j}{d}\binom{k}{e} (-1)^d (-t)^e q^{i+j+k-h+1} p^m. \end{aligned} \tag{50}$$

We set

$$E_{n-h} = F_{n-h}' - \sum_{0\le d\le j, 0\le e\le k, d+e=h} \binom{j}{d}\binom{k}{e} (-1)^d (-t)^e q^{i+j+k-h} p^m.$$

Eliminating F_{n-h}' from this expression and (50), we have an equation for E_{n-h}:

$$X_1' E_{n-h} = (\lambda qp + \rho) q E_{n-h}. \tag{51}$$

Here we claim that $E_{n-h} = 0$ $(1 \le h \le j + k)$. In fact, otherwise, there exist non-negative integers x and y, and a polynomial $A \in R_0'$ such that

$$E_{n-h} = A q^x p^y, \tag{52}$$

$$qp \nmid A, \tag{53}$$

$$x - y = n - h. \tag{54}$$

Substituting (52) into (51) and dividing the resulting equation by $q^x p^y$, we have

$$X_1' A = \{(\lambda - 2x + 3y) qp + \rho - x + 2y\} q A. \tag{55}$$

Since $qp | X_1' A$ and $qp \nmid A$, we have

$$\rho = x - 2y. \tag{56}$$

From (43), (47), (54) and (56), we have

$$x = i + j + k - 2h, \tag{57}$$

$$y = m - h. \tag{58}$$

On the other hand, by (56), the equation (55) is turned to

$$X_1' A = (\lambda - 2x + 3y) q^2 p A.$$

Since $(\lambda - 2x + 3y) + l - 1 = h + l - 1 \ge 1 \ne 0$ for every $l \ge 1$ by (26) in Section 3, (57) and (58), it follows from Sublemma 1 of Proposition 4.4 that

$A \equiv 0 \mod (qp+1)^e$ for every integer $e \geq 1$, that is, $A = 0$. This contradicts (53), and therefore we have $E_{n-h} = 0$. Thus $(48)_h$ holds for every integer h such that $0 \leq h \leq j+k$.

Next we show

$$(59)_h \qquad F'_{i-m-h} = 0$$

for every integer $h \geq 1$. We proceed again by induction on h. The polynomial F'_{i-1} satisfies the equation $(15)_{i-m-1}$ in Section 3:

$$(60) \qquad \begin{aligned} X'_1 F'_{i-m-1} &= (\lambda qp+\rho)qF'_{i-m-1} + (\mu qp+\sigma)F'_{i-m} \\ &\quad + \nu p F'_{i-m+1} - X'_0 F'_{i-m} - X'_{-1} F'_{i-m+1}. \end{aligned}$$

Here F'_{i-m} and F'_{i-m+1} are given by

$$(61) \qquad F'_{i-m} = (-1)^j(-t)^k q^i p^m$$

and

$$(62) \qquad F'_{i-m+1} = \{j(-1)^{j-1}(-t)^k + k(-1)^j(-t)^{k-1}\}q^{i+1}p^m$$

by $(48)_h$. Substituting (61) and (62) into (60), we have

$$X'_1 F'_{i-m-1} = (\lambda qp+\rho)qF'_{i-m-1}.$$

By the same argument as in the preceding paragraph, we have $F'_{i-m-1} = 0$, i.e., $(59)_1$. The polynomial F'_{i-m-2} satisfies the equation $(15)_{i-m-2}$ in Section 3:

$$(63) \qquad \begin{aligned} X'_1 F'_{i-m-2} &= (\lambda qp+\rho)qF'_{i-m-2} + (\mu qp+\sigma)F'_{i-m-1} \\ &\quad + \nu p F'_{i-m} - X'_0 F'_{i-m-1} - X'_{-1} F'_{i-m}. \end{aligned}$$

Substituting $(59)_1$ and (61) into (63), we have

$$X'_1 F'_{i-m-2} = (\lambda qp+\rho)qF'_{i-m-2}.$$

Thus we have $F'_{i-m-2} = 0$ again. By the same argument we have $(59)_h$ for every integer $h \geq 1$. From $(48)_h$ and $(59)_h$, we have $F = F'_n + F'_{n-1} + \cdots + F'_{i-m} = q^i(q-1)^j(q-t)^k p^m$. Proposition 4.7 is thus proved.

REMARK 4.9. We can determine all the invariant ideals on U_0 or V_1 or V_3 or V_4 for $\boldsymbol{a} = (0,0,0,0)$ or $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$ or $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. See Remarks 4.2 and 4.3.

PROPOSITION 4.8. *For arbitrary non-negative integers i, j and k such that $i+j+k \geq 1$, a principal ideal $(q^i(q-1)^j(q-t)^k)$ is $X(\frac{1}{2}, \frac{1}{2}, 0, 0)$-invariant. Conversely, if I is an $X(\frac{1}{2}, \frac{1}{2}, 0, 0)$-invariant principal ideal properly between the zero-ideal and $K[q,p]$, then there exist non-negative integers i, j and k such that $i+j+k \geq 1$ and $I = (q^i(q-1)^j(q-t)^k)$.*

PROOF. The first half is obvious. For the second half, the notation being as in Proposition 3.1, it is sufficient to prove that the $X(\frac{1}{2}, \frac{1}{2}, 0, 0)$-invariant polynomial F is equal to $q^i(q-1)^j(q-t)^k$ with some non-negative integers i, j and k such that $i+j+k \geq 1$. We set $(\frac{1}{2}, \frac{1}{2}, 0, 0) = (a_1, a_2, a_3, a_4)$. Although $a_1 = a_2$, we can use the results obtained in Steps 3 and 6 of the proof of Proposition 3.1 without assuming (31) and (32) in Section 3. From Step 3 we have

$$F_m = q^i(q-1)^j(q-t)^k p^m \tag{64}$$

with

$$i+j+k+m \geq 1. \tag{65}$$

From (60) in Section 3, we have

$$\rho = i+j+k-2m. \tag{66}$$

On the other hand, the polynomial F'_n satisfies the equation $(15)_n$ in Section 3:

$$X'_1 F'_n = (\lambda qp + \rho) q F'_n. \tag{67}$$

Here λ is given by (26) in Section 3, and the derivation X'_1 by

$$X'_1 = (2qp+1)q^2 \frac{\partial}{\partial q} - \left(3q^2p^2 + 2qp + \frac{1}{4}\right) \frac{\partial}{\partial p}.$$

Note that

$$X'_1(qp) = -\left(qp + \frac{1}{2}\right)^2 q. \tag{68}$$

Since $(a_2 - a_3)(a_1 - a_3) = \frac{1}{4} \neq 0$, the integer n is non-negative. Therefore we may set

$$F'_n = A\left(qp + \frac{1}{2}\right)^a q^n \tag{69}$$

with a non-negative integer a and a polynomial $A \in R'_0$ such that $qp + \frac{1}{2} \nmid A$. Substituting (69) into (67) and dividing the resulting equation by $(qp + \frac{1}{2})^a q^n$, we have

$$X'_1 A = \left\{(a - 2n + \lambda)qp + \frac{1}{2}a - n + \rho\right\} qA.$$

Since $(qp + \frac{1}{2})^2 | A$ by (67), it follows immediately that

$$a - 2n + \lambda = 0, \tag{70}$$

$$\frac{1}{2}a - n + \rho = 0, \tag{71}$$

and

$$A \in K. \tag{72}$$

From (70) and (71), we have

$$\lambda = 2\rho. \tag{73}$$

Substituting (26) in Section 3 and (66) into (73), we have $m = 0$. It follows from (64) and (65) that $F = F_0 = q^i(q-1)^j(q-t)^k$ and $i + j + k \geq 1$. Considering the Newton polygon of F, we have $a = 0$ and therefore $F'_n = q^n$ with $n = i + j + k$. Proposition 4.8 is proved.

REMARK 4.10. There exists a one-parameter family of classical solutions at $(\frac{1}{2}, \frac{1}{2}, 0, 0)$ defined by $q = \infty$. In fact, take the derivation $\bar{X}(\frac{1}{2}, \frac{1}{2}, 0, 0)$ introduced in Remark 4.1. By the same argument, we see that the $\bar{X}(\frac{1}{2}, \frac{1}{2}, 0, 0)$-invariant principal ideals of $K[Q_3, P_3]$ are of the form $(Q_3^L(Q_3-1)^M(Q_3-t)^N)$.

REMARK 4.11. From Proposition 4.8, it follows that the solution of E. Picard (e.g. [4], Example 3.1) is not classical in the sense of Umemura.

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