Birational rowmotion and Coxeter-motion on minuscule posets

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Submitted: May 1, 2020; Accepted: Jan 6, 2021; Published: Jan 29, 2021

Abstract

Birational rowmotion is a discrete dynamical system on the set of all positive real-valued functions on a finite poset, which is a birational lift of combinatorial rowmotion on order ideals. It is known that combinatorial rowmotion for a minuscule poset has order equal to the Coxeter number, and exhibits the file homomesy phenomenon for refined order ideal cardinality statistics. In this paper we generalize these results to the birational setting. Moreover, as a generalization of birational promotion on a product of two chains, we introduce birational Coxeter-motion on minuscule posets, and prove that it enjoys periodicity and file homomesy.

Mathematics Subject Classifications: 05E18, 06A11

1 Introduction

Rowmotion (at the combinatorial level) is a bijection R on the set $\mathcal{J}(P)$ of order ideals of a finite poset P, which assigns to $I \in \mathcal{J}(P)$ the order ideal $R(I)$ generated by the minimal elements of the complement $P \setminus I$. The map R can be also described in terms of toggles. For each $v \in P$, let $t_v : \mathcal{J}(P) \to \mathcal{J}(P)$ be the map given by

$$t_v(I) = \begin{cases} I \cup \{v\} & \text{if } v \notin I \text{ and } I \cup \{v\} \in \mathcal{J}(P), \\ I \setminus \{v\} & \text{if } v \in I \text{ and } I \setminus \{v\} \in \mathcal{J}(P), \\ I & \text{otherwise,} \end{cases} \tag{1}$$

and call it the *toggle* at v. Then the rowmotion map R is expressed as the composition

$$R = t_{v_1} \circ t_{v_2} \circ \cdots \circ t_{v_N}, \tag{2}$$

*This work was partially supported by JSPS Grants-in-Aid for Scientific Research No. 18K03208.

where $(v_1, v_2, \dots, v_N)$ is any linear extension of P, i.e., a list of all the elements of P such that $v_i < v_j$ in P implies $i < j$. This rowmotion has been studied from several perspectives and under various names. See [18] and [19] for the history and references.

Rowmotion exhibits nice properties such as periodicity and homomesy on special posets including root posets (see [11, 1]) and minuscule posets (see [15, 16]). In general, given a set S and a bijection $f : S \to S$, we say that a statistic $\theta : S \to \mathbb{R}$ is *homomesic* with respect to f if there exists a constant C such that for any $\langle f \rangle$-orbit T

$$\frac{1}{\#T} \sum_{x \in T} \theta(x) = C.$$

We refer the reader to [14] for the homomesy phenomenon. For a minuscule poset P and a simple root $\alpha \in \Pi$, we put

$$P^\alpha = \{v \in P : c(v) = \alpha\}, \tag{3}$$

where $c : P \to \Pi$ is the coloring of P with color set Π, the set of simple roots. This subset P^α is called the *file* corresponding to α. (See Section 3 for the definition of minuscule posets and related terminology.)

If P is a minuscule poset, then the associated rowmotion map R has the following properties:

Theorem 1. *Let P be a minuscule poset associated to a minuscule weight λ of a simple Lie algebra $\mathfrak{g}$. Then we have*

(a) (periodicity, Rush–Shi [15, Thoerem 1.4]) The rowmotion map R has finite order equal to the Coxeter number h of $\mathfrak{g}$.

(b) (file homomesy, Rush–Wang [16, Theorem 1.2]) For each simple root $\alpha \in \Pi$, the refined order ideal cardinality $\#(I \cap P^\alpha)$ is homomesic with respect to R. More precisely, for any $I \in \mathcal{J}(P)$, we have

$$\frac{1}{h} \sum_{k=0}^{h-1} \# \left(R^k(I) \cap P^\alpha \right) = \langle \varpi^\vee, \lambda \rangle,$$

where $\varpi^\vee$ is the fundamental coweight corresponding to α.

One motivation of this paper is to lift the results in the above theorem to the birational level.

Einstein–Propp [4] introduced birational rowmotion by lifting the notion of toggles from the combinatorial level to the piecewise-linear level, and then to the birational level. Given a finite poset P, let $\widehat{P} = P \sqcup \{\widehat{1}, \widehat{0}\}$ be the poset obtained from P by adjoining an extra maximum element $\widehat{1}$ and an extra minimum element $\widehat{0}$. For positive real numbers A and B, we put

$$\mathcal{K}^{A,B}(P) = \{F : \widehat{P} \to \mathbb{R}_{>0} \mid F(\widehat{1}) = A,\ F(\widehat{0}) = B\},$$

where $\mathbb{R}_{>0}$ denotes the set of positive real numbers. For $v \in P$, we define the *birational toggle* $\tau_v^{A,B} : \mathcal{K}^{A,B}(P) \to \mathcal{K}^{A,B}(P)$ at v by

$$\left(\tau_v^{A,B} F\right)(x) = \begin{cases} \dfrac{1}{F(v)} \cdot \dfrac{\sum_{w \in \widehat{P},\, w \lessdot v} F(w)}{\sum_{z \in \widehat{P},\, z \gtrdot v} 1/F(z)} & \text{if } x = v, \\ F(x) & \text{otherwise,} \end{cases} \tag{4}$$

where the symbol $x \gtrdot y$ means that x covers y, i.e., $x > y$ and there is no element z such that $x > z > y$. It is clear that $\tau_v^{A,B}$ is an involution. (See Equation (12) for a definition of piecewise-linear toggles.) Then we define *birational rowmotion* $\rho^{A,B} : \mathcal{K}^{A,B}(P) \to \mathcal{K}^{A,B}(P)$ by

$$\rho^{A,B} = \tau_{v_1}^{A,B} \circ \cdots \circ \tau_{v_N}^{A,B}, \tag{5}$$

where $(v_1, \dots, v_N)$ is a linear extension of P. It can be shown that the definition of $\rho^{A,B}$ is independent of the choice of linear extension. Since rowmotion is defined by toggling from top to bottom, we have a recursive formula for the values of the birational rowmotion map:

$$\left(\rho^{A,B} F\right)(v) = \frac{1}{F(v)} \cdot \frac{\sum_{w \in \widehat{P},\, w \lessdot v} F(w)}{\sum_{z \in \widehat{P},\, z \gtrdot v} 1/\left(\rho^{A,B} F\right)(z)}. \tag{6}$$

We omit the superscript ^{A,B} and simply write $\mathcal{K}(P)$, τ_v and ρ when there is no confusion.

For birational rowmotion on a product of two chains, periodicity and (multiplicative) file homomesy are obtained by Grinberg–Roby [7] and Einstein–Propp [4], Musiker–Roby [9] respectively. In this paper we generalize their results from products of two chains (type A minuscule posets) to arbitrary minuscule posets.

For a minuscule poset and a simple root $\alpha \in \Pi$, we define

$$\Phi_\alpha(F) = \prod_{v \in P^\alpha} F(v) \tag{7}$$

for $F \in \mathcal{K}^{A,B}(P)$. Our main results for birational rowmotion are summarized as follows:

Theorem 2. *Let P be the minuscule poset associated to a minuscule weight λ of a finite dimensional simple Lie algebra $\mathfrak{g}$. Let $\rho = \rho^{A,B}$ be the birational rowmotion map. Then we have*

(a) (periodicity) The map ρ has finite order equal to the Coxeter number h of $\mathfrak{g}$.

(b) (reciprocity) For any $v \in P$ and $F \in \mathcal{K}^{A,B}(P)$, we have

$$\left(\rho^{\mathrm{rk}(v)} F\right)(v) = \frac{AB}{F(\iota v)}, \tag{8}$$

where $\mathrm{rk} : P \to \{1, 2, \dots, h-1\}$ is the rank function of the graded poset P and $\iota : P \to P$ is the canonical involutive anti-automorphism of P (see Proposition 11).

(c) (file homomesy) For a simple root α, we have

$$\prod_{k=0}^{h-1} \Phi_\alpha(\rho^k F) = A^{h\langle \varpi^\vee, -w_0\lambda\rangle} B^{h\langle \varpi^\vee, \lambda\rangle} \tag{9}$$

for any $F \in \mathcal{K}^{A,B}(P)$, where w_0 is the longest element of the Weyl group W of $\mathfrak{g}$, and $\varpi^\vee$ is the fundamental coweight corresponding to α.

Part (a) of this theorem is established in [6, 7] except for the type E_7 minuscule poset. In this paper we provide a way to settle the E_7 case by using a computer. For a type A minuscule poset, Part (b) is obtained in [7, Theorem 32]. Our proof of Part (b) is based on a case-by-case analysis (with a help of computer in types E_6 and E_7). Part (c) in type A follows from Einstein–Propp [4, Theorems 5.3 and 6.6] and Musiker–Roby [9, Theorem 2.16]. We will give an almost uniform proof to Part (c). Also we can use tropicalization (or ultradiscretization) to deduce the results for piecewise-linear rowmotion as well as combinatorial rowmotion in Theorem 1 (see Section 2).

Another aim of this paper is to introduce and study birational Coxeter-motion on minuscule posets, which is regarded as a generalization of birational *promotion* on a product of two chains (see [4, Definition 4.3]). For a simple root $\alpha \in \Pi$, we define $\sigma_\alpha^{A,B} : \mathcal{K}^{A,B}(P) \to \mathcal{K}^{A,B}(P)$ as the composition

$$\sigma_\alpha^{A,B} = \prod_{v \in P_\alpha} \tau_v^{A,B}, \tag{10}$$

which is independent of the order of composition. Then a *Coxeter-motion map* is a product of all the $\sigma_\alpha^{A,B}$'s in any order. Our results for birational Coxeter-motion are stated as follows:

Theorem 3. *Let P be a minuscule poset. Let $\gamma = \gamma^{A,B}$ be a birational Coxeter-motion map. Then we have*

(a) (periodicity) The map γ has finite order equal to the Coxeter number h.

(b) (file homomesy) For each simple root $\alpha \in \Pi$, we have

$$\prod_{k=0}^{h-1} \Phi^\alpha(\gamma^k F) = A^{h\langle \varpi^\vee, -w_0\lambda\rangle} B^{h\langle \varpi^\vee, \lambda\rangle}. \tag{11}$$

If P is a type A minuscule poset and π is the birational promotion map (a special case of birational Coxeter-motion maps), then there is an explicitly defined "recombination map" $\mathfrak{R}$ such that $\mathfrak{R}\rho = \pi\mathfrak{R}$ (see [4, Theorem 6.2]), which, together with Theorem 2 (a), implies Part (a) of the above theorem. We prove Part (a) for arbitrary minuscule posets by showing that any birational Coxeter-motion map is conjugate to the birational rowmotion map in the birational toggle group (Theorem 15 below). By applying tropicalization to Part (a), we obtain the periodicity of piecewise-linear Coxeter-motion, which is proved in [5, Theorem 1.12] via quiver representation. Part (b) in type A is obtained in [4, Theorem 5.3].

Hopkins [8] obtains another example of homomesy for the birational rowmotion for a wider class of posets including minuscule posets.

Theorem 4. *(Hopkins [8, Theorem 4.43]) Let P be a minuscule poset and $\rho = \rho^{A,B}$ the birational rowmotion map. For $F \in \mathcal{K}^{A,B}(P)$, we define*

$$\Psi(F) = \prod_{x \in P} \frac{F(x)}{\sum_{y \in \widehat{P}, y \lessdot x} F(y)}.$$

Then we have

$$\prod_{k=0}^{h-1} \Psi(\rho^k F) = \left(\frac{A}{B} \right)^{\#P}.$$

Via tropicalization, this theorem reduces to the homomesy phenomenon of the antichain cardinality statistic, which was proved in [16, Theorem 1.4]. In a forthcoming paper [10], we use explicit formulas for iterations of the birational rowmotion map to give refinements of Theorem 4. Our refinement in type A provides a birational lift of the homomesy given in [13, Proof of Theorem 27].

The remaining of this paper is organized as follows. We collect some general facts concerning birational rowmotion in Section 2, and give a definition and properties of minuscule posets in Section 3. In Sections 4 to 6 we give a proof of our main theorems. The periodicity in Theorem 2 (a) and Theorem 3 (a) is proved in Section 4, and the reciprocity in Theorem 2 (b) is verified in Section 5. In Section 6, after investigating local properties around a file, we complete the proof of file homomesy in Theorem 2 (c) and Theorem 3 (b).

2 Generalities on rowmotion

In this section, we explain how combinatorial and birational rowmotion are related and give some general facts about birational rowmotion.

2.1 Combinatorial, piecewise-linear and birational rowmotion

We begin by recalling the definition of piecewise-linear toggles and rowmotion. Given a finite poset P and real numbers a, b, we put

$$\mathcal{P}^{a,b}(P) = \{ f : \widehat{P} \to \mathbb{R} : f(\widehat{1}) = a,\ f(\widehat{0}) = b \},$$

where $\widehat{P} = P \sqcup \{\widehat{1}, \widehat{0}\}$. We define the *piecewise-linear toggles* $\widetilde{t}_v^{\pm,a,b} : \mathcal{P}^{a,b}(P) \to \mathcal{P}^{a,b}(P)$ at $v \in P$ by the formulas

$$\begin{aligned} \left(\widetilde{t}_v^{+,a,b} f \right)(v) &= \max\{ f(w) : w \in \widehat{P},\ w \lessdot v \} + \min\{ f(z) : z \in \widehat{P},\ z \gtrdot v \} - f(v), \\ \left(\widetilde{t}_v^{-,a,b} f \right)(v) &= \min\{ f(w) : w \in \widehat{P},\ w \lessdot v \} + \max\{ f(z) : z \in \widehat{P},\ z \gtrdot v \} - f(v), \end{aligned} \tag{12}$$

and $\left(\widetilde{t}_v^{\pm,a,b} f\right)(x) = f(x)$ for $x \neq v$. For an order ideal $I \in \mathcal{J}(P)$, let $\chi_I^{\pm}$ be the characteristic functions defined by

$$\chi_I^+(v) = \begin{cases} 0 & \text{if } v \in I \text{ or } v = \widehat{0}, \\ 1 & \text{if } v \in P \setminus I \text{ or } v = \widehat{1}, \end{cases} \quad \chi_I^-(v) = \begin{cases} 1 & \text{if } v \in I \text{ or } v = \widehat{0}, \\ 0 & \text{if } v \in P \setminus I \text{ or } v = \widehat{1}. \end{cases}$$

Then it follows from definitions (1) and (12) that the toggle $\widetilde{t}_v^{\pm,a,b}$ is a piecewise-linear lift of the combinatorial toggle t_v in the following sense:

$$\widetilde{t}_v^{+,1,0}(\chi_I^+) = \chi_{t_v I}^+, \quad \widetilde{t}_v^{-,0,1}(\chi_I^-) = \chi_{t_v I}^-. \tag{13}$$

The *piecewise-linear rowmotion* map $\widetilde{R}^{\pm,a,b} : \mathcal{P}^{a,b}(P) \to \mathcal{P}^{a,b}(P)$ is defined by

$$\widetilde{R}^{\pm,a,b} = \widetilde{t}_{v_1}^{\pm,a,b} \circ \cdots \circ \widetilde{t}_{v_N}^{\pm,a,b},$$

where $(v_1, \dots, v_N)$ is a linear extension of P.

A rational function $F(X_1, \cdots, X_m) \in \mathbb{Q}(X_1, \cdots, X_m)$ is called *subtraction-free* if F is expressed as a ratio $F = G/H$ of two polynomials $G(X_1, \cdots, X_m)$ and $H(X_1, \cdots, X_m) \in \mathbb{Z}[X_1, \dots, X_m]$ with nonnegative integer coefficients. By using

$$\lim_{\varepsilon \to +0} \varepsilon \log(e^{a/\varepsilon} + e^{b/\varepsilon}) = \max\{a, b\}, \quad \lim_{\varepsilon \to -0} \varepsilon \log(e^{a/\varepsilon} + e^{b/\varepsilon}) = \min\{a, b\},$$

we can see that, if $F(X_1, \dots, X_m)$ is subtraction-free, then for any real numbers $x_1, \dots, x_m \in \mathbb{R}$ the limits

$$f^{\pm}(x_1, \cdots, x_m) = \lim_{\varepsilon \to \pm 0} \varepsilon \log F(e^{x_1/\varepsilon}, \cdots, e^{x_m/\varepsilon})$$

exist and $f^+(x_1, \dots, x_m)$ (resp. $f^-(x_1, \dots, x_m)$) is the piecewise-linear function in $x_1, \dots, x_m$ obtained from F by replacing the multiplication $\cdot$, the division $/$ and the addition $+$ with the addition $+$, the subtraction $-$ and the maximum max (resp. the minimum min). This procedure from F to $f^{\pm}$ is called the tropicalization (or ultradiscretization).

Proposition 5. *Let P be a finite poset. Let $R : \mathcal{J}(P) \to \mathcal{J}(P)$ and $\rho = \rho^{A,B} : \mathcal{K}^{A,B}(P) \to \mathcal{K}^{A,B}(P)$ be the combinatorial and birational rowmotion maps respectively. Let $m : P \times \mathbb{Z} \to \mathbb{Z}$ be a map with finite support. If there is a integers p and q such that*

$$\prod_{(v,k) \in P \times \mathbb{Z}} \left[\left(\rho^k F\right)(v) \right]^{m(v,k)} = A^p B^q \tag{14}$$

for any $F \in \mathcal{K}^{A,B}(P)$, then

$$\sum_{(v,k) \in P \times \mathbb{Z}} m(v,k) \chi[v \notin R^k(I)] = p, \quad \sum_{(v,k) \in P \times \mathbb{Z}} m(v,k) \chi[v \in R^k(I)] = q, \tag{15}$$

where $\chi[S] = 1$ if S is true and 0 if S is false.

Proof. By applying the tropicalization procedure to (14), we obtain

$$\sum_{(v,k)\in P\times\mathbb{Z}} m(v,k)\left(\widetilde{R}^{\pm,a,b} f\right)(v) = ap + bq$$

for any $f \in \mathcal{P}^{a,b}(P)$. Then specializing $f = \chi_I^{\pm}$ and using (13), we obtain (15). $\square$

Corollary 6. *(a) If $\left(\rho^h F\right)(v) = F(v)$ for any $F \in \mathcal{K}^{A,B}(P)$ and $v \in P$, then $R^h(I) = I$ any $I \in \mathcal{J}(P)$.*

(b) Let v and $w \in P$ and k be a positive integer. Suppose that $\left(\rho^k F\right)(v) \cdot F(w) = AB$ for any $F \in \mathcal{K}^{A,B}(P)$. Then, for any $I \in \mathcal{J}(P)$, we have $v \in R^k(I)$ if and only if $w \notin I$.

(c) Let M be a subset of P and h be a positive integer. If $\prod_{k=0}^{h-1} \prod_{v\in M} \left(\rho^k F\right)(v) = A^p B^q$ for any $F \in \mathcal{K}^{A,B}(P)$, then we have $\sum_{k=0}^{h-1} \# \left(R^k(I) \cap M\right) = q$ for any $I \in \mathcal{J}(P)$.

Similar statements hold for birational Coxter-motion.

2.2 Birational rowmotion on graded posets

In this subsection we present some properties of birational rowmotion on graded posets. A poset P is called *graded of height n* if there exists a rank function $\mathrm{rk} : P \to \{1, 2, \dots, n\}$ satisfying the following three conditions:

(i) If v is minimal in P, then $\mathrm{rk}(v) = 1$;

(ii) If v is maximal in P, then $\mathrm{rk}(v) = n$;

(iii) If v covers w, then $\mathrm{rk}(v) = \mathrm{rk}(w) + 1$.

Lemma 7. *If P is a graded poset of height n and the birational rowmotion map $\rho^{A,B}$ has finite order N, then N is divisible by $n + 1$.*

Proof. By Corollary 6 (a), we have $R^N(I) = I$ for all $I \in \mathcal{J}(P)$. On the other hand, it is easy to see that the $\langle R\rangle$-orbit of the empty order ideal $\emptyset$ has length $n + 1$. Hence we see that $n + 1$ divides N. $\square$

The following lemma gives a relation between $\rho^{A,B}$ and $\rho^{1,1}$.

Lemma 8. *Let P be a graded poset of height n. For a map $F : P \to \mathbb{R}_{>0}$ and positive real numbers A, $B \in \mathbb{R}_{>0}$, we denote by $F^{A,B} \in \mathcal{K}^{A,B}(P)$ the extension of F to $\widehat{P}$ such that $F^{A,B}(\widehat{1}) = A$ and $F^{A,B}(\widehat{0}) = B$. For $1 \leqslant k \leqslant n + 1$ and $v \in P$, we have*

$$\left(\left(\rho^{A,B}\right)^k F^{A,B}\right)(v) = \left(\left(\rho^{1,1}\right)^k F^{1,1}\right)(v) \times \begin{cases} A & \text{if } 1 \leqslant k \leqslant \mathrm{rk}(v) - 1, \\ AB & \text{if } k = \mathrm{rk}(v), \\ B & \text{if } \mathrm{rk}(v) + 1 \leqslant k \leqslant n, \\ 1 & \text{if } k = n + 1. \end{cases} \tag{16}$$

Proof. We can use the recursive formula (6) to proceed by double induction on k and $n - \mathrm{rk}(v)$. $\square$

2.3 Change of variables

Let P be a finite poset. Given an initial state $X \in \mathcal{K}^{A,B}(P)$, we regard $X(v)$ $(v \in P)$ as indeterminates. In the computation of $\left(\rho^k X\right)(v)$ $(v \in P)$ of iterations of the birational rowmotion map $\rho = \rho^{A,B}$, it is convenient to change variables from $\{X(v) : v \in P\}$ to $\{Z(v) : v \in P\}$ defined by the formula

$$Z(v) = \begin{cases} X(v) & \text{if } v \text{ is minimal,} \\ \dfrac{X(v)}{\sum_{w \in P,\, w \lessdot v} X(w)} & \text{otherwise.} \end{cases} \tag{17}$$

This change of variables is used in [9] to describe a lattice path formula for birational rowmotion on a type A minuscule poset. Then the inverse change of variables is given by

$$X(v) = \sum Z(v_1) Z(v_2) \cdots Z(v_r), \tag{18}$$

where the sum is taken over all saturated chains $v_1 \gtrdot \cdots \gtrdot v_r$ in P such that $v_1 = v$ and v_r is minimal in P. Note that this change of variables is a birational lift of Stanley's transfer map between the order polytope and the chain polytope of a poset (see [17, Section 3]).

3 Minuscule posets

In this section we review a definition and properties of minuscule posets.

3.1 Definition and properties of minuscule posets

Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra over the complex number field $\mathbb{C}$ of type X_n, where $X \in \{A, B, C, D, E, F, G\}$ and n is the rank of $\mathfrak{g}$. We fix a Cartan subalgebra $\mathfrak{h}$ and choose a positive root system Δ_+ of the root system $\Delta \subset \mathfrak{h}^*$. Let $\Pi = \{\alpha_1, \dots, \alpha_n\}$ be the set of simple roots, where we follow [2, Planche I–IX] for the numbering of simple roots. We denote by ϖ_i the fundamental weight corresponding to the ith simple root α_i. Let $\Delta_+^\vee \subset \mathfrak{h}$ be the positive coroot system. Let W be the Weyl group of $\mathfrak{g}$, which acts on $\mathfrak{h}$ and $\mathfrak{h}^*$. The simple reflections $\{s_\alpha : \alpha \in \Pi\}$ generate W.

For a dominant integral weight λ, we denote by $V_{X_n,\lambda}$ the irreducible $\mathfrak{g}$-module with highest weight λ and by $L_{X_n,\lambda}$ the set of weights of $V_{X_n,\lambda}$. We say that a nonzero dominant integral weight λ is *minuscule* if $L_{X_n,\lambda}$ is a single W-orbit. See [3, VIII, §7, n°3] for properties of minuscule weights. It is known that minuscule weights are fundamental weights. Table 1 is the list of minuscule weights.

Let λ be a minuscule weight of a simple Lie algebra $\mathfrak{g}$ of type X_n. We equip the set of weights $L_{X_n,\lambda}$ with a poset structure by defining $\mu \geqslant \nu$ if $\nu - \mu$ is a linear combination of simple roots with nonnegative integer coefficients. We note that λ is the minimum element of the poset $L_{X_n,\lambda}$.

Definition 9. Let $\mathfrak{g}$ be a simple Lie algebra of type X_n and λ a minuscule weight. Then the *minuscule poset* $P_{X_n,\lambda}$ is defined by

$$P_{X_n,\lambda} = \{\beta^\vee \in \Delta_+^\vee : \langle \beta^\vee, \lambda \rangle = 1\}, \tag{19}$$

Table 1: List of minuscule weights

type	minuscule weights	Coxeter number
A_n	$\varpi_1, \varpi_2, \ldots, \varpi_n$	$n+1$
B_n	ϖ_n	$2n$
C_n	ϖ_1	$2n$
D_n	$\varpi_1, \varpi_{n-1}, \varpi_n$	$2n-2$
E_6	ϖ_1, ϖ_6	12
E_7	ϖ_7	18
E_8	none	30
F_4	none	12
G_2	none	6

where the partial ordering on $P_{X_n,\lambda}$ is given by saying that $\alpha^\vee \geqslant \beta^\vee$ if $\alpha^\vee - \beta^\vee$ is a linear combination of simple coroots with nonnegative integer coefficients.

Proposition 10. *Let λ be a minuscule weight and $P_{X_n,\lambda}$ be the corresponding minuscule poset. Then we have*

(a) ([12, Propositions 3.2 and 4.1]) The poset $L_{X_n,\lambda}$ is a distributive lattice.

(b) ([12, Theorem 11]) There exists a unique map $c : P_{X_n,\lambda} \to \Pi$, called the coloring *of $P_{X_n,\lambda}$, such that the map*

$$\mathcal{J}(P_{X_n,\lambda}) \ni I \mapsto \lambda - \sum_{v \in I} c(v) \in L_{X_n,\lambda}$$

gives an isomorphism of posets.

If λ is a minuscule weight, then the stabilizer W_λ of λ in W is the maximal parabolic subgroup generated by $\{s_\beta : \beta \in \Pi \setminus \{\alpha\}\}$, where α is the simple root corresponding to the fundamental weight λ.

Proposition 11. *Let $P_{X_n,\lambda}$ be the minuscule poset corresponding to a minuscule weight λ, and w_λ the longest element of the stabilizer W_λ. Then the map*

$$\iota : P_{X_n,\lambda} \ni \beta^\vee \mapsto w_\lambda \beta^\vee \in P_{X_n,\lambda}$$

gives an involutive anti-automorphism of the poset $P_{X_n,\lambda}$.

Proof. It is enough to show that $\beta^\vee > \gamma^\vee$ implies $w_\lambda \beta^\vee < w_\lambda \gamma^\vee$ for $\beta^\vee$, $\gamma^\vee \in P_{X_n,\lambda}$. It follows from $\langle \beta^\vee, \lambda \rangle = \langle \gamma^\vee, \lambda \rangle = 1$ that $\beta^\vee - \gamma^\vee$ is a linear combination of $\Pi^\vee \setminus \{\alpha^\vee\}$ with nonnegative integer coefficients, where $\Pi^\vee$ is the set of simple coroots and $\alpha^\vee$ is the simple coroot dual to λ. Since $w_\lambda(\Pi^\vee \setminus \{\alpha^\vee\}) = -(\Pi^\vee \setminus \{\alpha^\vee\})$, we see that $w_\lambda \beta^\vee - w_\lambda \gamma^\vee$ is a linear combination of $\Pi^\vee \setminus \{\alpha^\vee\}$ with nonpositive integer coefficients. $\square$

The following properties of minuscule posets can be checked easily (e.g., by using a description given in the next subsection).

Proposition 12. *Let $P = P_{X_n,\lambda}$ be the minuscule poset corresponding to a minuscule weight λ, and $c : P \to \Pi$ the coloring.*

(a) The poset P is graded of height $h-1$, where h is the Coxeter number of $\mathfrak{g}$.

(b) The poset P has a unique minimal element $v_{\min}$ and a unique maximal element $v_{\max}$. Moreover, if we put $\alpha_{\min} = c(v_{\min})$ and $\alpha_{\max} = c(v_{\max})$, then the simple root $\alpha_{\min}$ corresponds to the fundamental weight λ and $\alpha_{\max} = -w_0\alpha_{\min}$ corresponds to $-w_0\lambda$, where w_0 is the longest element of W.

(c) If $v \lessdot w$ in P, then their colors $c(v)$ and $c(w)$ are adjacent in the Dynkin diagram of $\mathfrak{g}$.

(d) For each $\alpha \in \Pi$, the subposet $P^\alpha = \{v \in P : c(v) = \alpha\}$ is a chain.

(e) If v, $w \in P^\alpha$, then the difference $\mathrm{rk}(v) - \mathrm{rk}(w)$ is even.

3.2 Description of minuscule posets

In this subsection we give explicit descriptions of minuscule posets and their colorings. The minuscule posets can be embedded into the poset $\mathbb{Z}^2$, where $(i, j) \leqslant (i', j')$ in $\mathbb{Z}^2$ if and only if $i \leqslant i'$ and $j \leqslant j'$.

Type $\boldsymbol{A_n}$. The positive coroot system $\Delta_+^\vee$ of type A_n can be described as $\Delta_+^\vee = \{e_i - e_j : 1 \leqslant i < j \leqslant n+1\}$ with $e_1 + \cdots + e_{n+1} = 0$. Then we have

$$P_{A_n,\varpi_r} = \{e_i - e_j : 1 \leqslant i \leqslant r,\ r+1 \leqslant j \leqslant n+1\}$$

and the map $e_i - e_j \mapsto (r-i, j-r-1)$ gives an isomorphism of posets from P_{A_n,ϖ_r} to the subposet

$$\{(i,j) \in \mathbb{Z}^2 : 0 \leqslant i \leqslant r-1,\ 0 \leqslant j \leqslant n-r\} \subset \mathbb{Z}^2.$$

The poset P_{A_n,ϖ_r} is a product poset $[0, r-1] \times [0, n-r]$ of two chains, where $[0, m] = \{0, 1, \dots, m\}$ is a chain. We call this poset P_{A_n,ϖ_r} a *rectangle poset.* The involution ι is the 180° rotation of the Hasse diagram. For example, the Hasse diagrams and the colorings of P_{A_7,ϖ_1} and P_{A_7,ϖ_3} are given in Figure 1, where we label a vertex v with i to indicate that $c(v) = \alpha_i$.

Type $\boldsymbol{B_n}$. If we realize the positive coroot system $\Delta_+^\vee$ of type B_n as $\Delta_+^\vee = \{e_i \pm e_j : 1 \leqslant i < j \leqslant n\} \cup \{2e_i : 1 \leqslant i \leqslant n\}$, then we have

$$P_{B_n,\varpi_n} = \{e_i + e_j : 1 \leqslant i \leqslant j \leqslant n\},$$

and the map $e_i + e_j \mapsto (n-j, n-i)$ gives an poset isomorphism from P_{B_n,ϖ_n} to the subposet

$$\{(i,j) \in \mathbb{Z}^2 : 0 \leqslant i \leqslant j \leqslant n-1\} \subset \mathbb{Z}^2.$$

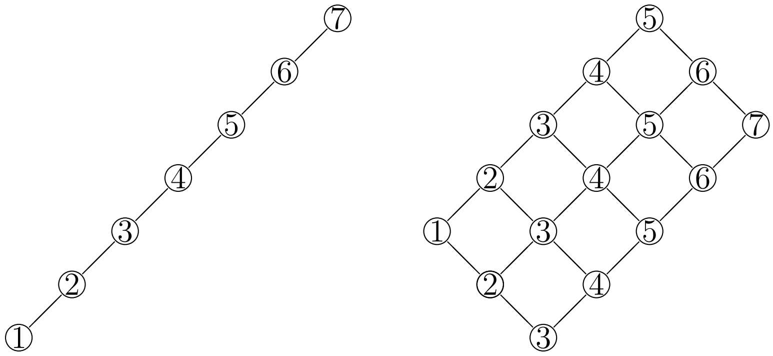

Figure 1: P_{A_7,ϖ_1} (left) and P_{A_7,ϖ_3} (right)

We call P_{B_n,ϖ_n} a *shifted staircase poset*. The involution ι is the horizontal flip of the Hasse diagram. For example the Hasse diagram of P_{B_4,ϖ_4} and its coloring are given in Figure 2.

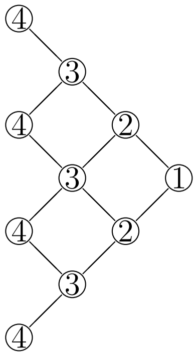

Figure 2: P_{B_4,ϖ_4}

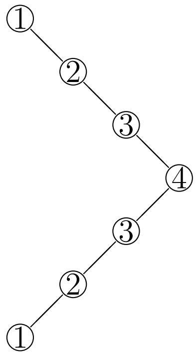

Figure 3: P_{C_4,ϖ_1}

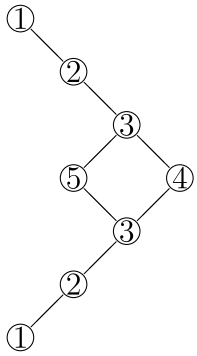

Figure 4: P_{D_5,ϖ_1}

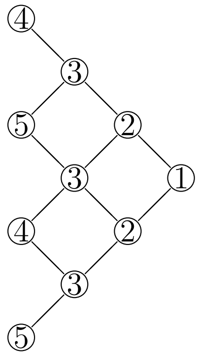

Figure 5: P_{D_5,ϖ_5}

Type C_n. If we realize the positive coroot system $\Delta_+^\vee$ of type C_n as $\Delta_+^\vee = \{e_i \pm e_j : 1 \leqslant i < j \leqslant n\} \cup \{e_i : 1 \leqslant i \leqslant n\}$, then we have

$$P_{C_n,\varpi_1} = \{e_1 - e_2, \dots, e_1 - e_n, e_1, e_1 + e_n, \dots, e_1 + e_2\}.$$

The poset P_{C_n,ϖ_1} is a chain, and isomorphic to the subposet

$$\{(1,1), \dots, (1, n-1), (1, n), (2, n), \dots, (n, n)\} \subset \mathbb{Z}^2.$$

For example the Hasse diagram of P_{C_4,ϖ_1} and its coloring are given in Figure 3. Note that P_{C_n,ϖ_1} is isomorphic to P_{A_{2n-1},ϖ_1}, but they have different colorings.

Type D_n. We realize the positive coroot system $\Delta_+^\vee$ of type D_n as $\Delta_+^\vee = \{e_i \pm e_j : 1 \leqslant i < j \leqslant n\}$.

For the minuscule weight ϖ_1, we have

$$P_{D_n,\varpi_1} = \{e_1 - e_2, \dots, e_1 - e_{n-1}, e_1 - e_n, e_1 + e_n, e_1 + e_{n-1}, \dots, e_1 + e_2\},$$

and it is isomorphic to the subposet

$$\{(1,1),\dots,(1,n-1),(1,n),(2,n-1),(2,n),\dots,(n,n)\} \subset \mathbb{Z}^2.$$

See Figure 4 for the Hasse diagram of P_{D_5,ϖ_1} and its coloring. The poset P_{D_n,ϖ_1} is called a *double-tailed diamond poset*. The involutive anti-automorphism ι is given by

$$\iota(e_1+e_k)=e_1-e_k \quad (1\leqslant k\leqslant n-1), \quad \iota(e_1+\varepsilon e_n)=e_1+(-1)^n\varepsilon e_n.$$

For the minuscule weights ϖ_n and ϖ_{n-1}, we have

$$P_{D_n,\varpi_n}=\{e_i+e_j : 1\leqslant i<j\leqslant n\}$$

and $P_{D_n,\varpi_{n-1}}$ is obtained from P_{D_n,ϖ_n} by replacing e_i+e_n with e_i-e_n for $1\leqslant i\leqslant n-1$. Both posets P_{D_n,ϖ_n} and $P_{D_n,\varpi_{n-1}}$ are isomorphic to $\{(i,j)\in\mathbb{Z}^2 : 0\leqslant i\leqslant j\leqslant n-2\}$. For example, the Hasse diagram and the coloring of P_{D_5,ϖ_5} are given in Figure 5. Note that $P_{D_n,\varpi_{n-1}}\cong P_{D_n,\varpi_n}$, and they are isomorphic to $P_{B_{n-1},\varpi_{n-1}}$, but they have different colorings.

Type E_6. The minuscule poset P_{E_6,ϖ_6} is isomorphic to the subposet

$$\left\{\begin{matrix}(1,1),(2,1),(3,1),(4,1),(5,1),(3,2),(4,2),(5,2),\\ (4,3),(5,3),(6,3),(4,4),(5,4),(6,4),(7,4),(8,4)\end{matrix}\right\}\subset\mathbb{Z}^2,$$

and the Hasse diagram and the coloring are given in Figure 6. The involution ι is the $180°$ rotation of the Hasse diagram. As posets, $P_{E_6,\varpi_1}\cong P_{E_6,\varpi_6}$.

Type E_7. The minuscule poset P_{E_7,ϖ_7} is isomorphic to the subposet

$$\left\{\begin{matrix}(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(2,4),(2,5),(2,6),\\ (3,5),(3,6),(3,7),(4,5),(4,6),(4,7),(5,5),(5,6),(5,7),\\ (4,8),(4,9),(5,8),(5,9),(6,8),(6,9),(7,9),(8,9),(9,9)\end{matrix}\right\}\subset\mathbb{Z}^2,$$

and the Hasse diagram and the coloring are given in Figure 7. The involution ι is the horizontal flip of the Hasse diagram.

4 Periodicity

The goal of this section is to prove the periodicity of birational rowmotion and Coxeter-motion (Theorem 2 (a) and Theorem 3 (a)).

4.1 Periodicity of birational rowmotion

For the birational rowmotion map on minuscule posets, periodicity has been established in [6, 7] except for the type E_7 minuscule poset. Let P be a minuscule poset associated to a Lie algebra $\mathfrak{g}$, and $\rho^{A,B} : \mathcal{K}^{A,B}(P)\to\mathcal{K}^{A,B}$ the birational rowmotion map. Since periodicity depends only on the poset structure, we may assume that $\mathfrak{g}$ is simply-laced. And by Proposition 12 (a), Lemmas 7 and 8, it is enough to show that $\rho=\rho^{1,1}$ satisfies $\rho^h=1$, where h is the Coxeter number of $\mathfrak{g}$.

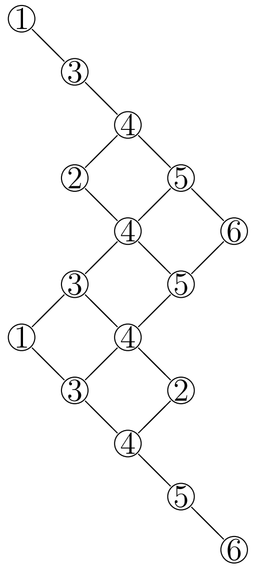

Figure 6: P_{E_6,ϖ_6}

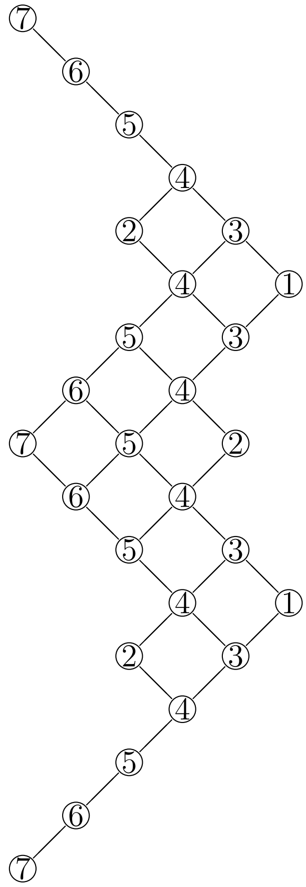

Figure 7: P_{E_7,ϖ_7}

- If P is a type A_n minuscule poset, i.e., if P is a rectangle poset $[0, r-1] \times [0, n-r]$, then it was shown that the birational rowmotion map ρ has order $n+1$ (Grinberg–Roby [7, Theorem 30], see [9, Corollary 2.12] for another proof).

- If $P = P_{D_n,\varpi_1}$ is a double-tailed diamond poset, then P is a skeletal poset of height $2n-3$, and it follows from [6, Propositions 61, 74 and 75] that ρ has order $2n-2$ (see [6, Section 10] for a definition of skeletal posets and details).

- If $P = P_{D_n,\varpi_n}$ is a shifted staircase poset, then Grinberg–Roby [7, Theorem 58] proved that ρ has order $2n$.

- If $P = P_{E_6,\varpi_6}$ is the minuscule poset of type E_6, then by using a computer we can verify that ρ has order 12.

- Let $P = P_{E_7,\varpi_7}$ be the minuscule poset of type E_7. Given an initial state $X \in \mathcal{K}^{1,1}(P)$, we regard $\{X(v) : v \in P\}$ as indeterminates and introduce new indeterminates $\{Z(v) : v \in P\}$ by (17). Also we use the realization of P as a subposet of $\mathbb{Z}^2$ given in Subsection 3.2. With the author's laptop, it takes about 20 seconds for Maple19 to compute all the values $\left(\rho^k X\right)(v)$ $(0 \leqslant k \leqslant 18,\ v \in P)$ as rational functions in $\{Z(v) : v \in P\}$ and check that $(\rho^{18} X)(v) = X(v)$ for all $v \in P$. The change of variables from $\{X(v) : v \in P\}$ to $\{Z(v) : v \in P\}$ makes computation much faster. (It takes almost three hours without using the Z-coordinates.)

This completes the proof of Theorem 2 (a).

4.2 Periodicity of birational Coxeter-motion

In order to prove the periodicity of birational Coxeter-motion (Theorem 3 (a)), we work with the birational toggle group and show that any birational Coxeter-motion maps are conjugate to the birational rowmotion map in this group.

Let P be a finite poset and fix positive real numbers A and B. We define the *birational toggle group*, denote by $G(P)$, to be the subgroup generated by birational toggles $\tau_v = \tau_v^{A,B}$ $(v \in P)$ in the group of all bijections on $\mathcal{K}^{A,B}(P)$.

A key tool here is the non-commutativity graph. Given elements $g_1, \dots, g_n$ of a group G, the *non-commutativity graph* $\Gamma(g_1, \dots, g_n)$ is defined as the graph with vertex set $\{1, 2, \dots, n\}$, in which two vertices i and j are joined if and only if $g_i g_j \neq g_j g_i$. The following lemma is useful.

Lemma 13. *([2, V, §6, n°1, Lemma 1]) Let $g_1, \dots, g_n$ be elements of a group G. If the non-commutativity graph $\Gamma(g_1, \dots, g_n)$ has no cycle, then $g_{\nu(1)} \dots g_{\nu(n)}$ is conjugate to $g_1 \dots g_n$ in G for any permutation ν of $1, 2, \dots, n$.*

First we prove that all birational Coxeter-motion maps are conjugate.

Proposition 14. *Let P be a minuscule poset. Then all birational Coxeter-motion maps are conjugate to each other in the birational toggle group $G(P)$.*

Proof. Note that birational toggles τ_v and τ_w are commutative unless $v \lessdot w$ ore $v \gtrdot w$. It follows from Proposition 12 (c) that, if simple roots α and β are not adjacent in the Dynkin diagram of $\mathfrak{g}$, then the corresponding elements σ_α and σ_β commute with each other in $G(P)$. Hence the non-commutativity graph $\Gamma(\sigma_{\alpha_1}, \dots, \sigma_{\alpha_n})$, where $\alpha_1, \dots, \alpha_n$ are the simple roots, is a subgraph (of the underlying simple graph) of the Dynkin diagram. Since the Dynkin diagram of $\mathfrak{g}$ has no cycle, we can use Lemma 13 to conclude that any two Coxeter-motion maps are conjugate in $G(P)$. ☐

The periodicity of birational Coxeter-motion maps (Theorem 3 (a)) immediately follows from the following thoerem and the periodicity of the birational rowmotion map (Theorem 2 (a)).

Theorem 15. *Let P be a minuscule poset. Then any birational Coxeter-motion map is conjugate to the birational rowmotion map $\rho = \rho^{A,B}$ in the birational toggle group $G(P)$.*

This theorem is a birational lift of [15, Theorem 1.3]. In order to prove this theorem, we use the notion of rc-poset, which was introduced by Striker–Williams [18, Section 4.2]. We put $\Lambda = \{(i,j) \in \mathbb{Z}^2 : i+j \text{ is even}\}$. A poset P is called a *rowed-and-columned poset* (*rc-poset* for short) if there is a map $\pi : P \to \Lambda$ such that, if v covers u in P and $\pi(v) = (i,j)$, then $\pi(u) = (i+1, j-1)$ or $(i-1, j-1)$. Minuscule posets $P = P_{X_n,\lambda}$ are rc-posets with respect to the composition map $\pi : P \to \Lambda$ of the embedding $P \hookrightarrow \mathbb{Z}^2$ given in Subsection 3.2 and the map $\mathbb{Z}^2 \ni (i,j) \mapsto (j-i, j+i) \in \Lambda$. A *row* (resp. *column*) of an RC-poset P is a subset M of P of the form

$$M = \{v \in P : \text{the second coordinate of } \pi(v) \text{ equals } r\},$$
$$(\text{resp. } M = \{v \in P : \text{the first coordinate of } \pi(v) \text{ equals } c\})$$

for some r (resp. c). If M is a subset of a row or a column of P, then the composition of toggles τ_v ($v \in M$) is independent of the order of composition, so we denote by $\tau[M]$ the resulting element of the toggle group $G(P)$. If $R_1, \dots, R_n$ are the non-empty rows of an rc-poset P from bottom to top, then the rowmotion map $\rho = \rho^{A,B}$ is given by

$$\rho = \tau[R_1] \circ \tau[R_2] \circ \cdots \circ \tau[R_n].$$

The following Lemma is proved by exactly the same argument as in [18].

Lemma 16. *([18, Theorem 5.2]) Let P be an rc-poset. Let $R_1, \dots, R_n$ be the non-empty rows of P from bottom to top, and $C_1, \dots, C_m$ the non-empty columns of P from left to right. Then the rowmotion map ρ is conjugate to $\tau[C_{\nu(1)}] \circ \cdots \circ \tau[C_{\nu(m)}]$ in $G(P)$ for any permutation ν of $1, 2, \dots, m$.*

We prove Theorem 15 by using this lemma.

Proof of Theorem 15. Let $\Pi = \{\alpha_1, \dots, \alpha_n\}$ be the set of simple roots, where we follow the numbering in [2], and $C_1, \dots, C_m$ the non-empty columns of P (see Figures 1–7). Then, by Lemmas 13 and 16, it is enough to prove that $\gamma = \sigma_{\alpha_1} \cdots \sigma_{\alpha_n}$ is conjugate to $\tau[C_1] \cdots \tau[C_m]$. We prove this claim by a case-by-case argument.

- If $P = P_{A_n,\varpi_r}$, then $\sigma_{\alpha_i} = \tau[C_i]$ for $1 \leqslant i \leqslant n$ and $\gamma = \tau[C_1] \cdots \tau[C_m]$.
- If $P = P_{B_n,\varpi_n}$, then $\sigma_{\alpha_i} = \tau[C_{n+1-i}]$ for $1 \leqslant i \leqslant n$, and γ is conjugate to $\sigma_{\alpha_n} \cdots \sigma_{\alpha_1} = \tau[C_1] \cdots \tau[C_m]$ by Lemma 13.
- If $P = P_{C_n,\varpi_1}$, then $\sigma_{\alpha_i} = \tau[C_i]$ for $1 \leqslant i \leqslant n$ and $\gamma = \tau[C_1] \cdots \tau[C_n]$.
- If $P = P_{D_n,\varpi_1}$, then $\tau[C_i] = \sigma_{\alpha_i}$ for $i \neq n-3$ and $\tau[C_{n-3}] = \sigma_{\alpha_{n-3}} \sigma_{\alpha_n}$. Hence $\tau[C_1] \cdots \tau[C_{n-1}] = \sigma_{\alpha_1} \cdots \sigma_{\alpha_{n-4}} \sigma_{\alpha_{n-3}} \sigma_{\alpha_n} \sigma_{\alpha_{n-2}} \sigma_{\alpha_{n-1}}$ is conjugate to γ by Lemma 13.
- If $P = P_{D_n,\varpi_n}$, then $\tau[C_1] = \sigma_{\alpha_{n-1}} \sigma_{\alpha_n}$ and $\tau[C_i] = \sigma_{\alpha_{n-i}}$ for $2 \leqslant i \leqslant n-1$. Hence $\tau[C_1] \cdots \tau[C_{n-1}] = \sigma_{\alpha_{n-1}} \sigma_{\alpha_n} \sigma_{\alpha_{n-2}} \cdots \sigma_{\alpha_1}$ is conjugate to γ by Lemma 13.

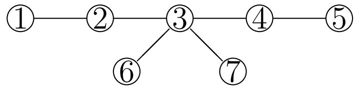

Figure 8: Non-commutativity graph for P_{E_6,ϖ_6}

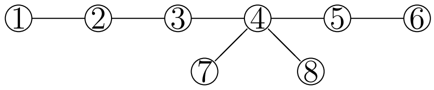

Figure 9: Non-commutativity graph for P_{E_7,ϖ_7}

- If $P = P_{E_6,\varpi_6}$, then we have

$$C_1 = P^{\alpha_1}, \quad C_2 = P^{\alpha_3} \sqcup (C_2 \cap P^{\alpha_2}), \quad C_3 = P^{\alpha_4},$$
$$C_4 = P^{\alpha_5} \sqcup (C_4 \cap P^{\alpha_2}), \quad C_5 = P^{\alpha_6}.$$

If we put

$$g_1 = \tau[P^{\alpha_1}], \quad g_2 = \tau[P^{\alpha_3}] \quad g_3 = \tau[P^{\alpha_4}], \quad g_4 = \tau[P^{\alpha_5}], \quad g_5 = \tau[P^{\alpha_6}],$$
$$g_6 = \tau[C_2 \cap P^{\alpha_2}], \quad g_7 = \tau[C_5 \cap P^{\alpha_2}],$$

then Figure 8 shows the non-commutativity graph $\Gamma(g_1, \ldots, g_7)$. Hence by applying Lemma 13, we see that

$$\tau[C_1] \cdots \tau[C_5] = \tau[P^{\alpha_1}]\tau[P^{\alpha_3}]\tau[C_2 \cap P^{\alpha_2}]\tau[P^{\alpha_4}]\tau[P^{\alpha_5}]\tau[C_5 \cap P^{\alpha_2}]\tau[P^{\alpha_6}]$$

is conjugate to

$$\gamma = \tau[P^{\alpha_1}]\tau[C_2 \cap P^{\alpha_2}]\tau[C_5 \cap P^{\alpha_2}]\tau[P^{\alpha_3}]\tau[P^{\alpha_4}]\tau[P^{\alpha_5}]\tau[P^{\alpha_6}].$$

- If $P = P_{E_7,\varpi_7}$, then we have

$$C_1 = P^{\alpha_7}, \quad C_2 = P^{\alpha_6}, \quad C_3 = (C_3 \cap P^{\alpha_2}) \sqcup P^{\alpha_5},$$
$$C_4 = P^{\alpha_4}, \quad C_5 = (C_5 \cap P^{\alpha_2}) \sqcup P^{\alpha_3}, \quad C_6 = P^{\alpha_1},$$

and $P^{\alpha_2} = (C_3 \cap P^{\alpha_2}) \sqcup (C_5 \cap P^{\alpha_2})$. If we put

$$g_1 = \tau[P^{\alpha_7}], \quad g_2 = \tau[P^{\alpha_6}], \quad g_3 = \tau[P^{\alpha_5}],$$
$$g_4 = \tau[P^{\alpha_4}], \quad g_5 = \tau[P^{\alpha_3}], \quad g_6 = \tau[P^{\alpha_1}],$$
$$g_7 = \tau[C_3 \cap P^{\alpha_2}], \quad g_8 = \tau[C_5 \cap P^{\alpha_2}]$$

then Figure 9 shows the non-commutativity graph $\Gamma(g_1, \ldots, g_8)$. Hence by applying Lemma 13, we see that

$$\tau[C_1] \cdots \tau[C_6] = \tau[P^{\alpha_7}]\tau[P^{\alpha_6}]\tau[C_3 \cap P^{\alpha_2}]\tau[P^{\alpha_5}]\tau[P^{\alpha_4}]\tau[C_5 \cap P^{\alpha_2}]\tau[P^{\alpha_3}]\tau[P^{\alpha_1}]$$

is conjugate to

$$\gamma = \tau[P^{\alpha_1}]\tau[C_3 \cap P^{\alpha_2}]\tau[C_5 \cap P^{\alpha_2}]\tau[P^{\alpha_3}]\tau[P^{\alpha_4}]\tau[P^{\alpha_5}]\tau[P^{\alpha_6}]\tau[P^{\alpha_7}].$$

This completes the proof of Theorem 15, and hence of Theorem 3 (a). □

5 Reciprocity

In this section we prove the reciprocity for birational rowmotion (Theorem 2 (b)) and propose a conjectural reciprocity for a particular birational Coxeter-motion map.

The proof of the reciprocity for birational rowmotion is based on a case-by-case analysis. Let P be a minuscule poset associated to a simple Lie algebra $\mathfrak{g}$ and $\rho^{A,B}$ the birational rowmotion map. Since the claim of Theorem 2 (b) depends only on the poset structure of P, we may assume that $\mathfrak{g}$ is simply-laced. By Lemma 8, it is enough to consider the case where $A = B = 1$. For a type A minuscule poset, the reciprocity was proved by Grinberg–Roby [7, Theorem 32] and Musiker–Roby [9, Corollary 2.13]. Also, with a help of computer, we can verify the reciprocity for the minuscule posets of types E_6 and E_7 by checking $\left(\rho^{\mathrm{rk}(w)} X\right)(w) = 1/X(\iota w)$ as rational functions in the variables $\{Z(v) : v \in P\}$ given by (17). The remaining minuscule posets are the shifted staircase posets P_{D_n,ϖ_n} and the double-tailed diamond posets P_{D_n,ϖ_1}.

5.1 Shifted staircase posets

Let $P = \{(i,j) \in \mathbb{Z}^2 : 0 \leqslant i \leqslant j \leqslant r\}$ be a shifted staircase poset, and $\rho = \rho^{1,1} : \mathcal{K}^{1,1}(P) \to \mathcal{K}^{1,1}(P)$ the birational rowmotion map on P. We derive the reciprocity for P from that for the rectangle poset $\widetilde{P} = \{(i,j) \in \mathbb{Z}^2 : 0 \leqslant i, j \leqslant r\}$. We denote by $\widetilde{\rho} : \mathcal{K}^{1,1}(\widetilde{P}) \to \mathcal{K}^{1,1}(\widetilde{P})$ the birational rowmotion map on $\widetilde{P}$ with $A = B = 1$. The following lemma is a consequence of [7, Lemma 59 (c)] and Lemma 8 (with $A = 1/2$ and $B = 2$).

Lemma 17. *For $F \in \mathcal{K}^{1,1}(P)$, we define $\widetilde{F} \in \mathcal{K}^{1,1}(\widetilde{P})$ by*

$$\widetilde{F}(i,j) = \begin{cases} F(i,j) & \text{if } i \leqslant j, \\ F(j,i) & \text{if } i > j. \end{cases}$$

Then we have

$$\left(\rho^k F\right)(i,j) = \left(\widetilde{\rho}^k \widetilde{F}\right)(i,j) \times \begin{cases} 1/2 & \text{if } 1 \leqslant k \leqslant i+j, \\ 1 & \text{if } k = i+j+1, \\ 2 & \text{if } i+j+2 \leqslant k \leqslant 2r+1, \\ 1 & \text{if } k = 2r+2 \end{cases}$$

for $1 \leqslant k \leqslant 2r+2$ and $(i,j) \in P$.

By using this lemma and the reciprocity for the rectangle poset $\widetilde{P}$, we have

$$\left(\rho^{i+j+1} F\right)(i,j) = \left(\widetilde{\rho}^{i+j+1} \widetilde{F}\right)(i,j) = \frac{1}{\widetilde{F}(r-i, r-j)} = \frac{1}{F(r-j, r-i)}.$$

This is the desired identity for a shifted staircase poset.

5.2 Double-tailed diamond posets

In this subsection, we prove the reciprocity for double-tailed diamond posets. Let $P = P_{D_n,\varpi_1}$ be the minuscule poset associated to the minuscule weight $\lambda = \varpi_1$ of the Lie algebra of type D_n. We label elements of P by

$$\begin{gathered} v_i = e_1 + e_{i+1} \quad (1 \leqslant i \leqslant n-2), \\ v_{n-1}^+ = e_1 + e_n, \quad v_{n-1}^- = e_1 - e_n, \\ v_i = e_1 - e_{2n-1-i} \quad (n \leqslant i \leqslant 2n-3). \end{gathered}$$

Note that v_1 is the maximum element and v_{2n-3} is the minimum element.

Fix an initial state $X \in \mathcal{K}^{1,1}(P)$. We regard $X(v)$ $(v \in P)$ as indeterminates and define $Z \in \mathcal{K}^{1,1}(P)$ by (17). We write

$$\begin{aligned} x_i &= X(v_i) \quad (1 \leqslant i \leqslant 2n-3,\, i \neq n-1), \quad x_{n-1}^\pm = X(v_{n-1}^\pm), \\ z_i &= Z(v_i) \quad (1 \leqslant i \leqslant 2n-3,\, i \neq n-1), \quad z_{n-1}^\pm = Z(v_{n-1}^\pm). \end{aligned}$$

Then we have

$$z_i = \begin{cases} \dfrac{x_i}{x_{i+1}} & \text{if } i \neq n-1, 2n-3, \\ \dfrac{x_{n-2}}{x_{n-1}^+ + x_{n-1}^-} & \text{if } i = n-1, \\ x_{2n-3} & \text{if } i = 2n-3, \end{cases} \qquad z_{n-1}^\pm = \frac{x_{n-1}^\pm}{x_n}.$$

For positive integers i and l satisfying $1 \leqslant i \leqslant 2n-3$ and $i+l-1 \leqslant 2n-3$, we define monomials $C(i;l)$ and $C^\pm(i;l)$ as follows:

(i) If $1 \leqslant i \leqslant n-2$ and $i+l-1 \leqslant n-2$, then we put

$$C(i;l) = z_i z_{i+1} \cdots z_{i+l-1}.$$

(ii) If $1 \leqslant i \leqslant n-1$ and $n-1 \leqslant i+l-1 \leqslant 2n-3$, then we put

$$C^\pm(i;l) = z_i z_{i+1} \cdots z_{n-2} z_{n-1}^\pm z_n \cdots z_{i+l-1}.$$

(iii) If $n+2 \leqslant i \leqslant 2n-3$, then we put

$$C(i;l) = z_i z_{i+1} \cdots z_{i+l-1}.$$

Then the original indeterminates $X(v)$ can be expressed in terms of $Z(v)$ as follows:

Lemma 18. *The values $X(v)$ $(v \in P)$ are expressed in terms of $C(i;l)$ and $C^\pm(i;l)$ as follows:*

$$\begin{cases} X(v_i) = C^+(i; 2n-i-2) + C^-(i; 2n-i-2) & \text{if } 1 \leqslant i \leqslant n-2, \\ X(v_{n-1}^\pm) = C^\pm(n-1; n-1) & \text{if } i = n-1, \\ X(v_i) = C(i; 2n-i-2) & \text{if } n \leqslant i \leqslant 2n-3. \end{cases}$$

Recall that P is a graded poset with rank function rk given by $\mathrm{rk}(v_i) = 2n - i - 2$ $(1 \leqslant i \leqslant 2n-3,\ i \neq n-1)$ and $\mathrm{rk}(v^{\pm}_{n-1}) = n-1$. Then it is straightforward to prove the following explicit formulas by using induction on k and i. (We omit the proof.)

Proposition 19. *Let $v \in P$ and k a positive integer. If $1 \leqslant k \leqslant \mathrm{rk}(v)$, then the value $\left(\rho^k X\right)(v)$ of iterations of birational rowmotion is expressed in terms of $C(i;l)$ and $C^{\pm}(i;l)$ as follows:*

(a) If $v = v_i$ with $1 \leqslant i \leqslant n-2$, we have

$$\left(\rho^k X\right)(v_i) = \begin{cases} \dfrac{1}{C(k;i)} & \text{if } 1 \leqslant k \leqslant n-i-1, \\ \dfrac{1}{C^+(k;i)} + \dfrac{1}{C^-(k+1;i)} & \text{if } n-i \leqslant k \leqslant n-1, \\ \dfrac{1}{C(k;i)} & \text{if } n \leqslant k \leqslant 2n-i-2. \end{cases}$$

(b) If $v = v^{\pm}_{n-1}$, we have

$$\left(\rho^k X\right)(v^{\pm}_{n-1}) = \frac{1}{C^{\varepsilon(-1)^{k-1}}(k;n-1)}.$$

(c) If $v = v_i$ with $n \leqslant i \leqslant 2n-3$, we have

$$\left(\rho^k X\right)(v_i) = \frac{1}{C^+(k;i) + C^-(k;i)}.$$

Since the involution $\iota : P \to P$ is given by

$$\iota(v_i) = v_{2n-i-2} \quad (1 \leqslant i \leqslant 2n-3,\ i \neq n-1), \quad \iota(v^{\varepsilon}_{n-1}) = v^{\varepsilon(-1)^n}_{n-1},$$

we obtain the desired reciprocity by comparing formulas in Lemma 18 and Proposition 19. This completes the proof of Theorem 2 (b) for all minuscule posets.

5.3 Reciprocity for birational Coxeter-motion

We have the following conjectural reciprocity for a particular birational Coxeter-motion map.

Conjecture 20. Let P be a minuscule poset. We decompose the simple root system Π into a disjoint union of two subsets Π_1 and Π_2 such that any roots in Δ_i are pairwise orthogonal for each i. We define γ_1 and γ_2 by

$$\gamma_1 = \prod_{\alpha \in \Pi_1} \sigma^{A,B}_{\alpha}, \quad \gamma_2 = \prod_{\beta \in \Pi_2} \sigma^{A,B}_{\beta},$$

and put

$$\delta = \underbrace{\gamma_1 \gamma_2 \gamma_1 \gamma_2 \gamma_1 \cdots}_{h \text{ factors}},$$

where h is the Coxeter number. Then we conjecture that

$$(\delta F)(v) = \frac{AB}{F(\iota v)} \tag{20}$$

for any $F \in \mathcal{K}^{A,B}(P)$ and $v \in P$.

The periodicity of birational Coxeter-motion maps is a consequence of this conjecture. In fact, $\gamma = \gamma_1 \gamma_2$ is a Coxeter-motion map and

$$\gamma^h = \begin{cases} \delta^2 & \text{if } h \text{ is even}, \\ \delta_{1,2}\delta_{2,1} & \text{if } n \text{ is odd}, \end{cases}$$

where $\delta_{1,2} = \gamma_1 \gamma_2 \gamma_1 \gamma_2 \gamma_1 \cdots \gamma_1$ and $\delta_{2,1} = \gamma_2 \gamma_1 \gamma_1 \gamma_2 \gamma_1 \cdots \gamma_2$. If h is even, then we have

$$\left(\gamma^h F\right)(v) = \left(\delta^2 F\right)(v) = \frac{AB}{(\delta F)(\iota v)} = \frac{AB}{AB/F(\iota^2 v)} = F(v).$$

If h is odd, we can derive $\left(\gamma^h F\right)(v) = F(v)$ from (20) in a similar manner.

6 File homomesy

This section is devoted to the proof of the file homomesy phenomenon (Theorem 2 (c) and Theorem 3 (b)).

6.1 Local properties

First we investigate local properties of birational rowmotion and Coxeter-motion around a given file.

Let P be a minuscule poset with coloring $c : P \to \Pi$. We regard the Hasse diagram of the poset $\widehat{P} = P \sqcup \{\widehat{1}, \widehat{0}\}$ as a directed graph, where a directed edge $u \to v$ corresponds to the covering relation $u \lessdot v$. For $\alpha \in \Pi$, let $\widehat{N}^\alpha$ be the *neighborhood* of $P^\alpha = \{x \in P : c(x) = \alpha\}$ given by

$$\widehat{N}^\alpha = \{x \in \widehat{P} : \text{there is an element } y \in P^\alpha \text{ such that } x \lessdot y \text{ or } x \gtrdot y\}.$$

We define G^α to be the bipartite directed subgraph of the Hasse diagram of $\widehat{P}$ with black vertex set P^α and white vertex set $\widehat{N}^\alpha$. It follows from Proposition 12 (c) that

$$\widehat{N}^\alpha = \bigsqcup_{\beta \sim \alpha} P^\beta \sqcup \begin{cases} \{\widehat{1}, \widehat{0}\} & \text{if } \alpha = \alpha_{\max} = \alpha_{\min}, \\ \{\widehat{1}\} & \text{if } \alpha = \alpha_{\max} \neq \alpha_{\min}, \\ \{\widehat{0}\} & \text{if } \alpha = \alpha_{\min} \neq \alpha_{\max}, \\ \emptyset & \text{otherwise}, \end{cases}$$

where β runs over all simple roots adjacent to α in the Dynkin diagram, and $\alpha_{\max}$ (resp. $\alpha_{\min}$) is the color of the maximum (resp. minimum) element of P.

To describe the graph structure of G^α, we introduce two sequences of posets G_m and H_m. For a positive integer m, let G_m be the poset consisting of $3m$ elements $x_1, \cdots, x_m, y_1, \cdots, y_{m-1}, z_1, \cdots, z_{m-1}, u, v$ with covering relations

$$u \lessdot x_1, \quad x_i \lessdot y_i \lessdot x_{i+1}, \quad x_i \lessdot z_i \lessdot x_{i+1}, \quad x_m \lessdot v.$$

Note that G_1 is the three-element chain. And, for an integer $m \geqslant 2$, let H_m be the $(2m+1)$-element chain

$$u \lessdot x_1 \lessdot y_1 \lessdot x_2 \lessdot y_2 \lessdot \cdots \lessdot y_{m-1} \lessdot x_m \lessdot v.$$

We regard the Hasse diagrams of G_m and H_m as bipartite directed graphs with black vertices $x_1, \dots, x_m$. For example, the Hasse diagrams of G_4 and H_4 are shown in Figures 10 and 11 respectively.

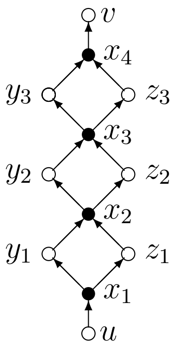

Figure 10: G_4

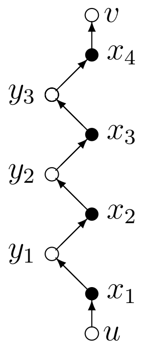

Figure 11: H_4

Lemma 21. *Each bipartite directed graph G^α is decomposed into a disjoint union of graphs of the form G_m or H_m as follows:*

- *If $P = P_{A_n, \varpi_r}$, then*
$$G^{\alpha_i} \cong \begin{cases} G_i & \text{if } 1 \leqslant i \leqslant r, \\ G_r & \text{if } r \leqslant i \leqslant s, \\ G_{n-i+1} & \text{if } s \leqslant i \leqslant l, \end{cases}$$
where $r + s = n + 1$ and $r \leqslant s$.

- *If $P = P_{B_n, \varpi_n}$, then*
$$G^{\alpha_i} \cong \begin{cases} G_i & \text{if } 1 \leqslant i \leqslant n-1, \\ H_l & \text{if } i = n. \end{cases}$$

- *If* $P = P_{C_n,\varpi_1}$, *then*

$$G^{\alpha_i} \cong \begin{cases} G_1 \sqcup G_1 & \text{if } 1 \leqslant i \leqslant n-2, \\ H_2 & \text{if } i = n-1, \\ G_1 & \text{if } i = n. \end{cases}$$

- *If* $P = P_{D_n,\varpi_1}$, *then*

$$G^{\alpha_i} \cong \begin{cases} G_1 \sqcup G_1 & \text{if } 1 \leqslant i \leqslant n-3, \\ G_2 & \text{if } i = n-2, \\ G_1 & \text{if } i = n-1,\ n. \end{cases}$$

- *If* $P = P_{D_n,\varpi_n}$, *then*

$$G^{\alpha_i} \cong \begin{cases} G_l & \text{if } 1 \leqslant i \leqslant n-2, \\ (G_1)^{\sqcup \lfloor (n-1)/2 \rfloor} & \text{if } i = n-1, \\ (G_1)^{\sqcup \lfloor n/2 \rfloor} & \text{if } i = n, \end{cases}$$

  where $G_1^{\sqcup m}$ *is the disjoint union of* m *copies of* G_1, *and* $\lfloor x \rfloor$ *stands for the largest integer not exceeding* x.

- *If* $P = P_{E_6,\varpi_6}$, *then*

$$G^{\alpha_i} \cong \begin{cases} G_1 \sqcup G_1 & \text{if } i = 1,\ 2,\ 6, \\ G_1 \sqcup G_2 & \text{if } i = 3,\ 5, \\ G_4 & \text{if } i = 4. \end{cases}$$

- *If* $P = P_{E_7,\varpi_7}$, *then*

$$G^{\alpha_i} \cong \begin{cases} G_1 \sqcup G_1 & \text{if } i = 1, \\ G_1 \sqcup G_1 \sqcup G_1 & \text{if } i = 2, \\ G_2 \sqcup G_2 & \text{if } i = 3, \\ G_6 & \text{if } i = 4, \\ G_1 \sqcup G_3 \sqcup G_1 & \text{if } i = 5, \\ G_1 \sqcup G_2 \sqcup G_1 & \text{if } i = 6, \\ G_1 \sqcup G_1 \sqcup G_1 & \text{if } i = 7. \end{cases}$$

The following relations are a key to the proof of the file homomesy phenomenon.

Lemma 22. *Let* $\rho = \rho^{A,B}$ *be the birational rowmotion map and* α *a simple root.*

(a) If the graph $G_m = \{x_1, \dots, x_m, y_1, \dots, y_{m-1}, z_1, \dots, z_{m-1}, u, v\}$ *appears as a connected component of* G^α, *then we have*

$$\prod_{i=1}^{m} (\rho^{i-1}F)(x_i) \cdot \prod_{i=1}^{m} (\rho^{i}F)(x_i) = F(u) \cdot \prod_{i=1}^{m-1} (\rho^{i}F)(y_i) \cdot \prod_{i=1}^{m-1} (\rho^{i}F)(z_i) \cdot (\rho^{m}F)(v), \quad (21)$$

where we use the same symbol to denote the corresponding vertices of G^α.

(b) If the graph $H_m = \{x_1, \ldots, x_m, y_1, \ldots, y_{m-1}, u, v\}$ appears as a connected component of G^α, then we have

$$\prod_{i=1}^{m}(\rho^{i-1}F)(x_i) \cdot \prod_{i=1}^{m}(\rho^{i}F)(x_i) = F(u) \cdot \prod_{i=1}^{m-1}(\rho^{i}F)(y_i)^2 \cdot (\rho^{m}F)(v), \tag{22}$$

where we use the same symbol to denote the corresponding vertices of G^α.

Proof. (a) It follows from (6) that

$$F(x_m) \cdot (\rho F)(x_m) = (\rho F)(v) \cdot (F(y_{m-1}) + F(z_{m-1})),$$
$$F(x_i) \cdot (\rho F)(x_i) = \frac{(\rho F)(y_i) \cdot (\rho F)(z_i) \cdot (F(y_{i-1}) + F(z_{i-1}))}{(\rho F)(y_i) + (\rho F)(z_i)} \quad (2 \leqslant i \leqslant m-1),$$
$$F(x_1) \cdot (\rho F)(x_1) = \frac{F(u) \cdot (\rho F)(y_1) \cdot (\rho F)(z_1)}{(\rho F)(y_1) + (\rho F)(z_1)}.$$

By replacing F with $\rho^{m-1}F$ (resp. $\rho^{i-1}F$) in the first (resp. second) equation, and then by multiplying the resulting equations together, we obtain (21).

(b) can be checked by a similar computation. □

Lemma 23. *Let α be a simple root and and $\sigma_\alpha = \prod_{v \in P^\alpha} \tau_v^{A,B}$ the product of birational toggles over P^α.*

(a) If G_m appears as a connected component of G^α, then we have

$$\prod_{i=1}^{m} F(x_i) \cdot \prod_{i=1}^{m}(\sigma_\alpha F)(x_i) = F(u) \cdot \prod_{i=1}^{m-1} F(y_i) \cdot \prod_{i=1}^{m-1} F(z_i) \cdot F(v). \tag{23}$$

(b) If H_m appears as a connected component of G^α, then we have

$$\prod_{i=1}^{m} F(x_i) \cdot \prod_{i=1}^{m}(\sigma_\alpha F)(x_i) = F(u) \cdot \prod_{i=1}^{m-1} F(y_i)^2 \cdot F(v). \tag{24}$$

Proof. (a) By the definition (4), we have

$$F(x_m) \cdot (\sigma_\alpha F)(x_m) = F(v) \cdot (F(y_{m-1}) + F(z_{m-1})),$$
$$F(x_i) \cdot (\sigma_\alpha F)(x_i) = \frac{F(y_i) \cdot F(z_i) \cdot (F(y_{i-1}) + F(z_{i-1}))}{F(y_i) + F(z_i)} \quad (2 \leqslant i \leqslant m-1),$$
$$F(x_1) \cdot (\sigma_\alpha F)(x_1) = \frac{F(y_1) \cdot F(z_1) \cdot F(u)}{F(y_1) + F(z_1)}.$$

Multiplying them together, we obtain (23).

(b) can be checked by a similar computation. □

6.2 File homomesy for birational rowmotion

In this subsection, we prove the file homomesy phenomenon for birational rowmotion (Theorem 2 (c)).

The following properties of Coxeter elements will be useful in the proof of Theorem 2 (c) and Theorem 3(b); the proof of the latter will be given in the next subsection. A *Coxeter element* in a Weyl group $W = \langle s_\alpha : \alpha \in \Pi \rangle$ is a product of all simple reflections s_α in any order. Then it is known that all Coxeter elements are conjugate. By definition, the Coxeter number is the order of any Coxeter element.

Lemma 24. *Let c be a Coxeter element and h the Coxeter number. Then we have*

(a) If $\mu \in \mathfrak{h}^*$ satisfies $c\mu = \mu$, then $\mu = 0$.

(b) As a linear transformation on $\mathfrak{h}^*$, we have

$$\sum_{k=0}^{h-1} c^k = 0 \tag{25}$$

(c) Let $\alpha \in \Pi$ be a simple root and ϖ the corresponding fundamental weight. If $c = s_{\alpha_1} \cdots s_{\alpha_n}$ is a Coxeter element with $\Pi = \{\alpha_1, \dots, \alpha_n\}$ and $\beta = s_{\alpha_1} \cdots s_{\alpha_{k-1}} \alpha_k$, where $\alpha = \alpha_k$, then we have

$$c\varpi = \varpi - \beta, \tag{26}$$

$$\sum_{k=1}^{h-1} \sum_{i=0}^{k-1} c^i(\beta) = h\varpi. \tag{27}$$

Proof. (a) See [2, V, §6, n°2].

(b) follows from $c^h = 1$ and (a).

(c) Since $s_\gamma \varpi = \varpi - \langle \gamma^\vee, \varpi \rangle \gamma = \varpi - \delta_{\alpha,\gamma} \alpha$ for $\gamma \in \Pi$, we have $c\varpi = \varpi - s_{\alpha_1} \cdots s_{\alpha_{k-1}} \alpha_k = \varpi - \beta$. Hence we see that

$$c^k \varpi = \varpi - \sum_{i=0}^{k-1} c^i \beta.$$

By using (25), we obtain

$$0 = \sum_{k=0}^{h-1} c^k \varpi = h\varpi - \sum_{k=1}^{h-1} \sum_{i=0}^{k-1} c^i \beta,$$

from which (27) follows. ◻

In order to prove Theorem 2 (c), we consider

$$\Phi'_\alpha(F) = \prod_{v \in P^\alpha} \left(\rho^{(\mathrm{rk}(v) - \mathrm{rk}(v_0^\alpha))/2} F\right)(v), \tag{28}$$

instead of $\Phi_\alpha(F) = \prod_{v \in P^\alpha} F(v)$. Here v_0^α is the minimum element of P^α. Note that P^α is a chain and $\mathrm{rk}(v) - \mathrm{rk}(v_0^\alpha)$ is an even integer (see Proposition 12 (d) and (e)). Since ρ has finite order h, we have

$$\prod_{k=0}^{h-1} \Phi_\alpha(\rho^k F) = \prod_{k=0}^{h-1} \Phi'_\alpha(\rho^k F). \tag{29}$$

Remark 25. It is worth mentioning that $\Phi'_\alpha(\rho^k X)$ are Laurent monomials in the variables $Z(v)$ defined by (17). In a forthcoming paper [10], we will give explicit formulas for $\Phi'_\alpha(\rho^k X)$ in classical types.

Remark 26. For a type A minuscule poset $P \cong [0,r] \times [0, n-r]$, we can express $\Phi'_\alpha(F)$ in terms of Einstein–Propp's recombination map $\mathfrak{R} : \mathcal{K}^{A,B}(P) \to \mathcal{K}^{A,B}(P)$ defined by

$$(\mathfrak{R}F)(i,j) = \left(\rho^j F\right)(i,j).$$

(See [4, Section 6].) For a simple root $\alpha_k = e_k - e_{k+1}$, we have

$$\Phi'_{\alpha_k}(F) = \begin{cases} \Phi_{\alpha_k}(\mathfrak{R}F) & \text{if } 1 \leqslant k \leqslant r, \\ \Phi_{\alpha_k}(\mathfrak{R}\rho^{r-k}F) & \text{if } r \leqslant k \leqslant n. \end{cases}$$

Proposition 27. *For $\alpha \in \Pi$ and $F \in \mathcal{K}^{A,B}(P)$, we have*

$$\Phi'_\alpha(F) \cdot \Phi'_\alpha(\rho F) = A^{\delta_{\alpha,\alpha_{\max}}} B^{\delta_{\alpha,\alpha_{\min}}} \prod_{\beta \sim \alpha} \Phi'_\beta(\rho^{m_{\alpha,\beta}} F)^{-\langle \alpha^\vee, \beta \rangle}, \tag{30}$$

where β runs over all simple roots adjacent to α in the Dynkin diagram and

$$m_{\alpha,\beta} = \begin{cases} 1 & \text{if } v_0^\beta > v_0^\alpha, \\ 0 & \text{if } v_0^\beta < v_0^\alpha. \end{cases}$$

Proof. We explain the proof in the case where $\mathfrak{g}$ is of type E_7, $\lambda = \varpi_7$ and $\alpha = \alpha_5$. (The other cases can be proved in a similar way.) We label elements of P^α as $v_0^\alpha, v_1^\alpha, v_2^\alpha, \dots$ from bottom to top. By definition (28), we have

$$\begin{aligned}
\Phi'_{\alpha_4}(F) &= F(v_0^{\alpha_4}) \cdot (\rho F)(v_1^{\alpha_4}) \cdot (\rho^2 F)(v_2^{\alpha_4}) \cdot (\rho^3 F)(v_3^{\alpha_4}) \cdot (\rho^4 F)(v_4^{\alpha_4}) \cdot (\rho^5 F)(v_5^{\alpha_4}), \\
\Phi'_{\alpha_5}(F) &= F(v_0^{\alpha_5}) \cdot (\rho^2 F)(v_1^{\alpha_5}) \cdot (\rho^3 F)(v_2^{\alpha_5}) \cdot (\rho^4 F)(v_3^{\alpha_5}) \cdot (\rho^6 F)(v_4^{\alpha_5}), \\
\Phi'_{\alpha_6}(F) &= F(v_0^{\alpha_6}) \cdot (\rho^3 F)(v_1^{\alpha_6}) \cdot (\rho^4 F)(v_2^{\alpha_6}) \cdot (\rho^7 F)(v_3^{\alpha_6}).
\end{aligned}$$

The subgraph G^{α_5} has three connected components

$$\begin{gathered}\{v_0^{\alpha_4}, v_0^{\alpha_5}, v_0^{\alpha_6}\} \cong G_2, \\ \{v_1^{\alpha_4}, v_2^{\alpha_4}, v_3^{\alpha_4}, v_4^{\alpha_4}, v_1^{\alpha_5}, v_2^{\alpha_5}, v_3^{\alpha_5}, v_1^{\alpha_6}, v_2^{\alpha_6}\} \cong G_3, \\ \{v_5^{\alpha_4}, v_4^{\alpha_5}, v_3^{\alpha_6}\} \cong G_1.\end{gathered}$$

By applying (21) to each of the three connected components of G^{α_5}, we obtain

$$\begin{aligned}&F(v_0^{\alpha_5}) \cdot (\rho F)(v_0^{\alpha_5}) = F(v_0^{\alpha_6}) \cdot (\rho F)(v_0^{\alpha_4}), \\ &F(v_1^{\alpha_5}) \cdot (\rho F)(v_2^{\alpha_5}) \cdot (\rho^2 F)(v_3^{\alpha_5}) \cdot (\rho F)(v_1^{\alpha_5}) \cdot (\rho^2 F)(v_2^{\alpha_5}) \cdot (\rho^3 F)(v_3^{\alpha_5}) \\ &\quad = F(v_1^{\alpha_4}) \cdot (\rho F)(v_1^{\alpha_6}) \cdot (\rho F)(v_2^{\alpha_4}) \cdot (\rho^2 F)(v_2^{\alpha_6}) \cdot (\rho^2 F)(v_3^{\alpha_4}) \cdot (\rho^3 F)(v_4^{\alpha_4}) \\ &F(v_4^{\alpha_5}) \cdot (\rho F)(v_4^{\alpha_5}) = F(v_5^{\alpha_4}) \cdot (\rho F)(v_3^{\alpha_6}).\end{aligned}$$

By replacing F with $\rho^2 F$ (resp. $\rho^6 F$) in the second (resp. third) equation, and then by multiplying the three resulting equations together, we have

$$\Phi'_{\alpha_5}(F) \cdot \Phi'_{\alpha_5}(\rho F) = \Phi'_{\alpha_6}(F) \cdot \Phi'_{\alpha_4}(\rho F).$$

Since $v_0^{\alpha_6} < v_0^{\alpha_5} < v_0^{\alpha_4}$ (see Figure 7), we obtain (30) in this case. □

Corollary 28. *For a simple root $\beta \in \Pi$, we put*

$$\widetilde{\Phi}_\beta(F) = \prod_{k=0}^{h-1} \Phi_\beta(\rho^k F).$$

Then we have for fixed $\alpha \in \Pi$,

$$\prod_{\beta \in \Pi} \widetilde{\Phi}_\beta(F)^{\langle \alpha^\vee, \beta \rangle} = A^{\delta_{h\alpha,\alpha_{\max}}} B^{h\delta_{\alpha,\alpha_{\min}}} \tag{31}$$

for any $F \in \mathcal{K}^{A,B}(P)$.

Proof. Since ρ has finite order h, Equation (29) implies $\widetilde{\Phi}_\beta(F) = \prod_{k=0}^{h-1} \Phi'_\beta(\rho^{k+m} F)$ for any integer m. Hence (31) follows from (30). □

Now we are ready to prove Theorem 2 (c).

Proof of Theorem 2 (c). We define an element $\widetilde{\mu}(F) \in \mathfrak{h}^*$ for $F \in \mathcal{K}^{A,B}(P)$ by putting

$$\widetilde{\mu}(F) = \sum_{\alpha \in \Pi} \log \widetilde{\Phi}_\alpha(F) \cdot \alpha.$$

Note that, if $\varpi^\vee$ is the fundamental coweight corresponding to α, then we have

$$\log \widetilde{\Phi}_\alpha(F) = \langle \varpi^\vee, \widetilde{\mu}(F) \rangle.$$

Since $\varpi_{\max} = -w_0\lambda$ (resp. $\varpi_{\min} = \lambda$) is the fundamental weight corresponding to the color $\alpha_{\max}$ (resp. $\alpha_{\min}$) of the maximum (resp. minimum) element of P (see Proposition 12 (b)), it it enough to show

$$\widetilde{\mu}(F) = ha \cdot \varpi_{\max} + hb \cdot \varpi_{\min}, \tag{32}$$

where $a = \log A$, $b = \log B$.

Since we have

$$\sum_{\beta \in \Pi} \langle \alpha^\vee, \beta \rangle \log \widetilde{\Phi}_\beta(F) = ha\delta_{\alpha,\alpha_{\max}} + hb\delta_{\alpha,\alpha_{\max}}$$

by Corollary 28, we see that for any $\alpha \in \Pi$

$$\begin{aligned} s_\alpha \widetilde{\mu}(F) &= \sum_{\beta \in \Pi} \log \widetilde{\Phi}_\beta(F) \cdot (\beta - \langle \alpha^\vee, \beta \rangle \alpha) \\ &= \sum_{\beta \in \Pi} \log \widetilde{\Phi}_\beta(F) \beta - \left(\sum_{\beta \in \Pi} \langle \alpha^\vee, \beta \rangle \log \widetilde{\Phi}_\beta(F) \right) \alpha \\ &= \widetilde{\mu}(F) - (\delta_{\alpha,\alpha_{\max}} ha + \delta_{\alpha,\alpha_{\min}} hb)\, \alpha. \end{aligned}$$

Let $c = s_{\alpha_1} \cdots s_{\alpha_n}$ be a Coxeter element and put

$$\beta_{\max} = s_{\alpha_1} \cdots s_{\alpha_{k-1}} \alpha_k, \quad \beta_{\min} = s_{\alpha_1} \cdots s_{\alpha_{m-1}} \alpha_m,$$

where $\alpha_k = \alpha_{\max}$, $\alpha_m = \alpha_{\min}$. Then we have

$$c\widetilde{\mu}(F) = \widetilde{\mu}(F) - (ha \cdot \beta_{\max} + hb \cdot \beta_{\min}).$$

By substituting $\beta_{\max} = \varpi_{\max} - c\varpi_{\max}$ and $\beta_{\min} = \varpi_{\min} - c\varpi_{\min}$ (see (26)), we have

$$c\,(\widetilde{\mu}(F) - ha \cdot \varpi_{\max} - hb \cdot \varpi_{\max}) = \widetilde{\mu}(F) - ha \cdot \varpi_{\max} - hb \cdot \varpi_{\max}.$$

Then it follows from Lemma 24 (a) that

$$\widetilde{\mu}(F) - ha \cdot \varpi_{\max} - hb \cdot \varpi_{\max} = 0.$$

This completes the proof of (32) and hence of Theorem 2 (c). □

6.3 File homomesy for birational Coxeter-motion

In this subsection we prove Theorem 3 (b). The following proposition is a consequence of Lemma 21 and Equations (23), (24).

Proposition 29. *Let $\sigma_\alpha = \prod_{v \in P^\alpha} \tau_v : \mathcal{K}^{A,B}(P) \to \mathcal{K}^{A,B}(P)$ be the product of toggles over P^α. Then*

(a) For a simple root α, we have

$$\Phi_\alpha(F) \cdot \Phi_\alpha(\sigma_\alpha F) = A^{\delta_{\alpha,\alpha_{\max}}} B^{\delta_{\alpha,\alpha_{\min}}} \prod_{\beta \sim \alpha} \Phi_\beta(F)^{-\langle \alpha^\vee, \beta \rangle}.$$

(b) For simple roots $\alpha \neq \beta$, we have $\Phi_\beta(\sigma_\alpha F) = \Phi_\beta(F)$.

By using this proposition, we can complete the proof of the file homomesy phenomenon for birational Coxeter-motion.

Proof of Theorem 3 (b). We define an element $\mu(F) \in \mathfrak{h}^*$ for $F \in \mathcal{K}^{A,B}(P)$ by putting

$$\mu(F) = \sum_{\beta \in \Pi} \log \Phi_\beta(F) \cdot \beta.$$

First we prove

$$\mu(\sigma_\alpha F) = s_\alpha \mu(F) + \left(\delta_{\alpha,\alpha_{\max}} a + \delta_{\alpha,\alpha_{\min}} b\right) \alpha \tag{33}$$

where $a = \log A$ and $b = \log B$. By using Proposition 29, we have

$$\begin{aligned}
\mu(\sigma_\alpha F) &= \sum_{\beta \neq \alpha} \log \Phi_\beta(\sigma_\alpha F)\beta + \log \Phi_\alpha(\sigma_\alpha F)\alpha \\
&= \sum_{\beta \neq \alpha} \log \Phi_\beta(F)\beta + \left(\delta_{\alpha,\alpha_{\max}} a + \delta_{\alpha,\alpha_{\min}} b - \sum_{\beta \neq \alpha} \langle \alpha^\vee, \beta \rangle \log \Phi_\beta(F) - \log \Phi_\alpha(F) \right) \alpha \\
&= \sum_{\beta \neq \alpha} \log \Phi_\beta(F)(\beta - \langle \alpha^\vee, \beta \rangle \alpha) - \log \Phi_\alpha(F)\alpha + \left(\delta_{\alpha,\alpha_{\max}} a + \delta_{\alpha,\alpha_{\min}} b\right) \alpha \\
&= \sum_{\beta \neq \alpha} \log \Phi_\beta(F) s_\alpha(\beta) + \log \Phi_\alpha(F) s_\alpha(\alpha) + \left(\delta_{\alpha,\alpha_{\max}} a + \delta_{\alpha,\alpha_{\min}} b\right) \alpha \\
&= s_\alpha(\mu(F)) + \left(\delta_{\alpha,\alpha_{\max}} a + \delta_{\alpha,\alpha_{\min}} b\right) \alpha.
\end{aligned}$$

Suppose that $\gamma = \sigma_{\alpha_1} \cdots \sigma_{\alpha_n}$, and let $c = s_{\alpha_1} \cdots s_{\alpha_n}$ be the corresponding Coxeter element. Then, by iteratively using (33), we obtain

$$\mu(\gamma F) = c(\mu(F)) + a \cdot \beta_{\max} + b \cdot \beta_{\min},$$

where $\beta_{\max}$ and $\beta_{\min}$ are defined by $\beta_{\max} = s_{\alpha_1} \cdots s_{\alpha_{k-1}} \alpha_k$, $\beta_{\min} = s_{\alpha_1} \cdots s_{\alpha_{m-1}} \alpha_m$ with $\alpha_k = \alpha_{\max}$ and $\alpha_m = \alpha_{\min}$. Hence by induction on k we see that

$$\mu(\gamma^k F) = c^k(\mu(F)) + a \sum_{i=0}^{k-1} c^i(\beta_{\max}) + b \sum_{i=0}^{k-1} c^i(\beta_{\min}).$$

Therefore we have

$$\sum_{k=0}^{h-1} \mu(\gamma^k F) = \sum_{k=0}^{h-1} c^k(\mu(F)) + a \sum_{k=1}^{h-1} \sum_{i=0}^{k-1} c^i(\beta_{\max}) + b \sum_{k=1}^{h-1} \sum_{i=0}^{k-1} c^i(\beta_{\min}).$$

Now it follows from (25) and (27) that

$$\sum_{k=0}^{h-1} \mu(\gamma^k F) = ah \cdot \varpi_{\max} + bh \cdot \varpi_{\min}.$$

By the definition of $\mu(F)$, we have

$$\sum_{\beta \in \Pi} \log \left(\prod_{k=0}^{h-1} \Phi_\beta(\gamma^k F) \right) \cdot \beta = ah \cdot \varpi_{\max} + bh \cdot \varpi_{\min}.$$

Then we can complete the proof by taking the pairing $\langle \varpi^\vee, \quad \rangle$. □

Acknowledgements

The author is grateful to Tom Roby for fruitful discussions.

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